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A Molecular Logic of Chords and Their Internal Harmony

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Abstract. Chords are not pure sets of tones or notes. They are mainly characterized by their *matrices*. A chord matrix is the pattern of all the lengths of intervals given without further context. Chords are well-structured invariants. They show their *inner* logical form. This opens up the possibility to develop a *molecular* logic of chords. Chords are our primitive, but, nevertheless, already interrelated expressions. The logical space of *internal* harmony is our well-known chromatic scale represented by an infinite line of integers. Internal harmony is nothing more than the pure interrelatedness of two or more chords. We consider three cases: (a) chords inferentially related to subchords, (b) pairs of chords in the space of major–minor tonality and (c) arbitrary chords as arguments of unary chord operators in relation to their outputs. One interesting result is that chord negation transforms any pure major chord into *its* pure minor chord and vice versa. Another one is the fact that the negation of chords with symmetric matrices does not change anything. A molecular logic of chords is mainly characterized by combining general rules for chord operators with the inner logical form of their arguments.

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1. Introduction

A *logic* can be understood as a network of codices. A *codex* is an ideal construction and contains the totality of those rules which completely determine concepts, languages, formal systems, models etc. (cf. [1]). If we look at a Hilbert-style deductive system of classical propositional logic, then the recursive def-

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inition of the concept *well-defined formula* is a codex which is decidable. We can call this concept an *internal* one. There is no context needed to decide whether a given sequence of symbols is a well-defined formula or not.

We consider any appropriate axiomatic basis. Then the concept *being a theorem* becomes an internal concept as well. If we add an (semantic) interpretation, the concept *being a tautology* will be the next codex-based candidate. In games like chess the concept *admissible move* can be characterized as an internal one with respect to a determinable sublist of the FIDE-Laws of Chess (cf. [2]).

To get a logic of chords we need the primitive symbols as well as the primitive expressions of our language and rules of combining them to more complex symbols. Should a logic of chords begin with atomic tones or primitive intervals? In classical propositional logic the decision is clear: propositional variables are primitive symbols and primitive formulas as well.

Chords are not pure sets of tones or notes. They are mainly characterized by their *matrices*. A chord matrix is the pattern of all the lengths of intervals within this chord free of context. Chords are well-structured invariants. They show their *inner* logical form. This opens up the possibility to develop a *molecular* logic of chords. Chords are our primitive, but, nevertheless, already interrelated expressions.

To show the inner form of chords we have to assume a general structure of our logical space. We take the *chromatic scale* for granted. This assumption is independent of the concrete tuning. We need only the assumption that we can investigate all possibilities within an infinite and discrete space of integers. This is our coordinate system.

No chord can be understood independently of its own logical space. Each chord represents an invariant proper segment of this space. A chord itself is a "fusion" of at least two intervals which leads necessarily to at least a third interval within this chord. In general, there is no restriction to major or minor chords.

Harmony is a relational concept. *Internal* harmony is the totality of the interrelations between chords as well-structured segments of the logical space. Each such structure is admissible. The internal major–minor tonality, the dodecaphonic harmony etc. can be considered as special cases of internal harmony. Of course, this does not give us the whole harmony of music! There is a lot of inspiring research regarding context-dependent (external) harmony.

Internal harmony is nothing more than the pure interrelatedness of two or more chords. We consider three cases:

(a) Chords inferentially related to sub-chords,
(b) Pairs of chords in the subspace of major–minor tonality and
(c) Arbitrary chords as arguments of unary chord operators in relation to their outputs.

One interesting result of (c) is that chord negation transforms any pure major chord into *its* pure minor chord and vice versa. Another one is the fact that the negation of chords with symmetric matrices does not change anything. A molecular logic of chords is mainly characterized by combining general rules for chord operators with the inner logical form of their arguments.[1]

2. Declaration of Our Logical Space and Its Invariant Basic Structures

Logical space is the space of the whole game (e.g., geometry of a soccer field, geometry of a chessboard or the syntax of a formal language). I.e., our game has a *range* limited by its space; range of calculus/theory. We can name phenomena/symbols/structures which cannot be explicated within such a formal/empirical theory. Inconsistency proofs show that we are not able to prove every formula. Formation rules show that not any sequence of symbols is allowed: "$\vee)p\forall$" is not well-formed with respect to the syntax of propositional logic. Phenomena of performance cannot be explicated in a theory of competence.

Our logical space of representing all possible chords and the internal harmonic relations between them has an underlying formal structure. It is the *chromatic scale* of tone pitches explicated by a discrete scale of integers. Each integer t_i can be used to represent a (different) simple *tone pitch*. The chromatic scale—like the scale of integers—is to be thought as open in both directions and, therefore, infinite. Independently from our hearing capacities we have—from our logical point of view—an infinite number of tones, intervals and chord patterns.

Each *interval* can be represented by an ordered pair of integers t_i and t_j. But we must distinguish between a *melodic* and a *harmonic* understanding of intervals.

Melodically intervals are ordered sequences of two tones without any restriction. We have two separate (possibly overlapping) sounds; the sound of t_i and the sound of t_j. Both tones can have the same pitch (repetition of tones). This is usually called *prime* (interval): $\langle t_i, t_j\rangle$ with $t_i = t_j$ or simply $\langle t_i, t_i\rangle$. The pitch of the first tone can be higher (decreasing interval: $\langle t_i, t_j\rangle$ with $t_i > t_j$) or lower (increasing interval: $\langle t_i, t_j\rangle$ with $t_i < t_j$). We call the relation between t_i and t_j *interval distance* $D\langle t_i, t_j\rangle$ or d^i_j for short. The interval distance $t_i - t_j$ can be positive (decreasing), negative (increasing) or zero (prime).

Harmonically an interval is exactly *one* sound "consisting of" two tones. In this respect there is neither pitch repetition nor an increasing/decreasing melody. There is simply one sound $\{t_i, t_j\}$ covering two different tones t_i and t_j with t_i representing the tone with the higher pitch and t_j representing the tone with the lower pitch. An interval sound $\{t_i, t_j\}$ does not have a distance but it has its characteristic *length* $L\{t_i, t_j\}$ (l^i_j for short) which is always positive: $L\{t_i, t_j\} = t_i - t_j > 0$. Each interval has its characteristic distance (melodically) or length (harmonically).

[1] The reader could miss a long list of references. There maybe remote predecessors but no really close forerunners of the presented logic of chords.

Within a logic of harmony each interval sound is as good as any other interval sound. We do not assume any kind of an a priori distinction between consonant and dissonant intervals. Our goal is to develop a logical theory of all the possibilities within the chromatic space, of all possible compositions playable on the piano: tonal, atonal, serial, jazz, folk, pop etc.

We declare *chords* as the basic (minimal) expressions of our logic of harmony. A chord is neither a sequence of three or more tones, nor a sequence of a tone and an interval sound in any order, nor a sequence of two or more interval sounds.[2] Each chord is context-freely identifiable by its characteristic *inner structure*. The number of tones is not important: chords are not pure sets of tones. Chords are primarily well-structured with respect to interval lengths. The minimal condition for being a chord is that the lengths of at least two intervals $\{t_i, t_j\}$ and $\{t_k, t_l\}$ are *directly connected* in such a way that $t_j = t_k$ (the lower pitch of the first interval is equal to the higher pitch of the second interval) and there is only one sound event of t_j/t_k. As a consequence each chord contains at least three tones t_1 (lowest pitch), t_2 (intermediate pitch) and t_3 (highest pitch). Another consequence is that each chord contains at least three internal lengths: $L\{t_3, t_2\}$, $L\{t_3, t_1\}$, $L\{t_2, t_1\}$. A chord is not a sequence of two or more of the intervals $\{t_3, t_2\}$, $\{t_3, t_1\}$, $\{t_2, t_1\}$, but all three intervals are emancipated parts of one sound. The general pattern of a 3-tone chord is

$$\left\{ \begin{matrix} t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix}^1 \big[\{t_3, t_1\} \big]^2 \right\}$$

The whole structure is the pattern of one sound. The vertical order of the three tone pitches t_3, t_2 and t_1 is in accordance with the usual notation in sheet music: higher position means higher pitch. Additionally, in our notation higher index of t means higher pitch. The superscripts indicate the grades of the intervals within a chord. The superscript "1" says that the intervals within the square brackets are *basic* intervals. Basic intervals of a chord are the intervals of directly adjacent tones. The superscript "2" or, generally, the square brackets containing exactly one interval shows the *frame* interval. The frame interval of a chord is the interval consisting of the tones with the highest and the lowest pitch.

Each chord can be uniquely identified solely by its *inner* structure. You can cut an arbitrary chord out of any sheet music and you know already whether this chord is a 3-tone-major-chord in root position, a 3-tone-minor

[2] Of course, there are *broken* chords and we can try to hear melodies consisting of three or more tones or containing tones and interval sounds as chords. The other way around: we can analyze a complex sound as representing a melody. In the context of analyzing compositions it can be hard to decide whether such sequences/sounds are melodies or chords. In any case our decision to interpret such structures/sounds either melodically or harmonically is a decision regarding their logical form. It could be shown that there are interesting cases where neglecting this distinction leads to confusion. It is possible to look at/search for melodies (tone sequences) and/or interval sequences throughout chord sequences. But the logical distinctions remain unaffected!

chord in first inversion, a 4-tone-major-seventh-chord in third inversion etc. But you do not know its harmonic function as a tonic, subdominant, dominant, double dominant etc. chord. Otherwise, you know that a chord with its characteristic inner structure can be used in any logically possible harmonic function and you know already that any minor chord cannot be used as a major dominant.

A chord is a *molecular* expression characterized not primarily by its tones but mainly by its *matrix of interval lengths*:

$$\left\{ \begin{bmatrix} l_2^3 \\ l_1^2 \end{bmatrix}^1 \begin{bmatrix} l_1^3 \end{bmatrix}^2 \right\}$$

Again: l_2^3 and l_1^2 are the basic interval lengths and l_1^3 is the frame interval length.

A *class* of (partially or totally tone-different) chords— e.g., the class of 3-tone-major-chords in root position—can be identified simply by knowing its characteristic *matrix of interval lengths* common to all of its elements.

