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Natural Deduction for Quantum Logic

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Abstract. This paper presents a natural deduction system for orthomodular quantum logic. The system is shown to be provably equivalent to Nishimura's quantum sequent calculus. Through the Curry–Howard isomorphism, quantum λ-calculus is also introduced for which strong normalization property is established.

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1. Introduction

This paper presents a natural deduction system for orthomodular quantum logic. The main contributions of the system are as follows. First, thanks to its intrinsic and straightforward appearance, we can understand the meaning of inference in quantum logic deeply by comparing the system with those for other logics (e.g. intuitionistic logic or classical logic). Second, we can also introduce the corresponding quantum λ-calculus, which allows us to further investigate computational theories based on quantum logic, via the Curry–Howard isomorphism. Third, we can establish a desirable property regarding normalization of proofs, or equivalently, termination of computation.

As an earlier study for quantum natural deduction, we need to mention Delmas-Rigoutsos's *double deduction system* [5], which incorporates a concept of compatibility into a natural deduction system for classical logic. Differently from this approach, we define a natural deduction system that directly corresponds to **GOM**, Nishimura's quantum sequent calculus [13].

Besides **GOM**, a few systems for quantum sequent calculus have been proposed by Cutland and Gibbins [3] and Nishimura [14]. Furthermore, an extended logical system containing quantum logic called *basic logic* along with its sequent calculus has been studied by Sambin et al. [19], Faggian and Sambin [7], and Dalla Chiara and Giuntini [4]. These systems, however, are all inadequate for being translated into natural deduction forms due to their complex treatment of negation ($\neg$) and cut.

It is known that quantum logic has no satisfactory implication operation ($\rightarrow$) as in the case of intuitionistic logic or classical logic. Indeed, Nishimura's **GOM** only adopts conjunction ($\wedge$) and negation ($\neg$) as the basic set of operations. On the other hand, it is almost inevitable to include implication in the basic set of operations for the purpose of developing a natural deduction system and the corresponding λ-calculus. To handle such a contradiction, we employ the *Sasaki hook*, a kind of quasi-implication, as one of the basic operations of our system. Although it fails to satisfy the deduction theorem, the Sasaki hook still enjoys some expected properties of implication such as *modus ponens*. The problem of implication in quantum logic is what has been largely addressed in the literature; for details, the reader is referred to Hardegree [9], Herman et al. [11], Pavičić [15], Malinowski [12], Pavičić and Megill [16], Roman and Zuazua [18], Ying [22], Chajda and Halaš [1], Harding [10], Chajda [2], and Younes and Schmitt [23].

Another problem that we encounter when associating **GOM** with a natural deduction system is how to treat assumptions in a deduction process. In the usual natural deduction system for intuitionistic logic or classical logic, assumptions that are not used in the application of a rule may be omitted. That is, for example:

$$\dfrac{\begin{array}{c}[\alpha]\\ \vdots \\ \beta\end{array}}{\alpha \rightarrow \beta} \tag{1}$$

In this case, it is legitimate that assumptions other than α are not explicitly stated even if they exist. The point becomes clear when we express this situation in the sequent calculus form:

$$\frac{\Gamma, \alpha \vdash \beta}{\Gamma \vdash \alpha \rightarrow \beta}$$

Here, all (undischarged) assumptions other than α are explicitly written as Γ, a (possibly empty) set of formulas.

In quantum logic, however, the introduction rule of implication (the Sasaki hook) is subject to a restriction due to the failure of the deduction theorem. That is, the introduction rule of implication (1) can only be applied if there exist no assumptions other than α. Again, we view this as a sequent calculus form:

$$\frac{\alpha \vdash \beta}{\vdash \alpha \rightarrow \beta}$$

Taking this restriction into account, we will use the following convention in defining and applying rules of our natural deduction system: *the assumptions that are not explicitly stated must not exist.* When written in this style, the introduction rule of implication (1) in intuitionistic logic or classical logic would become as follows:

$$\dfrac{\begin{array}{c}[\alpha], \Gamma \\ \vdots \\ \beta\end{array}}{\alpha \to \beta} \tag{2}$$

Let us give an example of a proof diagram. The following proof is legitimate in intuitionistic or classical logic:

$$\dfrac{\dfrac{\dfrac{\dfrac{[\mathrm{b} : \beta \to \gamma] \qquad \dfrac{[\mathrm{c} : \alpha \to \beta] \qquad [\mathrm{a} : \alpha]}{\beta}}{\gamma}}{\alpha \to \gamma}\,\text{(a)}}{(\beta \to \gamma) \to (\alpha \to \gamma)}\,\text{(b)}}{(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))}\,\text{(c)}$$

Here, a, b, and c are labels, which are used to identify assumptions to be discharged.

However, the proof above is illegitimate in quantum logic, as the rule labeled by (a) must not be applied due to the existence of assumptions b and c remaining undischarged at this point:

$$\dfrac{\mathrm{b} : \beta \to \gamma \qquad \dfrac{\mathrm{c} : \alpha \to \beta \qquad \mathrm{a} : \alpha}{\beta}}{\gamma}$$

Indeed, the proposition $(\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$ is not a theorem of quantum logic. It can therefore be said that quantum logic is non-monotonic in the sense that adding assumptions might interfere with a deduction process.

A difference in the treatment of assumptions between natural deduction and sequent calculus has been mentioned in Restall [17]. Monotonicity in quantum logic has been pointed out in the context of quantum uncertainty in Engesser et al. [6].

Once a natural deduction system for quantum logic is obtained, the corresponding *quantum λ-calculus* can be introduced via the Curry–Howard isomorphism: the proofs of the natural deduction system can be reversibly translated into the terms of the λ-calculus, respectively. Finally, we will prove the *strong normalization property* for the quantum λ-calculus, which claims that any computation in the quantum λ-calculus eventually terminates.

The quantum λ-calculus introduced in this paper is based on orthomodular quantum logic, while several other systems based on intuitionistic linear logic have also been studied under the name *quantum λ-calculus* [20,21]. The proof of the strong normalization property presented in this paper follows that of Girard et al. [8].

The organization of this paper is as follows: Sect. 2 provides a formal definition of **NQ**, a natural deduction system for quantum logic. Section 3 shows the equivalence between **NQ** and **GOM**. Section 4 introduces the corresponding quantum λ-calculus together with the Curry–Howard isomorphism from **NQ**. Section 5 establishes a rigorous proof of the strong normalization property for the quantum λ-calculus. The last section draws a conclusion.

2. Formal Definition of NQ

This section provides a formal definition of **NQ**, a natural deduction system for quantum logic. To begin with, we construct formulas over a countable set of propositional variables and a propositional constant $\bot$.

Definition 2.1 (*Formula*). *Formulas* are inductively defined as follows:

- α is a formula if α is a propositional variable.
- α is a formula if α is $\bot$.
- $(\alpha \wedge \beta)$ is a formula if α and β are formulas.
- $(\alpha \to \beta)$ is a formula if α and β are formulas.
- Only strings obtained by finitely many applications of the above rules are formulas.

Parentheses are omitted if there is no ambiguity.

We employ $\wedge$ (conjunction) and $\to$ (implication) as the basic connectives, and introduce the following as abbreviations:

- $\alpha \vee \beta \equiv \neg(\neg\alpha \wedge \neg\beta)$ (disjunction)
- $\neg\alpha \equiv \alpha \to \bot$ (negation)

The lower-case Greek letters $\alpha, \beta, \gamma, \delta$ and those with subscripts like $\alpha_1, \alpha_2, \ldots$ are used to denote metavariables representing formulas; the upper-case Greek letters $\Gamma, \Delta, \Pi, \Sigma$ and those with subscripts like $\Gamma_1, \Gamma_2, \ldots$ are used to denote metavariables representing finite (possibly empty) sets of formulas. $\{\alpha\}$ and $\{\alpha\} \cup \Gamma$ are usually abbreviated to α and α, Γ, respectively. $\neg\Gamma$ is a shorthand notation for $\{\neg\alpha \mid \alpha \in \Gamma\}$.

Definition 2.2 (*Subformula*). Let α be a formula. β is said to be a *subformula* of α if β is a substring of α and β itself is a formula.

Definition 2.3 (*Label*). *Labels* are used to indicate assumptions in a proof. $\mathrm{a} : \alpha$ denotes an assumption where a is a label and α is a formula.

We fix a countable set of labels, so that different formulas correspond to different labels.

Definition 2.4 (*Proof of* **NQ**). In **NQ**, a conclusion α is said to be *provable* from a set Γ of assumptions if there exists a proof diagram that derives α from Γ. *Proof diagrams* (or simply *proofs*) of **NQ** are inductively defined by the following rules.

- Assumption rule

$$\mathrm{a} : \alpha$$

or

$$\alpha$$

when the label is omitted, is a (trivial) proof that derives α from α. This proof is referred to as **ASM**(a: α/α).

- $\bot$-rule

  If

$$\begin{array}{c}\Gamma\\ \vdots\\ \bot\end{array}$$

is a proof, say P, that derives $\bot$ from Γ, then

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \bot\end{array}}{\alpha}$$

is a proof that derives α from Γ. This proof is referred to as $\mathbf{EFQ}(P/\alpha)$.

- $\wedge$El-rule

  If

$$\begin{array}{c}\mathrm{a}:\alpha,\Gamma_1\\ \vdots\\ \gamma\end{array}$$

is a proof that derives γ from $\{\mathrm{a}:\alpha\}\cup\Gamma_1$, and if

$$\begin{array}{c}\Gamma_2\\ \vdots\\ \alpha\wedge\beta\end{array}$$

is a proof that derives $\alpha\wedge\beta$ from Γ_2, then

$$\dfrac{\begin{array}{c}[\mathrm{a}:\alpha],\Gamma_1\\ \vdots\\ \gamma\end{array}\qquad\begin{array}{c}\Gamma_2\\ \vdots\\ \alpha\wedge\beta\end{array}}{\gamma}\,(\mathrm{a})$$

is a proof that derives γ from $\Gamma_1\cup\Gamma_2$. Here, $[\mathrm{a}:\alpha]$ represents a discharged assumption. This rule can be replaced by the following simpler one. If

$$\begin{array}{c}\Gamma\\ \vdots\\ \alpha\wedge\beta\end{array}$$

is a proof, say P, that derives $\alpha\wedge\beta$ from Γ, then

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \alpha\wedge\beta\end{array}}{\alpha}$$

is a proof that derives α from Γ. This proof is referred to as $\wedge\mathbf{El}(P/\alpha)$.

