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Varieties of Cubes of Opposition

Claudio E. A. Pizzi

Abstract. The objects called cubes of opposition have been presented in the literature in discordant ways. The aim of the paper is to offer a survey of such various kinds of cubes and evaluate their relation with an object, here called "Aristotelian cube", which consists of two Aristotelian squares and four squares which are semiaristotelian, i.e. are such that their vertices are linked by some so-called Aristotelian relation. Two paradigm cases of Aristotelian squares are provided by propositions written in the language of the logic of consequential implication, whose distinctive feature is the validity of two formulas, $A \rightarrow B \supset \neg (A \rightarrow \neg B)$ and $A \rightarrow B \supset \neg (\neg A \rightarrow B)$, expressing two different forms of contrariety. Part of section 1 is devoted to define the notions of rotation and of r-Aristotelian square, i.e. a square resulting from some rotation of an Aristotelian square. In section 2 this notion is extended to the one of a r-Aristotelian cube, i.e. of a cube resulting from some rotation of some square of an Aristotelian cube. This notion is used in the sequel to analyze various cubes of oppositions which can be found in the literature: (1) the one used by W. Lenzen to reconstruct Caramuel's Octagon; (2) the one used by D. Luzeaux to represent the implicative relation among S5-modalities; (3) the one introduced by D. Dubois to represent the relations between quantified propositions containing positive predicates and their negations; (4) the one called Moretti cube. None of such cubes is strictly speaking Aristotelian but each of them may be proved to be r-Aristotelian. Section 5 discusses the assertion that Dubois cube was anticipated in a paper published by Reichenbach in 1952. Actually Dubois' construction was anticipated by the so-called Johnson–Keynes cube, while the Reichenbach cube, unlike Dubois cube, is an instance of an Aristotelian cube in the sense defined in this paper. The dominance of such notion is confirmed by J.F. Nilsson's cube, representing relations between propositions with nested quantifiers, and also by a cube introduced by S. Read to treat quantifiers with existential import. A cube similar to Read's cube, introduced by Chatti and Schang, is shown to be r-Aristotelian. In section 6 the author remarks that the logic of the formulas occurring in the cubes of Chatti–Schang and Read have the drawback of not satsfying the law of Identity. He then proposes a definition of non-standard quantifiers which satisfies Identity, are independent of existential assumptions and such that their interrelations are represented by an Aristotelian cube.

Mathematics Subject Classification. 03Bxx.

Keywords. Square of opposition, Aristotle's square, Cube of opposition, Aristotelian diagrams, Rotation, Contrary proposition, Consequential implication, Connexive implication, Strict implication, Existential import.

1. Aristotelian Squares and their Rotations

In the wide literature about the square of opposition many papers in the last decades have been devoted to a notion which appears to be a natural generalization of the one of a square of opposition: the notion of a cube of opposition. Unfortunately in many of these contributions the reader comes across notions of a cube of opposition which appear to be discordant and *prima facie* irreconcilable. It seems then that it is useful to examine such different notions and to see which relations are devisable between them, in order to prepare the ground to a possible unified treatment of the whole subject.

In the paper [28] the present author introduced the notion of an Aristotelian cube as a particular construction based on a couple of Aristotelian squares. This presupposes having a clear notion of what an Aristotelian square is. In the mentioned paper and in other ones by the same author[1] this notion is defined as follows:

An Aristotelian *square* Γ with respect to a given logic **S** is a an ordered set of four propositions <W, X, Y, Z>, with the following properties:

(i) W and X are contraries in **S** (i.e. in **S** W logically implies ¬X and X logically implies ¬W)
(ii) W and Y on the one hand and X and Z on the other hand are contradictories in **S** (i.e. W ≡ ¬Y and X ≡ ¬Z are **S**-theorems)
(iii) Y and Z are subcontraries in **S** (i.e. their disjunction Z ∨ X is a **S**-theorem)
(iv) Z is subalternant of W and Y is subalternant of X in **S** (i.e. W logically implies Z in **S**, X logically implies Y in **S**).

If Z is subalternant of W or W is subalternant of Z in **S**, W and Z are said to be *connected*.

The relations described in (i)–(iv) will be said *Aristotelian relations*.[2] In Γ = <W, X, Y, Z> the pairs of wffs <W, X>, <X,Y>, <Y, Z>, <Z, W> are called *corner edges* or simply *edges* of Γ. W will be called the *origin* of Γ and Γ will be said *originated* by W.

Let us remark that if Γ is an Aristotelian square all permutations of Γ are not strictly speaking Aristotelian squares. For instance, if Γ is <W, X, Y, Z>, its permutation <W,Y, Z, X> is not an Aristotelian square since W e Z

[1] See [30–32].
[2] For an analogous terminology cf. [12], p.3 ff.

are not contradictories, as required by the above definition of an Aristotelian square.

A *semiaristotelian square* Γ with respect to a given logic **S** is an ordered 4-tuple $\Gamma = <W, X, Y, Z>$ such that each one of the edges of Γ consists in an ordered pair of sentences which have between them some Aristotelian relation.

From the preceding definitions it turns out (1) that every Aristotelian square is also semiaristotelian, but not *vice versa*. (2) that, if Γ is Aristotelian, every permutation of elements of Γ is a semiaristotelian square.

In this section we choose as a reference logic the logic called CI.0 whose axioms describe the behaviour of kind of implication called *consequential implication* (here symbolized by the arrow $\rightarrow$).[3] Being a variant of so-called connexive implication, such a logic has the following properties: Boethius' Thesis: $A \rightarrow B \supset \neg(A \rightarrow \neg B)$; Aristotle's Thesis: $\neg(A \rightarrow \neg A)$; Secondary Boethius: $A \rightarrow B \supset \neg (\neg A \rightarrow B)$; Contraposition: $A \rightarrow B \supset (\neg B \rightarrow \neg A)$.

The symbols for the relations which we call of *cotenability* and *secondary cotenability* are defined as follows: $A/^\circ B =_{df} \neg(A \rightarrow \neg B)$; $A \backslash^\circ B =_{df} \neg(\neg A \rightarrow B)$.

From the last two definitions it turns out that $A \backslash^\circ B$ is equivalent to $\neg A/^\circ\neg B$ and that $A/^\circ B$ is equivalent to $\neg A\backslash^\circ\neg B$.

In [27] CI.0 is proved to be definitionally equivalent to the modal logic KT, so decidable thanks to the well-known decidability procedures of KT. While $\Box A$ is definable as $T \rightarrow A$ (T being an arbitrary tautology) the arrow is definable in the following way:

(Def $\rightarrow$) $A \rightarrow B = \Box(A \supset B) \wedge (\Box A \equiv \Box B) \wedge (\Diamond A \equiv \Diamond B)$

Due to Boethius' Thesis and to Secondary Boethius $A \rightarrow B$ turns out to be contrary to both $A \rightarrow \neg B$ and $\neg A \rightarrow B$. The latter form of contrariety will be called *oblique contrariety.* As a consequence, we have not one but two Aristotelian squares of opposition originated by $A \rightarrow B$ whose wffs contain only, A, B, $\neg$and $\rightarrow$, as shown in Figs. 1 and 2.

The notion of a *rotation* of a figure will have a relevant place in what follows. Rotation is a geometrical operation which may be applied to monodimensional, bidimensional and three-dimensional objects. The simplest form of rotation of a monodimensional object is provided by the rotation of a segment around its midpoint. The rotations in which we are interested here are of course the rotations of bidimensional figures and especially of a square, which may be or not be an Aristotelian square.

The most simple rotations of a square whose vertices are labelled by wffs are the rotations around the horizontal axis and the vertical axis. In the following figures the symbol $^{(*)}$ marks the wff which originates the square to which rotation is applied, that will be called conventionally *basic square.* Considering the square in Fig. 1 as the basic square, the results of the two rotations are represented in Figs. 3 and 4.

