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On the Existence of an Optimal Control of Ship Automatic Steering Instruments

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Abstract The existence of linear quadratic optimal control of ship automatic steering instruments is studied. Firstly, the sufficient conditions for the quadratic integrability of the solutions of linear second order time-variant differential equations are developed. Secondly, the optimal control form of the ship automatic steering instrument is obtained by using the dynamic programming method, which guarantees a minimal ship sway range, during long-distance navigation, by using as little energy as possible. Finally, based on the above mentioned sufficient conditions, the conditions for the realization of optimal control are obtained, which provides a foundation for choosing the weighted coefficients for optimal control in engineering.

Key Words ship automatic steering instrument; optimal control; quadratic integrability

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1 Introduction

Izosimov and Utkin (1981) studied the following model of ship automatic steering instruments by using the method of variable structure control

$$Jx'' + Dx' = -T \tag{1}$$

where x denotes the angle by which the ship deviates from the specified navigation course, x' the first order derivative of x with respect to the time variable t, x'' the second order derivative of x with respect to the time variable t, J the moment of inertia of the ship, D the damping coefficient, and T the moment of force generated by the rudder. This paper is to generalize the above model, to work out the optimal state feedback control by using the dynamic programming method and to give sufficient conditions to guarantee that a closed-loop system functions normally.

Suppose that the moment of inertia $J(t)$, damping coefficient $D(t)$, and the moment of force $T(t)$ are functions of the time variable t. Let

$$p(t) = \frac{D(t)}{J(t)}, \quad u(t) = \frac{T(t)}{J(t)}. \tag{2}$$

Then we have

$$x'' + p(t)x' = -u(t).$$

Consider the optimal control problem of the following system

$$x'' + p(t)x' = -u(t),$$
$$x(0) = x_0, \quad x'(0) = x_1, \quad x_0^2 + x_1^2 \neq 0. \tag{3}$$

The index function is

$$J(u(t)) = \int_0^\infty (2q^2(t)x^2(t) + \frac{1}{2}u^2(t))\mathrm{d}t, \tag{4}$$

where $q(t)$ is bounded on $[0, +\infty)$, and $u(t) \in L^2[0, +\infty)$. Suppose that $u(t)$ is unrestricted, we try to seek for an optimal state feedback control $u^*(t) \in L^2[0, +\infty)$ so that

$$J(u^*(t)) = \min_{u(\cdot)\in L^2[0,+\infty)} J(u(t)). \tag{5}$$

The physical meaning of problem (3)-(5) is that during a long range navigation the sway range of the ship is minimized by using as little energy as possible. In addition, this problem is also of practical meaning for automatic navigation instruments on cosmos detectors.

2 Lemmas and Theorems

From Ouyang (1988) and Zheng (1986) we can see that in the problem of linear quadratic optimal control, we usually require that the solutions of a closed loop system are quadratic integrable. Ouyang (1983) had studied the quadratic integrability of solutions of second order differential equations and got good re-

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sults.

In the following, we shall give sufficient conditions for quadratic integrability of a class of differential equation.

Lemma 1 Suppose that

$$x''(t) + p(t)x' + q(t)x = 0,$$
$$x(0) = x_0,\ x'(0) = x_1,\ x_0^2 + x_1^2 \neq 0, \quad (6)$$

where $q(t)$ and $p(t)$ satisfy

(i) $p(t) \in C[0, +\infty)$, $p(t) \geqslant 0$; $q(t) > 0$, $q'(t) > 0$, $q(t) \in C^1[0, +\infty)$; $q'(t) = O(q(t))$, $t \to +\infty$.

(ii) $\int_0^\infty \frac{\mathrm{d}t}{\sqrt{q(t)}} < \infty$,

$\int_0^\infty \sqrt{q(t)} \left| \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{q'(t)}{q(t)^{3/2}}\right) \right| \mathrm{d}t < \infty$,

$\int_0^\infty \frac{p(t)}{\sqrt{q(t)}} < \infty$.