- $\wedge$Er-rule

  If

$$\begin{array}{c}\mathrm{a}:\beta,\Gamma_1\\ \vdots\\ \gamma\end{array}$$

is a proof that derives γ from $\{\mathrm{a} : \beta\} \cup \Gamma_1$, and if

$$\begin{array}{c}\Gamma_2 \\ \vdots \\ \alpha \wedge \beta\end{array}$$

is a proof that derives $\alpha \wedge \beta$ from Γ_2, then

$$\dfrac{\begin{array}{c}[\mathrm{a} : \beta], \Gamma_1 \\ \vdots \\ \gamma\end{array} \qquad \begin{array}{c}\Gamma_2 \\ \vdots \\ \alpha \wedge \beta\end{array}}{\gamma}\ \text{(a)}$$

is a proof that derives γ from $\Gamma_1 \cup \Gamma_2$. This rule can be replaced by the following simpler one. If

$$\begin{array}{c}\Gamma \\ \vdots \\ \alpha \wedge \beta\end{array}$$

is a proof, say P, that derives $\alpha \wedge \beta$ from Γ, then

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \alpha \wedge \beta\end{array}}{\beta}$$

is a proof that derives β from Γ. This proof is referred to as $\wedge\mathbf{Er}(P/\beta)$.

- $\wedge$I-rule

  If

$$\begin{array}{c}\Gamma_1 \\ \vdots \\ \alpha\end{array}$$

is a proof, say P_1, that derives α from Γ_1, and if

$$\begin{array}{c}\Gamma_2 \\ \vdots \\ \beta\end{array}$$

is a proof, say P_2, that derives β from Γ_2, then

$$\dfrac{\begin{array}{c}\Gamma_1 \\ \vdots \\ \alpha\end{array} \qquad \begin{array}{c}\Gamma_2 \\ \vdots \\ \beta\end{array}}{\alpha \wedge \beta}$$

is a proof that derives $\alpha \wedge \beta$ from $\Gamma_1 \cup \Gamma_2$. This proof is referred to as $\wedge\mathbf{I}(P_1, P_2/\alpha \wedge \beta)$.

- $\rightarrow$ E-rule (Modus Ponens)

  If

$$\begin{array}{c}\Gamma_1 \\ \vdots \\ \alpha \rightarrow \beta\end{array}$$

is a proof, say P_1, that derives $\alpha \to \beta$ from Γ_1, and if

$$\begin{array}{c}\Gamma_2\\ \vdots\\ \alpha\end{array}$$

is a proof, say P_2, that derives α from Γ_2, then

$$\dfrac{\begin{array}{c}\Gamma_1\\ \vdots\\ \alpha \to \beta\end{array} \qquad \begin{array}{c}\Gamma_2\\ \vdots\\ \alpha\end{array}}{\beta}$$

is a proof that derives β from $\Gamma_1 \cup \Gamma_2$. This proof is referred to as $\to \mathbf{E}(P_1, P_2/\beta)$.

- $\to$ I-rule

  If

$$\begin{array}{c}\alpha\\ \vdots\\ \beta\end{array}$$

is a proof, say P, that derives β from α, then

$$\dfrac{\begin{array}{c}[\mathrm{a} : \alpha]\\ \vdots\\ \beta\end{array}}{\alpha \to \beta}$$

is a proof that derives $\alpha \to \beta$ from no assumptions. This proof is referred to as $\to \mathbf{I}(P[\mathrm{a} : \alpha]/\alpha \to \beta)$. As noted in Sect. 1, no assumptions other than $\mathrm{a} : \alpha$ are allowed to be made when this rule is applied.

- MT (Modus Tollens)

  If

$$\begin{array}{c}\emptyset\\ \vdots\\ \alpha \to \beta\end{array}$$

is a proof, say P_1, that derives $\alpha \to \beta$ from no assumptions, and if

$$\begin{array}{c}\Gamma\\ \vdots\\ \beta \to \bot\end{array}$$

is a proof, say P_2, that derives $\beta \to \bot$ from Γ, then

$$\dfrac{\begin{array}{c}\emptyset\\ \vdots\\ \alpha \to \beta\end{array} \qquad \begin{array}{c}\Gamma\\ \vdots\\ \beta \to \bot\end{array}}{\alpha \to \bot}$$

is a proof that derives $\alpha \to \bot$ from Γ. This proof is referred to as $\mathbf{MT}(P_1, P_2/\alpha \to \bot)$.

- $\neg\neg$E-rule

  If

$$\begin{array}{c} \mathrm{a}:\alpha, \Gamma_1 \\ \vdots \\ \gamma \end{array}$$

is a proof that derives γ from $\{\mathrm{a}:\alpha\} \cup \Gamma_1$, and if

$$\begin{array}{c} \Gamma_2 \\ \vdots \\ (\alpha \to \bot) \to \bot \end{array}$$

is a proof that derives $(\alpha \to \bot) \to \bot$ from Γ_2, then

$$\dfrac{\begin{array}{c} [\mathrm{a}:\alpha], \Gamma_1 \\ \vdots \\ \gamma \end{array} \qquad \begin{array}{c} \Gamma_2 \\ \vdots \\ (\alpha \to \bot) \to \bot \end{array}}{\gamma}\ \text{(a)}$$

is a proof that derives γ from $\Gamma_1 \cup \Gamma_2$. This rule can be replaced by the following simpler one. If

$$\begin{array}{c} \Gamma \\ \vdots \\ (\alpha \to \bot) \to \bot \end{array}$$

is a proof, say P, that derives $(\alpha \to \bot) \to \bot$ from Γ, then

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ (\alpha \to \bot) \to \bot \end{array}}{\alpha}$$

is a proof that derives α from Γ. This proof is referred to as $\neg\neg\mathbf{E}(P/\alpha)$.

- $\neg\neg$I-rule

  If

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}$$

is a proof, say P, that derives α from Γ, then

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}}{(\alpha \to \bot) \to \bot}$$

is a proof that derives $(\alpha \to \bot) \to \bot$ from Γ. This proof is referred to as $\neg\neg\mathbf{I}(P/(\alpha \to \bot) \to \bot)$.

3. Equivalence Between NQ and GOM

This section shows the equivalence between **NQ** and Nishimura's **GOM** (Theorems 3.5 and 3.6).

Definition 3.1 (**GOM** [13]).[1]

- Axioms

$$\alpha \vdash \alpha$$

- Rules

$$\frac{\Gamma \vdash \Delta}{\Pi, \Gamma \vdash \Delta, \Sigma} \text{ (extension)}$$

$$\frac{\Gamma_1 \vdash \Delta_1, \alpha \qquad \alpha, \Gamma_2 \vdash \Delta_2}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2} \text{ (cut)}$$

$$\frac{\alpha, \Gamma \vdash \Delta}{\alpha \wedge \beta, \Gamma \vdash \Delta} \text{ } (\wedge \mathrm{l}_1)$$

$$\frac{\beta, \Gamma \vdash \Delta}{\alpha \wedge \beta, \Gamma \vdash \Delta} \text{ } (\wedge \mathrm{l}_2)$$

$$\frac{\Gamma \vdash \Delta, \alpha \qquad \Gamma \vdash \Delta, \beta}{\Gamma \vdash \Delta, \alpha \wedge \beta} \text{ } (\wedge \mathrm{r})$$

$$\frac{\Gamma \vdash \Delta, \alpha}{\neg\alpha, \Gamma \vdash \Delta} \text{ } (\neg \mathrm{l})$$

$$\frac{\alpha \vdash \Delta}{\neg\Delta \vdash \neg\alpha} \text{ } (\neg \mathrm{r})$$

$$\frac{\alpha, \Gamma \vdash \Delta}{\neg\neg\alpha, \Gamma \vdash \Delta} \text{ } (\neg\neg \mathrm{l})$$

$$\frac{\Gamma \vdash \Delta, \alpha}{\Gamma \vdash \Delta, \neg\neg\alpha} \text{ } (\neg\neg \mathrm{r})$$

$$\frac{\neg\beta \vdash \neg\alpha \qquad \neg\alpha, \beta \vdash}{\neg\alpha \vdash \neg\beta} \text{ (O-modular)}$$

In **GOM**, we regard $\alpha \to \beta$ as a shorthand for $\neg(\alpha \wedge \neg(\alpha \wedge \beta))$ (the Sasaki hook). Then, the following proposition holds:

Proposition 3.2. *O-modular can be replaced by the following MP rule.*

$$\frac{\Gamma \vdash \Delta, \alpha \to \beta \qquad \Gamma \vdash \Delta, \alpha}{\Gamma \vdash \Delta, \beta} \text{ (MP)}$$

Proof. First, the following diagram verifies that O-modular can be derived from MP:

[1] We are using slightly different symbols and names than those used in [13].