Please note that the 4-tuple $<A/^\circ B, A/^\circ\neg B, ^{(*)} A \rightarrow \neg B, A \rightarrow B>$ represented in Fig. 3 is not an Aristotelian square according to the definition

[3] Cf. [27,28].

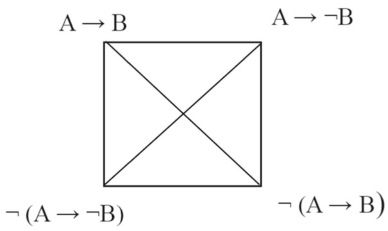

Figure 1. Square of opposition of consequential conditionals with standard contrariety

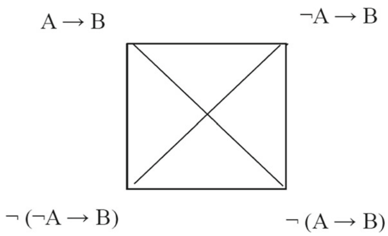

Figure 2. Square of opposition of consequential conditionals with oblique contrariety

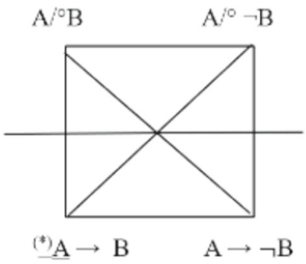

Figure 3. Horizontal rotation of the square of Fig. 1

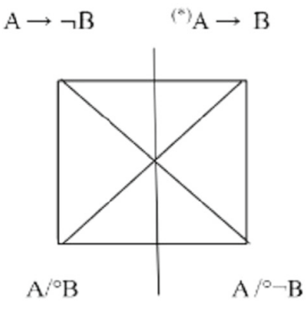

Figure 4. Vertical rotation of the square of Fig. 1

of p.1, while $<A \to \neg B, {}^{(*)}A \to B, A/^{\circ}B, A/^{\circ}\neg B>$ represented in Fig. 4 is an Aristotelian square in the defined sense, but its origin is not coincident with the origin of the basic square.

In [31] the author introduced the concept of a circular rotation of an arbitrary square. Such rotations are rotations of the basic square around a punctiform axis located at the cross of the diagonals.

For every square $\Gamma = <A, B, C, D>$ there are three not degenerate[4] clockwise rotations. The squares obtained from Γ by non degenerate clockwise rotations will be named cr1: $<D, {}^{(*)}A, B, C>$, cr2: $<C, D, {}^{(*)}A, B>$, cr3: $<B, C, D, {}^{(*)}A>$ (see Fig. 5).

An useful remark is that every circular rotation has two variants which are its vertical rotation (i.e. with the contraries and subcontraries in inverted position) and its horizontal rotation. By converse, circular rotation may be applied to the vertical and horizontal rotation of the basic square. In some cases, but not in all, the iterated and combined application of the three defined rotations may yield a square which may be reached by a one-step rotation.

In this connection something should be said about the diagonal rotation, i.e.the rotation around one of two diagonals of the square. Suffice it to consider the following figure, which is obtained by rotating the basic square around the diagonal which connects the vertices $A/^{\circ}B$ and $A \to \neg B$.
Let us now look at cr1. If we apply the horizontal rotation to cr1 we obtain the square of Fig. 6, and an analogous result for the other diagonal may be reached by applying sequentially two circular rotations to the square of Fig. 6. To sum up, the diagonal rotation is redundant since it can be replaced by a sequential combination of horizontal rotations and circular rotations.

Something should be said also about anticlockwise rotations. The anticlockwise rotations coincide with the clockwise rotations in reverse order. As one can see from Fig. 5, for instance, the last clockwise rotation cr3 is coincident with the one-step anticlockwise rotation, cr2 coincides with the two-step anticlockwise rotations and so on.

In the light of the preceding considerations and observing that a double application of vertical (as well as horizontal) rotation takes back to the starting square, it must be concluded that if a sequence of circular or vertical/horizontal rotations brings you from some square Q_i to some square Q_j, there is also some sequence of circular or vertical/horizontal rotations that bring you back from Q_j to Q_i.

The examples of rotations presented up to now concern the rotation of Aristotelian squares. But rotation may be applied to every kind of square, intended as an ordered 4-tuple of wffs. If the basic square is a semiaristotelian square, every rotation of it is obviously a semiaristotelian square since its elements are connected by some Aristotelian relation.

All the squares which are obtained by one or more rotations of an Aristotelian square Q_i will be called *r-Aristotelian squares.*

[4] By a degenerate rotation of Γ we intend a rotation, or a sequence of rotations, whose output is Γ itself.

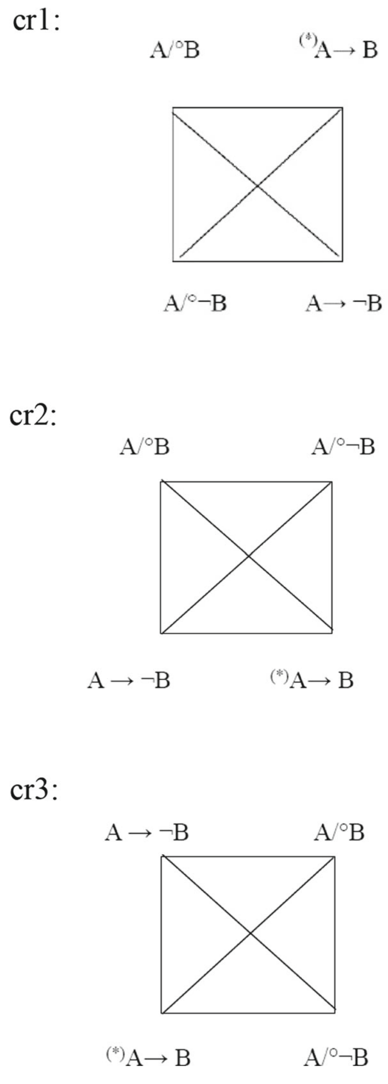

Figure 5. Three subsequent circular rotations of the basic square of Fig. 1

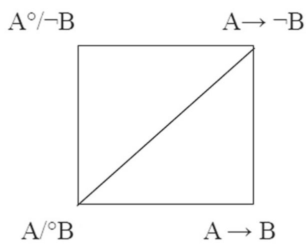

Figure 6. Diagonal rotation of the square of Fig. 1

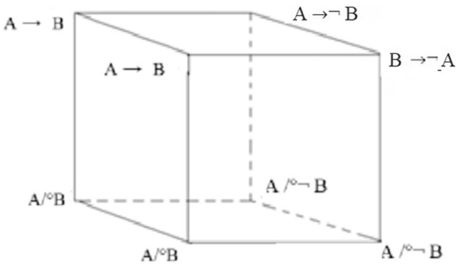

Figure 7. Example of a degenerate cube, i.e. of a cube containing two equal squares

We have now to qualify the relations between Aristotelian squares and a r-Aristotelian square. We agree to say that an Aristotelian square Q_i:<A, B, C, D > is *identical* to a square Q_j:<P, R, S,T> iff A and P, B and R, C and S, D and T are the same wffs; it is *equal* to Q_jw.r.t. the background logic **S** when $\vdash_S A \equiv P$, $\vdash_S B \equiv R$, $\vdash_S C \equiv S$, $\vdash_S D \equiv T$; it is *equivalent* to Q_j w.r.t. ***S*** when every member of Q_i is **S**-equivalent to some member of Q_j. When two squares are not identical they are said to be *distinct.*

From the preceding definitions it turns out that any rotation Q_j of a given Aristotelian square Q_i is *equivalent* to Q_i but not *equal* to Q_j.

2. Aristotelian Cubes and r-Aristotelian Cubes

Now let us move on to define an Aristotelian cube. An Aristotelian cube is an object consisting of six squares: two Aristotelian squares and four semiaristotelian squares.

