A rigorous proof of MIMO channel capacity's increase with antenna number *

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It is well known that adding more antennas at the transmitter or at the receiver may offer larger channel capacity in the multiple-input multiple-output(MIMO) communication systems. In this letter, a simple proof is presented for the fact that the channel capacity increases with an increase in the number of receiving antennas. The proof is based on the famous capacity formula of Foschini and Gans with matrix theory.

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A multiple-input multiple-output (MIMO) system has multiple antennas at the transmitter and the receiver. Using multiple antennas at the locations makes it possible to form multiple channels and, therefore, to increase the channel capacity significantly. The capacity of MIMO systems has been intensively studied. For uncorrelated Rayleigh fading, there are many exact results, for example, the channel capacity formula [1], the mean capacity [2], the capacity variance [3] and the characteristic function [4],etc. For the correlated Rayleigh case, there are also many results available: the characteristic function for the correlation only among the receiver or the transmitter [5], the fading correlation in wireless MIMO communication systems [6], the effect of the fading correlation on the capacity [7], and the second order statistics of non line-of-sight indoor MIMO channels. It is generally believed that the capacity of MIMO systems increases with the increase in the number of antennas at either the transmitter or the receiver. However, a strict mathematical proof is lack for this fact. In this letter, we give a rigorous proof using matrix theory. We briefly introduce the mathematical model for MIMO channels, and we show that the channel capacity increases with the number of antennas at the receiver.

In this letter, we use the same mathematical model as Foschini and Gans [1]. We assume n transmitting antennas and m receiving antennas. The channel between the transmitting antennas and receiving antennas is single point-to-point, and it is assumed to be quasi-static Rayleigh flat fading, i.e. the channel is fixed during a burst and changes randomly from burst to burst. A complex baseband model involving a fixed linear matrix channel with additive white Gaussian noise is used.

The capacity of a MIMO wireless system with n transmitting antennas, m receiving antennas, and an average received signal-to-noise ratio (SNR), ρ (independent of n), at each receiving antenna has been obtained in Ref.[1] as

$$C_{mm,n} = \log_2 \det[I_m + (\rho / n)HH^*]\,(\text{bps}/\text{Hz}) \quad , \tag{1}$$

where det[·] denotes determinant, I_m is the $m \times m$ identity matrix, H is the channel transfer matrix with m rows and n columns, and the " * " indicates the conjugate transpose. We denote a system with n transmitting and m receiving antennas by (n, m).

The determinant in equation(1) can be expressed as

$$D_{mm,n} = \det[I_m + A] \quad , \tag{2}$$

where $D_{mm,n}$ denotes the determinant of a square matrix of order (m, m) with n transmitting antennas and m receiving antennas, $A=\bar{H}\bar{H}^*$, where $\bar{H} = (\rho/n)^{1/2} H$, A is a square matrix, and H is the channel transfer matrix of the (n, m) system with each element being the normalized complex Gaussian random variable.

To facilitate the discussion, we introduce the following definition (see Ref.[9]): when the r rows and columns struck out of a square matrix A have the same indices, the remaining submatrix is called the r^{th} principal minor of A.

In terms of equation (2), we have

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$$D_{mm,n}=\begin{vmatrix} 1+\sum_{j=1}^{n}\left|\overline{H}_{1j}\right|^2 & \sum_{j=1}^{n}\overline{H}_{1j}\overline{H}_{2j}^{*} & \cdots\cdots & \sum_{j=1}^{n}\overline{H}_{1j}\overline{H}_{\mathrm{m}\text{-}1j}^{*} & 0 \\ \sum_{j=1}^{n}\overline{H}_{2j}\overline{H}_{1j}^{*} & 1+\sum_{j=1}^{n}\left|\overline{H}_{2j}\right|^2 & \cdots\cdots & \sum_{j=1}^{n}\overline{H}_{2j}\overline{H}_{\mathrm{m}\text{-}1j}^{*} & 0 \\ . & & & & \\ . & & & & \\ \sum_{j=1}^{n}\overline{H}_{m-1j}\overline{H}_{1j}^{*} & \sum_{j=1}^{n}\overline{H}_{m-1j}\overline{H}_{2j}^{*} & \cdots\cdots & 1+\sum_{j=1}^{n}\left|\overline{H}_{m-1j}\right|^2 & 0 \\ \sum_{j=1}^{n}\overline{H}_{mj}\overline{H}_{1j}^{*} & \sum_{j=1}^{n}\overline{H}_{mj}\overline{H}_{2j}^{*} & \cdots\cdots & \sum_{j=1}^{n}\overline{H}_{mj}\overline{H}_{\mathrm{m}\text{-}1j}^{*} & 1 \end{vmatrix}+$$