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\beta\vdash\neg\alpha \qquad \neg\beta\vdash\neg\beta}{\neg\beta\vdash\neg\alpha\wedge\neg\beta}}{\neg(\neg\alpha\wedge\neg\beta)\vdash\neg\neg\beta} \qquad \dfrac{\beta\vdash\beta}{\neg\neg\beta\vdash\beta}}{\neg(\neg\alpha\wedge\neg\beta)\vdash\beta} \qquad \neg\alpha,\beta\vdash}{\neg(\neg\alpha\wedge\neg\beta),\neg\alpha\vdash}}{\neg\alpha\wedge\neg(\neg\alpha\wedge\neg\beta),\neg\alpha\vdash\cdots(\mathrm{A})}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\alpha\vdash\neg\alpha}{\neg\alpha\wedge\neg(\neg\alpha\wedge\neg\beta)\vdash\neg\alpha} \qquad (\mathrm{A})}{\neg\alpha\wedge\neg(\neg\alpha\wedge\neg\beta)\vdash}}{\neg\alpha\wedge\neg(\neg\alpha\wedge\neg\beta)\vdash\alpha}}{\neg\alpha\vdash\neg\alpha\to\neg\beta} \qquad \neg\alpha\vdash\neg\alpha}{\neg\alpha\vdash\neg\beta}\,(\mathrm{MP})$$

Second, the following diagram verifies that MP can be derived from O-modular:

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha\vdash\alpha}{\alpha\wedge\beta\vdash\alpha} \qquad \dfrac{\dfrac{\alpha\wedge\beta\vdash\alpha\wedge\beta}{\alpha\wedge\beta\vdash\neg\neg(\alpha\wedge\beta)} \qquad \dfrac{\dfrac{\neg(\alpha\wedge\beta)\vdash\neg(\alpha\wedge\beta)}{\alpha\wedge\neg(\alpha\wedge\beta)\vdash\neg(\alpha\wedge\beta)}}{\neg\neg(\alpha\wedge\beta)\vdash\alpha\to\beta}}{\alpha\wedge\beta\vdash\alpha\to\beta}}{\alpha\wedge\beta\vdash\alpha\wedge(\alpha\to\beta)}}{\neg\neg(\alpha\wedge\beta)\vdash\alpha\wedge(\alpha\to\beta)}}{\neg\neg(\alpha\wedge\beta)\vdash\neg\neg(\alpha\wedge(\alpha\to\beta))\cdots(\mathrm{B})}$$

$$\dfrac{\dfrac{\dfrac{\alpha\vdash\alpha}{\alpha\wedge(\alpha\to\beta)\vdash\alpha} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha\vdash\alpha}{\alpha,\neg(\alpha\wedge\beta)\vdash\alpha} \qquad \dfrac{\neg(\alpha\wedge\beta)\vdash\neg(\alpha\wedge\beta)}{\alpha,\neg(\alpha\wedge\beta)\vdash\neg(\alpha\wedge\beta)}}{\alpha,\neg(\alpha\wedge\beta)\vdash\alpha\wedge\neg(\alpha\wedge\beta)}}{\alpha\to\beta,\alpha,\neg(\alpha\wedge\beta)\vdash}}{\alpha\wedge(\alpha\to\beta),\alpha,\neg(\alpha\wedge\beta)\vdash}}{\alpha\wedge(\alpha\to\beta),\neg(\alpha\wedge\beta)\vdash}}{\neg\neg(\alpha\wedge(\alpha\to\beta)),\neg(\alpha\wedge\beta)\vdash\cdots(\mathrm{C})}$$

$$\dfrac{\dfrac{(\mathrm{B}) \qquad (\mathrm{C})}{\neg\neg(\alpha\wedge(\alpha\to\beta))\vdash\neg\neg(\alpha\wedge\beta)}\,(\text{O-modular}) \qquad \dfrac{\dfrac{\beta\vdash\beta}{\alpha\wedge\beta\vdash\beta}}{\neg\neg(\alpha\wedge\beta)\vdash\beta}}{\neg\neg(\alpha\wedge(\alpha\to\beta))\vdash\beta\cdots(\mathrm{D})}$$

$$\dfrac{\dfrac{\dfrac{\Gamma\vdash\Delta,\alpha \qquad \Gamma\vdash\Delta,\alpha\to\beta}{\Gamma\vdash\Delta,\alpha\wedge(\alpha\to\beta)}}{\Gamma\vdash\Delta,\neg\neg(\alpha\wedge(\alpha\to\beta))} \qquad (\mathrm{D})}{\Gamma\vdash\Delta,\beta}$$

□

Additionally, we regard $\bot$ as a shorthand for $\alpha\wedge\neg\alpha$. The following proposition assures that α can be an arbitrary formula.

Proposition 3.3. *In* **GOM**, $\alpha \wedge \neg\alpha$ *and* $\beta \wedge \neg\beta$ *are provably equivalent.*

Proof. The following diagram verifies that $\alpha \wedge \neg\alpha \vdash \beta \wedge \neg\beta$ is provable in **GOM**.

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \wedge \neg\alpha \vdash \alpha} \qquad \dfrac{\dfrac{\alpha \vdash \alpha}{\neg\alpha, \alpha \vdash}}{\alpha \wedge \neg\alpha, \alpha \vdash}}{\alpha \wedge \neg\alpha \vdash}}{\alpha \wedge \neg\alpha \vdash \beta \wedge \neg\beta}$$

Similarly, $\beta \wedge \neg\beta \vdash \alpha \wedge \neg\alpha$ is provable in **GOM**. □

Proposition 3.4. *In* **GOM**, $\alpha \to \bot$ *and* $\neg\alpha$ *are provably equivalent.*

Proof. First, the following diagram verifies that $\alpha \to \bot \vdash \neg\alpha$ is provable in **GOM**.

$$\dfrac{\dfrac{\alpha \vdash \alpha \qquad \dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \vdash \neg\neg\alpha} \qquad \dfrac{\dfrac{\dfrac{\neg\alpha \vdash \neg\alpha}{\bot \vdash \neg\alpha}}{\alpha \wedge \bot \vdash \neg\alpha}}{\neg\neg\alpha \vdash \neg(\alpha \wedge \bot)}}{\alpha \vdash \neg(\alpha \wedge \bot)}}{\alpha \vdash \alpha \wedge \neg(\alpha \wedge \bot)}}{\alpha \to \bot \vdash \neg\alpha}$$

Second, the following diagram verifies that $\neg\alpha \vdash \alpha \to \bot$ is provable in **GOM**.

$$\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \wedge \neg(\alpha \wedge \bot) \vdash \alpha}}{\neg\alpha \vdash \alpha \to \bot}$$

□

Theorem 3.5 (Admissibility of the rules of **NQ** in **GOM**). *The rules of* **NQ** *are provable in* **GOM**.

Proof. Suppose α is provable from Γ in **NQ**:

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}$$

We then show that $\Gamma \vdash \alpha$ is provable in **GOM** by induction on n, the number of rules applied.

- Case $n = 1$. The only rule applied is Assumption rule, that is,

  $$\mathrm{a} : \alpha$$

  Then, $\alpha \vdash \alpha$ is provable in **GOM**. Indeed,

$$\alpha \vdash \alpha$$

is a proof in **GOM**.

- Case $n > 1$. We proceed by case analysis on the last rule applied.
  - Case $\bot$-rule. By the induction hypothesis, $\Gamma \vdash \beta \wedge \neg\beta$ is provable in **GOM**. Then, $\Gamma \vdash \alpha$ is provable in **GOM**. Indeed,

$$\dfrac{\dfrac{\Gamma \vdash \beta \wedge \neg\beta \qquad \dfrac{\dfrac{\beta \vdash \beta}{\beta \wedge \neg\beta \vdash \beta} \qquad \dfrac{\dfrac{\beta \vdash \beta}{\neg\beta, \beta \vdash}}{\beta \wedge \neg\beta, \beta \vdash}}{\beta \wedge \neg\beta \vdash}}{\Gamma \vdash}}{\Gamma \vdash \alpha}$$

  is a proof in **GOM**.
  - Case $\wedge$El-rule. By the induction hypothesis, $\alpha, \Gamma_1 \vdash \gamma$ and $\Gamma_2 \vdash \alpha \wedge \beta$ are provable in **GOM**. Then, $\Gamma_1, \Gamma_2 \vdash \gamma$ is provable in **GOM**. Indeed,

$$\dfrac{\dfrac{\Gamma_2 \vdash \alpha \wedge \beta \qquad \dfrac{\alpha \vdash \alpha}{\alpha \wedge \beta \vdash \alpha}}{\Gamma_2 \vdash \alpha} \qquad \alpha, \Gamma_1 \vdash \gamma}{\Gamma_1, \Gamma_2 \vdash \gamma}$$

  is a proof in **GOM**.
  - Case $\wedge$Er-rule. Similar to Case $\wedge$El-rule.
  - Case $\wedge$I-rule. By the induction hypothesis, $\Gamma_1 \vdash \alpha$ and $\Gamma_2 \vdash \beta$ are provable in **GOM**. Then, $\Gamma_1, \Gamma_2 \vdash \alpha \wedge \beta$ is provable in **GOM**:

$$\dfrac{\dfrac{\Gamma_1 \vdash \alpha}{\Gamma_1, \Gamma_2 \vdash \alpha} \qquad \dfrac{\Gamma_2 \vdash \beta}{\Gamma_1, \Gamma_2 \vdash \beta}}{\Gamma_1, \Gamma_2 \vdash \alpha \wedge \beta}$$

  - Case $\rightarrow$ E-rule. By the induction hypothesis, $\Gamma_1 \vdash \alpha \rightarrow \beta$ and $\Gamma_2 \vdash \alpha$ are provable in **GOM**. Then, $\Gamma_1, \Gamma_2 \vdash \beta$ is provable in **GOM**:

$$\dfrac{\dfrac{\Gamma_1 \vdash \alpha \rightarrow \beta}{\Gamma_1, \Gamma_2 \vdash \alpha \rightarrow \beta} \qquad \dfrac{\Gamma_2 \vdash \alpha}{\Gamma_1, \Gamma_2 \vdash \alpha}}{\Gamma_1, \Gamma_2 \vdash \beta}\ \text{(MP)}$$

  - Case $\rightarrow$ I-rule. By the induction hypothesis, $\alpha \vdash \beta$ is provable in **GOM**. Then, $\vdash \alpha \rightarrow \beta$ is provable in **GOM**:

$$\dfrac{\dfrac{\dfrac{\alpha \vdash \alpha}{\alpha \wedge \neg(\alpha \wedge \beta) \vdash \alpha} \qquad \dfrac{\dfrac{\dfrac{\alpha \vdash \alpha \qquad \alpha \vdash \beta}{\alpha \vdash \alpha \wedge \beta}}{\neg(\alpha \wedge \beta), \alpha \vdash}}{\alpha \wedge \neg(\alpha \wedge \beta), \alpha \vdash}}{\alpha \wedge \neg(\alpha \wedge \beta) \vdash}}{\vdash \alpha \rightarrow \beta}$$

  - Case MT. By the induction hypothesis, $\vdash \alpha \rightarrow \beta$ and $\Gamma \vdash \neg\beta$ are provable in **GOM**. Then, $\Gamma \vdash \neg\alpha$ is provable in **GOM**:

$$\dfrac{\Gamma \vdash \neg\beta \qquad \dfrac{\dfrac{\dfrac{\vdash \alpha \to \beta}{\alpha \vdash \alpha \to \beta} \quad \alpha \vdash \alpha}{\alpha \vdash \beta}\ (\text{MP})}{\neg\beta \vdash \neg\alpha}}{\Gamma \vdash \neg\alpha}$$

– Case $\neg\neg$I-rule. By the induction hypothesis, $\alpha, \Gamma \vdash \gamma$ is provable in **GOM**. Then, $\neg\neg\alpha, \Gamma \vdash \gamma$ is provable in **GOM**:

$$\dfrac{\alpha, \Gamma \vdash \gamma}{\neg\neg\alpha, \Gamma \vdash \gamma}$$

– Case $\neg\neg$E-rule. By the induction hypothesis, $\Gamma \vdash \alpha$ is provable in **GOM**. Then, $\Gamma \vdash \neg\neg\alpha$ is provable in **GOM**:

$$\dfrac{\Gamma \vdash \alpha}{\Gamma \vdash \neg\neg\alpha}$$

□

Theorem 3.6 (Admissibility of the axioms and rules of **GOM** in **NQ**). *The axioms and the rules of* **GOM** *are provable in* **NQ**.