More rigorously, an Aristotelian *cube* in **S** is a set K $=\{Q_1 \ldots Q_6\}$ such that

(i) each Q_i $(1 \leq i \leq 6)$ in K is a semiaristotelian square in **S** and
(ii) two of the squares in K are Aristotelian squares in **S**
(iii) each edge of each square in K is coincident with the edge of some other square in K.

When an Aristotelian cube contains two squares which are equal w.r.t. to the background logic **S** it will be called a *degenerate* Aristotelian cube in **S**.[5] The is an example of a degenerate Aristotelian cube in CI.0, given that there are two squares consisting of three identical wffs in the same position and of two homologous wffs $A \rightarrow \neg B$ e $B \rightarrow \neg A$ which are are equivalent in CI.0 cube in Fig. 7.

We will agree to qualify the faces of the cube as *anterior, posterior, top, bottom, left, right* from the viewpoint of an observer which looks at the cube

[5]In the special case in which the two Aristotelian squares are identical the cube contains a reduplication of the same Aristotelian square, and such square could conventionally be identified with such degenerate cube. The well-known Blanchè's hexagon may also be identified with a special kind of a degenerate cube (see [30], p. 205).

in frontal position.[6] When two squares have no edge in common they are said to be *opposite* squares, otherwise they are said *adjacent* squares. As a consequence of what has been said, two Aristotelian squares in the same non-degenerate cube cannot have common edges. Suppose in fact that <A, B> is such a common edge. Then at the opposite side of the diagonals starting from A and B we should have wffs C, D such that $\vdash A \equiv \neg C$ and$\vdash B \equiv \neg D$, so two squares in the same cube would turn out be equal, contrary to the hypothesis that the cube is non-degenerate. It follows from this also that two Aristotelian squares in a cube cannot be adjacent but are always opposite squares.[7]

If Q_1 and Q_2 are two opposite Aristotelian squares in an Aristotelian cube K, K could also be represented as the ordered couple $< <Q_1, Q_2>, \{Q_3, Q_4, Q_5, Q_6\}>$. Q_1 and Q_2 may be the top and bottom squares, the left and right squares, the anterior and posterior squares. As for the remaining squares, each one of them has a common edge with Q_1 and Q_2, so their position is univocally determined by Q_1 and Q_2.

Special cases of Aristotelian cubes are what we call *connected* cubes: in such cubes the homologous vertices of the two opposite Aristotelian squares have a relation of subalternance.

Beyond the six squares which constitute the cube we can identify also the squares which may be called *cross-sectional squares,* i.e. the squares in which two edges are the diagonals of two opposite squares. Since there are three pairs of opposite squares and each one of them has two diagonals, in total there are six cross-sectional squares. If a cube is an Aristotelian cube this does not imply that its cross-sectional square are Aristotelian squares. However, it may happen that a non-Aristotelian cube may have cross-sectional squares that are Aristotelian squares. This point will be illustrated in the next section.

A question of some interest is the following. How should we consider a cube where the two basic squares are one the rotation of the other, so they are equivalent, even if not equal squares? We cannot say that this cube is a degenerate cube, since the two squares are not equal. We agree to call it a *self-generated cube*, given that it contains two copies of the same wffs which are however located in different positions inside the square to which they belong. The following is an example of a self-generated cube since the posterior face is the rotation cr3 (i.e. the third circular rotation) of the anterior face. Note that all the other squares of the cube are semiaristotelian (Fig. 8).

[6] It is well known, considering the so-called Necker's cube, that by a Gestalt phenomenon the anterior face may be seen as posterior and viceversa. This is a psychological, not a logical ambiguity, due to the fact the figures representing a cube are 2-dimensional representation of a 3-dimensional object. Anyway, to avoid confusion, in the drawing the difference is graphically marked by the difference between dotted lines and continuous lines. The posterior face is conventionally identified by being drawn with dotted lines, while the anterior face is drawn with continuous lines.

[7] In [1] Béziau proves that there is no cube of opposition such that each side of it is a square of opposition. In [28] Pizzi proves an akin result reported also in the present text: it is impossible that there exists more than two Aristotelian squares in the same non-degenerate cube.

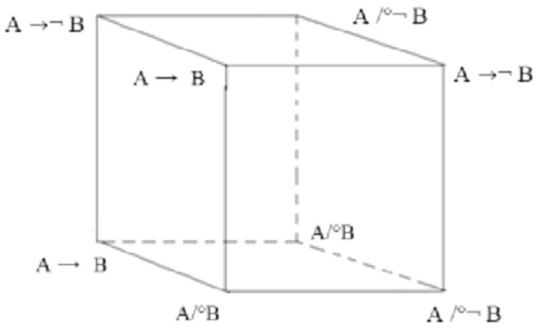

FIGURE 8. Example of a self-generated cube, i.e. of a cube containing two equivalent squares

In what follows we will call *r-Aristotelian cube* every cube which is derived from an Aristotelian cube by one or more rotations of some of its Aristotelian or semiaristotelian squares. A r-Aristotelian cube may be an Aristotelian cube, but generally it is not such.

If a cube K' is such that each one of its squares is equivalent to some of the squares of a cube K", K' and K" will be said *equivalent cubes*. Notice that this definition implies that if a cube K' is a r-Aristotelian cube resulting from the rotation of some square in an Aristotelian cube K", K' and K" are equivalent cubes.

The notion of a r-Aristotelian cube should not be confused with the notion of a rotation of a cube. A cube can be rotated around some edge of its squares or by a concomitant circular rotation of two opposite squares, but this has no impact on the internal interrelations of its squares. In what follows we will neglect the rotations of a cube, being irrelevant in this context that a square which appears frontal in some representation may, for instance, appears top or bottom in some different representation of the same cube.

3. Lenzen–Caramuel's Cube and Luzeaux Cube

Let us now examine various kinds of Aristotelian cubes which can be found in the literature and that appear to be different from what we have defined as an Aristotelian cube.

It would be useful—but it will not be made here—to summarize an historical reconstruction of the idea of generalizing Aristotle'squares to more complex figures. An interesting example is so-called Buridan's Octagon.[8] When an Octagon contains two squares of oppositions it may represented as a cube, i.e. as a 3-dimensional objects. An Octagon representable as a cube may be found

[8] See S. Read's analysis in [33].

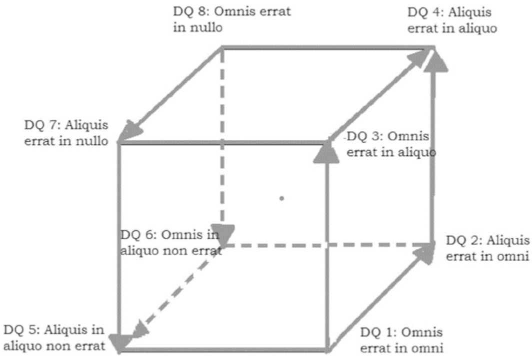

Figure 9. The Lenzen-Caramuel cube

in Johnson's [18] Part 1, p. 142. Other octagons also representable as cubes are the Moretti–Pellisier–Octagon and the Bèziau Octagon.[9]

In various papers written by Prof. Wolfgang Lenzen one can find what he calls "cube of opposition" (see [20,21]). In [20] Lenzen reconstructs the octagon of oppositions introduced in 1654 by the Spanish philosopher and theologian Juan Caramuel. Lenzen represents the octagon as a cube having the same structure of a cube he introduced in another paper to represent Leibniz's theory of oppositions. The arrows represent subalternance (i.e. logical implication) while the non directed lines connect pairs of contradictory propositions (See Fig. 9).

An intuitive variant of this cube could be expressed in the language of temporal modalities. In other words the variant would be one in which the propositions make reference not to the object of error but to the time of error. DQ1 would then become "Everyone always errs", DQ2 "Someone always errs" and so on.[10] Anyway, if we use first order language to formalize the propositions of the above cube, the wffs of the right square, for instance, result as follows:

(DQ 1) $\forall x \forall y$ E$(x,\ y)$ (Everyone errs in everything)
(DQ 2) $\exists x \forall y$ E$(x,\ y)$ (Someone errs in everything)
(DQ 3) $\forall x \exists y$ E$(x,\ y)$ (Everyone errs in something)
(DQ 4) $\exists x \exists y$ E(x,y) (Someone errs in something).

