# Beam conditions for radiation generated by an electromagnetic Hermite-Gaussian model source*

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Within the framework of the correlation theory of electromagnetic laser beams, the far field cross-spectral density matrix of the light radiated from an electromagnetic Hermite-Gaussian model source is derived. By utilizing the convergence property of Hermite polynomials, the conditions of the matrices for the source to generate an electromagnetic Hermite-Gaussian beam are obtained. Furthermore, in order to generate a scalar Hermite-Gaussian model beam, it is required that the source should be locally rather coherent in the spatial domain.

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In recent years, the paraxial diffraction theories for the optical beams such as the fundamental Gaussian beam, Hermite-Gaussian beam and Laguerre-Gaussian beam have been well developed. Since then, as a class of the commonly utilized higher-order laser beams, the Hermite-Gaussian beam has attracted particular interests, in both theoretical and experimental aspects. The alternative forms of the Hermite-Gaussian functions within complex arguments have been widely studied[1,2]. The scattering problems, which are induced by the incidence of the Hermite-Gaussian beam, have also attracted attention of many researchers[3,4]. Other than these investigations, the Hermite-Gaussian beam has been shown to have potential applications in free space optical communications, in which it has been frequently utilized to determine the atmospheric scintillation index[5]. Cai et al[6] introduced the partially coherent Hermite-Gaussian model beam and investigated its propagation properties through optical systems. By utilizing an elongated Hermite-Gaussian $TEM_{01}$ mode beam, the high frequency optical method has been created in experiments to trap atoms and small particles[7]. On the other hand, highly directional laser beams are commonly required in laser guiding, aiming, remote sensing and free-space optical communications. Among these applications, the condition for a Gaussian Schell-model source to generate a beamlike field has been recorded[8]. Subsequently, this condition has been well extended to the case related with the electromagnetic Gaussian Schell-model beam by considering the effect of polarization[9]. Very recently, Wu et al[10] have proposed the electromagnetic $J_0$-correlated Schell-model source. Besides this, they also obtained the condition for such a model source to generate a beamlike field in the far zone. However, the condition for an electromagnetic Hermite-Gaussian model source to generate a beam has not been studied. In this paper, the far field cross-spectral density matrix of the light radiated from an electromagnetic Hermite-Gaussian model source is derived. The condition for such a source to generate a beamlike field is also obtained. These results may provide potential applications in generation of $TEM_{mn}$ mode beams, design of laser sources and optical laser modulations, etc.

Considering a planar, secondary and fluctuating electromagnetic source, which is located in the plane $z$=0 and radiates into the half space $z$>0, the second-order correlation properties of such a source can be presented by the 2 × 2 cross-spectral density matrix[9-11]

$$\boldsymbol{W}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\begin{bmatrix} W_{xx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) & W_{xy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) \\ W_{yx}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) & W_{yy}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega) \end{bmatrix}, \quad (1)$$

where $\omega$ is the frequency, and $\boldsymbol{\rho}_1$, $\boldsymbol{\rho}_2$ are two position vectors in the source plane $z$ = 0. Each matrix element in Eq.(1) can be decomposed into

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$$W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)=\left\langle E_i^*(\boldsymbol{\rho}_1,\omega)E_j(\boldsymbol{\rho}_2,\omega)\right\rangle,\ (i,j=x,y)\,, \tag{2}$$

where $E_x$ and $E_y$ represent the electric field components along the $x$ and $y$ directions in the initial plane, respectively. The asterisk means the complex conjugation and the angle bracket denotes taking over the statistical ensemble of the optical field. While generating or modulating a laser model in optical methods, correlation properties between the electric field components always should be considered. In this paper, the electromagnetic Hermite-Gaussian model source is represented in the rectangular Cartesian coordinate at the plane $z$=0, similar to the analytical form for the electromagnetic Gaussian Schell-model source[6,9,10]