Proof. We say that a sequent $\Gamma \vdash \Delta$ is *normal* if Δ consists of at most one formula. Then, the following holds (Theorem 2.5.a in [13]): *If* $\Gamma \vdash \Delta$ *is provable in* **GOM**, *then there exists a proof of* $\Gamma \vdash \Delta'$ *for some* $\Delta' \subseteq \Delta$ *and all the sequents occurring in that proof are normal.* Using this theorem, we may only consider normal sequents. Suppose $\Gamma \vdash \Delta$ is normal and provable in **GOM**. Then, letting α be

$$\begin{cases} \text{the single formula in } \Delta \text{ if } \Delta \not\equiv \emptyset \\ \bot \text{ if } \Delta \equiv \emptyset \end{cases}$$

we show that α is provable from Γ in **NQ**:

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}$$

by induction on n, the number of rules applied.

- Case $n = 0$. The proof is of the form:

$$\alpha \vdash \alpha$$

  Then, α is provable from $\{\alpha\}$ in **NQ**:

$$\mathrm{a} : \alpha$$

  where a is arbitrarily chosen.
- Case $n > 0$. We proceed by case analysis on the last rule applied.
  – Case extension. First, we consider the case where the succedent is weakened:

$$\dfrac{\Gamma \vdash}{\Gamma \vdash \alpha}$$

By the induction hypothesis, $\bot$ is provable from Γ in **NQ**:

$$\begin{array}{c}\Gamma\\ \vdots\\ \bot\end{array}$$

Then, α is provable from Γ in **NQ**:

$$\begin{array}{c}\Gamma\\ \vdots\\ \dfrac{\bot}{\alpha}\end{array}$$

Second, we consider the case where the antecedent is weakened:

$$\frac{\Gamma \vdash \Delta}{\Pi, \Gamma \vdash \Delta}$$

where $\Pi \not\equiv \emptyset$. By the induction hypothesis, γ is provable from Γ in **NQ**:

$$\begin{array}{c}\Gamma\\ \vdots\\ \gamma\end{array}$$

where γ is either the single formula in Δ, or $\bot$. Let $\Pi \equiv \{\delta_1, \delta_2, \ldots, \delta_n\}$. By applying $\wedge$I-rule and $\wedge$El-rule alternately:

$$\dfrac{\dfrac{\dfrac{\dfrac{\delta_1 \quad \delta_2}{\delta_1 \wedge \delta_2}}{\delta_1} \qquad \delta_3}{\delta_1 \wedge \delta_3}}{\begin{array}{c}\delta_1\\ \vdots\end{array}}$$

we finally observe that δ_1 is provable from Π in **NQ**:

$$\begin{array}{c}\Pi\\ \vdots\\ \delta_1\end{array}$$

Then, γ is provable from $\Gamma \cup \Pi$ in **NQ**:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \gamma\end{array} \qquad \begin{array}{c}\Pi\\ \vdots\\ \delta_1\end{array}}{\gamma \wedge \delta_1}}{\gamma}$$

We omit the other cases where both/neither the succedent and/nor the antecedent are weakened.

– Case cut. By the induction hypothesis, α is provable from Γ_1 in **NQ**:

$$\begin{array}{c}\Gamma_1\\ \vdots\\ \alpha\end{array}$$

and γ is provable from $\{\alpha\} \cup \Gamma_2$ in **NQ**:

$$\begin{array}{c} \alpha, \Gamma_2 \\ \vdots \\ \gamma \end{array}$$

Then, γ is provable from $\Gamma_1 \cup \Gamma_2$ in **NQ**:

$$\begin{array}{c} \Gamma_1 \\ \vdots \\ \alpha, \Gamma_2 \\ \vdots \\ \gamma \end{array}$$

– Case $\wedge l_1$. By the induction hypothesis, γ is provable from $\{\alpha\} \cup \Gamma$ in **NQ**:

$$\begin{array}{c} \alpha, \Gamma \\ \vdots \\ \gamma \end{array}$$

Then, γ is provable from $\Gamma \cup \{\alpha \wedge \beta\}$ in **NQ**:

$$\dfrac{\begin{array}{c} [\mathrm{a} : \alpha], \Gamma \\ \vdots \\ \gamma \end{array} \qquad \alpha \wedge \beta}{\gamma}\,(\mathrm{a})$$

– Case $\wedge l_2$. Similar to Case $\wedge l_1$.

– Case $\wedge r$. By the induction hypothesis, α is provable from Γ in **NQ**:

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}$$

and β is provable from Γ in **NQ**:

$$\begin{array}{c} \Gamma \\ \vdots \\ \beta \end{array}$$

Then, $\alpha \wedge \beta$ is provable from Γ in **NQ**:

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array} \qquad \begin{array}{c} \Gamma \\ \vdots \\ \beta \end{array}}{\alpha \wedge \beta}$$

– Case $\neg l$. By the induction hypothesis, α is provable from Γ in **NQ**:

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}$$

Then, $\bot$ is provable from $\{\alpha \to \bot\} \cup \Gamma$ in **NQ**:

$$\dfrac{\alpha \to \bot \qquad \begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}}{\bot}$$

– Case ¬r. By the induction hypothesis, γ is provable from $\{\alpha\}$ in **NQ**:

$$\begin{array}{c}\alpha\\ \vdots\\ \gamma\end{array}$$

Then, $\alpha \to \bot$ is provable from $\{\gamma \to \bot\}$ in **NQ**:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{a} : \alpha]\\ \vdots\\ \gamma\end{array}}{\alpha \to \gamma}\,(\mathrm{a}) \qquad \gamma \to \bot}{\alpha \to \bot}$$

– Case ¬¬l. By the induction hypothesis, γ is provable from $\{\alpha\} \cup \Gamma$ in **NQ**:

$$\begin{array}{c}\alpha, \Gamma\\ \vdots\\ \gamma\end{array}$$

Then, γ is provable from $\{(\alpha \to \bot) \to \bot\} \cup \Gamma$ in **NQ**:

$$\dfrac{\begin{array}{c}[\mathrm{a} : \alpha], \Gamma\\ \vdots\\ \gamma\end{array} \qquad (\alpha \to \bot) \to \bot}{\gamma}\,(\mathrm{a})$$

– Case ¬¬r. By the induction hypothesis, α is provable from Γ in **NQ**:

$$\begin{array}{c}\Gamma\\ \vdots\\ \alpha\end{array}$$

Then, $(\alpha \to \bot) \to \bot$ is provable from Γ in **NQ**:

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ \alpha\end{array}}{(\alpha \to \bot) \to \bot}$$

– Case O-modular. By Proposition 3.2, O-modular can be replaced by MP. By the induction hypothesis, $\alpha \to \beta$ is provable from Γ in **NQ**:

$$\begin{array}{c}\Gamma\\ \vdots\\ \alpha \to \beta\end{array}$$

and α is provable from Γ in **NQ**:

$$\begin{array}{c}\Gamma\\ \vdots\\ \alpha\end{array}$$

Then, β is provable from Γ in **NQ**:

$$\dfrac{\begin{matrix}\Gamma \\ \vdots \\ \alpha \to \beta\end{matrix} \qquad \begin{matrix}\Gamma \\ \vdots \\ \alpha\end{matrix}}{\beta}$$

□

4. Quantum λ-Calculus

This section introduces the corresponding quantum λ-calculus together with the Curry–Howard isomorphism from **NQ**: the proofs of **NQ** can be expressed as the typed λ-terms, and vice versa (Theorem 4.7). To begin with, we construct types over a countable set of type variables and a type constant $\bot$.

Definition 4.1 (*Type*). *Types* of λ-terms are inductively defined as follows:

- α is a type of rank 0 if α is a type variable.
- α is a type of rank 0 if α is $\bot$.
- $(\alpha \times \beta)$ is a type of rank $m + n + 1$ if α and β are types of rank m and n, respectively.
- $(\alpha \to \beta)$ is a type of rank $m + n + 1$ if α and β are types of rank m and n, respectively.
- Only strings obtained by finitely many applications of the above rules are types.

Parentheses are omitted if there is no ambiguity. The rank of α is written as $\mathbf{rank}(\alpha)$.

Next, we construct λ-terms over a countable set of term variables.

Definition 4.2 (*Typed Variable*). For a term variable x and a type α, $x : \alpha$ is said to be a *typed variable*.

Definition 4.3 (*Environment*). A finite (possibly empty) set of typed variable is said to be an *environment*, and is denoted by the upper-case Greek letters $\Gamma, \Delta, \Pi, \Sigma$ and those with subscripts like $\Gamma_1, \Gamma_2, \ldots$.

Definition 4.4 (*Typed λ-Term*). *Typed λ-terms* (or simply *λ-terms*), are inductively defined as follows:

- $x : \alpha$ is a λ-term under Γ if $x : \alpha \in \Gamma$.
- $\varepsilon(M) : \alpha$ is a λ-term under Γ if $M : \bot$ is a λ-term under Γ (α is an arbitrary type).
- $\pi_1(M) : \alpha$ and $\pi_2(M) : \beta$ are λ-terms under Γ if $M : \alpha \times \beta$ is a λ-term under Γ.
- $\langle M, N\rangle : \alpha \times \beta$ is a λ-term under $\Gamma_1 \cup \Gamma_2$ if $M : \alpha$ is a λ-term under Γ_1 and $N : \beta$ is a λ-term under Γ_2, where the set of typed free variables occurring in M and N are consistent, that is, different types are not assigned to the same variables.