Let us assume that $\left\{ \begin{bmatrix} l_2^3 = 3 \\ l_1^2 = 4 \end{bmatrix}^1 \begin{bmatrix} l_1^3 = 7 \end{bmatrix}^2 \right\}$ is the characteristic matrix of interval lengths of the chord class 3-tone-major-chord in root position: $\left\{ \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 \begin{bmatrix} +7 \end{bmatrix}^2 \right\}$ for short. Positive natural numbers of the Form $+i$ express interval lengths. Then the chords $\left\{ \begin{matrix} 7 \\ 4 \\ 0 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 \begin{bmatrix} +7 \end{bmatrix}^2 \right\}$, $\left\{ \begin{matrix} 8 \\ 5 \\ 1 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 \begin{bmatrix} +7 \end{bmatrix}^2 \right\}$ and $\left\{ \begin{matrix} 11 \\ 8 \\ 4 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 \begin{bmatrix} +7 \end{bmatrix}^2 \right\}$ are all 3-tone-major chords in root position. Pairwise they can have a tone in common (4 or 8) or not.[3]

The smallest logical structures of our logic of harmony are 3-tone-chords consisting of three tones, two basic intervals (basic interval lengths) and one frame interval (one interval length). But structures containing more than three tones and two basic interval lengths are elementary formulas as well. The *general form* of chords[4] as elementary structures of our logic of harmony with respect to all given interval lengths is:

[3] We use integers as name of tones/pitches. Later we will use an exponential notation of integers.

[4] In this paper I avoid to speak of chord *types*. But each concrete chord—i.e., that we know all the tones and, therefore, all the concrete intervals of this chord—is ultimately a chord *type*. The given general form is the general form of chord types. We develop a logic of chord types and their forms. The logic of chords is a logic of *abstract* entities, not of concrete occurrences in sheet music or in music performances.

$$\left\{\begin{matrix} t_{n+1} \\ t_n \\ t_{n-1} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} l_n^{n+1} \\ l_{n-1}^n \\ \cdot \\ \cdot \\ \cdot \\ l_2^3 \\ l_1^2 \end{bmatrix}^1 \begin{bmatrix} l_{n-1}^{n+1} \\ l_{n-2}^n \\ \vdots \\ l_2^4 \\ l_1^3 \end{bmatrix}^2 \cdots \begin{bmatrix} l_2^{n+1} \\ l_1^n \end{bmatrix}^{n-1} \begin{bmatrix} l_1^{n+1} \end{bmatrix}^n \right\}$$

Each chord contains $n + 1$ ($n \geq 2$) distinct tones (tone pitches). Each chord shows $\frac{n\times(n+1)}{2}$ intervals/interval lengths. The superscript i of square brackets (with $1 \leq i \leq n$) indicates the *grade* of all the interval lengths occurring in these brackets. We call the interval lengths of grade i with $2 \leq i \leq n-1$ *intermediate* interval lengths. If we consider chords with more than 3 tones, we have at least one grade of intermediate interval lengths. As stated above basic interval lengths are of grade 1. The frame interval length is of grade n. We call the structure of all basic interval lengths $[]^1$ of a chord its *matrix of basic interval lengths*: its *basic matrix* for short. We call the structure

$$\left\{\begin{bmatrix} l_n^{n+1} \\ l_{n-1}^n \\ \cdot \\ \cdot \\ \cdot \\ l_2^3 \\ l_1^2 \end{bmatrix}^1 \begin{bmatrix} l_{n-1}^{n+1} \\ l_{n-2}^n \\ \vdots \\ l_2^4 \\ l_1^3 \end{bmatrix}^2 \cdots \begin{bmatrix} l_2^{n+1} \\ l_1^n \end{bmatrix}^{n-1} \begin{bmatrix} l_1^{n+1} \end{bmatrix}^n \right\}$$

of an arbitrary chord its *matrix*.

Each chord matrix characterizes a *chord class* uniquely. Given any concrete basic matrix of a chord class we can compute its matrix. Given all tones of a chord in our chromatic space we can compute its basic matrix and, therefore, its matrix. Because we assume that our logical space (the chromatic scale) is infinite there are infinitely many chords realizing the same matrix. If we know the basic matrix of a chord and at least one concrete tone t_i and its position in the form of the chord, we can compute the complete form of this chord. E.g.:

$$\left\{ 8 \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 []^2 \right\} \text{ leads to } \left\{ \begin{matrix} 11 \\ 8 \\ 4 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 \begin{bmatrix} +7 \end{bmatrix}^2 \right\}$$

3. Independence of Chords and Inference Relations Between Chords

With respect to propositional variables in classical logic—which is our prototype of an atomic codex—we have two kinds of independence: (I1) Independence in the sense that any combination of propositional variables is possible, that a propositional variable can occur at any argument position of any n-ary sentence operator. There is no constraint to use propositional variables within the whole logical space of classical logic. (I2) There are no inference relations between such atomic propositional variables. Let p_i and p_j be two arbitrary different propositional variables then p_j does not follow from p_i and p_i does not follow from p_j: $\forall p_i \forall p_j$ with $i \neq j$: (a) $p_i \nvdash p_j$ and (b) $p_j \nvdash p_i$. The same holds if we put a negation sign in front of one or both propositional variables.

In classical logic it is impossible to find inference relations between a formula A and its classical negation $\neg A$: $\forall A$: (a) $A \nvdash \neg A$ and (b) $\neg A \nvdash A$. Primitive n-ary first order formulas of the form $F^n i_1 \dots i_n$ are not atomic like propositional variables because they are composed by an n-ary function F^n and really atomic terms $i_1, \dots, i_n$. Nevertheless we keep (I2): without further context there is no inference between two syntactically different primitive first order formulas: negated or unnegated.

If we look at chords as the primitive forms of our logic of harmony in analogy to atomic propositional variables, we keep independence (I1): each combination of chords is possible, not only combinations that sound "nice" or "pleasant". Chord operators are like sentence operators: an n-ary chord operators takes n chords as input and yields one chord as output. Each chord can occur at any argument position of any n-ary chord operator without restriction.[5]

If we take all instances of our general form of chords as primitive expressions, then there are interesting *inference relations* between chords. We can differentiate between two types of inference relations:

(1) Inference relations with respect to chords and
(2) Inference relations with respect to chord types.

Definition 1 (*Inference to chords*). Let C_1 be a chord containing the tones $t_1, \dots, t_n$ and C_2 be the chord containing the tones $s_1, \dots, s_m$. $C_1 \vdash_t C_2$ iff $\{s_1, \dots, s_m\} \subseteq \{t_1, \dots, t_n\}$.

This inference relation is reflexive: $C \vdash_t C$.

Because of our assumption that a chord contains at least 3 tones there is the following restriction: if C_1 is a 3-tone-chord, then there is exactly one chord which can be inferred: C_1 itself. Intervals are not allowed as consequences. Each consequence C_2 of a chord C which is not identical with C has at least one tone less than C. To get real transitivity we have to start with at least a 5-tone-chord.

There is a special case of inferences between chords.
(1*) Inferences with respect to connected tone-sequences: i.e., that the basic matrix of the consequent is a subpart of the basic matrix of the antecedent:

Definition 2. Let C_1 be a chord containing the tone-sequence $\langle t_1, \dots, t_n \rangle$ and C_2 be the chord containing the tone-sequence $\langle s_1, \dots, s_m \rangle$. $C_1 \vdash_{t^*} C_2$ iff $\{\langle s_1, \dots, s_m \rangle\}$ is part of the tone-sequence $\langle t_1, \dots, t_n \rangle$.

[5] If we try to develop a special logic of tonality or atonality, it could be done by postulating such restrictions.

$$\left\{\begin{matrix}9\\6\\2\\0\end{matrix}\begin{bmatrix}+3\\+4\\+2\end{bmatrix}^1\begin{bmatrix}+7\\+6\end{bmatrix}^2\begin{bmatrix}+9\end{bmatrix}^3\right\}\vdash_t\left\{\begin{matrix}9\\6\\0\end{matrix}\begin{bmatrix}+3\\+6\end{bmatrix}^1\begin{bmatrix}+9\end{bmatrix}^2\right\}\text{ but}$$

$$\left\{\begin{matrix}9\\6\\2\\0\end{matrix}\begin{bmatrix}+3\\+4\\+2\end{bmatrix}^1\begin{bmatrix}+7\\+6\end{bmatrix}^2\begin{bmatrix}+9\end{bmatrix}^3\right\}\not\vdash_{t^*}\left\{\begin{matrix}9\\6\\0\end{matrix}\begin{bmatrix}+3\\+6\end{bmatrix}^1\begin{bmatrix}+9\end{bmatrix}^2\right\}$$

There are several possibilities to define inference relations between chord classes. The concrete tones does not matter here. Let us take the following example:

Definition 3 (*Inference to chord types*). Let BM_{C_1} be the basic matrix of an $n+1$-tone-chord C_1 consisting of the sequence $\langle l_n^{n+1}, \ldots, l_1^2\rangle$ of basic interval lengths. Let BM_{C_2} be the basic matrix of an $m+1$-tone-chord C_2 consisting of the sequence $\langle l_m^{m+1}, \ldots, l_1^2\rangle$ of basic interval lengths. $C_1 \vdash_T C_2$ iff BM_{C_2} is a proper or an improper sub-sequence of BM_{C_1}.

We can have inferences between two chords which do not have any tone in common:

$$\left\{\begin{matrix}9\\6\\2\\0\end{matrix}\begin{bmatrix}+3\\+4\\+2\end{bmatrix}^1\begin{bmatrix}+7\\+6\end{bmatrix}^2\begin{bmatrix}+9\end{bmatrix}^3\right\}\vdash_T\left\{\begin{matrix}7\\3\\1\end{matrix}\begin{bmatrix}+4\\+2\end{bmatrix}^1\begin{bmatrix}+6\end{bmatrix}^2\right\}$$

These inference relations between chords or chord types seem to be very strange from a classical, an atomistic point of view. We can preserve the atomistic position if we move on to an understanding of possible worlds as complete and consistent sets of literals (unnegated/negated propositional variables). With respect to the propositional variables p and q we get four consistent and complete worlds: $\{p, q\}$, $\{\neg p, q\}$, $\{p, \neg q\}$ and $\{\neg p, \neg q\}$. No such set is a proper subset of any other set.