Then, the solution $x(t)$ of Eq. (6) satisfies

$$x(t) \in L^2[0, +\infty).$$

Proof Let

$$v(t) = \sqrt{q(t)}x^2 + \frac{(x')^2}{\sqrt{q(t)}},\ t \in [0, +\infty),$$

then,

$$\frac{\mathrm{d}v(t)}{\mathrm{d}t} = \frac{q'(t)}{2\sqrt{q(t)}}x^2 + 2\sqrt{q(t)}xx' + \frac{2x'x''\sqrt{q(t)} - \frac{1}{2\sqrt{q(t)}}q'(t)(x')^2}{q(t)}$$
$$= \frac{q'(t)}{2\sqrt{q(t)}}x^2 + 2\sqrt{q(t)}xx' + \frac{2x'(-p(t)x' - q(t)x)}{\sqrt{q(t)}} - \frac{q'(t)}{2q(t)^{3/2}}(x')^2$$
$$= \frac{q'(t)}{2\sqrt{q(t)}}x^2 - \frac{4p(t)q(t) + q'(t)}{2q(t)^{3/2}}(x')^2$$
$$\leqslant \frac{q'(t)}{2\sqrt{q(t)}}x^2$$
$$\leqslant \frac{q'(t)}{2q(t)}v(t), \quad \left(\text{because } x^2 \leqslant \frac{v(t)}{\sqrt{q(t)}}\right)$$

therefore

$$0 \leqslant v(t) \leqslant \frac{v(0)\sqrt{q(t)}}{\sqrt{q(0)}}.$$

However, as

$$x^2 \leqslant \frac{v(t)}{\sqrt{q(t)}},$$

we have

$$x(t) = O(1).$$

In view of

$$v'(t) \leqslant \frac{q'(t)}{2\sqrt{q(t)}}\left[x^2 - \frac{(x')^2}{q(t)}\right],$$
$$(x')^2 = (xx')' - xx''$$
$$= (xx')' - x(-p(t)x' - q(t)x)$$
$$= (xx')' + p(t)xx' + q(t)x^2,$$

hence we obtain

$$v'(t) \leqslant \frac{q'}{2\sqrt{q}}\left[-\frac{(xx')'}{q} - \frac{p}{q}xx'\right].$$

Integration of the above inequality gives

$$v(t) - v(0) \leqslant -\int_0^t \frac{q'(xx')'}{2q^{3/2}}\mathrm{d}\tau - \int_0^t \frac{pq'}{2q^{3/2}}xx'\mathrm{d}\tau.$$

Let

$$I_1 = -\int_0^t \frac{q'(xx')'}{2q^{3/2}}\mathrm{d}\tau,\ I_2 = -\int_0^t \frac{pq'}{2q^{3/2}}xx'\mathrm{d}\tau,$$

then we have

$$|I_1| = \left| \frac{q'(0)x(0)x'(0)}{2q^{3/2}(0)} - \frac{q'(t)x(t)x'(t)}{2q^{3/2}(t)} + \int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{q'(\tau)}{2q^{3/2}(\tau)}\right)\right]x(\tau)x'(\tau)\mathrm{d}\tau \right|$$
$$\leqslant \frac{q'(0)x(0)x'(0)}{2q^{3/2}(0)} + \frac{1}{4}\left|\frac{q'(t)}{q(t)}\right| \frac{v(t)}{\sqrt{q(t)}3} + \left| \int_0^t \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{q'(\tau)}{2q^{3/2}(\tau)}\right)\right] \frac{v(\tau)}{2}\mathrm{d}\tau \right|$$
$$\left(\text{because } |xx'| \leqslant \frac{v(t)}{2}\right)$$
$$\leqslant O(1) + \frac{v(0)}{4\sqrt{q(0)}}\int_0^\infty \left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{q'(\tau)}{q^{3/2}(\tau)}\right)\right] \times \sqrt{q(\tau)}\mathrm{d}\tau$$
$$= O(1),$$

$$I_2 = -\int_0^t \frac{pq'}{2q^{3/2}}xx'\mathrm{d}\tau \leqslant \int_0^t \frac{pq'}{2q^{3/2}}|xx'|\mathrm{d}\tau$$
$$\leqslant \frac{1}{4}\int_0^t \frac{p(\tau)}{\sqrt{q(\tau)}} \frac{q'(\tau)}{q(\tau)}v(\tau)\mathrm{d}\tau$$
$$= O(1)\int_0^t \frac{p}{\sqrt{q}}v\mathrm{d}\tau.$$