In what follows, for sake of simplicity, every square will be identified by the ordered 4-tuple of numbers occurring in the names of its vertices: <7, 3, 1, 5> for instance will denote the anterior square. It is easy to realize that all

[9]For a survey of such Octagons see the site https://logicalgeometry.org/diagrams/two_dimensional. In the case of what is called here Buridan's Octagon (not found in Read's paper) the two squares constituting the cube are $< \Diamond p \wedge \Diamond \neg p, \Box \neg p, \Box p \vee \Box \neg p, \Diamond p >$ and $<p \wedge \neg \Box p, \neg \Diamond p, \neg p \vee \Box p, \Diamond p >$.

[10] The modal octagon due to Buridan and reproduced at p. 17 of [33] contains the same propositions but with alethic modalities in place of temporal modalities. The analogy between Caramuel's cube and Buridan's octagon is highligted by Lenzen at p. 23 of his paper.

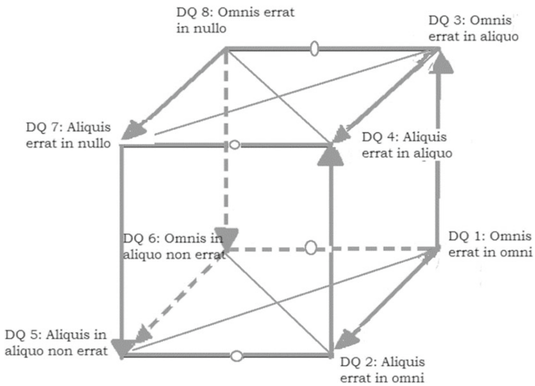

FIGURE 10. An Aristotelian cube equivalent to the Lenzen-Caramuel cube of Fig. 9

the faces of the cube are semiaristotelian squares, but none is an Aristotelian square. Even the cross-section squares are not Aristotelian squares.

A first remark about Lenzen–Caramuel cube is that interchanging, in the top square, the diagonal <8, 3> with the edge <8, 4> and interchanging, in the bottom square, the diagonal <6, 1> with the edge <6, 2>, and consequently changing the direction of the involved arrows, the result is the cube of Fig. 10.

Looking at the top square <8, 3, 4, 7> and to the bottom square <6,1, 2, 5>, one can see that the uninterrupted diagonal lines connect contradictory statement, while the lines with a circlet connect contrary or subcontrary statements. Thus <8, 3, 4, 7> and <6, 1, 2, 5> are Aristotelian squares, while the other squares are semiaristotelian. The cross section squares <8, 4, 2, 6> and <7, 3, 1, 5> are the two squares that in Lenzen original cube were in anterior and posterior position.

The cube of Fig. 10 is then an Aristotelian cube, and with a one-step rotation of the cube the top square may be moved in frontal position so to give an Aristotelian cube in standard configuration.

Now we may observe that in place of performing an interchange among diagonals and edges there is a more direct way to describe the relevant transformation. We may say in fact that, given the cube of Fig. 9, the cube of Fig. 10 is the result of applying the vertical rotation to the (semiaristotelian) right square <3, 4, 2, 1>, which after rotation becomes <4, 3, 1, 2> i.e. the right square of the cube of Fig. 10. Reversing the procedure, i.e. applying to cube of Fig. 10 the vertical rotation of the right face, we obtain the Lenzen–Caramuel cube of Fig. 9. Such cube is then a r-Aristotelian cube, given that it is derived by rotation of a square in an Aristotelian cube, and is equivalent to it. Furthermore note that the cube of Fig. 10 is a connected cube, since the homologous vertices of the two Aristotelian squares are connected by a relation of logical implication.

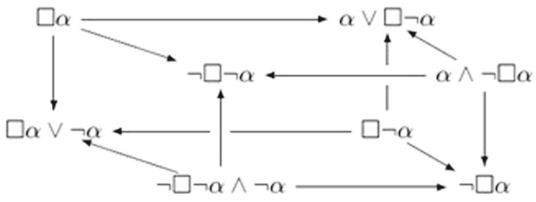

FIGURE 11. A cube representing the logical relatons among the first-degree modal propositions in S5

Lenzens' cube in Fig. 9 is based on relations of logical implication and contradiction. It is to be noted however that logical cubes have been frequently used in logic to visualize simply relations of logical implication among formulas—relations which may subsist or not subsist w.r.t. the background logic. The cube of Fig. 11 may be found in Luzeaux et al. [22] (p.175) and is intended to exhibit the logical relations among the first-degree modal propositions in the modal logic S5.

The authors remark that the four vertices that are at the origin of the implicative arrows are reciprocally contraries and form a tethraedon. Another tethraedon is formed by subcontrary formulas. Furthermore, it is to be noted that $\Box A$ contradicts $\neg\Box A$, which is located in a vertex which is at the maximum distance from it.[11] Idem for $A \vee\Box\neg A$ with respect to its contradictory $\neg A \wedge \Diamond A$. Note that the two cross-section squares $< \Box A, A \wedge\neg\Box A, \neg\Box A, \Box A \vee \neg A>$ and $< \Diamond A \wedge \neg A, \Box\neg A, A \vee \Box\neg A, \Diamond A>$ on the contrary are Aristotelian. The cube is not obviously an Aristotelian cube since none of its squares is Aristotelian.

But the horizontal rotation of one of its squares, the right square $<A \vee \Box\neg A, A \wedge \neg\Box A, \Box\neg A, \neg\Box A>$ yields a r- Aristotelian cube. In fact, inverting $\neg\Box A$ with $A \wedge\neg\Box A$ and $\Box\neg A$ with $A \vee\Box\neg A$, as a result we have at the top and at the bottom two squares which are respectively Aristotelian and r-Aristotelian. At the top we have in fact the classical square $< \Box A, \Box\neg A, \neg\Box A, \Diamond A >$ and at the bottom $< \Box A \vee \neg A, A \vee\Box \neg A, {}^{(*)}A \wedge\neg\Box A, \Diamond A \wedge\neg A>$. To obtain an Aristotelian cube we have simply to apply a double anticlockwise rotation to the bottom square, so to move $A\wedge\neg\Box A$ at the origin of the square in order to obtain a standard Aristotelian square. It is trivial to check that all other squares are semiaristotelian.

4. Dubois Cube, Johnson–Keynes Cube and Moretti Cube

In various papers authored by D. Dubois in cooperation with other researchers[12] we can find the following figure, written in the language of first-order

[11] There is a correspondence between strength of the logical opposition and geometrical distance, as noticed by Demey and Smessaert [9].
[12] See [15–17].

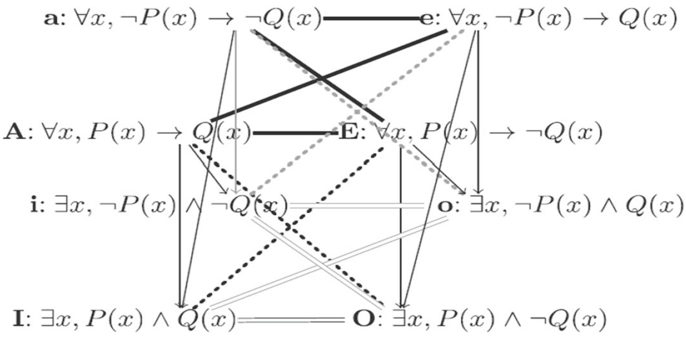

FIGURE 12. The Dubois cube written in first-order language

logic. The letters A, E, O, I stand for the traditional names Universal Affirmative, Universal Negative, Particular Affirmative, Particular Negative. The minor letters *a, e, o, i* stand for wffs which are obtained from the wffs denoted by the corresponding capital letters by prefixing a negation sign to their atomic subformulas (Fig. 12).