$$\begin{vmatrix} 1+\sum_{j=1}^{n}\left|\overline{H}_{1j}\right|^2 & \sum_{j=1}^{n}\overline{H}_{1j}\overline{H}_{2j}^{*} & \cdots\cdots & \sum_{j=1}^{n}\overline{H}_{1j}\overline{H}_{\mathrm{m}\text{-}1j}^{*} & \sum_{j=1}^{n}\overline{H}_{1j}\overline{H}_{mj}^{*} \\ \sum_{j=1}^{n}\overline{H}_{2j}\overline{H}_{1j}^{*} & 1+\sum_{j=1}^{n}\left|\overline{H}_{2j}\right|^2 & \cdots\cdots & \sum_{j=1}^{n}\overline{H}_{2j}\overline{H}_{\mathrm{m}\text{-}1j}^{*} & \sum_{j=1}^{n}\overline{H}_{2j}\overline{H}_{mj}^{*} \\ . & & & & \\ . & & & & \\ \sum_{j=1}^{n}\overline{H}_{m-1j}\overline{H}_{1j}^{*} & \sum_{j=1}^{n}\overline{H}_{m-1j}\overline{H}_{2j}^{*} & \cdots\cdots & 1+\sum_{j=1}^{n}\left|\overline{H}_{m-1j}\right|^2 & \sum_{j=1}^{n}\overline{H}_{m-1j}\overline{H}_{mj}^{*} \\ \sum_{j=1}^{n}\overline{H}_{mj}\overline{H}_{1j}^{*} & \sum_{j=1}^{n}\overline{H}_{mj}\overline{H}_{2j}^{*} & \cdots\cdots & \sum_{j=1}^{n}\overline{H}_{mj}\overline{H}_{\mathrm{m}\text{-}1j}^{*} & \sum_{j=1}^{n}\left|\overline{H}_{mj}\right|^2 \end{vmatrix}. \tag{3}$$

The first term in equation(3) equals to $D_{m\text{-}1m\text{-}1,n}$. Repeating the above process for the second term and its followings until there is no "1" appearing in the entry of any determinant, we obtain

$$D_{mm,n}=D_{m-1m-1,n}+\det A+\sum_{1}^{\binom{m-1}{m-2}}\det A_{p,p}+\sum_{1}^{\binom{m-1}{m-3}}\det A_{pq,pq}+\cdots+\sum_{1}^{\binom{m-1}{1}}\det A_{pq..r,pq..r}+\sum_{j=1}^{n}\left|H_{mj}\right|^2, \tag{4}$$

where $A_{P,P}$ is the first principal minor of A obtained by striking out the p^{th} row and column of A with p ranging from 1 to m-1, $A_{pq,pq}$ is the second principal minor of A obtained by striking out the p^{th} and q^{th} rows and columns of A with p and q ranging from 1 to m-1, and $A_{pq\ldots r,\,pq\ldots r}$ is the $(m\text{-}2)^{th}$ principal minor of A obtained by striking out p^{th}, q^{th},… and r^{th} rows and columns of A with p,q,… and r ranging from 1 to m-1.

As it is defined, $A=\overline{H}\,\overline{H}{}^*$ where $\overline{H}$ is a matrix of order (m, n), and $\overline{H}{}^*$ is of order (n, m). $A_{p,q}$ can be written as $A_{p,p}=\overline{H}_p\times\overline{H}_p{}^*$, where $\overline{H}_p$ is obtained by deleting the p^{th} row with p ranging from 1 to m-1,and $A_{pq\ldots r,\,pq\ldots r}=\overline{H}_{pq\ldots r}\overline{H}{}^*_{pq\ldots r}$, where $\overline{H}_{pq\ldots r}$ is obtained by deleting m-2 rows with p, q, …,r ranging from 1 to m-1.