We can switch to a non-classical point of view if we allow incomplete or inconsistent *set-ups* like $\{p\}$, $\{\neg p\}$, $\{q\}$, $\{\neg q\}$ (consistent, but incomplete), $\{p, \neg p, q\}$, $\{p, \neg p, \neg q\}$, $\{p, q, \neg q\}$, $\{\neg p, q, \neg q\}$, $\{p, \neg p, q, \neg q\}$ (complete, but inconsistent) and $\{p, \neg p\}$, $\{q, \neg q\}$ (incomplete and inconsistent). Now we get a lot of inferences between set-ups with respect to $\subseteq$.[6]

But this non-classical step keeps the atomic character of propositional variables. Atoms, atomic symbols are free of logical form. They do not have an inner structure which could be relevant for harmonic relations between them. There is nothing that makes p resembling q or r.

Chords are independent of each other in the sense that any sequence of chords is allowed without any restriction (independence I1). But they are *harmonically dependent* in the sense that each pair of chords due to the *inner*

[6] We get a similar situation if we interpret the values of a 4-valued logic as sets with respect to the elements 1 and 0: $\emptyset$, $\{1\}$, $\{0\}$ and $\{1, 0\}$.

structure of both chords constitutes *internal* harmony which can be described by using chord operators. Chords are not points with respect to our logical space, but structured segments of this space. *Internal harmony* is nothing more than the relation between two or more chords based solely on the inner structure of these chords. Internal harmony does not depend on further context. One subgoal of presenting our logic of chords is to show which relations that we describe intuitively as harmonic are *harmonic in itself*. We will see that some relations are internal (X-dominant, relative minor), but other well-known relations are not (tonic, dominant, subdominant).

In our sense concepts like "chord" as well as "harmony" are *formal* concepts. Euphony is not necessary. E.g., we have of course chords and harmony in our sense not only in tonal music but also in twelve-tone music (dodecaphony) and free jazz located in the chromatic space. A key feature of the logic of chords is that this formal theory is not an atomistic but a *molecular* one.

4. Syntax of Chords in More Detail

4.1. Proper Names for Tones (Tone Pitches) Using Integers

The general form of names for tones with respect to their pitches is t^y. "t" gives us the position of the tone in the octave space "y": $0 \leq t \leq 11$. The exponent y can be any integer. The computation of the integer t^y runs as follows: $t^y = t + (12 \times y)$. We call t the basic number of a tone name. Examples are:

$$\begin{aligned} 0^0 &= 0 + (12 \times 0) = 0 \\ 4^1 &= 4 + (12 \times 1) = 16 \\ 7^{-1} &= 7 + (12 \times -1) = 7 - 12 = -5 \\ 9^{-3} &= 9 + (12 \times -3) = 9 - 36 = -27 \,\text{etc.} \end{aligned}$$

We use the following correlation between traditional tone (pitch) names and our notation:

...	,,,A$^\sharp$	,,,B	,,C	...	,,B	,C	...	,B	C	...	B
...	10^{-4}	11^{-4}	0^{-3}	...	11^{-3}	0^{-2}	...	11^{-2}	0^{-1}	...	11^{-1}
c	...	b	c'	...	b'	c"	...	b"	c"'	...	...
0^0	...	11^0	0^1	...	11^1	0^2	...	11^2	0^3	...	...

4.2. Complex Proper Names of Intervals and Interval Lengths Using Integers

4.2.1. Intervals. Intervals as sounds will be coded by complex names of the form $\{t_i, t_j\}$. $\{t_i, t_j\}$ is a pair of the two tones t_i and t_j with t_i and t_j representing integers and $t_i > t_j$. $\{t_i, t_j\}$ is the form of *complex names* for intervals as sounds. But to know an interval as sound one has to know the concrete values of t_i and t_j. $t_i = t_j$ is not allowed with respect to intervals as sounds because the sound represented by t_i and the sound represented by $\{t_i, t_i\}$ would be indistinguishable. There is no perfect unison as a "new" sound.

4.2.2. Lengths of Intervals and Interval Classes. Each interval as sound has simply a length which is represented by a positive natural number l^i_j: $l^i_j > 0$. L is a (length-)function which takes an interval as argument and yields the length of this interval simply as the difference between t_i and t_j: $L\{t_i, t_j\} = (t_i - t_j) = l^i_j$. In order to differentiate between integers and their exponential form t^y denoting tones and positive integers and their exponential form $l^i_j = i^z$ we write the latter in isolation in the form $+\,i^z$ with $11 \geq i \geq 0$ and $z > 0$ if $i = 0$, $z \geq 0$ else. $+\,0^0$ is not allowed. The computation is the same as for t^y. $+\,i^z$ is the name of the class of intervals with the same length l:
Example

0^1 denotes the tone c'.

$+\,0^1$ denotes the class of intervals with the length 12 called "(perfect) octave".

$L\{3^2, 3^1\}$ gives us the length of the concrete interval (as sound) $\{3^2, 3^1\}$ which is $3^2 - 3^1 = 27 - 15 = +\,0^1 = +\,12$.

4.2.3. Traditional Class Names of Interval Lengths and Their Formal Counterparts.

MINOR SECOND	$+\,1^0$ (+ 1 for short)
MAJOR SECOND	$+\,2^0$ (+ 2)
MINOR THIRD	$+\,3^0$ (+ 3)
MAJOR THIRD	$+\,4^0$ (+ 4)
PERFECT FOURTH	$+\,5^0$ (+ 5)
TRITONE	$+\,6^0$ (+ 6
PERFECT FIFTH	$+\,7^0$ (+ 7)
MINOR SIXTH	$+\,8^0$ (+ 8
MAJOR SIXTH	$+\,9^0$ (+ 9)
MINOR SEVENTH	$+\,10^0$ (+ 10
MAJOR SEVENTH	$+\,11^0$ (+ 11)
(PERFECT) OCTAVE	$+\,0^1$ (+ 12
MINOR NINTH	$+\,1^1$ (+ 13)
MAJOR NINTH	$+\,2^1$ (+ 14
MINOR TENTH	$+\,3^1$ (+ 15)
MAJOR TENTH	$+\,4^1$ (+ 16
etc.	

4.2.4. Definition of Identity of the Lengths of Two (Distinct) Invervals.

Definition 4. Two intervals $I_i = \{t_{i_1}, t_{i_2}\}$ and $I_j = \{t_{j_1}, t_{j_2}\}$ are *identical with respect to their interval lengths* (*IL-identical*): $I_i =_{IL} I_j$ or $\{t_{i_1}, t_{i_2}\} =_{IL} \{t_{j_1}, t_{j_2}\}$ iff $L\{t_{i_1}, t_{i_2}\} = L\{t_{j_1}, t_{j_2}\}$.

4.2.5. Overview of Some Tonal Chord Matrices.

Chord pattern	(Completed) Traditional name
$\left\{\begin{bmatrix}+3\\+4\end{bmatrix}^1 \begin{bmatrix}+7\end{bmatrix}^2\right\}$	3-Tone-major-chord root position
$\left\{\begin{bmatrix}+4\\+3\end{bmatrix}^1 \begin{bmatrix}+7\end{bmatrix}^2\right\}$	3-Tone-minor-chord root position
$\left\{\begin{bmatrix}+5\\+3\end{bmatrix}^1 \begin{bmatrix}+8\end{bmatrix}^2\right\}$	3-Tone-major-chord first inversion [sixth chord]
$\left\{\begin{bmatrix}+3\\+5\end{bmatrix}^1 \begin{bmatrix}+8\end{bmatrix}^2\right\}$	3-Tone-minor-chord second inversion [six-four chord]
$\left\{\begin{bmatrix}+4\\+5\end{bmatrix}^1 \begin{bmatrix}+9\end{bmatrix}^2\right\}$	3-Tone-major-chord second inversion [six-four chord]
$\left\{\begin{bmatrix}+5\\+4\end{bmatrix}^1 \begin{bmatrix}+9\end{bmatrix}^2\right\}$	3-Tone-minor-chord first inversion [sixth chord]
$\left\{\begin{bmatrix}+5\\+3\\+4\end{bmatrix}^1 \begin{bmatrix}+8\\+7\end{bmatrix}^2 \begin{bmatrix}+0^1\end{bmatrix}^3\right\}$	4-Tone-major-chord root position
$\left\{\begin{bmatrix}+5\\+4\\+3\end{bmatrix}^1 \begin{bmatrix}+9\\+7\end{bmatrix}^2 \begin{bmatrix}+0^1\end{bmatrix}^3\right\}$	4-Tone-minor-chord root position
$\left\{\begin{bmatrix}+4\\+5\\+3\end{bmatrix}^1 \begin{bmatrix}+9\\+8\end{bmatrix}^2 \begin{bmatrix}+0^1\end{bmatrix}^3\right\}$	4-Tone-major-chord first inversion
$\left\{\begin{bmatrix}+3\\+5\\+4\end{bmatrix}^1 \begin{bmatrix}+8\\+9\end{bmatrix}^2 \begin{bmatrix}+0^1\end{bmatrix}^3\right\}$	4-Tone-minor-chord first inversion
$\left\{\begin{bmatrix}+3\\+4\\+5\end{bmatrix}^1 \begin{bmatrix}+7\\+9\end{bmatrix}^2 \begin{bmatrix}+0^1\end{bmatrix}^3\right\}$	4-Tone-major-chord second inversion
$\left\{\begin{bmatrix}+3\\+3\\+4\end{bmatrix}^1 \begin{bmatrix}+6\\+7\end{bmatrix}^2 \begin{bmatrix}+10\end{bmatrix}^3\right\}$	4-Tone-major-seventh-chord root position
$\left\{\begin{bmatrix}+2\\+3\\+3\end{bmatrix}^1 \begin{bmatrix}+5\\+6\end{bmatrix}^2 \begin{bmatrix}+8\end{bmatrix}^3\right\}$	4-Tone-major-seventh-chord first inversion
$\left\{\begin{bmatrix}+4\\+2\\+3\end{bmatrix}^1 \begin{bmatrix}+6\\+5\end{bmatrix}^2 \begin{bmatrix}+9\end{bmatrix}^3\right\}$	4-Tone-major-seventh-chord second inversion
$\left\{\begin{bmatrix}+3\\+4\\+2\end{bmatrix}^1 \begin{bmatrix}+7\\+6\end{bmatrix}^2 \begin{bmatrix}+9\end{bmatrix}^3\right\}$	4-Tone-major-seventh-chord third inversion

5. Inner Structure of Chords: Major Chords, Minor Chords and Their Roots

Each chord C consists of its tones and has a characteristic *inner* structure which is called its *matrix* M_C. This is the characteristic pattern of interval lengths of chord C. For any pair of distinct chords holds that they bear in some characteristic relation of *internal harmony* due to their inner structure alone. No external context is needed!