- $(\lambda x.M) : \alpha \to \beta$ is a λ-term under Γ if $M : \beta$ is a λ-term under $\{x : \alpha\}$, where the set of typed free variables occurring in M and $\{x : \alpha\}$ are consistent.
- $(MN) : \beta$ is a λ-term under $\Gamma_1 \cup \Gamma_2$ if $M : \alpha \to \beta$ is a λ-term under Γ_1 and $N : \alpha$ is a λ-term under Γ_2.
- $\tau(M, N) : \alpha \to \bot$ is a λ-term under Γ if $M : \alpha \to \beta$ is a λ-term under $\emptyset$ and $N : \beta \to \bot$ is a λ-term under Γ.
- $\eta(M) : \alpha$ is a λ-term under Γ if $M : (\alpha \to \bot) \to \bot$ is a λ-term under Γ.
- $\theta(M) : (\alpha \to \bot) \to \bot$ is a λ-term under Γ if $M : \alpha$ is a λ-term under Γ.
- Only strings obtained by finitely many applications of the above rules are λ-terms.

Parentheses are omitted if there is no ambiguity; types and environments may be omitted if there is no need to specify them.

Definition 4.5 (*Subterm*). Let M be a λ-term. N is said to be a *subterm* of M if N is a substring of M and N itself is a λ-term.

Theorem 4.6 (Curry–Howard isomorphism between the formulas and the types). *There exists an isomorphism between the formulas of* **NQ** *and the types of the quantum λ-calculus.*

Proof. Let **PropVar** be the set of all propositional variables, and let **TypeVar** be the set of all type variables. Recall that both sets are countable. Then, there exists an isomorphism Φ : **PropVar** $\to$ **TypeVar** and its reverse map Φ^{-1} : **TypeVar** $\to$ **PropVar**. Let **Form** be the set of all formulas, and let **Type** be the set of all types. Then, Φ and Φ^{-1} can be extended to an isomorphism between **Form** and **Type** as follows:

- $\Phi(\bot) := \bot$
- $\Phi(\alpha \wedge \beta) := \Phi(\alpha) \times \Phi(\beta)$
- $\Phi(\alpha \to \beta) := \Phi(\alpha) \to \Phi(\beta)$
- $\Phi^{-1}(\bot) := \bot$
- $\Phi^{-1}(\alpha \times \beta) := \Phi^{-1}(\alpha) \wedge \Phi^{-1}(\beta)$
- $\Phi^{-1}(\alpha \to \beta) := \Phi^{-1}(\alpha) \to \Phi^{-1}(\beta)$ □

Theorem 4.7 (Curry–Howard isomorphism between the proofs and the terms). *There exists an isomorphism between the proofs of* **NQ** *and the λ-terms of the quantum λ-calculus.*

Proof. Let **Label** be the set of all labels. Recall that this set is also countable. Then, there exists an isomorphism Ψ : **Label** $\to$ **TypeVar** and its reverse map Ψ^{-1} : **TypeVar** $\to$ **Label**. Let **Proof** be the set of all proofs, and let **Term** be the set of all λ-terms. Then, Ψ and Ψ^{-1} can be extended to an isomorphism between **Proof** and **Term** as follows:

- $\Psi(\mathbf{ASM}(\mathrm{a} : \alpha/\alpha)) = \Psi(\mathrm{a}) : \Phi(\alpha)$
- $\Psi(\mathbf{EFQ}(P/\alpha)) = \varepsilon(\Psi(P)) : \Phi(\alpha)$
- $\Psi(\wedge\mathbf{El}(P/\alpha)) = \pi_1(\Psi(P)) : \Phi(\alpha)$
- $\Psi(\wedge\mathbf{Er}(P/\beta)) = \pi_2(\Psi(P)) : \Phi(\beta)$

- $\Psi(\wedge\mathbf{I}(P_1, P_2/\alpha \wedge \beta)) = \langle \Psi(P_1), \Psi(P_2)\rangle : \Phi(\alpha) \times \Phi(\beta)$
- $\Psi(\rightarrow \mathbf{E}(P_1, P_2/\beta)) = (\Psi(P_1)\Psi(P_2)) : \Phi(\beta)$
- $\Psi(\rightarrow \mathbf{I}(P[\mathrm{a} : \alpha]/\alpha \rightarrow \beta)) = \lambda\Psi(\mathrm{a}).\Psi(P) : \Phi(\alpha) \rightarrow \Phi(\beta)$
- $\Psi(\mathbf{MT}(P_1, P_2/\alpha \rightarrow \bot)) = \tau(\Psi(P_1), \Psi(P_2)) : \Phi(\alpha) \rightarrow \bot$
- $\Psi(\neg\neg\mathbf{E}(P/\alpha)) = \eta(\Psi(P)) : \Phi(\alpha)$
- $\Psi(\neg\neg\mathbf{I}(P/\alpha \rightarrow \bot)) = \theta(\Psi(P)) : (\Phi(\alpha) \rightarrow \bot) \rightarrow \bot$

- $\Psi^{-1}(x : \alpha) = \mathbf{ASM}(\Psi^{-1}(x) : \Phi^{-1}(\alpha)/\Phi^{-1}(\alpha))$
- $\Psi^{-1}(\varepsilon(M : \bot) : \alpha) = \mathbf{EFQ}(\Psi^{-1}(M : \bot)/\Phi^{-1}(\alpha))$
- $\Psi^{-1}(\pi_1(M : \alpha \times \beta) : \alpha) = \wedge\mathbf{El}(\Psi^{-1}(M : \alpha \times \beta)/\Phi^{-1}(\alpha))$
- $\Psi^{-1}(\pi_2(M : \alpha \times \beta) : \beta) = \wedge\mathbf{Er}(\Psi^{-1}(M : \alpha \times \beta)/\Phi^{-1}(\beta))$
- $\Psi^{-1}(\langle M : \alpha, N : \beta\rangle : \alpha \times \beta) = \wedge\mathbf{I}(\Psi^{-1}(M : \alpha), \Psi^{-1}(N : \beta)/\Phi^{-1}(\alpha) \wedge \Phi^{-1}(\beta))$
- $\Psi^{-1}(((M : \alpha \rightarrow \beta)(N : \beta)) : \beta) = \rightarrow \mathbf{E}(\Psi^{-1}(M : \alpha \rightarrow \beta), \Psi^{-1}(N : \alpha)/\Phi^{-1}(\beta))$
- $\Psi^{-1}(\lambda x.(M : \beta) : \alpha \rightarrow \beta) = \rightarrow \mathbf{I}(\Psi^{-1}(M : \beta)[\Psi^{-1}(x) : \Phi^{-1}(\alpha)]/\Phi^{-1}(\alpha) \rightarrow \Phi^{-1}(\beta))$
- $\Psi^{-1}(\tau((M : \alpha \rightarrow \bot), (N : \beta \rightarrow \bot)) : \alpha \rightarrow \bot) = \mathbf{MT}(\Psi^{-1}(M : \alpha \rightarrow \beta), \Psi^{-1}(N : \beta \rightarrow \bot)/\Phi^{-1}(\alpha) \rightarrow \bot)$
- $\Psi^{-1}(\eta(M : (\alpha \rightarrow \bot) \rightarrow \bot)) : \alpha) = \neg\neg\mathbf{E}(\Psi^{-1}(M : (\alpha \rightarrow \bot) \rightarrow \bot)/\Phi^{-1}(\alpha))$
- $\Psi^{-1}(\theta(M : \alpha) : (\alpha \rightarrow \bot) \rightarrow \bot) = \neg\neg\mathbf{I}(\Psi^{-1}(M : \alpha)/(\Phi^{-1}(\alpha) \rightarrow \bot) \rightarrow \bot)$ □

Definition 4.8 (*Conversion*). A λ-term M is said to *be converted* if a subterm of M is substituted with another λ-term in the following way:

- $\pi_1(\langle N : \alpha, L : \beta\rangle)$ is substituted with $N : \alpha$.

  In **NQ**, a section of a proof:

$$\cfrac{\cfrac{\begin{matrix}\Gamma_1 \\ \vdots \\ \alpha\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots \\ \beta\end{matrix}}{\alpha \wedge \beta}}{\alpha}$$

  is converted to

$$\begin{matrix}\Gamma_1 \\ \vdots \\ \alpha\end{matrix}$$

- $\pi_2(\langle N : \alpha, L : \beta\rangle)$ is substituted with $L : \beta$.

  In NQ, a section of a proof:

$$\cfrac{\cfrac{\begin{matrix}\Gamma_1 \\ \vdots \\ \alpha\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots \\ \beta\end{matrix}}{\alpha \wedge \beta}}{\beta}$$

is converted to

$$\begin{array}{c} \Gamma_2 \\ \vdots \\ \beta \end{array}$$

- $(\lambda x : \alpha.N : \beta)(L : \alpha)$ is converted to $(N[x := L : \alpha]) : \beta$. Here, $N[x := L]$ represents the substitution instance of N where all the free occurrences of x are substituted with L. The bound variables are assumed to have been renamed consistently, in order to avoid capturing free variables.

  In **NQ**, a section of a proof:

$$\dfrac{\dfrac{\begin{array}{c}[\mathrm{a} : \alpha] \\ \vdots \\ \beta\end{array}}{\alpha \to \beta} \qquad \begin{array}{c}\Gamma \\ \vdots \\ \alpha\end{array}}{\beta}$$

is converted to

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \\ \vdots \\ \beta \end{array}$$

- $\eta(\theta(M : \alpha))$ is converted to $M : \alpha$.

  In **NQ**, a section of a proof:

$$\dfrac{\dfrac{\begin{array}{c}\Gamma \\ \vdots \\ \alpha\end{array}}{(\alpha \to \bot) \to \bot}}{\alpha}$$

is converted to

$$\begin{array}{c} \Gamma \\ \vdots \\ \alpha \end{array}$$

Henceforth, we write $M \Rightarrow M'$ to mean that M is converted to M' in one step. Note that converting a λ-term can be seen as executing a computation, while converting a proof can be seen as removing a detour in it. A conversion sequence $M_0 \Rightarrow M_1 \Rightarrow \cdots \Rightarrow M_n$ of λ-terms (or $P_0 \Rightarrow P_1 \Rightarrow \cdots \Rightarrow P_n$ of proofs) is denoted by $M_0 \Rightarrow^* M_n$ ($P_0 \Rightarrow^* P_n$). Here, $n \geq 0$, that is, $\Rightarrow^*$ includes the possibility that no conversion is performed.