Assuming as a premise that (i) some P and some not-P exist ($\exists x \mathrm{P}x$ and $\exists x\ \neg \mathrm{P}x$) and (ii) some Q and some not-Q exist ($\exists x \mathrm{Q}x$ and $\exists x\ \neg \mathrm{Q}x$) the thick non-directed segments connect contraries, the double thick non-directed segments connect subcontraries, the diagonal dotted non-directed segments connect contradictories, the vertical arrows connect subalternants. (Warning: the arrow occurring in the figure has not the same meaning of the arrow intoduced as symbol of consequential implication in the first section but stands for the traditional horseshoe $\supset$, i.e. the symbol for material conditional).

As Dubois remarks, the anterior and posterior faces are Aristotelian squares. But unfortunately this is not an Aristotelian cube in the sense defined in §3 since the remaining faces are not semiaristotelian squares. In particolar the wff $\forall x\ (\neg \mathrm{P}x \supset \neg \mathrm{Q}x)$ has no Aristotelian relation with $\forall x\ (\mathrm{P}x \supset \mathrm{Q}x)$, even if it is equivalent to its contrapostive $\forall x\ (\mathrm{Q}x \supset \mathrm{P}x)$.

In one of papers written by Dubois et al. an analogous cube is formulated in set-theoretical language, i.e. in a language that describes the interrelations between the sets designed by the predicate variables (Fig. 13).

Making a further translation, the cube written in set-theoretical language might be translated in the language of propositional modal logic assigning to the variables sets of possible worlds (propositions) and translating complementation into negation, intersection and union in conjunction and disjunction respectively, set-inclusion into so-called strict implication symbolized by $⥽\ q$. The assertion that a set P is not-empy may be translated in the statement asserting that the proposition that p is possible ($\Diamond p$), its meaning being that p is true in at least one possible world. For sake of simplicity one may assume that the background modal logic is S5 (which as well-known, is translationally equivalent to the fragment of first order logic containing monadic predicates).

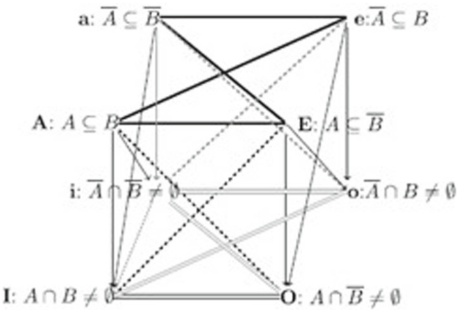

Figure 13. The Dubois cube written in set-theoretical language

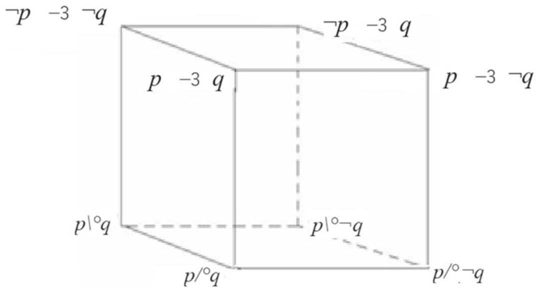

Figure 14. A modal translation of the cube of Fig. 12 and Fig. 13

Under the presuppostion that $\Diamond p$ and $\Diamond\neg p$ are true and that $\Diamond q$ and $\Diamond\neg q$ are also true, the cube of Fig. 14 may be seen as a translation, *mutatis mutandis*, of the preceding ones.

Just as before, the two anterior and posterior faces are two Aristotelian squares, but the other squares cannot be said semiaristotelian: so this is not an Aristotelian cube, even if its cross-sectional squares are Aristotelian. However we can show that it is an r-Aristotelian cube since it may be transformed into an Aristotlian cube and *vice versa* with some rotation: more specifically, with the vertical rotation of the posterior square (Fig. 15).

The anterior square is, as before, the classical square for the implication, while the posterior square $< \neg p \mathrel{-\!\!3} q, \neg p \mathrel{-\!\!3} \neg q, \neg(\neg p \mathrel{-\!\!3} q), \neg(\neg p \mathrel{-\!\!3} \neg q) >$ is also Aristotelian. The reader will note that the two kinds of contrariety described in §2 as a property of consequential implication allow for the construction of an analogous cube. The lateral squares are semiaristotelian but they are anyway interesting squares. In fact, given that $\Diamond\neg p$ is among the presuppositions and that thanks to $\neg p \mathrel{-\!\!3} -q$ it implies $\Diamond\neg q$, $\neg p \mathrel{-\!\!3} q$ is the oblique contrary of $p \mathrel{-\!\!3} q$ (to understand this point it is enough to consider that $\neg p \mathrel{-\!\!3} q \wedge p \mathrel{-\!\!3} q$ implies $\Box q$, i.e. the negation of the presupposition $\Diamond\neg q$), while $\neg p \mathrel{-\!\!3} \neg q$ is oblique contrary of $p \mathrel{-\!\!3} \neg q$ (by the same argument their conjunction implies $\Box\neg q$, which is the negation of the presupposition $\Diamond q$). The reader may check

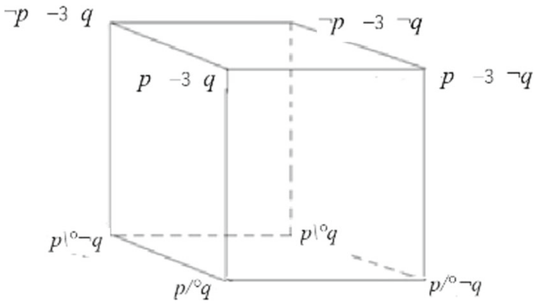

FIGURE 15. An Aristotelian cube equivalent to the one of Fig. 14

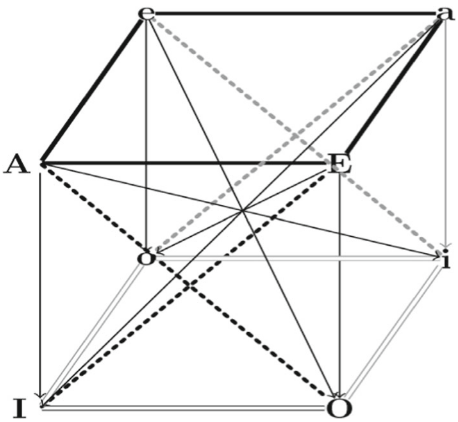

FIGURE 16. An Aristotelian cube equivalent to the Dubois cube of Fig. 12

that the top and the bottom squares are also semiaristotelian. ($\Diamond(p \wedge q)$ and $\Diamond(\neg p \wedge q)$ are subcontraries in the light of $\Diamond p$, $\Diamond\neg p$, $\Diamond q$, $\Diamond\neg q$).

Applying the same operation, i.e. vertical rotation of the posterior square, to the Dubois Cube of Fig. 12, such cube after this trasformation will have the shape of the cube in Fig. 16, i.e. the shape of a perfect Aristotelian cube in the sense defined at p. 7. Dubois cube is then a r-Aristotelian cube.