$$W_{ij}^{(0)}(x_1,y_1,x_2,y_2)=\sqrt{S_i^{(0)}(x_1,y_1,\omega)}\sqrt{S_j^{(0)}(x_2,y_2,\omega)}\mu_{ij}^{(0)}\times (x_2-x_1,y_2-y_1,\omega),\ (i,j=x,y)\,, \tag{3}$$

where $S_i^{(0)}$ and $S_j^{(0)}$ are the spectral densities of $i$ and $j$ components of the electric field in the source plane. $\mu_{ij}^{(0)}$ denotes the degree of correlation between two components. According to Ref.[6], the spectral densities and the degree of correlation for the electromagnetic Hermite-Gaussian model source can be expressed as

$$S_i^{(0)}(x,y,\omega)=A_i^2\exp\left(-\frac{x^2+y^2}{\sigma_0^2}\right)H_m\left(\frac{\sqrt{2}}{\sigma_0}x\right)H_n\left(\frac{\sqrt{2}}{\sigma_0}y\right), \tag{4}$$

$$\mu_{ij}^{(0)}(x_2-x_1,y_2-y_1,\omega)=B_{ij}\exp\left[-\frac{(x_2-x_1)^2+(y_2-y_1)^2}{2\delta_{ij}^2}\right],\ (i,j=x,y)\,, \tag{5}$$

where $A_i$, $B_{ij}$, $\sigma$ and $\delta_{ij}$ are independent of positions but may depend on frequency. $H_n(.)$ is the Hermite polynomial with order $n$. All above quantities should obey the limitation which has been derived (see Eq.(9) of Ref.[10]). The far field cross-spectral density matrix of the light at two position points $\boldsymbol{r}_1=r_1\boldsymbol{s}_1$ and $\boldsymbol{r}_2=r_2\boldsymbol{s}_2$ ($\boldsymbol{s}_1^2=\boldsymbol{s}_2^2=1$)[8-10] is introduced

$$W_{ij}^{(\infty)}(r_1\boldsymbol{s}_1,r_2\boldsymbol{s}_2,\omega)=(2\pi k)^2\cos\theta_1\cos\theta_2\times \left\{\widetilde{W}_{ij}^{(0)}(-k\boldsymbol{s}_{1\perp},k\boldsymbol{s}_{2\perp},\omega)\frac{\exp[\mathrm{i}\,k(\boldsymbol{r}_2-\boldsymbol{r}_1)]}{\boldsymbol{r}_1\boldsymbol{r}_2}\right\},\ (i,j=x,y), \tag{6}$$

where

$$\widetilde{W}_{ij}^{(0)}(\boldsymbol{f}_1,\boldsymbol{f}_2,\omega)=\frac{1}{(2\pi)^4}\iint W_{ij}^{(0)}(\boldsymbol{\rho}_1,\boldsymbol{\rho}_2,\omega)\times \exp[-\mathrm{i}(\boldsymbol{f}_1\cdot\boldsymbol{\rho}_1+\boldsymbol{f}_2\cdot\boldsymbol{\rho}_2)]\mathrm{d}^2\rho_1\,\mathrm{d}^2\rho_2,\ (i,j=x,y), \tag{7}$$

is the four-dimensional Fourier transform of $W_{ij}^{(0)}$. $\boldsymbol{s}_{i\perp}=\boldsymbol{s}_{ix}\cdot\boldsymbol{x}+\boldsymbol{s}_{iy}\cdot\boldsymbol{y}$ $(i=1,2)$ is projections of unit vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ onto the transversal plane $z$=0, respectively. $\boldsymbol{f}_i=\boldsymbol{f}_{ix}\cdot\boldsymbol{x}+\boldsymbol{f}_{iy}\cdot\boldsymbol{y}$ ($i$=1, 2) is two-dimensional vectors represented in the frequency domain. $\theta_1$ and $\theta_2$ are the angles that unit vectors $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ make with the $z$ axis.