There are well-known traditional concepts of harmonics which can be understood as cases of internal harmony: (1) relative minor, (2) opposite relative minor and (3) X-dominant with $X \in \{\emptyset, sub\}$.

(3) is a new concept: $C_1 \overset{X}{\Longleftrightarrow} C_2$ means that two arbitrary major chords C_1 and C_2 are X-dominants of each other. I.e., that the distance between their basic tones is $\pm 5^i$ or $\pm 7^j$ (with i, j being arbitrary integers) but we do not know whether C_1 is the dominant of C_2 ($X = \emptyset$) or C_1 is the subdominant of C_2 ($X = sub$).

Conjecture 1. Typical harmonic characterizations which are related to tonality like dominant, subdominant, dominant seventh, tonic are cases of harmony not representable in $\mathcal{L}$.

Conjecture 2. To characterize *tonality* we have to modify the logical space by fixing a designated point or structure on our chromatic scale—traditionally realized by a key like A-minor or C-major. From a logical perspective a designated point is a fixed number of our form t^z of integers representing roots of tonic chords.

5.1. Pure Major and Minor Chords

Pure major and minor chords contain a finite number of tone names of three forms: t_1^i, t_2^j, t_3^k. Such chords contain exactly three basic numbers of tones: t_1, t_2 and t_3: 0, 4 and 7 in the following example (1), 0, 3 and 7 in example (2) and 0, 5 and 9 in example (3).

$$(1) \left\{ \begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix}^1 \begin{bmatrix} +7 \end{bmatrix}^2 \right\} : \text{3-tone-C-major-chord in root position}$$

$$(2) \left\{ \begin{matrix} 0^1 \\ 7^0 \\ 3^0 \end{matrix} \begin{bmatrix} +3 \\ +5 \end{bmatrix}^1 \begin{bmatrix} +8 \end{bmatrix}^2 \right\} : \text{3-tone-C-minor-chord in first inversion}$$

$$(3) \left\{ \begin{matrix} 5^2 \\ 0^1 \\ 9^0 \\ 5^{-1} \end{matrix} \begin{bmatrix} +5^1 \\ +3^0 \\ +4^1 \end{bmatrix}^1 \begin{bmatrix} +8^1 \\ +7^1 \end{bmatrix}^2 \begin{bmatrix} +0^3 \end{bmatrix}^3 \right\} : \text{4-Tone-F-major-chord.}$$

Chords with a distinct fourth basic number can be seen as major chords, but they are not pure in our sense. Famous examples are major seventh chords or minor seventh chords.

If C^J is any pure major chord or C^N is any pure minor chord, then all interval lengths within their matrices have one of these seven possible forms:

$+\,0^{i_1}$ with $i_1 > 0$, $+\,3^{i_2}$, $+\,4^{i_3}$, $+\,5^{i_4}$, $+\,7^{i_5}$, $+\,8^{i_6}$ and $+\,9^{i_7}$. But the converse does not hold:

$$\left\{\begin{matrix} 0^1 \\ 7^0 \\ 0^0 \end{matrix} \begin{bmatrix} +\,5 \\ +\,7 \end{bmatrix}^1 \left[+\,0^1\right]^2\right\} \quad \left\{\begin{matrix} 8^1 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +\,4 \\ +\,4 \end{bmatrix}^1 \left[+\,8\right]^2\right\}$$

are counterexamples.They contain exclusively admissible forms of interval lengths, but they are neither major nor minor chords.

5.2. Roots of Major and Minor Chords

There are precisely six matrices of 3-tone-major-chords:

$$\left\{\begin{bmatrix} +\,3 \\ +\,4 \end{bmatrix}^1 \left[+\,7\right]^2\right\} \quad \left\{\begin{bmatrix} +\,5 \\ +\,3 \end{bmatrix}^1 \left[+\,8\right]^2\right\} \quad \left\{\begin{bmatrix} +\,4 \\ +\,5 \end{bmatrix}^1 \left[+\,9\right]^2\right\}$$

$$\left\{\begin{bmatrix} +\,9 \\ +\,7 \end{bmatrix}^1 \left[+\,4^1\right]^2\right\} \quad \left\{\begin{bmatrix} +\,7 \\ +\,8 \end{bmatrix}^1 \left[+\,3^1\right]^2\right\} \quad \left\{\begin{bmatrix} +\,8 \\ +\,9 \end{bmatrix}^1 \left[+\,5^1\right]^2\right\}.$$

We can generalize this observation to the general case by characterizing the basic matrices (patterns of the lengths of basic intervals) of chords: We start with the following six incomplete forms:

$$\begin{bmatrix} \vdots \\ +\,3^i \\ \vdots \\ +\,4^j \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ +\,5^i \\ \vdots \\ +\,3^j \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ +\,4^i \\ \vdots \\ +\,5^j \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ +\,9^i \\ \vdots \\ +\,7^j \\ \vdots \end{bmatrix} . \begin{bmatrix} \vdots \\ +\,7^i \\ \vdots \\ +\,8^j \\ \vdots \end{bmatrix} . \begin{bmatrix} \vdots \\ +\,8^i \\ \vdots \\ +\,9^j \\ \vdots \end{bmatrix} .$$

Let $l_{i_2}^{i_3}$ be the length of an arbitrary interval $\{t_{i3}, t_{i2}\}$ of any of these incomplete forms. Let $l_{i_3}^{i_4}$ be the length of the interval that follows immediately upwards and let $l_{i_1}^{i_2}$ be the length of the interval that follows immediately downwards. Then each finite completion of these forms yields a major chord if the following conditions are fulfilled:

(a) $\forall l_{i_1}^{i_2} \forall l_{i_2}^{i_3} \forall l_{i_3}^{i_4}$:
If $l_{i_2}^{i_3} \in \{+\,3^{j_1}, +\,7^{j_2}\}$ then $l_{i_3}^{i_4} \in \{+\,5^{j_3}, +\,9^{j_4}\}$.
If $l_{i_2}^{i_3} \in \{+\,5^{k_1}, +\,8^{k_2}\}$ then $l_{i_3}^{i_4} \in \{+\,4^{k_3}, +\,7^{k_4}\}$.
If $l_{i_2}^{i_3} \in \{+\,4^{l_1}, +\,9^{l_2}\}$ then $l_{i_3}^{i_4} \in \{+\,3^{l_3}, +\,8^{l_4}\}$.
If $l_{i_2}^{i_3} \in \{+\,3^{j_5}, +\,8^{j_6}\}$ then $l_{i_1}^{i_2} \in \{+\,4^{j_7}, +\,9^{j_8}\}$.
If $l_{i_2}^{i_3} \in \{+\,4^{k_5}, +\,7^{k_6}\}$ then $l_{i_1}^{i_2} \in \{+\,5^{k_7}, +\,8^{k_8}\}$.
If $l_{i_2}^{i_3} \in \{+\,5^{l_5}, +\,9^{l_6}\}$ then $l_{i_1}^{i_2} \in \{+\,3^{l_7}, +\,7^{l_8}\}$.

(b) You can add to each of these (still possibly incomplete) forms a finite number of interval lengths of the form $+\,0^k$ with $k > 0$ at the top, between any pair of basic interval lengths and at the bottom.

The totality of all of these constructions yields the class of all pure n-tone-major-chord-types and, therefore, the (basic) matrices of all logically possible pure major chords. We abbreviate the class of all possible pure major chords by $\mathcal{J}$.

Let C^J be any pure major chord: $C^J \in \mathcal{J}$. Each major chord has one or more roots (basic tones, not tonic notes) represented by integers $b_1, \ldots, b_n$. If C^J contains more than one root, then for the interval distance D between any two of them holds:
$D\langle b_i, b_j\rangle = 0^z$ with $D\langle b_i, b_j\rangle = b_i - b_j$ and z any integer: $D\langle b_i, b_j\rangle$ can of course be negative. I.e., the base numbers of our form of integers for b_i and b_j are identical: Let s^i be the integer form of b_i and t^j be the integer form of b_j. If b_i and b_j are basic tones of C^J, then $s = t$.

To be a major root is an *inner* property of major chords. To be a root of C^J depends exclusively on the position of this tone in C^J relative to the basic matrix of C^J. Warning: to be a root in C^J does not mean to be a tonic note in the sense of being the root of a tonic. Being a tonic note is *not* an inner property of major or minor chords.

Let t_{i+1} and t_i be two tones in any basic interval $\{t_{i+1}, t_i\}$ of C^J. Then we can find the basic tones of the form b_i of C^J simply by looking at the length of the basic interval under consideration:

(i) If $L\{t_{i+1}, t_i\} = +4^j$ or $L\{t_{i+1}, t_i\} = +7^k$, then $b_i = t_i$.
(ii) If $L\{t_{i+1}, t_i\} = +5^l$ or $L\{t_{i+1}, t_i\} = +8^m$, then $b_i = t_i$.
(iii) If b_i is a basic tone of C^J, then all tones t of C^J with $D\langle b_i, t\rangle = 0^z$ (octave-distinct tones with $z \neq 0$) are basic tones as well.
(iv) $l_k^{k+1} = +3^i$ or $l_l^{l+1} = +9^j$ does not allow to compute a basic tone of any major chord.