Definition 4.9 (*Normal form*). A λ-term (or a proof) is said to be in its *normal form* if it cannot be further converted. A λ-term is said to be *normalizable* if there exists a conversion sequence that starts with itself and ends with its normal form.

5. Strong Normalization

This section establishes a rigorous proof of the strong normalization property for the quantum λ-calculus (Theorem 5.2).

Definition 5.1 (*Strongly normalizing*). A λ-term is said to be *strongly normalizing* if there exists no infinite conversion sequence that starts with itself.

Theorem 5.2 *(Strong normalization). The λ-terms of the quantum λ-calculus are strongly normalizing.*

To prove this theorem, we need some preliminaries. Let **Term** be the set of all λ-terms, and let **SNTerm** be the set of all strongly normalizing λ-terms. For $M \in \mathbf{SNTerm}$, the maximum length of a conversion sequence that starts with M is written as $|M|$.

Definition 5.3 ($\mathbf{RED}_\alpha$). For each type α, a set $\mathbf{RED}_\alpha$ of λ-terms is inductively defined as follows:

- Case $\mathbf{rank}(\alpha) = 0$. $\mathbf{RED}_\alpha$ is **SNTerm**. In particular, $\mathbf{RED}_\perp$ is **SNTerm**.
- Case $\mathbf{rank}(\alpha) > 0$.
  - Case $\alpha \equiv \beta \times \gamma$. $\mathbf{RED}_{\beta\times\gamma}$ is the set of all those λ-terms M that satisfy the following condition: $\pi_1(M) \in \mathbf{RED}_\beta$ and $\pi_2(M) \in \mathbf{RED}_\gamma$.
  - Case $\alpha \equiv \beta \to \gamma$. $\mathbf{RED}_{\beta\to\gamma}$ is the set of all those λ-terms M that satisfy the following conditions:
    (a) $MN \in \mathbf{RED}_\gamma$ for any $N \in \mathbf{RED}_\beta$.
    (b) $\eta(M) \in \mathbf{RED}_\delta$ if $\beta \equiv \delta \to \gamma$.

Proposition 5.4. *For each type α, $\mathbf{RED}_\alpha$ satisfies the following conditions:*

(C1) $\mathbf{RED}_\alpha \subseteq \mathbf{SNTerm}$.
(C2) If $M \in \mathbf{RED}_\alpha$ and $M \Rightarrow^* M'$, then $M' \in \mathbf{RED}_\alpha$.
(C3) If M is of the form other than $N \times L$ (product), $\lambda x.N$ (λ-abstraction), or $\theta(N)$ (double-negation), and if $M' \in \mathbf{RED}_\alpha$ for any M' with $M \Rightarrow M'$, then $M \in \mathbf{RED}_\alpha$.

Note that (C3) implies the following: *if M is of the form other than product, λ-abstraction, or double-negation, and if M is in its normal form, then $M \in \mathbf{RED}_\alpha$ for any α.*

Proof. By induction on $\mathbf{rank}(\alpha)$.

- Case $\mathbf{rank}(\alpha) = 0$. By definition, $\mathbf{RED}_\alpha$ is **SNTerm**.
  (C1) Immediate.
  (C2) Suppose $M \in \mathbf{SNTerm}$ and $M \Rightarrow^* M'$. Then, it is immediate that $M' \in \mathbf{SNTerm}$.
  (C3) Suppose $M \Rightarrow M'$ and $M' \in \mathbf{RED}_\alpha$ (= **SNTerm**). Letting $m := \max\{|M'| \mid M \Rightarrow M'\}$, we have $|M| = m+1$. Hence $M \in \mathbf{SNTerm}$.
- Case $\mathbf{rank}(\alpha) > 0$. We proceed by case analysis on the form of α.
  - Case $\alpha \equiv \beta \times \gamma$.
    (C1) Suppose $M \in \mathbf{RED}_{\beta\times\gamma}$. We will show $M \in \mathbf{SNTerm}$. From the assumption that $M \in \mathbf{RED}_{\beta\times\gamma}$, it follows that $\pi_1(M) \in$

$\mathbf{RED}_\beta$. Then, by the induction hypothesis (C1), we have $\pi_1(M) \in \mathbf{SNTerm}$. Applying π_1 to a conversion sequence that starts with M, say $M \Rightarrow M_1 \Rightarrow M_2 \Rightarrow \dots$, we get the sequence $\pi_1(M) \Rightarrow \pi_1(M_1) \Rightarrow \pi_1(M_2) \Rightarrow \dots$, which means that $|M| \leq |\pi_1(M)|$. Hence, we have $M \in \mathbf{SNTerm}$.

(C2) Suppose that $M \in \mathbf{RED}_{\beta\times\gamma}$ and $M \Rightarrow^* M'$. We will show $M' \in \mathbf{RED}_{\beta\times\gamma}$. From the assumption that $M \in \mathbf{RED}_{\beta\times\gamma}$, it follows that $\pi_1(M) \in \mathbf{RED}_\beta$ and $\pi_2(M) \in \mathbf{RED}_\gamma$. Moreover, from the assumption that $M \Rightarrow^* M'$, it follows that $\pi_1(M) \Rightarrow^* \pi_1(M')$ and $\pi_2(M) \Rightarrow^* \pi_2(M')$. Then, by the induction hypothesis (C2), we have that $\pi_1(M') \in \mathbf{RED}_\beta$ and $\pi_2(M') \in \mathbf{RED}_\gamma$. Hence, by the definition of $\mathbf{RED}_{\beta\times\gamma}$, we have $M' \in \mathbf{RED}_{\beta\times\gamma}$.

(C3) Suppose that M is of the form other than product, λ-abstraction, or double-negation, and suppose $M' \in \mathbf{RED}_\alpha$ for any M' with $M \Rightarrow M'$. We will show $M \in \mathbf{RED}_{\beta\times\gamma}$. Since M is of the form other than product, the possible one-step conversion from $\pi_1(M)$ is of the form $\pi_1(M) \Rightarrow \pi_1(M')$ with $M \Rightarrow M'$. From the assumption that $M' \in \mathbf{RED}_{\beta\times\gamma}$, it follows that $\pi_1(M') \in \mathbf{RED}_\beta$. Then, by the induction hypothesis (C3), we have $\pi_1(M) \in \mathbf{RED}_\beta$. Similarly, $\pi_2(M) \in \mathbf{RED}_\gamma$. Hence, by the definition of $\mathbf{RED}_{\beta\times\gamma}$, we have $M \in \mathbf{RED}_{\beta\times\gamma}$.

– Case $\alpha \equiv \beta \to \gamma$.

(C1) Suppose $M \in \mathbf{RED}_{\beta\to\gamma}$. We will show $M \in \mathbf{SNterm}$. From the assumption that $M \in \mathbf{RED}_{\beta\to\gamma}$, it follows that:
  (a) $MN \in \mathbf{RED}_\gamma$ for any $N \in \mathbf{RED}_\beta$.
  (b) $\eta(M) \in \mathbf{RED}_\delta$ if $\beta \equiv \delta \to \gamma$.

Let $x : \beta$ be a typed variable. Note that x is in its normal form. Then, by the induction hypothesis (C3), we have $x \in \mathbf{RED}_\beta$. Hence, by (a), we have $Mx \in \mathbf{RED}_\gamma$. Then, by the induction hypothesis (C1), we have $Mx \in \mathbf{SNTerm}$. Applying a conversion sequence that starts with M, say $M \Rightarrow M_1 \Rightarrow M_2 \Rightarrow \dots$, to x, we get the sequence $Mx \Rightarrow M_1x \Rightarrow M_2x \Rightarrow \dots$, which means that $|M| \leq |Mx|$. Hence, we have $M \in \mathbf{SNTerm}$.

(C2) Suppose that $M \in \mathbf{RED}_{\beta\to\gamma}$ and $M \Rightarrow^* M'$. We will show $M' \in \mathbf{RED}_{\beta\to\gamma}$. To show this, we need to verify:
  (a) $M'N \in \mathbf{RED}_\gamma$ for any $N \in \mathbf{RED}_\beta$.
  (b) $\eta(M') \in \mathbf{RED}_\delta$ if $\beta \equiv \delta \to \gamma$.

For (a): From the assumption that $M \in \mathbf{RED}_{\beta\to\gamma}$, it follows that $MN \in \mathbf{RED}_\gamma$ for any $N \in \mathbf{RED}_\beta$. Moreover, from the assumption that $M \Rightarrow^* M'$, it follows that $MN \Rightarrow^* M'N$. Then, by the induction hypothesis (C2), we have $M'N \in \mathbf{RED}_\gamma$.

For (b): Let $\beta \equiv \delta \to \gamma$. From the assumption that $M \in \mathbf{RED}_{\beta\to\gamma}$, it follows that $\eta(M) \in \mathbf{RED}_\delta$. Moreover, from the assumption that $M \Rightarrow^* M'$, it follows that $\eta(M) \Rightarrow^* \eta(M')$.

Then, by the induction hypothesis (C2), we have $\eta(M') \in \mathbf{RED}_\delta$.

(C3) Suppose that M is of the form other than product, λ-abstraction, or double-negation, and suppose $M' \in \mathbf{RED}_{\beta\to\gamma}$ for any M' with $M \Rightarrow M'$. We will show $M \in \mathbf{RED}_{\beta\to\gamma}$. To show this, we need to verify:

(a) $MN \in \mathbf{RED}_\gamma$ for any $N \in \mathbf{RED}_\beta$.
(b) $\eta(M) \in \mathbf{RED}_\delta$ if $\beta \equiv \delta \to \gamma$.