Therefore

$$v(t) \leqslant O(1) + O(1)\int_0^t \frac{p(\tau)}{\sqrt{q(\tau)}}v(\tau)\mathrm{d}\tau.$$

By using the Gronwall-Bellman inequality,

$$v(t) \leqslant O(1)\exp(O(1)\int_0^t \frac{p(\tau)}{\sqrt{q(\tau)}}d\tau) \leqslant O(1)\exp(O(1)\int_0^\infty \frac{p(t)}{\sqrt{q(t)}}dt) = O(1).$$

Therefore

$$\int_0^\infty x^2(t)dt \leqslant \int_0^\infty \frac{v(t)}{\sqrt{q(t)}}dt \leqslant O(1)\int_0^\infty \frac{1}{\sqrt{q(t)}}dt = O(1).$$

The proof of the lemma is completed.
According to Fu (1998) we have

Lemma 2 Suppose that the state equation of the following control system is

$$x'(t) = f[x(t), u(t), t], \quad x \in \mathfrak{R}^n, \ f \in \mathfrak{R}^n, \ u \in \mathfrak{R}^m.$$

The restrictive conditions are

$$x(t_0) = x_0, \ \xi[x(t_f), t_f] = 0.$$

The optimal index function is

$$J[u(t)] = \Phi[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t]dt.$$

Define the Hamilton function:

$$H[x(t), u(t), \lambda(t), t] = L[x(t), u(t), t] + \lambda^T(t)f[x(t), u(t), t],$$

where

$$\lambda(t) = \frac{\partial J^*[x(t), t]}{\partial x(t)}.$$

When $u(t)$ is unrestricted and satisfies the following partial differential equation:

$$-\frac{\partial J^*}{\partial t} = H^*[x(t), \frac{\partial J^*}{\partial t}, t],$$

$$H^*[x(t), \frac{\partial J^*}{\partial t}, t] = \min_{u(t)\in u} H[x(t), u(t), \frac{\partial J^*}{\partial t}, t],$$

we can obtain $u^*(t)$ to minimize $H[x(t), u(t), \frac{\partial J^*}{\partial x}, t]$ when $x(t)$, $\frac{\partial J^*}{\partial t}$, and t remain invariant and realize a closed-loop control easily.

Reconsider the optimal control problem (3)–(5). Let $x = x_1$ and $x'_1 = x_2$; then the optimal control problem (3)–(5) is equivalent to the following

$$\begin{cases} x'_1 = x_2, \\ x'_2 = -p(t)x_2 - u(t), \\ x_1(0) = x_0, \\ x_2(0) = x_1, \end{cases}$$

$$J(u(t)) = \int_0^\infty (2q^2(t)x_1^2(t) + \frac{1}{2}u^2(t))dt.$$

According to Lemma 2, let Hamilton function

$$\begin{aligned} H &= L + \lambda(t)^T f(x(t), u(t), t) \\ &= 2q^2(t)x_1^2 + \frac{1}{2}u^2 + \left(\frac{\partial J}{\partial x_1} \ \frac{\partial J}{\partial x_2}\right)\begin{pmatrix} x_2 \\ -p(x)x_2 - u(t) \end{pmatrix} \\ &= 2q^2(t)x_1^2 + \frac{1}{2}u^2 + \frac{\partial J}{\partial x_1}x_2 - \frac{\partial J}{\partial x_2}p(t)x_2 - \frac{\partial J}{\partial x_2}u(t). \end{aligned}$$