Substituting Eqs.(3),(4) and (5) into Eq.(7) and changing the variables of integration as follows

$$x_2-x_1=u,\ x_2+x_1=u',\ y_2-y_1=v,\ y_2+y_1=v', \tag{8}$$

$$\boldsymbol{f}_{2x}-\boldsymbol{f}_{1x}=\boldsymbol{f}_x,\ \boldsymbol{f}_{2x}+\boldsymbol{f}_{1x}=\boldsymbol{f}_x',\ \boldsymbol{f}_{2y}-\boldsymbol{f}_{1y}=\boldsymbol{f}_y,\ \boldsymbol{f}_{2y}+\boldsymbol{f}_{1y}=\boldsymbol{f}_y', \tag{9}$$

then Eq.(7) can be rewritten as

$$\widetilde{W}_{ij}^{(0)}(\boldsymbol{f}_x,\boldsymbol{f}_x',\boldsymbol{f}_y,\boldsymbol{f}_y',\omega)=\frac{A_iA_jB_{ij}}{(2\pi)^4}F(m,n,p_1,p_2,p_3,p_4)\times$$
$$\int\exp\left[-\left(\frac{1}{2\sigma_0^2}+\frac{1}{2\delta_{ij}^2}\right)u^2-\mathrm{i}\cdot\boldsymbol{f}_x\frac{u}{2}\right]H_{m-p_1}\left(-\frac{u}{\sigma_0}\right)H_{n-p_2}\left(\frac{u}{\sigma_0}\right)\mathrm{d}u\times$$
$$\int\exp\left[-\left(\frac{1}{2\sigma_0^2}+\frac{1}{2\delta_{ij}^2}\right)v^2-\mathrm{i}\cdot\boldsymbol{f}_y\frac{v}{2}\right]H_{m-p_3}\left(-\frac{v}{\sigma_0}\right)H_{n-p_4}\left(\frac{v}{\sigma_0}\right)\mathrm{d}v\times$$
$$\int\exp\left[-\frac{1}{2\sigma_0^2}u'^2-\mathrm{i}\cdot\boldsymbol{f}_x'\frac{u'}{2}\right]H_{p_1}\left(\frac{u'}{\sigma_0}\right)H_{p_2}\left(\frac{u'}{\sigma_0}\right)\mathrm{d}u'\times$$
$$\int\exp\left[-\frac{1}{2\sigma_0^2}v'^2-\mathrm{i}\cdot\boldsymbol{f}_y'\frac{v'}{2}\right]H_{p_3}\left(\frac{v'}{\sigma_0}\right)H_{p_4}\left(\frac{v'}{\sigma_0}\right)\mathrm{d}v', \tag{10}$$

where

$$F(m,n,p_1,p_2,p_3,p_4)=\sum_{p_1=0}^{m}\sum_{p_2=0}^{n}\sum_{p_3=0}^{m}\sum_{p_4=0}^{n}\frac{(-1)^{m+n-p_2-p_4}}{2^{m+n}}\times \binom{m}{p_1}\binom{n}{p_2}\binom{m}{p_3}\binom{n}{p_4}\,. \tag{11}$$

Substituting Eq.(10) into Eq.(6) and letting $\boldsymbol{s}_1=\boldsymbol{s}_2$, after tedious but straightforward integral calculations, the spectral density matrix at a single vector point in the far zone is

$$W_{ij}^{(\infty)}(r\boldsymbol{s},\omega)=\frac{k^2\cos^2\theta}{4r^2}A_iA_jB_{ij}\exp\left[-\frac{k^2}{2(1/\sigma_0^2+1/\delta_{ij}^2)}\right]\times$$
$$F(m,n,p_1\sim p_4)\times G(m,n,l_1\sim l_4)\times$$
$$P(\sigma_0,\delta_{ij})\times H_{2m-p_1-p_2-2l_1}\left[\frac{\mathrm{i}k\cdot s_x}{2\sigma_0\sqrt{(1/\sigma_0^2+1/\delta_{ij}^2)^2-2(1/\sigma_0^2+1/\delta_{ij}^2)\delta_{ij}^2}}\right]\times$$