Basic tones can occupy arbitrary positions within a chord. "Basic" does not mean "deepest"!

It is up to the reader to formulate the incomplete forms of basic matrices of minor chords C^N and all possible completions of them. For all of these forms it is then possible to identify all positions of roots within any minor chord analogously to major chords.

The concepts *major chord* and *minor chord* are characterizations with respect to the *inner* structure of chords alone. If we know the concrete pattern of interval lengths of such a chord (its matrix), we already know the position(s) of its roots. If we know the chord completely, we know the concrete tones being roots.

6. Internal Harmony

Our logic of chords is a formal theory of *molecules*, not of atoms. Each chord has its characteristic *inner* structure with respect to pitch levels. Without further context we can identify each chord simply by seeing its inner logical form. The minimal condition to get *internal* harmony is that we consider the relation between at least two chords. We can say that internal harmony is the togetherness of chords and nothing else. In this section we will consider three cases of internal harmony between two chords without looking for chord operators which transform one chord into the other.

6.1. Relative Minor

Let us start with any major chord C^J and any minor chord C^N. We can define the relation *is-a-relative-minor-of*: $C^N \overset{RM}{\Longrightarrow} C^J$ as our first case of internal harmony:

Definition 5. Let b_{C^J} be any one of the basic tones of C^J and b_{C^N} be any one of the basic tones of C^N. We can say that C^N is *a relative minor of* C^J iff the interval distance between b_{C^J} and b_{C^N} is $+3^i$ (i is any integer):

$$C^N \overset{RM}{\Longrightarrow} C^N \text{iff}\, D\langle b_{C^J}, b_{C^N}\rangle = +3^i$$
$$\text{with}\, D\langle b_{C^J}, b_{C^N}\rangle = b_{C^J} - b_{C^N}.$$

An example of the case that any A-minor chord is a relative minor chord to any given C-major chord:

$$\left\{\begin{matrix} 9^1 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +5^1 \\ +4^0 \end{bmatrix}^1 \left[+9^1\right]^2\right\} \overset{RM}{\Longrightarrow} \left\{\begin{matrix} 7^1 \\ 0^1 \\ 4^0 \end{matrix} \begin{bmatrix} +7^0 \\ +8^0 \end{bmatrix}^1 \left[+3^1\right]^2\right\}$$

because of $D\langle 0^1, 9^1\rangle = 0^1 - 9^1 = +3^{-1}$

We can define the inverse:

Definition 6. $C^N \overset{RM}{\Longleftarrow} C^J$ iff $D\langle b_{C^N}, b_{C^J}\rangle = b_{C^N} - b_{C^J} = +9^i$.

$$\left\{\begin{matrix} 7^1 \\ 0^1 \\ 4^0 \end{matrix} \begin{bmatrix} +7^0 \\ +8^0 \end{bmatrix}^1 \left[+3^1\right]^2\right\} \overset{RM}{\Longleftarrow} \left\{\begin{matrix} 9^1 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +5^1 \\ +4^0 \end{bmatrix}^1 \left[+9^1\right]^2\right\}$$

because of $D\langle 9^1, 0^1\rangle = 9^1 - 0^1 = +9^0$

6.2. Opposite Relative Minor

The relation *is-an-opposite-relative-minor-of* is the second kind of internal harmony. Here is the definition of this relation: $C^N \overset{ORM}{\Longrightarrow} C^J$:

Definition 7. Let b_{C^J} be any one of the basic tones of C^J and b_{C^N} be any one of the basic tones of C^N. We can say that C^N is an opposite relative minor of C^J iff the interval distance between b_{C^J} and b_{C^N} is $+8^i$:

$$C^N \overset{ORM}{\Longrightarrow} C^J \text{iff}\, D\langle b_{C^J}, b_{C^N}\rangle = +8^i$$
$$\text{with}\, D\langle b_{C^J}, b_{C^N}\rangle = b_{C^J} - b_{C^N}.$$

An example of the cases that any E-minor chord is an opposite relative minor chord to any given C-major chord is:

$$\left\{\begin{matrix} 7^1 \\ 4^0 \\ 11^{-1} \end{matrix} \begin{bmatrix} +3^1 \\ +5^0 \end{bmatrix}^1 [+9^1]^2 \right\} \overset{ORM}{\Longrightarrow} \left\{\begin{matrix} 7^1 \\ 0^1 \\ 4^0 \end{matrix} \begin{bmatrix} +7^0 \\ +8^0 \end{bmatrix}^1 [+3^1]^2 \right\}$$

because of $D\langle 0^1, 4^0\rangle = 0^1 - 4^0 = +8^0$

Again, we can define the inverse:

Definition 8. $C^N \overset{ORM}{\Longleftarrow} C^J$ iff $D\langle b_{C^N}, b_{C^J}\rangle = b_{C^N} - b_{C^J} = +4^i$.

6.3. *X*-Dominant with $X \in \{\emptyset, sub\}$

Let us start with two major chords C_1^J and C_2^J. An example could be

$$C_1^J : \left\{\begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3^0 \\ +4^0 \end{bmatrix}^1 [+7^0]^2 \right\} : \text{3-tone-C-major chord in root position and}$$

$$C_2^J : \left\{\begin{matrix} 9^0 \\ 5^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4^0 \\ +5^0 \end{bmatrix}^1 [+9^0]^2 \right\} : \text{3-tone-F-major chord in second inversion.}$$

Traditionally (with respect to tonality) we could say that C_2^J is the *subdominant of* C_1^J, presupposing that C_1^J is the *tonic*. But we could also say that C_1^J is the *dominant of* C_2^J, presupposing that C_2^J is the tonic. With respect to our understanding of internal harmony it is not decidable which reading should be adopted. But it is internally clear that we have only these two options. We can combine both cases under the label "C_1^J and C_2^J are X-dominants of each other" with $X \in \{\emptyset, sub\}$. For $X = \emptyset$ we get the dominant-reading from C_2^J to C_1^J. For $X = sub$ we get the subdominant-reading from C_1^J to C_2^J. But we describe the same situation between these two chords from two different tonal points of view. Within a fixed tonal frame we cannot have both cases at once. Cp. the following theorem-like statements:

Theorem 1. *If C_2^J is the subdominant of C_1^J, then C_1^J is the tonic (and not the dominant) of C_2^J.*

Theorem 2. *If C_1^J is the dominant of C_2^J, then C_2^J is the tonic (and not the subdominant) of C_1^J.*

Let us define the relation that C_1^J and C_2^J are X-dominats of each other: $C_1^J \overset{XD}{\Longleftrightarrow} C_2^J$

Definition 9. Let $b_{C_1^J}$ be any one of the basic tones of C_1^J and $b_{C_2^J}$ be any one of the basic tones of C_2^J. We say that C_1^J and C_2^J are *X-dominants of each other* iff the interval distance between $b_{C_1^J}$ and $b_{C_2^J}$ is $+7^i$ or $+5^j$ (i and j are integers).

$$C_1^J \overset{XD}{\Longleftrightarrow} C_2^J \text{iff} D\langle b_{C_1^J}, b_{C_2^J}\rangle \in \{+7^i, +5^j\}.$$

Example 1.

$$\left\{\begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3^0 \\ +4^0 \end{bmatrix}^1 [+7^0]^2 \right\} \overset{XD}{\Longleftrightarrow} \left\{\begin{matrix} 9^0 \\ 5^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4^0 \\ +5^0 \end{bmatrix}^1 [+9^0]^2 \right\}$$

with $D\langle 0^0, 5^0\rangle = +\,7^{-1}$.

$$\left\{\begin{matrix} 9^0 \\ 5^0 \\ 0^0 \end{matrix} \begin{bmatrix} +\,4^0 \\ +\,5^0 \end{bmatrix}^1 \left[+\,9^0\right]^2\right\} \overset{XD}{\Longleftrightarrow} \left\{\begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +\,3^0 \\ +\,4^0 \end{bmatrix}^1 \left[+\,7^0\right]^2\right\}$$

with $D\langle 5^0, 0^0\rangle = +\,5^0$. Therefore, this 3-tone-C-major-chord and this 3-tone-F-major-chord are X-dominants of each other.

Example 2.

$$\left\{\begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +\,3^0 \\ +\,4^0 \end{bmatrix}^1 \left[+\,7^0\right]^2\right\} \overset{XD}{\Longleftrightarrow} \left\{\begin{matrix} 7^0 \\ 2^0 \\ 11^{-1} \end{matrix} \begin{bmatrix} +\,5^0 \\ +\,3^0 \end{bmatrix}^1 \left[+\,3^0\right]^2\right\}$$

with $D\langle 0^0, 7^0\rangle = +\,5^{-1}$.

$$\left\{\begin{matrix} 7^0 \\ 2^0 \\ 11^{-1} \end{matrix} \begin{bmatrix} +\,5^0 \\ +\,3^0 \end{bmatrix}^1 \left[+\,8^0\right]^2\right\} \overset{XD}{\Longleftrightarrow} \left\{\begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +\,3^0 \\ +\,4^0 \end{bmatrix}^1 \left[+\,7^0\right]^2\right\}$$

with $D\langle 7^0, 0^0\rangle = +\,7^0$. Therefore, this 3-tone-C-major-chord and this 3-tone-G-major-chord are X-dominants of each other.

(a) $\overset{XD}{\Longleftrightarrow}$ is irreflexive: No chord is an X-dominant of itself.
(b) $\overset{XD}{\Longleftrightarrow}$ is symmetric: If $C_1^J \overset{XD}{\Longleftrightarrow} C_2^J$, then $C_2^J \overset{XD}{\Longleftrightarrow} C_1^J$
(c) $\overset{XD}{\Longleftrightarrow}$ is not transitive: From $C_1^J \overset{XD}{\Longleftrightarrow} C_2^J$ and $C_2^J \overset{XD}{\Longleftrightarrow} C_3^J$ we do not get $C_1^J \overset{XD}{\Longleftrightarrow} C_3^J$.