It is of some interest to observe that in the literature one can find other cubes which have a kinship with some the ones listed in this section. A first reference is the so-called Johnson–Keynes cube which is reproduced in Boffa [2], p. 190 and has its source in Neville's Keynes [19] (See Fig. 17). But also, as already said, an octagon of implications and oppositions reconstruable as a 3- dimensional cube was introduced in Johnson 1921, part I, p. 142.[13]

[13]It is remarkable that Johnson introduced also a distinction between the Universal Affirmative A_n without existence presuppositions and the Universal Affirmative A_f with existence presuppositions. A_f logically implies A_n but not viceversa. This gives origin to a pair of Aristotelian squares, so the diagram presented by Johnson at p. 136 of Part I of [18] lends

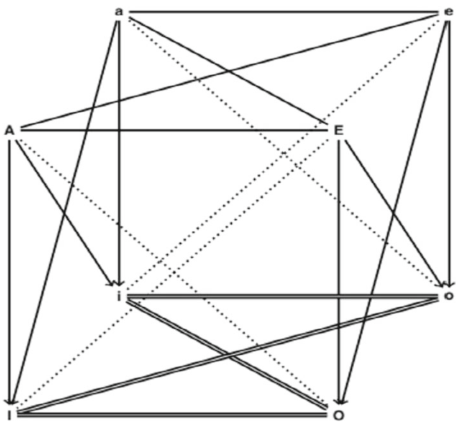

Figure 17. The Johnson-Keynes cube

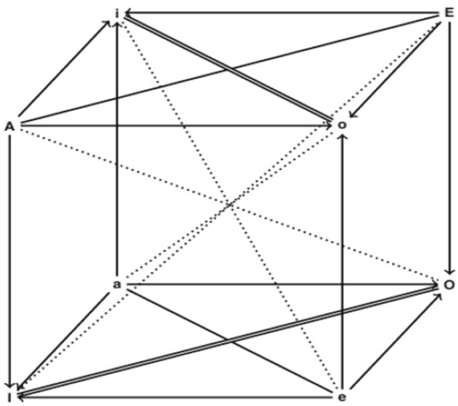

Figure 18. The Moretti cube

We have to remark that this cube is structurally similar to Dubois cube of Fig. 12. Boffa comments that in order to consider the cross-section squares AeiO and aEIo as Aristotelian squares (they are not such) the additional condition $(A \vee a) \supset (\neg E \wedge \neg e)$ should be satisfied.

A different construction, represented in Fig. 18, is what Boffa et al. ([2], p. 189) call Moretti cube. The meaning of the letters is the same as before, but one can note that the two squares <A, E, O, I> and <i, o, e, $^{(*)}$a> (one Aristotelian and the other r-Aristotelian) are cross-section squares of the cube and not opposite faces. Applying the horizontal rotation to the square <i, o, e, $^{(*)}$a> we obtain two cross-section squares which are perfect Aristotelian squares.

itself to be reconstructed as an Aristotelian cube. For a recent development of this idea see Chatti–Schang [3] and the last section of this paper.

Moretti cube has some kinship not with Dubois cube but with Lenzen-Caramuel's cube of Fig. 9, in the sense that performing a vertical rotation of the right side of the cube of Fig. 18, i.e. <o, E, O, e>, we obtain <A, E, O, I> as anterior square of the cube and <i, o, e, a> as posterior square, i.e. two squares which are respectively Aristotelian and r-Aristotelian. The further horizontal rotation of <i, o, e, a> transforms this square into the Aristotelian square <a, e, o, i>. So Moretti cube is transformed by a double rotation into the Johnson–Keynes cube.

5. Reichenbach's Cube as a Supposed Predecessor of Dubois Cube

Given the strong analogies between some of the above discussed cubes, it is of interest to read the remarks that Dubois et al. wrote as a comment to the cube of Fig. 12[14]: "This cube, rediscovered in [20], is rarely mentioned; it apparently appeared for the first time in Reichenbach's modern study of syllogisms [36] in the middle of last century."

The item [20] mentioned in this quotation is listed in the bibliography of the present paper as [14], while [36] is here listed as [35]. When the authors of the quotation speak of having "rediscovered" the cube strangely they make no reference to the Johnson–Keynes cube, which is structurally identical to their own, but make a reference to an historically more recent Reichenbach's paper by saying that "apparently" their cube appeared for the first time in this work.[15] Let us see, however, what Reichenbach actually stated in the paper that in their bibliography Dubois et al. list as [36] and that in the bibliography at the end of this paper is listed as [35].

Reichenbach writes SAP to say "all the S are P" and SIP to say "some S is P". In his construction he also takes for granted that the sets referred to by the predicates S and P are non-empty. The letters AEIO and aeio are to be read as illustrated in the preceding pages. In Reichenbach's paper we find the image of Fig. 19.

In order to understand the relations among the wffs belonging to this cube it is important to spell out the restrictions which Reichenbach formulates at p. 4 of his paper: the cube holds "on the condition that none of the four classes S, P, not-S, not-P, is empty and no two of them are identical"(p. 4).

By *opposite* propositions Reichenbach means that "not both are true. (Condition: classes are not identical)". For instance, two converse propositions such as "All men are mortal" and "All mortal are men" are not joinly true due to the fact that the two classes have not the same members. By s*ubopposite* propositions he means that "not both are false". (Condition: the first class of one expression must not be identical with the second class of the other

[14] [17], p. 170.
[15] An anonymous referee remarked that this wrong information was already noticed also in the paper [11].

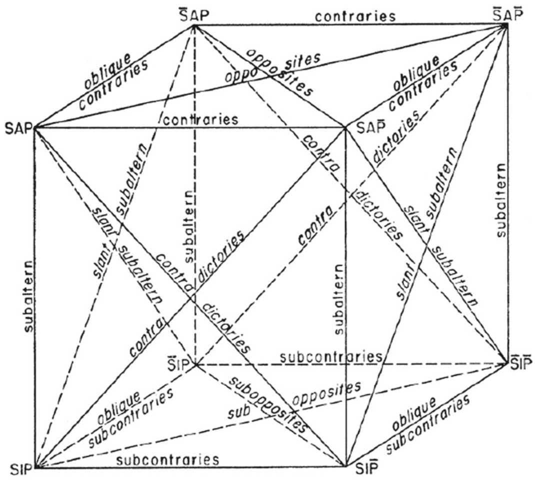

FIGURE 19. The Reichenbach cube

expression)". The background logic which a neopositivist like Reichebach took for granted was obviously FOL (First Order Logic)

It is straightforward to note that the anterior and the posterior faces are two Aristotelian squares, and that the others are semiaristotelian squares. So Reichenbach's cube is an Aristotelian Cube exactly in the sense which has been defined at the beginning of the present paper, and the reader will also note that it is analogous to the cube of Fig. 16. The problem is, as already noted, that Reichenbach's cube is not the Dubois cube but a cube which is obtained from it by rotation of one of its faces.[16]

A plausible conclusion which may be drawn from the preceding remarks is that the primary notion of a cube of opposition is Reichenbach's one, which is essentially coincident with the one proposed in the present paper (save for a detail which will be evidenced in the last section), while the others notions are secondary since the cubes they define may be derived from Reichenbach's one by means of some kind of rotation. Of course, since any rotation can be

[16]Reichenbach distinguished, as is standardly made, between contrary propositions A and B (in proper and oblique sense) and opposite propositions A and B on the basis of the different presuppositions which grant their truth: non-vacuity of the sets [**A**] and [**B**] and their complements for the former and non-identity of **A** and **B** for the latter. In case we want to simplify terminology and unify all such relations under the name of relations of contrariety, we would have six presuppositions for contrariety instead of five. In such a case, in order to make a comparison with the cubes treated in the preceding and following sections, the terminology and the presuppositions should be suitably extended to the kinds of cubes taken into consideration for comparison. In order to avoid unnecessary complications, in the present papers the only presuppositions which are judged to be interesting for the central topic are the presuppositions of existence (see next section). Furthermore, the relation of opposition defined by Reichenbach is not an "Aristotelian relation" as defined at p. 2, so it cannot be used to define a semiaristotelian square and indirectly to define an Aristotelian cube.