The determinant of $A=\bar{H}\,\bar{H}^*$ can be either zero when $m>n$ or positive when $m<n$ according to the theorems of 2.11.1 and 2.11.3 in Ref.[9]. The same method is true for the determinants of the $A_{p,p}=\bar{H}_p\,\bar{H}_p{}^*$ for p=1,2,…m-1, and the followings. Thus, equation(4) can be expressed as

$$D_{mm,n}=D_{m-1m-1,n}+P_m+\sum_{j=1}^{n}\left|H_{mj}\right|^2, \tag{5}$$

where P_m is positive, which is the sum of zero and positive terms except the first term and the last term in equation(4).

Therefore, $P_m+\sum_{j=1}^{n}\left|H_{mj}\right|^2$ is positive and we can conclude that

$$D_{mm,n}>D_{m-1m-1,n}\,. \tag{6}$$

Consequently, we can say that $C_{mm,n}$ for (n, m) is greater than $C_{m\text{-}1m\text{-}1,n}$ for (n, m-1).

A similar proof for the fact that the channel capacity increases with the increase in the number of the transmitting antennas is very difficult. The reason is that: when we add one more antenna to the transmitter, we add one more diversity to the transmitter. The capacity is therefore increased. However, on the other hand, the transmitted power by every transmitting antenna is reduced because the total power is fixed but the number of transmitting antennas is increased. The reduced transmitted power for each transmitting antenna will decrease the channel capacity while the capacity is increased as a result of the increase in diversity. Nevertheless, the proof can be straightforward if we use the above result of equation(6) and the duality theorem of Ref. [5].

In order to have a better view of this phenomenon, we use the Monte-Carlo method to compute the outage capacity. We set the outage probability of 10% and give the results of Fig.1 and Fig.2 for the case of capacity increasing with the receivion antennas and the case of capacity increasing with the transmitting antennas, respectively.

From the two figures, it is obviously that the channel capacities increase with the receiving antennas and transmitting antennas although the increase of the capacities tends to be slow after certain turning points of receiving antennas and transmitting antennas. It can be also seen from the two figures that the same number of receiving antennas can provide larger capacity than the same number of transmitting antennas with the same signal-to-noise ratio. The reason is mentioned above.

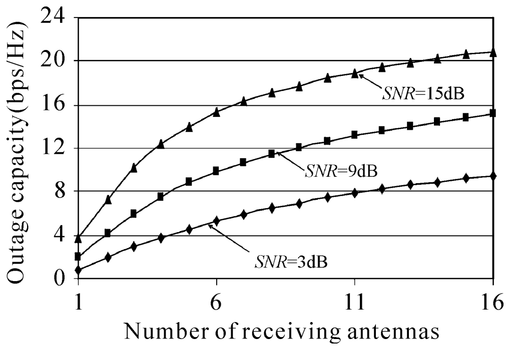

Fig.1 The capacity increases with the number of receiving antennas for three transmitting antennas.

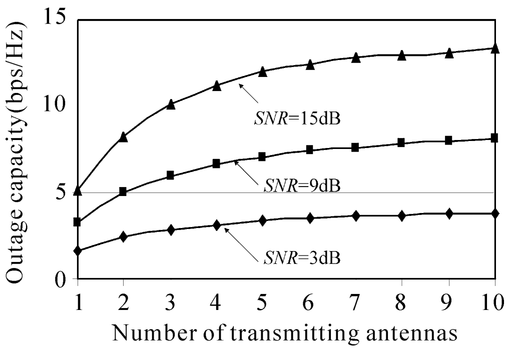

Fig.2 The capacity increases with the number of transmitting antennas for three receiving antennas.

Based on the famous capacity formula of Foschini and Gans with matrix theory, we have presented a simple proof of the fact that the channel capacity increases with the increase in the number of receiving antennas. A similar proof concerning the increases of the channel capacity with an increase in the number of transmitting antennas is much more difficult. However, the proof for the case of increasing transmitting antennas can be also straightforward based on the result for the case of increasing receiving antennas and the duality theorem.

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