Let $\mathcal{C}_1$ be the class of all major chords which contains at least one basic tone represented by an integer of the form $b_1^{i_1}$. E.g., the class of all C-major-chords contains exactly all major chords with at least one basic tone represented by an integer of the form 0^k. The class of all C$^\sharp$-major-chords contains exactly all major chords with at least one basic tone represented by an integer of the form 1^l etc. Let $\mathcal{C}_2$ be another class of all these major chords which contains at least one basic tone represented by an integer of the form $b_2^{i_2}$:

Theorem 3. $\forall C_1^J \in \mathcal{C}_1\, \forall C_2^J \in \mathcal{C}_2$*:*

$$D < b_1^{i_1}, b_2^{i_2} > \in \{+\,7^i, +\,5^j\} \Leftrightarrow (C_1^J \overset{XD}{\Longleftrightarrow} C_2^J).$$

E.g., each element of the class of all C-major-chords and each element of the class of all F-major-chords are X-dominants of each other and vice versa. The same holds for each element of the class of all C-major-chords and each element of the class of all G-major-chords.

Theorem 4. *There exist a class* $\mathcal{C}_3$ *with* $\mathcal{C}_2 \cap \mathcal{C}_3 = \emptyset$ *and* $b_3^{i_3} \neq b_2^{i_2}$ *such that* $\forall C_1^J \in \mathcal{C}_1\, \forall C_3^J \in \mathcal{C}_3$:

$$D < b_1^{i_1}, b_3^{i_3} > \in \{+\,7^i, +\,5^j\} \Leftrightarrow (C_1^J \overset{XD}{\Longleftrightarrow} C_3^J).$$

I.e., each major cord of class $\mathcal{C}_1$ has all elements of two different classes $\mathcal{C}_2$ and $\mathcal{C}_3$ as its X-dominants. This gives us the logical space for using this constellation as a *cadence*.
But we can define C_1^J and C_3^J as $2X$-dominants of each other:

Definition 10. $C_1^J \overset{2XD}{\Longleftrightarrow} C_3^J$ iff $\exists C_2^J[(C_1^J \overset{XD}{\Longleftrightarrow} C_2^J) \,\&\, (C_2^J \overset{XD}{\Longleftrightarrow} C_3^J)]$.

7. Unary Chord Operators

We can explicate internal harmony by using unary and binary O^1 *chord operators* O^2 which take one chord or two chords as argument (input) and gives another chord as output. Our *formation rules* look like the corresponding ones of classical propositional logic:

$$R1 : \frac{}{C} \qquad R2 : \frac{C}{O^1 C} \qquad R3 : \frac{C_1\ ,\ C_2}{O^2 C_1 C_2}$$

R1 says that any chord standing alone is a formula.
R2 says that given a chord and applying an arbitrary unary chord operator to it yields a chord.
R3 says that given two chords and applying a binary chord operator to them yields a chord.

Remark. If we restrict our chord syntax to R1 and R2 only, we get our very weak formal language $\mathcal{L}^1$. But it is a very interesting task to determine the resources and limits of representing internal harmony using only $\mathcal{L}^1$.

There are two interesting types of unary chord operators:

(O_{IL}): The logical behavior of such operators can be characterized by referring exclusively to the lengths of the (basic) intervals of the argument chord with respect to one or more fixed tones or one fixed interval. We can formulate general rules characterizing these operators. Applying these rules results in a systematic movement of tones if any. The concrete result depends not only on the general rule but also on the inner structure of the chord in argument position. We will look only at operators permuting the lengths of basic intervals. Examples are

(1) *Negation*/matrix-inversion: This operator inverts the order of the lengths of basic intervals within a fixed frame interval. As a consequence we get the inversion of interval lengths of any grade.
(2) *Reversions*: Given the fixed length of the frame interval (not the interval itself) these operators reverse the order of the lengths of basic intervals with respect to a fixed center tone or center interval. We will consider only the reversion of triads with a fixed middle tone.
(3) *Cyclic permutation*: It takes the basic interval length from the highest (lowest position) to the lowest (highest) position in the basic matrix and moves all the other basic interval lengths one step further.

(O_T): The logical behavior of such operators can be characterized only by using arithmetical operations applied to integers representing tones. Examples are:

(1) Chord inversion operators
(2) Barré operators

7.1. Operators Permuting the Lengths of Basic Intervals

7.1.1. Chord Negation. Our *negation* is a chord operator which inverses the order of the lengths of basic intervals within a fixed frame interval completely. Any $n+1$-tone-chord C can be characterized by its tones and its basic matrix in the following way:

$$\left\{\begin{matrix} t_{n+1} \\ t_n \\ t_{n-1} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ . \\ . \\ . \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix}^1 \right\}$$

The negation "−" can be applied to any chord of this form. We will see that our negation "−" has some classical and some non-classical properties. It acts within a fixed frame interval $\{t_{n+1}, t_1\}$ indicating that the highest tone t_{n+1} and the lowest tone t_1 remain unchanged and, therefore, the frame interval $\{t_{n+1}, t_1\}$ is fixed under negation.
The general rule for $-C$ runs as follows:

$$-\left\{\begin{matrix} t_{n+1} \\ t_n \\ t_{n-1} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ . \\ . \\ . \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix}^1 \right\} \Longrightarrow \left\{\begin{matrix} t_{n+1} \\ s_n \\ s_{n-1} \\ \vdots \\ s_3 \\ s_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, s_n\} \\ \{s_n, s_{n-1}\} \\ . \\ . \\ . \\ \{s_3, s_2\} \\ \{s_2, t_1\} \end{bmatrix}^1 \right\}$$

with the following conditions

(i) For even-numbered n: $\{t_{n+1}, s_n\} =_{IL} \{t_2, t_1\}$, $\{s_n, s_{n-1}\} =_{IL} \{t_3, t_2\}$, $\ldots$, $\{s_{\frac{n}{2}+2}, s_{\frac{n}{2}+1}\} =_{IL} \{t_{\frac{n}{2}+1}, t_{\frac{n}{2}}\}$, $\{s_{\frac{n}{2}+1}, s_{\frac{n}{2}}\} =_{IL} \{t_{\frac{n}{2}+2}, t_{\frac{n}{2}+1}\}$, $\ldots$, $\{s_2, t_1\} =_{IL} \{t_{n+1}, t_n\}$

(ii) For odd-numbered n: $\{t_{n+1}, s_n\} =_{IL} \{t_2, t_1\}$, $\{s_n, s_{n-1}\} =_{IL} \{t_3, t_2\}$, $\ldots$, $\{s_{\frac{n+1}{2}+2}, s_{\frac{n+1}{2}+1}\} =_{IL} \{t_{\frac{n+1}{2}}, t_{\frac{n+1}{2}-1}\}$, $\{s_{\frac{n+1}{2}+1}, s_{\frac{n+1}{2}}\} =_{IL} \{t_{\frac{n+1}{2}+1}, t_{\frac{n+1}{2}}\}$, $\{s_{\frac{n+1}{2}}, s_{\frac{n+1}{2}-1}\} =_{IL} \{t_{\frac{n+1}{2}+2}, t_{\frac{n+1}{2}+1}\}$, $\ldots$, $\{s_2, t_1\} =_{IL} \{t_{n+1}, t_n\}$

- Remember: $\{t_i, t_j\} =_{IL} \{t_k, t_l\} =_{df} L\{t_i, t_j\} = L\{t_k, t_l\}$.
- In case (ii) the length of the middle interval remains unchanged. But it can be represented by other tones.
- The deepest (lowest-pitched) tone t_1 remains unchanged.
- The sharpest (highest-pitched) tone t_{n+1} remains unchanged.
- The frame interval $\{t_{n+1}, t_1\}$ with its length $L\{t_{n+1}, t_1\}$ is fixed.

– From a phenomenological point of view this usually leads to precisely rule-governed "movements" of the tones $t_2, \ldots, t_n$ within the frame interval, but not necessarily so [cp. below].

If we negate tonal structures, we get well-known tonal correlations for free. Some examples:[7]

Argument	Result of negation "–"	Tonal correlation
C-major root position $\left\{\begin{matrix}7^0\\4\\0^0\end{matrix}\begin{bmatrix}+3^0\\+4^0\end{bmatrix}\begin{bmatrix}+7^0\end{bmatrix}\right\}$	C-minor root position $\left\{\begin{matrix}7^0\\3^0\\0^0\end{matrix}\begin{bmatrix}+4^0\\+3^0\end{bmatrix}\begin{bmatrix}+7^0\end{bmatrix}\right\}$	Major/minor same position
C-major first inversion $\left\{\begin{matrix}0^1\\7^0\\4^0\end{matrix}\begin{bmatrix}+5^0\\+3^0\end{bmatrix}\begin{bmatrix}+8^0\end{bmatrix}\right\}$	A-minor second inversion $\left\{\begin{matrix}0^1\\9^0\\4^0\end{matrix}\begin{bmatrix}+3^0\\+5^0\end{bmatrix}\begin{bmatrix}+8^0\end{bmatrix}\right\}$	Relative minor in next position
C-major second inversion $\left\{\begin{matrix}4^1\\0^1\\7^0\end{matrix}\begin{bmatrix}+4^0\\+5^0\end{bmatrix}\begin{bmatrix}+9^0\end{bmatrix}\right\}$	E-minor first inversion $\left\{\begin{matrix}4^1\\11^0\\7^0\end{matrix}\begin{bmatrix}+5^0\\+4^0\end{bmatrix}\begin{bmatrix}+9^0\end{bmatrix}\right\}$	Opposite relative minor former inversion

If we add an octave to the deepest tone and put the resulting chords under negation, we get other cases of internal harmony:

C-major root position $\left\{\begin{matrix}0^1\\7^0\\4^0\\0^0\end{matrix}\begin{bmatrix}+5^0\\+3^0\\+4^0\end{bmatrix}\begin{bmatrix}+8^0\\+7^0\end{bmatrix}\begin{bmatrix}+0^1\end{bmatrix}\right\}$	F-minor second inversion $\left\{\begin{matrix}0^1\\8^0\\5^0\\0^0\end{matrix}\begin{bmatrix}+4^0\\+3^0\\+5^0\end{bmatrix}\begin{bmatrix}+7^0\\+8^0\end{bmatrix}\begin{bmatrix}+0^1\end{bmatrix}\right\}$ Minor X-dominant 1
C-major first inversion $\left\{\begin{matrix}4^1\\0^1\\7^0\\4^0\end{matrix}\begin{bmatrix}+4^0\\+5^0\\+3^0\end{bmatrix}\begin{bmatrix}+9^0\\+8^0\end{bmatrix}\begin{bmatrix}+0^1\end{bmatrix}\right\}$	C$^\sharp$-minor first inversion $\left\{\begin{matrix}4^1\\1^1\\8^0\\4^0\end{matrix}\begin{bmatrix}+3^0\\+5^0\\+4^0\end{bmatrix}\begin{bmatrix}+8^0\\+9^0\end{bmatrix}\begin{bmatrix}+0^1\end{bmatrix}\right\}$
C-major second inversion $\left\{\begin{matrix}7^1\\4^1\\0^1\\7^0\end{matrix}\begin{bmatrix}+3^0\\+4^0\\+5^0\end{bmatrix}\begin{bmatrix}+7^0\\+9^0\end{bmatrix}\begin{bmatrix}+0^1\end{bmatrix}\right\}$	G-minor root position $\left\{\begin{matrix}7^1\\2^1\\10^0\\7^0\end{matrix}\begin{bmatrix}+5^0\\+4^0\\+3^0\end{bmatrix}\begin{bmatrix}+9^0\\+7^0\end{bmatrix}\begin{bmatrix}+0^1\end{bmatrix}\right\}$ Minor X-dominant 2

[7] Here and in other cases we omit the superscripts indicating the grade of interval lengths.

Theorem 5. *One reason to name "–" "negation" lies in the fact that double negation yields exactly the starting chord:*

$$- - A = A.$$

Definition 11. $\begin{bmatrix} l_n \\ \vdots \\ l_1 \end{bmatrix}^1$ is a symmetric basic matrix of chord $C =_{df}$

(i) For even-numbered n: $l_1 = l_n, \ldots, l_{\frac{n}{2}} = l_{\frac{n}{2}+1}$

(ii) For odd-numbered n: $l_1 = l_n, \ldots, l_{\frac{n+1}{2}} - 1 = l_{\frac{n+1}{2}} + 1$ and $l_{\frac{n+1}{2}}$ fixed.

Observation: For each chord C with a symmetric basic matrix holds that the patterns on the lengths of intervals within each grade m of C are symmetric as well. I.e., the matrix of C is symmetric. We abbreviate a chord with a symmetric matrix C^S

Theorem 6. *If the argument of our negation "–" is an arbitrary C^S, then applying negation shows no effect.*

$$-C^S = C^S.$$

This situation is analogous to the situation in several many-valued logics in general where negation maps at least one value on itself (e.g., a third value *indeterminate*) and in 4-valued semantics of the logic of first degree entailments in particular: If we could negate expressions which show only the valuations *both* and *neither*, then the negation of such expressions does not have any effect at all. From a classical point of view this situation is very strange. We can show that our negation behaves classically with respect to tonal chords but non-classically with respect to symmetrical diminished chords:[8]

$$-\begin{bmatrix} +3^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +6^0 \end{bmatrix} = \begin{bmatrix} +3^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +6^0 \end{bmatrix}$$

$$-\begin{bmatrix} +3^0 \\ +6^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +9^0 \\ +9^0 \end{bmatrix} \begin{bmatrix} +0^1 \end{bmatrix} = \begin{bmatrix} +3^0 \\ +6^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +9^0 \\ +9^0 \end{bmatrix} \begin{bmatrix} +0^1 \end{bmatrix}$$

Let us summarize some essential properties of our negation "–":

(1) Its general formal characterization is completely independent of the concrete tones of the argument chord.
(2) It can be applied to any chord of any complexity without any restriction.
(3) For each chord C—independently of its internal structure—double negation of C gives C back: $- - C = C$.
(4) The relation between the expressions $-C$ and C is a very global case of internal harmony.
(5) Nevertheless: The concrete result of this negation depends highly on the concrete inner structure of the argument chord.

[8] In the first 8 bars of Ludwig van Beethoven's Opus 132 (String Quartet No. 15 in A minor (1825)) you can find an amazing number of different diminished chords with a symmetric matrix.

(6) If we negate structures of major-minor tonality, we get interesting concrete cases of internal harmony like relative minor and kinds of chord inversion (see below).
(7) An extreme case is the negation of chords with symmetric matrices C^S: $-C^S = C^S$.

7.1.2. Major and Minor Chords Under Negation. Our negation "−" has the interesting property that it converts pure major chords in pure minor chords and vice versa without any exception. To be more precise: It produces to each pure major chord *its* corresponding pure minor chord and the next application gives the original major chord back. This resembles the situation of the Routley Star "*": We start in some world w. w^* is *its* counterpart. And we have $w^{**} = w$.

An analogy: We can think of W as the set of all pure major chords in the logical space of an infinite scale of integers. W^* is then the set of all pure minor chords in the same logical space. $W \cup W^*$ would be then the set of all pure tonal chords, i.e. the union of the set of all pure major chords and the set of all pure minor chords. Finally, "−" corresponds to the Routley Star "*" in the sense, that one application of "−" to an element of W switches from the set of pure major chords to *its* element of the set of all minor chords (W^*). The second application of "−" switches back not only to the first set but exactly to the same element, the same chord. This holds for all elements of both sets in both directions. Our negation "−" is an interesting way of a bijective mapping between the set of all pure major chords and the set of all minor chords. With respect to the set of all pure tonal chords it characterizes the whole space in this respect.

Another analogy: If you would like to think about the pure major chords as the true chords and the minor chords as the false chords (or vice versa!), and the space of pure tonal chords as the total space, then the situation is similar to the situation in classical logic. But be aware that the set of all pure tonal chords is only a small proper subset of the set all logically possible chords!
The underlying meta-theorem of the remarks above reads as follows:

Theorem 7. *Let $\mathcal{J}$ be the set of all pure major chords and $\mathcal{N}$ be the set of all pure minor chords:*

$$\forall C[(C \in \mathcal{J} \Leftrightarrow -C \in \mathcal{N}) \;\&\; (-C \in \mathcal{J} \Leftrightarrow C \in \mathcal{N})].$$

Because of $- - C = C$ it is clear that we have

$$\forall C[C \in \mathcal{J} \Leftrightarrow - - C \in \mathcal{J}]\text{and}$$
$$\forall C[C \in \mathcal{N} \Leftrightarrow - - C \in \mathcal{N}]$$

The proof of the theorem presupposes the explication of the general form of basic interval lengths of major chords as well as the general form of basic interval lengths of minor chords (cp. above). Then it is easy to see that the complete inversion of the order of the lengths of basic intervals transforms one form into the other and vice versa.

Because of the last theorem we know that C^J and $-C^J$ (C^N and $-C^N$) have different patterns of interval lengths. If C would be an C^S, then A and $-A$ have the same pattern of interval lengths. This leads to the following meta-theorem:

Theorem 8. *There are neither pure major chords nor pure minor chords with symmetric patterns of interval lengths: let $\mathcal{T}$ be the set of all pure tonal chords. Then*

$$\forall C[C \in \mathcal{T} \Rightarrow C \neq C^S].$$

7.1.3. Other Reversion Operators with Respect to Triads. Fixing the frame interval means fixing its interval length *together* with the pitches of the two tones realizing that interval. We get alternative reversion operators by fixing the length of the frame interval together with at most one tone of that interval. If we look at triads (3-tone-chords) of any kind, it is possible to fix (a) only the highest tone of the frame interval, (b) only the lowest tone of the frame interval or (c) the middle tone of this chord. Here is the rule of the reversion operator $-_2$ with respect to a fixed middle tone:

Definition 12. (R^m)

$$-_2\left\{\begin{matrix} t_3\\ t_2\\ t_1\end{matrix}\begin{bmatrix}\{t_3,t_2\}\\ \{t_2,t_1\}\end{bmatrix}\big[\{t_3,t_1\}\big]\right\} \Longrightarrow \left\{\begin{matrix} s_3\\ t_2\\ s_1\end{matrix}\begin{bmatrix}\{s_3,t_2\}\\ \{t_2,s_1\}\end{bmatrix}\big[\{s_3,s_1\}\big]\right\}$$

with

(1) $\{s_3, t_2\} =_{IL} \{t_2, t_1\}$
(2) $\{t_2, s_1\} =_{IL} \{t_3, t_2\}$ and, therefore,
(3) $\{s_3, s_1\} =_{IL} \{t_3, t_1\}$: length of frame interval is fixed but normally realized by other tones.[9]

Argument	Result of negation "$-_2$"	Tonal correlation
C-major root position $\left\{\begin{matrix} 7^0\\ 4\\ 0^0\end{matrix}\begin{bmatrix}+3\\ +4\end{bmatrix}[+7]\right\}$	C♯-minor root position $\left\{\begin{matrix} 8^0\\ 4^0\\ 1^0\end{matrix}\begin{bmatrix}+4\\ +3\end{bmatrix}[+7]\right\}$	Major/1+minor same position
C-major first inversion $\left\{\begin{matrix} 0^1\\ 7^0\\ 4^0\end{matrix}\begin{bmatrix}+5\\ +3\end{bmatrix}[+8]\right\}$	G-minor second inversion $\left\{\begin{matrix} 10^1\\ 7^0\\ 2^0\end{matrix}\begin{bmatrix}+3\\ +5\end{bmatrix}[+8]\right\}$	X-dominant minor next inversion
C-major second inversion $\left\{\begin{matrix} 4^1\\ 0^1\\ 7^0\end{matrix}\begin{bmatrix}+4\\ +5\end{bmatrix}[+9]\right\}$	F-minor first inversion $\left\{\begin{matrix} 5^1\\ 0^1\\ 8^0\end{matrix}\begin{bmatrix}+5\\ +4\end{bmatrix}[+9]\right\}$	X-dominant former inversion

[9] We omit the superscripts "0" of the numbers indicating the length of intervals.