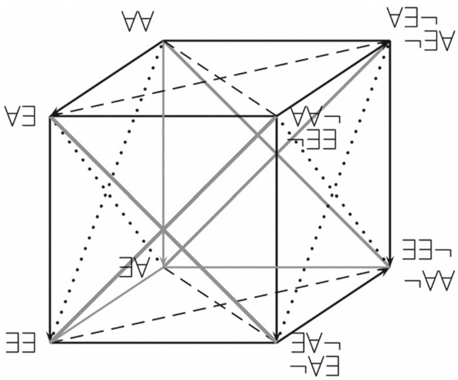

Figure 20. The Nilsson's cube

performed in two directions, one could reverse the proposed order of priority and maintain, for instance, that Dubois cube may be considered primary and Reichenbach's cube secondary. The problem is however that in Dubois cube of Fig. 12 the faces that are not Aristotelian are not even semiaristotelian. The figure which appears in Johnson's [18] at page 142 is construed just to stress that the vertices a and A, e and E i and I, o and O are logically independent (or, as Johnson say, complementary), while it seems to be a positive quality of a cube that the vertices connecting the edges of any square inside it should have between them some Aristotelian relation.

The assunption that the notion of a cube defined here should be considered primary receives support by the recent paper by Nilsson (see [25]), in which the author tries to provide a more general notion of a cube written in first order language, taking FOL as a background logic. Nilsson's key idea is the one of building a cube written a language containing both monadic and dyadic predicates and nested quantifiers. In the cube of Fig. 20 the non-dotted diagonals symbolize the contradictory propositions.

Here it is straighforward to realize that the anterior face and the posterior face consist of two Aristotelian squares, while the other squares are semiaristotelian squares. Nilsson's cube is then a cube in the sense defined at the beginning of the present paper. It also turns out that it is a connected cube, in the sense that the homologous vertices of the two Aristotelian squares are related by implication relations.

The symbols used by Nilsson in his diagram are explained so:

$\forall\forall)\forall x(Cx \supset \forall y\ (Dy \supset Rxy))$ every CRs every D
$\forall\exists)\ \forall x(Cx \supset \exists y\ (Dy \wedge Rxy))$ every CRs some D
$\exists\forall)\exists x\ (Cx \wedge \forall y(Dy \supset Rxy))$ some CRs every D
$\exists\exists)\exists x\ (Cx \wedge \exists y\ (Dy \wedge Rxy))$ some CRs some D
$\forall\forall\neg)\ \forall x\ (Cx \supset \forall y\ (Dy \supset \neg Rxy))$ every Cnot Rs (every) any D
$\neg\exists\exists)\ \neg\exists x\ (Cx \wedge \exists y(Dy \wedge Rxy))$ not some CRs some D
$\forall\exists\neg)\ \forall x(Cx \supset \exists y\ (Dy \wedge \neg Rxy))$ every Cnot Rs some D

$\neg\exists\forall$) $\neg\exists x\ (Cx \wedge \forall y(Dy \supset Rxy))$ not some CRs every D
$\exists\forall\neg$) $\exists x\ (Cx \wedge \forall y(Dy \supset \neg Rxy))$ some C not Rs (some) any D
$\neg\forall\exists$) $\neg\forall x\ (Cx \supset \exists y(Dy \wedge Rxy))$ not every CRs some D
$\exists\exists\neg$) $\exists x\ (Cx \wedge \exists y(Dy \wedge \neg Rxy))$ some Cnot Rs some D
$\neg\forall\forall$) $\neg\forall x\ (Cx \supset \forall y(Dy \supset Rxy))$ not every CRs every D

An interesting special case of this kind of cube is the one in which R is the identity predicate (=). This gives the following set of eight subject-copula-predicate propositions:

$$\left\{\begin{matrix}\text{every}\\ \text{some}\end{matrix}\right\} C \left\{\begin{matrix}\text{is}\\ \text{is not}\end{matrix}\right\} \left\{\begin{matrix}\text{every}\\ \text{some}\end{matrix}\right\} D$$

The reader can then check that the anterior Aristotelian square boils down to the standard square of non-nested quantified statement whose prefixes are $\forall$, $\forall\ \neg$, $\exists$, $\exists\neg$, while the four remaining unorthodox cases form the posterior square.

6. The Problem of Existential Import in the Costruction of Cubes of Opposition

To conclude this survey, it is important to recall that the logical relations established among the propositions belonging to the cubes analyzed in the preceding pages (Lenzen, Dubois, Reichenbach, Johnson–Keynes, Nilsson) are not unconditionally valid but depend on a set of existential propositions which are assumed to be true. In the case of strict implication the presuppositions concern of course the existence of possible worlds with certain properties, so they consist of possibility statements. The presence of these restrictions in a sense casts a shadow about the presentability of all such relations by means of squares and cubes. It is difficult to disagree with Chatti and Schang when in [3], p. 101 write: "The problem of existential import might be seen as a challenge to the theory of oppositions expressed by the traditional square of oppositions". Their paper actually tries to give an answer to the problem by introducing a non-standard formalization of the propositions normally occurring in traditional syllogistic.

(1) Every S is P: ∃xSx ∧∀x(Sx ⊃ Px) **A**imp!
(2) Every S is not-P: ∃xSx ∧∀x(Sx ⊃ ¬Px) **E**imp!
(3) Some S is P: ∃x (Sx ∧ Px) **I**imp!
(4) Some S is not-P: ∃x (Sx ∧ ¬Px) **O**imp!
(5) No S is not-P: ∀x(Sx ⊃ Px) **A**imp?
(6) No S is P: ∀x (Sx ⊃ ¬Px) **E**imp?
(7) Not every S is not-P: ¬∃x Sx ∧∀ x (Sx ⊃ ¬Px)] **I**imp?
(8) Not every S is P: ¬(∃x Sx ∧∀x (Sx ⊃ Px)) **O**imp?

Using the denominations introduced in the third column above, the relations between the mentioned propositions may be represented by the following cube, introduced at p. 122 of the paper under discussion.

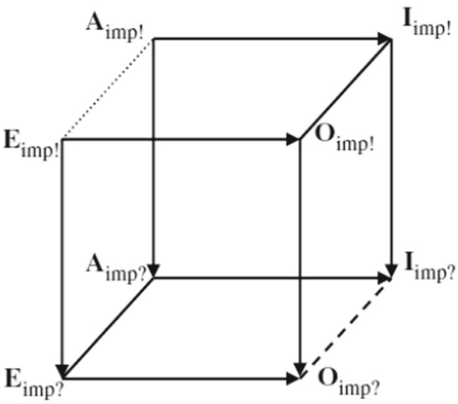

Figure 21. The Chatti-Schang cube

Taking FOL as a background logic as the authors do, the reader can check that the cross-sectional squares <Aimp! Oimp! Oimp? A imp?> and <Eimp! Iimp! Iimp? Eimp?> are Aristotelian squares. Thanks to the vertical rotation of the left square, as already made with the Lenzen–Caramuel cube at page 10, we obtain a cube with has <Aimp! Oimp! Oimp? A imp?> as anterior square and <Eimp! Iimp! Iimp? Eimp?> as posterior square. Both are Aristotelian squares and since all the other squares are semiaristotelian, the result of the operation is a perfect Aristotelian cube. To sum up, the above cube is a r-Aristotelian cube which has the important merit of presenting interrelations which do not depend on premises of existential form.

Contrary to the cube of Fig. 21, the cube presented by Read [34] (see Fig. 22) and quoted by Chatti and Schang in their paper, has the shape of what is called here an Aristotelian cube: in fact the opposite faces <A, A*, I*, I> and <E, E*, O*, O> where the starred letters denote propositions with negative predicates, are Aristotelian squares, while the other squares are semiaristotelian.[17] The affirmative propositions have existential import, the negative lack it. A, A*, E*, E correspond to the propositions above named as Aimp!, Eimp!, Aimp? E imp? respectively. Every affirmative proposition has then two contraries, as in the cube of Fig. 22.

There is however a problem which is not discussed by Chatti, Schang and Read, and concerns the logical properties of the propositions formalized in the proposed way. Without opening a wider discussion, let us simply remark that the assertion of Identity *Id*: "Every P is P", according to the authors' proposal, is formalized as $\exists x \mathrm{P}x \wedge \forall x\ (\mathrm{P}x \supset \mathrm{P}x)$. This means that if *Id* is a logical truth also $\exists x \mathrm{P}x$ should be such, but $\exists x \mathrm{P}x$ is a non-theorem in FOL. An analogous criticism applies to other formulas of this list.