Because $u(t)$ is unrestricted, let $\frac{\partial H}{\partial u} = 0$, *i.e.*

$$\frac{\partial H}{\partial u} = u - \frac{\partial J}{\partial x_2} = 0, \quad u = \frac{\partial J}{\partial x_2}.$$

By the Hamilton-Jacobi-Bellman equation and notice that J is independent of t, we have

$$2q^2(t)x_1^2 + [\frac{\partial J}{\partial x_1} - \frac{\partial J}{\partial x_2}p(t)]x_2 - \frac{1}{2}\left(\frac{\partial J}{\partial x_2}\right)^2 = 0. \quad (7)$$

Let

$$J = A(t)x_1^2 + 2B(t)x_1x_2 + C(t)x_2^2,$$

then

$$\frac{\partial J}{\partial x_1} = 2A(t)x_1 + 2B(t)x_2, \quad (8)$$

$$\frac{\partial J}{\partial x_2} = 2C(t)x_2 + 2B(t)x_1. \quad (9)$$

After substitution of Eqs.(8) and (9) in Eq.(7), we obtain

$$(2q^2(t) - 2B^2(t))x_1^2 + (2A(t) - 2B(t)p(t) - 4B(t)C(t))x_1x_2 + (2B(t) - 2p(t)C(t) - 2C^2(t))x_2^2 = 0,$$

therefore

$$\begin{cases} q^2(t) = B^2(t), \\ A(t) - B(t)p(t) - 2B(t)C(t) = 0, \\ B(t) - p(t)C(t) - C^2(t) = 0. \end{cases}$$

From the above, we have

$$\begin{cases} A(t) = q(t)\sqrt{p^2(t) + 4q(t)}, \\ B(t) = q(t), \\ C(t) = \frac{-p(t) + \sqrt{p^2(t) + 4q(t)}}{2}, \end{cases}$$

therefore

$$J = q(t)\sqrt{p^2(t) + 4q(t)}x_1^2 + 2q(t)x_1x_2 +$$

$$\frac{-p(t)+\sqrt{p^2(t)+4q(t)}}{2}x_2^2,$$

$$u(t)=\frac{\partial J}{\partial x_2}=2q(t)x_1+$$
$$(-p(t)+\sqrt{p^2(t)+4q(t)})x_2, \quad (10)$$

the closed-loop control system is

$$x''+\sqrt{p^2(t)+4q(t)}x'+2q(t)x=0.$$

Theorem 1 Suppose that functions $p(t)$ and $q(t)\in C[0,+\infty)$ are not constant functions

$$x''+\sqrt{p^2(t)+4q(t)}x'+2q(t)x=0,$$
$$x(0)=x_0,\ x'(0)=x_1,\ x_0^2+x_1^2\neq 0. \quad (11)$$

Suppose that $x(t)$ is a solution of (11); if

$$x(t)\in L^2[0,+\infty),$$

then system (11) is the optimal control system of problem (3)–(5); the optimal control is (10).

By using Lemma 1 and Theorem 1 we have

Theorem 2 If $p(t)$ and $q(t)$ in (10) satisfy

(i) $p(t)\in C[0,+\infty)$, $p(t)\geqslant 0$;
$q(t)>0$, $q'(t)>0$, $q(t)\in C^1[0,+\infty)$;
$q'(t)=O(q(t))$, $t\to+\infty$.

(ii) $\int_0^\infty \frac{dt}{\sqrt{q(t)}}<\infty$,

$$\int_0^\infty \sqrt{q(t)}\left|\frac{d}{dt}\left(\frac{q'(t)}{q^{3/2}(t)}\right)\right|dt<\infty,$$

$$\int_0^\infty \frac{\sqrt{p^2(t)+4q(t)}}{\sqrt{q(t)}}dt<\infty,$$

then system (11) is the optimal control system of problem (3)–(5); the optimal state feedback control is (10).

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