$$H_{2n-p_3-p_4-2l_2}\left[\frac{\mathrm{i}k\cdot s_y}{2\sigma_0\sqrt{\left(1/\sigma_0^2+1/\delta_{ij}^2\right)^2-2\left(1/\sigma_0^2+1/\delta_{ij}^2\right)\delta_{ij}^2}}\right]\times H_{p_1+p_2-2l_3}(0)\times H_{p_3+p_4-2l_4}(0)\ , \tag{12}$$

where

$$G(m,n,l_1\sim l_4)=\sum_{l_1=0}^{\min(m-p_1,m-p_2)}\sum_{l_2=0}^{\min(n-p_3,n-p_4)}\sum_{l_3=0}^{\min(p_1,p_2)}\sum_{l_4=0}^{\min(p_3,p_4)}\binom{m-p_1}{l_1}\binom{n-p_3}{l_2}\binom{p_1}{l_3}\binom{p_3}{l_4}\times$$

$$\frac{(m-p_2)!(n-p_4)!p_2!p_3!}{(m-p_2-l_1)!(n-p_4-l_2)!(p_2-l_3)!(p_3-l_4)!}2^{l_1+l_2+l_3+l_4}\times(-1)^{(p_1+p_2+p_3+p_4)/2-l_3-l_4}, \tag{13}$$

$$P(\sigma_0,\delta_{ij})=\frac{\sigma_0^2}{1/2\sigma_0^2+1/2\delta_{ij}^2}\times\left(1-\frac{2}{1+\sigma_0^2/\delta_{ij}^2}\right)^{m+n-(p_1+p_2+p_3+p_4)/2-l_1-l_2}. \tag{14}$$

The spectral density radiated by an electromagnetic model beam can be expressed as[9-12]

$$S^{(\infty)}(\boldsymbol{r},\omega)=\left\langle E_x^*(\boldsymbol{r},\omega)E_x(\boldsymbol{r},\omega)\right\rangle+\left\langle E_y^*(\boldsymbol{r},\omega)E_y(\boldsymbol{r},\omega)\right\rangle=\mathrm{Tr}\left[W_{ij}^{(\infty)}(\boldsymbol{r},\boldsymbol{r},\omega)\right]\ , \tag{15}$$

where the sign "Tr" denotes the trace of the cross-spectral density matrix $W_{ij}^{(\infty)}$. The inequation related to the Hermite polynomial[13] is introduced

$$|H_n(x)|\le\gamma_0\sqrt{n!}2^{n/2}\exp\left(x^2/2\right),\ \ \gamma_0\approx1.086435\ . \tag{16}$$

Substituting Eqs.(12) and (16) into Eq.(15), the spectral density of the light in the far zone is

$$S^{(\infty)}(\boldsymbol{r},\omega)\le\mathrm{Tr}\left\{\frac{k^2\cos^2\theta}{4\boldsymbol{r}^2}A_iA_jB_{ij}\times\gamma_0^2F(m,n,p_1\sim p_4)\times G(m,n,l_1\sim l_4)\times P(\sigma_0,\delta_{ij})\times\right.$$

$$\left[(2m-p_1-p_2-2l_1)!(2n-p_3-p_4-2l_2)!\right]^{1/2}2^{m+n-(p_1+p_2+p_3+p_4)/2-l_1-l_2}\times$$

$$\left.H_{p_1+p_2-2l_3}(0)H_{p_3+p_4-2l_4}(0)\times\exp\left[-\frac{k^2}{Q(\sigma_0,\delta_{ij})}\right]\right\}, \tag{17}$$

where

$$Q(\sigma_0,\delta_{ij})=\frac{\delta_{ij}^2-\sigma_0^4\delta_{ij}^6}{\sigma_0^4/2-\sigma_0^4\delta_{ij}^4}. \tag{18}$$