[17] The difference between the two cubes is highlighted by Chatti and Schang at p. 24 of their paper.

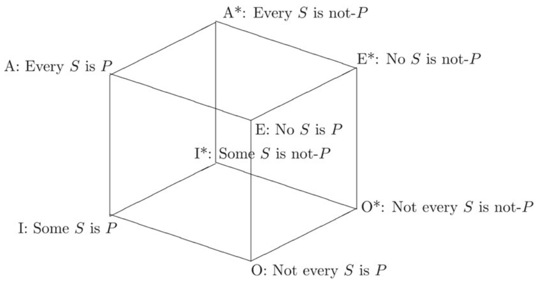

FIGURE 22. The Read cube

Obviously if the quantificational logic assumed as a background logic is not FOL the preceding considerations may be inappliable. De Kerk, Vignero, Demey in [4], for instance, introduce a logic named SYL which is FOL extended with $\exists x\mathrm{P}x$. This move restores the basic Aristotelian square of quantified formulas by removing the main presupposition of existence, but a drawback of SYL is that it imposes a restriction on Uniform Substitution in order to prevent the derivation of the contradiction $\exists x(\mathrm{P}x \wedge \neg \mathrm{P}x)$.

An open problem is then to define some non-standard notion a quantifer which is such as (i) to avoid the dependency on existence presuppositions (ii) to grant the unrestricted validity of Identity (iii) to keep FOL as a background logic.

A suggestion in the proposed direction comes from the remark that no presupposition is necessary for the logic of consequential implication outlined at the beginning of the present paper. We recall that in such logic A $\rightarrow$ B implies $\neg$(A $\rightarrow$ $\neg$B) (so that A $\rightarrow$ B and A $\rightarrow$ $\neg$B are contraries) without the presupposition that A is possibile. A merit of consequential implication and of its close relative, connexive implicative, is that it allows the costruction of squares and cubes without the dependency on some extra presupposition.

This has a consequence on the construction of first order squares and cubes. In order to remove the dependency of universally quantified statements on such presuppostions as $\exists$xPx, $\exists x\mathrm{P}x, \exists x\ \neg\mathrm{P}x\ \exists x\mathrm{Q}x$ and $\exists x\ \neg \mathrm{Q}x$ one could introduce a definition of the special quantifiers modelled on the definition of consequential implication in terms of monadic modal statements mentioned at p. 3 of the present paper. In [29] Pizzi suggests not one but two definitions of non standard quantifiers of different strength in the framework of FOL. We recall here the definition of the stronger universal quantifier which is the following:

(Def$\forall^{\rightarrow}$) $\alpha(\mathrm{x})\ \forall^{\rightarrow}\beta(\mathrm{x}) =_{\mathrm{df}} \forall\mathrm{x}(\alpha(\mathrm{x}) \supset \beta(\mathrm{x})) \wedge (\forall\mathrm{x}\alpha(\mathrm{x}) \equiv \forall\mathrm{x}\beta(\mathrm{x})) \wedge (\exists\mathrm{x}\alpha(\mathrm{x}) \equiv \exists\mathrm{x}\beta(\mathrm{x}))$

The symbol $\alpha(\mathrm{x})\ \exists/\circ\ \beta(\mathrm{x})$ is defined as $\neg(\alpha(\mathrm{x})\ \forall^{\rightarrow}\neg\beta(\mathrm{x})$.

An obvious consequence of (Def$\forall^{\rightarrow}$) is that the Identity $\alpha(\mathrm{x})\ \forall^{\rightarrow}\alpha(\mathrm{x})$ is preserved as a logical property of the defined operator. The property of $\forall^{\rightarrow}$are then coincident, *mutatis mutandis* (i.e. replacing modal operators with the homologous quantifiers) with the ones of the implication symbolized by the

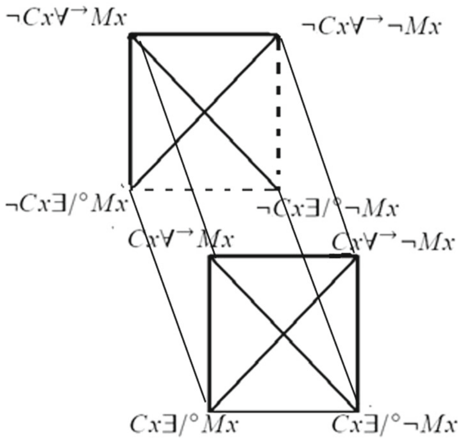

Figure 23. An Aristotelian cube of opposition independent from existential assumptions

arrow $\rightarrow$ (see page 3). In particular $\forall^{\rightarrow}$ satisfies both Boethius' Thesis and Secondary Boethius. Consequently the Aristotelian cube for $\forall^{\rightarrow}$ will have the shape of Fig. 23.

As the reader may check, this cube has the same structure of Reichenbach's cube of Fig. 19, with the remarkable difference that none of the represented relations depends on any non-logically true assumption.

7. Conclusion

In the present paper squares and cubes of opposition are not treated as geometrical objects but as logical objects, more precisely as sets of propositions endowed with specific logical properties. This should be clear from the fact that such properties depend on the properties of some logic which is assumed as a background logic and in some cases on presuppositions which are not logical truths w.r.t. the given background logic (in the first place existence presuppositions). In [31] the present author proposed to highlight graphically such presuppositions by locating them at the cross of the diagonals of the square. Furthermore—to mention the main point—he proposed to call "*Subaristotelian square*" any square which turns out to be Aristotelian thanks to the essential use of such non logical presuppositions. According to this restrictive notion of an Aristotelian square (so implicitly of an Aristotelian cube) most of the squares and cubes examined in the present paper should not properly be considered Aristotelian. It is however remarkable that a lot of the recent inquiries mentioned in the preceding pages are oriented toward the removal of the dependency on non-logical presuppositions. This means trying to avoid treating with the so-called Subaristotelian squares and favoring Aristotelian squares in proper sense. As the reader will notice, this progress has been realized by various authors along two different lines: (i) embodying the needed presuppositions in the definition of some non standard operator with the required properties (ii) modifying the set of axioms of the background systems normally used for this kind of analysis.

After this premise, there is no denying that all the notions used in this paper, beginning from the very notions of a square and of a cube, are primarily geometrical notions whose interest goes beyond their ability of providing didactically useful visualizations. In the last decades an important progress in mathematics has been provided by two fields of inquiry known as "Logical Geometry"[18] and "Oppositional Geometry".[19] In both paradigms Aristotelian squares and cubes turn out to be special cases of multidimensional structures which are studied for their intrinsic geometrical qualities. In Logical Geometry much interesting work has been devoted to the study of morphisms between such structures, while one of the founders of Oppositional Geometry, A. Moretti, has successfully worked on a geometrical representation of oppositions occurring in extra-logical contexs (linguistics, formal ontology, AI, philosophy and humanities). There is no doubt that the strictly logical approach followed in the present paper may receive a positive impact from the actual developments of Logical and Oppositional Geometry. Conversely, however, it may be that the logical inquiry is able to offer non trivial suggestions to researchers in the mathematical field: an example could be given the notion of r-Aristotelian cube, which in this paper is introduced as a key concept for a comprehensive understanding of the notion of a cube of opposition.[20]

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[18] See in the bibliography, beyond the others mentioned in the text, the papers by Demey, Smessaert, Vignero [5],[6],[7],[8],[10],[13],[37]. The website of Logical Geometry may be found at link https://logicalgeometry.org.

[19] A basic reference for oppositional geometry is in the PhD dissertation by Moretti [23], which has an important antecedent in the work of Pellissier (cf. [26]). See also Moretti [24].

[20] The author wishes to thank the two anonymous reviewers for concretely contributing to improve the quality of the original article.

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Claudio E. A. Pizzi
Emeritus Siena University
Siena
Italy
e-mail: pizzi4@gmail.com

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