For (a): Suppose $N \in \mathbf{RED}_\beta$. Note that by the induction hypothesis (C1), we have $N \in \mathbf{SNTerm}$. Then, we can show $MN \in \mathbf{RED}_\gamma$ by induction on $|N|$:

- Case $|N| = 0$. Suppose $MN \Rightarrow L$. Since N is in its normal form, the possible form of L is $L \equiv M'N$ with $M \Rightarrow M'$. From the assumption that $M' \in \mathbf{RED}_{\beta\to\gamma}$, it follows that $M'N \in \mathbf{RED}_\gamma$. Then, by the induction hypothesis (C3), we have $MN \in \mathbf{RED}_\gamma$.
- Case $|N| > 0$. Suppose $MN \Rightarrow L$. Since M is of the form other than λ-abstraction, the possible forms of L are the following:
  - Case $L \equiv M'N$ with $M \Rightarrow M'$. Then, we have $MN \in \mathbf{RED}_\gamma$ as described above.
  - Case $L \equiv MN'$ with $N \Rightarrow N'$. By the hypothesis (C2) of the induction on $\mathbf{rank}(\alpha)$, we have $N' \in \mathbf{RED}_\beta$. Note that $|N'| = |N| - 1 < |N|$. Then, by the hypothesis of the induction on $|N|$, we have $MN' \in \mathbf{RED}_\gamma$.

  Thus, by the hypothesis (C3) of the induction on $\mathbf{rank}(\alpha)$, we have $MN \in \mathbf{RED}_\gamma$.

For (b): Let $\beta \equiv \delta \to \gamma$. Since M is of the form other than double-negation, the possible one-step conversion that starts with $\eta(M)$ is of the form $\eta(M) \Rightarrow \eta(M')$ with $M \Rightarrow M'$. From the assumption that $M' \in \mathbf{RED}_{\beta\to\gamma}$, it follows that $\eta(M') \in \mathbf{RED}_\delta$. Then, by the induction hypothesis (C3), we have $\eta(M) \in \mathbf{RED}_\delta$. □

Proposition 5.5. *The followings hold:*

1. *If $M \in \mathbf{RED}_\perp$, then $\varepsilon(M) \in \mathbf{RED}_\alpha$ for any α.*
2. *If $M \in \mathbf{RED}_\alpha$ and $N \in \mathbf{RED}_\beta$, then $\langle M, N\rangle \in \mathbf{RED}_{\alpha\times\beta}$.*
3. *If $M[x := N] \in \mathbf{RED}_\beta$ for any $N \in \mathbf{RED}_\alpha$, then $\lambda x.M \in \mathbf{RED}_{\alpha\to\beta}$.*
4. *If $M \in \mathbf{RED}_{\alpha\to\beta}$ and $N \in \mathbf{RED}_{\beta\to\perp}$, then $\tau(M, N) \in \mathbf{RED}_{\alpha\to\perp}$.*
5. *If $M \in \mathbf{RED}_\alpha$, then $\theta(M) \in \mathbf{RED}_{(\alpha\to\perp)\to\perp}$.*

Proof. 1. Suppose $M \in \mathbf{RED}_\perp$ ($= \mathbf{SNTerm}$). We will show $\varepsilon(M) \in \mathbf{RED}_\alpha$ by induction on $\mathbf{rank}(\alpha)$.

- Case $\mathbf{rank}(\alpha) = 0$. From the assumption that $M \in \mathbf{SNTerm}$, it follows that any conversion sequence that starts with M is finite.

Then, any sequence starts with $\varepsilon(M)$ is also finite. Hence, we have $\varepsilon(M) \in \mathbf{SNTerm} = \mathbf{RED}_\alpha$.
- Case $\mathbf{rank}(\alpha) > 0$. We will show $\varepsilon(M) \in \mathbf{RED}_\alpha$ by induction on $|M|$.
  - Case $|M| = 0$. Note that M is in its normal form, and so is $\varepsilon(M)$. Hence, by (C3), we have $\varepsilon(M) \in \mathbf{RED}_\alpha$.
  - Case $|M| > 0$. Suppose $\varepsilon(M) \Rightarrow L$. The possible form of L is $L \equiv \varepsilon(M')$ with $M \Rightarrow M'$. From the assumption that $M \in \mathbf{RED}_\bot$ and (C2) , we have $M' \in \mathbf{RED}_\bot$. Note that $|M'| = |M| - 1 < |M|$. Then, by the hypothesis of the induction on $|M|$, we have $L \in \mathbf{RED}_\alpha$. Hence, by (C3), we have $\varepsilon(M) \in \mathbf{RED}_\alpha$.

2. Suppose that $M \in \mathbf{RED}_\alpha$ and $N \in \mathbf{RED}_\beta$. We will show $\pi_1(\langle M, N\rangle) \in \mathbf{RED}_\alpha$ by induction on $|M| + |N|$.
   - Case $|M| + |N| = 0$. Suppose $\pi_1(\langle M, N\rangle) \Rightarrow L$. Since M and N are in their normal forms, the possible form of L is $L \equiv M$. From the assumption that $M \in \mathbf{RED}_\alpha$, we have $L \in \mathbf{RED}_\alpha$. Hence, by (C3), we have $\pi_1(\langle M, N\rangle) \in \mathbf{RED}_\alpha$.
   - Case $|M| + |N| > 0$. Suppose $\pi_1(\langle M, N\rangle) \Rightarrow L$. The possible forms of L are the following:
     - Case $L \equiv \pi_1(\langle M', N\rangle)$ with $M \Rightarrow M'$. From the assumption that $M \in \mathbf{RED}_\alpha$ and (C2), it follows that $M' \in \mathbf{RED}_\alpha$. Note that $|M'| = |M| - 1 < |M|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_\alpha$.
     - Case $L \equiv \pi_1(\langle M, N'\rangle)$ with $N \Rightarrow N'$. From the assumption that $N \in \mathbf{RED}_\beta$ and (C2), it follows that $N' \in \mathbf{RED}_\beta$. Note that $|N'| = |N| - 1 < |N|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_\alpha$.
     - Case $L \equiv M$. From the assumption that $M \in \mathbf{RED}_\alpha$, it follows that $L \in \mathbf{RED}_\alpha$.

     Hence, by (C3), we have $\pi_1(\langle M, N\rangle) \in \mathbf{RED}_\alpha$.

   Similarly, $\pi_2(\langle M, N\rangle) \in \mathbf{RED}_\beta$. Hence, we have $\langle M, N\rangle \in \mathbf{RED}_{\alpha\times\beta}$.
3. Suppose $M[x := N] \in \mathbf{RED}_\beta$ for any $N \in \mathbf{RED}_\alpha$. We will show $\lambda x.M \in \mathbf{RED}_{\alpha\to\beta}$. To show this, we need to verify:
   (a) $(\lambda x.M)N \in \mathbf{RED}_\beta$ for any $N \in \mathbf{RED}_\alpha$.
   (b) $\eta(\lambda x.M) \in \mathbf{RED}_\gamma$ if $\alpha \equiv \gamma \to \beta$.

   For (a): Suppose $N \in \mathbf{RED}_\alpha$. We will show $(\lambda x.M)N \in \mathbf{RED}_\beta$ by induction on $|M| + |N|$.
   - Case $|M|+|N| = 0$. Suppose $(\lambda x.M)N \Rightarrow L$. Since M and N are in their normal forms, the possible form of L is $L \equiv M[x := N]$. From the assumption that $M[x := N] \in \mathbf{RED}_\beta$ for any $N \in \mathbf{RED}_\alpha$, and the assumption that $N \in \mathbf{RED}_\alpha$, it follows that $L \in \mathbf{RED}_\beta$. Hence, by (C3), we have $(\lambda x.M)N \in \mathbf{RED}_\beta$.
   - Case $|M| + |N| > 0$. Suppose $(\lambda x.M)N \Rightarrow L$. The possible forms of L are the following:
     - Case $L \equiv (\lambda x.M')N$ with $M \Rightarrow M'$. From the assumption that $M[x := N] \in \mathbf{RED}_\beta$ for any $N \in \mathbf{RED}_\alpha$, and the fact that

$x \in \mathbf{RED}_\alpha$, it follows that $M \equiv M[x := x] \in \mathbf{RED}_\beta$. Then, by (C2), we have $M' \in \mathbf{RED}_\beta$. Note that $|M'| = |M| - 1 < |M|$. Then, by the hypothesis of the induction on $|M|+|N|$, we have $L \in \mathbf{RED}_\beta$.

– Case $L \equiv (\lambda x.M)N'$ with $N \Rightarrow N'$. From the assumption that $N \in \mathbf{RED}_\alpha$ and (C2), it follows that $N' \in \mathbf{RED}_\alpha$. Note that $|N'| = |N| - 1 < |N|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_\beta$.

– Case $L \equiv M[x := N]$. From the assumption that $M[x := N] \in \mathbf{RED}_\beta$ for any $N \in \mathbf{RED}_\alpha$ and the assumption that $N \in \mathbf{RED}_\alpha$, it follows that $L \in \mathbf{RED}_\beta$.

Hence, by (C3), we have $(\lambda x.M)N \in \mathbf{RED}_\beta$.

For (b): Let $\alpha \equiv \gamma \to \beta$. We will show $\eta(\lambda x.M) \in \mathbf{RED}_\gamma$ by induction on $|M|$.

– Case $|M| = 0$. Note that M is in its normal form, and so is $\eta(\lambda x.M)$. Hence, by (C3), we have $\eta(\lambda x.M) \in \mathbf{RED}_\gamma$.

– Case $|M| > 0$. Suppose $\eta(\lambda x.M) \Rightarrow L$. The possible form of L is $L \equiv \eta(\lambda x.M')$ with $M \Rightarrow M'$. Note that $|M'| = |M| - 1 < |M|$. Then, by the hypothesis of the induction on $|M|$, we have $\eta(\lambda x.M') \in \mathbf{RED}_\gamma$. Hence, by (C3), we have $\eta(\lambda x.M) \in \mathbf{RED}_\gamma$.

4. Suppose that $M \in \mathbf{RED}_{\alpha\to\beta}$ and $N \in \mathbf{RED}_{\beta\to\bot}$. We will show $\tau(M, N) \in \mathbf{RED}_{\alpha\to\bot}$ by induction on $|M| + |N|$.
   - Case $|M| + |N| = 0$. Since M and N are in their normal forms, so is $\tau(M, N)$. Then, by (C3), we have $\tau(M, N) \in \mathbf{RED}_{\alpha\to\bot}$.
   - Case $|M| + |N| > 0$. Suppose $\tau(M, N) \Rightarrow L$. The possible forms of L are the following:
     – Case $L \equiv \tau(M', N)$ with $M \Rightarrow M'$. From the assumption that $M \in \mathbf{RED}_{\alpha\to\beta}$ and (C2), it follows that $M' \in \mathbf{RED}_{\alpha\to\beta}$. Note that $|M'| = |M| - 1 < |M|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_{\alpha\to\bot}$.
     – Case $L \equiv \tau(M, N')$ with $N \Rightarrow N'$. From the assumption that $N \in \mathbf{RED}_{\beta\to\bot}$ and (C2), it follows that $N' \in \mathbf{RED}_{\beta\to\bot}$. Note that $|N'| = |N| - 1 < |N|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_{\alpha\to\bot}$.