For an electromagnetic Hermite Gaussian model source, in order to generate a beamlike field in the far zone, the spectral density $S^{(\infty)}(\boldsymbol{r},\omega)$ must only contain very low spatial-frequency $k$ components, which essentially means that it should be similar to the behavior of the scalar case[8]. Therefore, the factor $\cos\theta$ in Eq.(17) may be approximated by unity. Subsequently, through observing Eqs.(17) and (18), the following requirements should be simultaneously satisfied

$$Q(\sigma_0,\delta_{xx})<<k^2,\ \ Q(\sigma_0,\delta_{yy})<<k^2\ . \tag{19}$$

Eq.(19) is equivalent to the form

$$\frac{\delta_{xx}^2-\sigma_0^4\delta_{xx}^6}{\sigma_0^4-2\sigma_0^4\delta_{xx}^4}<<\frac{2\pi^2}{\lambda^2},\ \ \frac{\delta_{yy}^2-\sigma_0^4\delta_{yy}^6}{\sigma_0^4-2\sigma_0^4\delta_{yy}^4}<<\frac{2\pi^2}{\lambda^2}. \tag{20}$$

Eq.(20) indicates that in order to retain a beamlike field radiated from an electromagnetic Hermite Gaussian model source, certain limitations to the beam width $\sigma_0$ and the correlation length $\delta_{ij}$ are essential. This condition is independent of the Hermite modes $m(n)$ and the state of polarization of sources. Analytical forms of Eq.(20) are totally different from the ones for the electromagnetic Gaussian Schell-model beam[9] and the $J_0$-correlated Schell-model beam[10]. Therefore, it can be concluded that for an electromagnetic Hermite-Gaussian model source, in order to generate a beamlike field in the far zone, specific beam conditions (Eq.(20)) should be required. For the case of the scalar Hermite-Gaussian model source, the beam condition may be simplified to

$$\frac{\delta_{\mathrm{scalar}}^2-\sigma_0^4\delta_{\mathrm{scalar}}^6}{\sigma_0^4-2\sigma_0^4\delta_{\mathrm{scalar}}^4}<<\frac{2\pi^2}{\lambda^2}, \tag{21}$$

where $\delta_{\mathrm{scalar}}$ denotes the spectral degree for coherence for the scalar Hermite Gaussian model source in the plane $z = 0$.

For a scalar Hermite Gaussian model source, the beam condition (see Eq.(21)) is of particular interest. Some special cases can be correspondingly analyzed. Observing Eq.(21), it is found that for an arbitrary value of the beam width $\sigma_0$ and the spectral degree of coherence $\delta_{\mathrm{scalar}}\to\infty$ (completely coherent in the spatial domain), Eq.(21) may be approximated to the following form

$$\frac{\delta_{\mathrm{scalar}}^2-\sigma_0^4\delta_{\mathrm{scalar}}^6}{\sigma_0^4-2\sigma_0^4\delta_{\mathrm{scalar}}^4}\approx\frac{\delta_{\mathrm{scalar}}^2}{2}<<\frac{2\pi^2}{\lambda^2}\to\delta_{\mathrm{scalar}}<<\frac{2\pi}{\lambda}=k. \tag{22}$$

This conflict means that the beam condition is invalidate for the Hermite Gaussian model within complete coherence in the whole source plane. Here, we specifically prove that in order to generate a scalar Hermite Gaussian model beam, the source should be locally rather coherent. Firstly, the parameter representing the coherence property of source is introduced[8]

$$\tau_0=\delta_{\mathrm{scalar}}/\sigma_0, \tag{23}$$

where $\tau_0$<<1 represents the spatially incoherent case. Inversely, $\tau_0$>>1 represents the spatially coherent case. Secondly, by substituting Eq.(23) into Eq.(21), the beam condition can be rewritten as

$$\frac{\tau_0^2-\tau_0^6\sigma_0^8}{\sigma_0^2-2\tau_0^4\sigma_0^6}<<\frac{2\pi^2}{\lambda^2}. \tag{24}$$

If the scalar Hermite Gaussian model source is spatially incoherent ($\tau_0$<<1) in the initial plane, the following approximation can be made

$$\frac{\tau_0^2-\tau_0^6\sigma_0^8}{\sigma_0^2-2\tau_0^4\sigma_0^6}\approx\frac{\tau_0^2}{\sigma_0^2}<<\frac{2\pi^2}{\lambda^2}\to\sigma_0>>\frac{\tau_0}{\sqrt{2}\,\pi}\lambda\ . \tag{25}$$

If the scalar Hermite Gaussian model source is spatially coherent ($\tau_0$>>1) in the initial plane, Eq.(24) can be approximated to the form

$$\frac{\tau_0^2-\tau_0^6\sigma_0^8}{\sigma_0^2-2\tau_0^4\sigma_0^6}\approx\frac{\tau_0^2\sigma_0^2}{2}<<\frac{2\pi^2}{\lambda^2}\to\sigma_0<<\frac{2\pi}{\tau_0}\lambda^{-1}\ . \tag{26}$$

Eqs.(25) and (26) clearly demonstrate that in order to generate the scalar Hermite Gaussian beam in the far field, the source should be locally rather coherent. Eqs.(20), (21), (25) and (26) are the main analytical results of this paper, which may provide a convenient approach to manipulate source parameters fulfilling certain requirements such as Eqs. (25) and (26). Other than these, Eqs.(20) and (21) further indicate that the beam condition for generating the Hermite Gaussian model beam is totally different from previous ones for stochastic electromagnetic beams[9]. Additionally, the obtained beam condition is independent of Hermite modes or the state of polarization of the source. These results may provide some practical applications in generation of TEM$_{mn}$ mode beams, design of laser sources and optical laser modulations, etc.

In conclusion, based on the correlation theory of electromagnetic beams, the far field cross-spectral density matrix of the light radiated from an electromagnetic Hermite Gaussian model source is derived. By utilizing the convergence property of Hermite polynomials, the beam condition for a source to generate electromagnetic and scalar Hermite Gaussian beams is obtained. An essential proof is also made that in order to generate a scalar Hermite Gaussian beam, the source should be locally rather coherent in the spatial domain. The limitation to the beam width $\sigma_0$ is also given.

## References

[1] A. E. Siegman, J. Opt. Soc. Am. **63**, 1093 (1973).

[2] E. Zauderer, J. Opt. Soc. Am. A **3**, 465 (1986).

[3] M. Yokota, T. Takenaka and O. Fukumitsu, J. Opt. Soc. Am. A **3**, 580 (1986).

[4] M. Yokota, S. He and T. Takenaka, J. Opt. Soc. Am. A **18**, 1681 (2001).

[5] C. Y. Young, Y. V. Gilchrest and B. R. Macon, Opt. Eng. **41**, 1097 (2002).

[6] Y. Cai and C. Chen, J. Opt. Soc. Am. A **24**, 2394 (2007).

[7] T. P. Meyrath, F. Schreck, J. L. Hanssen, C. S. Chuu and M. G. Raizen, Opt. Express **13**, 2843 (2005).

[8] L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, Cambridge U. Press, New York, 1995.

[9] O. Korotkova, M. Salem and E. Wolf, Opt. Lett. **29**, 1173 (2004).

[10] G. Wu, Q. Lou, J. Zhou, H. Guo, H. Zhao and Z. Yuan, Opt. Lett. **33**, 2677 (2008).

[11] E. Wolf, Opt. Commun. **265**, 60 (2006).

[12] E. Wolf, Phys. Lett. A **312**, 263 (2003).

[13] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Dover, 1970.