   Hence, by (C3), we have $\tau(M, N) \in \mathbf{RED}_{\alpha\to\bot}$.
5. Suppose $M \in \mathbf{RED}_\alpha$. We will show $\theta(M) \in \mathbf{RED}_{(\alpha\to\bot)\to\bot}$. To show this, we need to verify:
   (a) $\theta(M)N \in \mathbf{RED}_\bot$ for any $N \in \mathbf{RED}_{\alpha\to\bot}$.
   (b) $\eta(\theta(M)) \in \mathbf{RED}_\alpha$.

   For (a) : Suppose $N \in \mathbf{RED}_{\alpha\to\bot}$. We will show $\theta(M)N \in \mathbf{RED}_\bot$ by induction on $|M| + |N|$.
   - Case $|M| + |N| = 0$. Since M and N are in their normal forms, so is $\theta(M)N$. Then, by (C3), we have $\theta(M)N \in \mathbf{RED}_\bot$.
   - Case $|M| + |N| > 0$. Suppose $\theta(M)N \Rightarrow L$. The possible forms of L are the following:

- Case $L \equiv \theta(M')N$ with $M \Rightarrow M'$. From the assumption that $M \in \mathbf{RED}_\alpha$ and (C2), it follows that $M' \in \mathbf{RED}_\alpha$. Note that $|M'| = |M|-1 < |M|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_\perp$.
- Case $L \equiv \theta(M)N'$ with $N \Rightarrow N'$. From the assumption that $N \in \mathbf{RED}_{\alpha\to\perp}$ and (C2), it follows that $N' \in \mathbf{RED}_{\alpha\to\perp}$. Note that $|N'| = |N| - 1 < |N|$. Then, by the hypothesis of the induction on $|M| + |N|$, we have $L \in \mathbf{RED}_\perp$.

Hence, by (C3), we have $\theta(M)N \in \mathbf{RED}_\perp$.

For (b): We will show $\eta(\theta(M)) \in \mathbf{RED}_\alpha$ by induction on $|M|$.

- Case $|M| = 0$. Suppose $\eta(\theta(M)) \Rightarrow L$. Since M is in its normal form, the possible form of L is $L \equiv M$. From the assumption that $M \in \mathbf{RED}_\alpha$, it follows that $L \in \mathbf{RED}_\alpha$. Hence, by (C3), we have $\eta(\theta(M)) \in \mathbf{RED}_\alpha$.
- Case $|M| > 0$. Suppose $\eta(\theta(M)) \Rightarrow L$. The possible forms of L are the following:
  - Case $L \equiv M$. From the assumption that $M \in \mathbf{RED}_\alpha$, it follows that $L \in \mathbf{RED}_\alpha$.
  - Case $L \equiv \eta(\theta(M'))$ with $M \Rightarrow M'$. From the assumption that $M \in \mathbf{RED}_\alpha$ and (C2), it follows that $M' \in \mathbf{RED}_\alpha$. Note that $|M'| = |M|-1 < |M|$. Then, by the hypothesis of the induction on $|M|$, we have $L \in \mathbf{RED}_\alpha$.

Hence, by (C3), we have $\eta(\theta(M)) \in \mathbf{RED}_\alpha$. □

Proof of Theorem 5.2. Let $M : \alpha$ be a λ-term. We will show $M \in \mathbf{RED}_\alpha$. Then, by (C1), we will have $M \in \mathbf{SNTerm}$. More generally, the following holds: If M has no free variables other than $x_1 : \alpha_1, \dots, x_n : \alpha_n$, and if $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \dots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$, then $M[x_1 := M_1, \dots, x_n := M_n] \in \mathbf{RED}_\alpha$. We will show this by induction on the construction of M.

- Case $M \equiv x_i$. Immediate.
- Case $M \equiv \varepsilon(N)$. Let $N : \perp$ be a λ-term. Suppose that N has no free variables other than $x_1 : \alpha_1, \dots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \dots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the induction hypothesis, we have $N[x_1 := M_1, \dots, x_n := M_n] \in \mathbf{RED}_\perp$. Then, by 1 of Proposition 5.5, we have $(\varepsilon(N))[x_1 := M_1, \dots, x_n := M_n] \equiv \varepsilon(N[x_1 := M_1, \dots, x_n := M_n]) \in \mathbf{RED}_\alpha$.
- Case $M \equiv \pi_1(N)$. Let $N : \alpha \times \beta$ be a λ-term. Suppose that N has no free variables other than $x_1 : \alpha_1, \cdots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \dots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the induction hypothesis, we have $N[x_1 := M_1, \dots, x_n := M_n] \in \mathbf{RED}_{\alpha\times\beta}$. Then, by the definition of $\mathbf{RED}_{\alpha\times\beta}$, we have $(\pi_1(N))[x_1 := M_1, \dots, x_n := M_n] \equiv \pi_1(N[x_1 := M_1, \dots, x_n := M_n]) \in \mathbf{RED}_\alpha$.
- Case $M \equiv \pi_2(N)$. Similar to Case $\pi_1(N)$.
- Case $M \equiv \langle N, L\rangle$. Let $N : \beta$ and $L : \gamma$ be λ-terms. Accordingly, $\alpha \equiv \beta\times\gamma$. Suppose that N and L have no free variables other than $x_1 : \alpha_1, \dots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \dots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the

induction hypothesis, we have $N[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_\beta$ and $L[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_\gamma$. Then, by 2 of Proposition 5.5, we have $\langle N, L\rangle[x_1 := M_1, \ldots, x_n := M_n] \equiv \langle N[x_1 := M_1, \ldots, x_n := M_n], L[x_1 := M_1, \ldots, x_n := M_n]\rangle \in \mathbf{RED}_\alpha$.

- Case $M \equiv NL$. Let $N : \beta$ and $L : \gamma$ be λ-terms. Accordingly, $\beta \equiv \gamma \to \alpha$. Suppose that N and L have no free variables other than $x_1 : \alpha_1, \ldots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \ldots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the induction hypothesis, we have $N[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_\beta$ and $L[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_\gamma$. Then, by the definition of $\mathbf{RED}_{\gamma\to\alpha}$, we have $(NL)[x_1 := M_1, \ldots, x_n := M_n] \equiv N[x_1 := M_1, \ldots, x_n := M_n](L[x_1 := M_1, \ldots, x_n := M_n]) \in \mathbf{RED}_\alpha$.
- Case $M \equiv \lambda y.N$. Let $y : \beta$ be a typed variable and let $N : \gamma$ be a λ-term. Accordingly, $\alpha \equiv \beta \to \gamma$. Suppose that N has no free variables other than $x_1 : \alpha_1, \ldots, x_n : \alpha_n, y : \beta$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \ldots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}, L : \beta \in \mathbf{RED}_\beta$. By the induction hypothesis, we have $N[x_1 := M_1, \cdots, x_n := M_n, y := L] \in \mathbf{RED}_\gamma$. Then, by 3 of Proposition 5.5, we have $(\lambda y.N)[x_1 := M_1, \ldots, x_n := M_n] \equiv \lambda y.(N[x_1 := M_1, \ldots, x_n := M_n]) \in \mathbf{RED}_\alpha$.
- Case $M \equiv \tau(N, L)$. Let $N : \beta \to \gamma$ and $L : \gamma \to \perp$ be λ-terms. Accordingly, $\alpha \equiv \beta \to \perp$. Supose that N and L have no free variables other than $x_1 : \alpha_1, \ldots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \ldots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the induction hypothesis, we have $N[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_{\beta\to\gamma}$ and $L[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_{\gamma\to\perp}$. Then, by 4 of Proposition 5.5, we have $\tau(N, L)[x_1 := M_1, \ldots, x_n := M_n] \equiv \tau(N[x_1 := M_1, \ldots, x_n := M_n], L[x_1 := M_1, \ldots, x_n := M_n]) \in \mathbf{RED}_\alpha$.
- Case $M \equiv \eta(N)$. Let $N : (\alpha \to \perp) \to \perp$ be a λ-term. Suppose that N has no free variables other than $x_1 : \alpha_1, \cdots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \ldots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the induction hypothesis, we have $N[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_{(\alpha\to\perp)\to\perp}$. Then, by the definition of $\mathbf{RED}_{(\alpha\to\perp)\to\perp}$, we have $(\eta(N))[x_1 := M_1, \ldots, x_n := M_n] \equiv \eta(N[x_1 := M_1, \ldots, x_n := M_n]) \in \mathbf{RED}_\alpha$.
- Case $M \equiv \theta(N)$. Let $N : \beta$ be a λ-term. Accordingly, $\alpha \equiv (\beta \to \perp) \to \perp$. Suppose that N has no free variables other than $x_1 : \alpha_1, \ldots, x_n : \alpha_n$, and suppose $M_1 : \alpha_1 \in \mathbf{RED}_{\alpha_1}, \ldots, M_n : \alpha_n \in \mathbf{RED}_{\alpha_n}$. By the induction hypothesis, we have $N[x_1 := M_1, \ldots, x_n := M_n] \in \mathbf{RED}_\beta$. Then, by 5 of Proposition 5.5, we have $(\theta(N))[x_1 := M_1, \ldots, x_n := M_n] \equiv \theta(N[x_1 := M_1, \ldots, x_n := M_n]) \in \mathbf{RED}_\alpha$. □

6. Conclusion

In this paper, we have presented a natural deduction system for orthomodular quantum logic and the corresponding λ-calculus.

Proof theory and computational theory for quantum logic have not been thoroughly studied so far. One of the reasons for this is that quantum logic lacks a satisfactory implication operation. By treating the Sasaki hook as a quasi-implication and adopting it as a basic operation, we have obtained a straightforward formalization of natural deduction for quantum logic and the corresponding λ-calculus. We hope that both systems will contribute to the study of proof theory and computational theory for orthomodular quantum logic.

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