Theorem 9. *Let* C^3 *be any triad:* $-_2 -_2 C^3 = C^3$.

Theorem 10. *Let* C^{3S} *be any triad with a symmetrical matrix:* $-_2 C^{3S} = C^{3S}$.

Theorem 11. *Let* C *be a triad with a matrix of the form* $\left\{ \begin{bmatrix} l_2^3 \\ l_1^2 \end{bmatrix} \begin{bmatrix} l_1^3 \end{bmatrix} \right\}$*:*

$$- \left\{ \begin{bmatrix} l_2^3 \\ l_1^2 \end{bmatrix} \begin{bmatrix} l_1^3 \end{bmatrix} \right\} = -_2 \left\{ \begin{bmatrix} l_2^3 \\ l_1^2 \end{bmatrix} \begin{bmatrix} l_1^3 \end{bmatrix} \right\}.$$

But remember that the last theorem only means that all interval lengths are identical, but normally the frame interval will be realized by other tones. Compare:

$$- \left\{ \begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3^0 \\ +4^0 \end{bmatrix} \begin{bmatrix} +7^0 \end{bmatrix} \right\} = \left\{ \begin{matrix} 7^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +7^0 \end{bmatrix} \right\}$$

$$-_2 \left\{ \begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3^0 \\ +4^0 \end{bmatrix} \begin{bmatrix} +7^0 \end{bmatrix} \right\} = \left\{ \begin{matrix} 8^0 \\ 4^0 \\ 1^0 \end{matrix} \begin{bmatrix} +4^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +7^0 \end{bmatrix} \right\}$$

By alternately iterating "$-$" and "$-_2$" we can walk through the circle of fifths chromatically switching between major- and minor-chords and keeping root position. We start with C-major-triad in root position:

(a) Upwards

$\emptyset$	C-major $\left\{ \begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix} \begin{bmatrix} +7 \end{bmatrix} \right\}$	$-_2$	C♯-minor $\left\{ \begin{matrix} 8^0 \\ 4^0 \\ 1^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} \begin{bmatrix} +7 \end{bmatrix} \right\}$
$--_2$	C♯-major $\left\{ \begin{matrix} 8^0 \\ 5^0 \\ 1^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix} \begin{bmatrix} +7 \end{bmatrix} \right\}$	$-_2 --_2$	D-minor $\left\{ \begin{matrix} 9^0 \\ 5^0 \\ 2^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} \begin{bmatrix} +7 \end{bmatrix} \right\}$
$- -_2 --_2$	D-major $\left\{ \begin{matrix} 9^0 \\ 6^0 \\ 2^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix} \begin{bmatrix} +7 \end{bmatrix} \right\}$	...	...

(b) Downwards

$\emptyset$	C-major $\left\{\begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix} [+7] \right\}$	$-$	C-minor $\left\{\begin{matrix} 7^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\}$
$^-2^-$	B-major $\left\{\begin{matrix} 6^0 \\ 3 \\ 11^-1 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix} [+7] \right\}$	$^{--}2^-$	B-minor $\left\{\begin{matrix} 6^0 \\ 2^0 \\ 11^- \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\}$
$^-2^{--}2^-$	B-flat-major $\left\{\begin{matrix} 5^0 \\ 2 \\ 10^-2 \end{matrix} \begin{bmatrix} +3 \\ +4 \end{bmatrix} [+7] \right\}$	...	...

7.1.4. Cyclic Chord Operators. In many-valued logics with linearly ordered truth values represented by natural numbers, *cyclic negation* is a unary truth function that takes a truth value n and returns $n-1$ as value if n is not the lowest value; otherwise it returns the highest value. Following this idea by Emil Post we can characterize *cyclic* chord operators upwards $\uparrow$ and downwards $\downarrow$ by the following general rules:

$$\uparrow \left\{ \begin{matrix} t_{n+1} \\ t_n \\ t_{n-1} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ . \\ . \\ . \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix}^1 \right\} = \left\{ \begin{matrix} t_{n+1} \\ s_n \\ s_{n-1} \\ \vdots \\ s_3 \\ s_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, s_n\} \\ \{s_n, s_{n-1}\} \\ . \\ . \\ . \\ \{s_3, s_2\} \\ \{s_2, t_1\} \end{bmatrix}^1 \right\}$$

with $\{t_{n+1}, s_n\} =_{IL} \{t_2, t_1\}$, $\{s_n, s_{n-1}\} =_{IL} \{t_{n+1}, t_n\}$, $\ldots$, $\{s_3, s_2\} =_{IL} \{t_4, t_3\}$, $\{s_2, t_1\} =_{IL} \{t_3, t_2\}$

$$\downarrow \left\{ \begin{matrix} t_{n+1} \\ t_n \\ t_{n-1} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ . \\ . \\ . \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix}^1 \right\} = \left\{ \begin{matrix} t_{n+1} \\ s_n \\ s_{n-1} \\ \vdots \\ s_3 \\ s_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, s_n\} \\ \{s_n, s_{n-1}\} \\ . \\ . \\ . \\ \{s_3, s_2\} \\ \{s_2, t_1\} \end{bmatrix}^1 \right\}$$

with $\{s_2, t_1\} =_{IL} \{t_{n+1}, t_n\}$, $\{s_3, s_2\} =_{IL} \{t_2, t_1\}$, $\ldots$, $\{s_n, s_{n-1}\} =_{IL} \{t_{n-1}, t_{n-2}\}$, $\{t_{n+1}, s_n\} =_{IL} \{t_n, t_{n-1}\}$

Theorem 12. *With respect to triads "–", "↑" and "↓" are indistinguishable: Let C^3 be any triad. Then*

$$-C^3 = \uparrow C^3 = \downarrow C^3.$$

Theorem 13. *Let C^4 be any 4-tone-chord. Then* $\uparrow\uparrow\uparrow C^4 = \downarrow\downarrow\downarrow C^4 = C^4$.

Theorem 14. *Let C^{n+1} be any $n+1$-tone-chord:*

$$\underbrace{\uparrow \ldots \uparrow}_{n-times} C^{n+1} = \underbrace{\downarrow \ldots \downarrow}_{n-times} C^{n+1} = C^{n+1}.$$

Let us look at two examples of applying "↑" to 4-tone-chords C^4 and the internal harmonic relations.

Example 1. C^4 = 4-Tone-C-major-chord in root position:

C-major root position

$$\uparrow \left\{ \begin{matrix} 0^1 \\ 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +5 \\ +3 \\ +4 \end{bmatrix} \begin{bmatrix} +8 \\ +7 \end{bmatrix} \left[+0^1 \right] \right\}$$

A-flat-major first inversion

$$\left\{ \begin{matrix} 0^1 \\ 8^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +5 \\ +3 \end{bmatrix} \begin{bmatrix} +9 \\ +8 \end{bmatrix} \left[+0^1 \right] \right\}$$

Another major: root distance +4

A-flat-major first inversion

$$\uparrow \left\{ \begin{matrix} 0^1 \\ 8^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +5 \\ +3 \end{bmatrix} \begin{bmatrix} +9 \\ +8 \end{bmatrix} \left[+0^1 \right] \right\}$$

F-major second inversion

$$\left\{ \begin{matrix} 0^1 \\ 9^0 \\ 5^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \\ +5 \end{bmatrix} \begin{bmatrix} +7 \\ +8 \end{bmatrix} \left[+0^1 \right] \right\}$$

Another major: root distance +3

Applying ↑ to any 4-tone-major-chord in root position produces only major chords. The introduction of double ↑ yields the X-dominant of the argument chord.

Example 2. C^4 = 4-tone-C-seventh-major-chord in root position:

C-major seventh root position

$$\uparrow \left\{ \begin{matrix} 10^0 \\ 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3 \\ +3 \\ +4 \end{bmatrix} \begin{bmatrix} +6 \\ +7 \end{bmatrix} \left[+10 \right] \right\}$$

E♯-minor root position

$$\left\{ \begin{matrix} 10^0 \\ 6^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \\ +3 \end{bmatrix} \begin{bmatrix} +7 \\ +6 \end{bmatrix} \left[+10 \right] \right\}$$

and Major sixth as deepest tone

E♯-minor & major sixth

$$\uparrow \left\{ \begin{matrix} 10^0 \\ 6^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \\ +3 \end{bmatrix} \begin{bmatrix} +7 \\ +6 \end{bmatrix} \left[+10 \right] \right\}$$

C-minor seventh root position

$$\left\{ \begin{matrix} 10^0 \\ 7^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +3 \\ +4 \\ +3 \end{bmatrix} \begin{bmatrix} +7 \\ +7 \end{bmatrix} \left[+10 \right] \right\}$$

Symmetrical matrix pattern

7.2. Arithmetic, Tone-Related Chord Operators

Tone-related chord operators are unary operators where we have to use *arithmetic* operations to characterize the logical properties. We look at two examples.

7.2.1. Chord Inversion I^u. The general characterization of the (tonal) chord inversion operator upwards I^u is[10]

$$I^u \left\{ \begin{matrix} t_{n+1} \\ t_n \\ t_{n-1} \\ \vdots \\ t_3 \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ . \\ . \\ . \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix} \right\} \Rightarrow \left\{ \begin{matrix} t_1 + m \times 12 \\ t_{n+1} \\ t_n \\ \vdots \\ t_4 \\ t_3 \\ t_2 \end{matrix} \begin{bmatrix} \{t_1 + m \times 12, t_{n+1}\} \\ \{t_{n+1}, t_n\} \\ . \\ . \\ . \\ \{t_4, t_3\} \\ \{t_3, t_2\} \end{bmatrix} \right\}$$

with the following conditions

(i) $m = 1$, if $L\{t_{n+1}, t_1\} < 12$
(ii) $m = 2$, if $12 \leq L\{t_{n+1}, t_1\} < 24$.
(iii) in general: $m = i$, if $((i-1) \times 12) \leq L\{t_{n+1}, t_1\} < (i \times 12), 1 < i$.

The new highest tone has the same basic number as the former deepest tone. And it is the closest possible tone to the former highest tone.

$$I^u \left\{ \begin{matrix} 7^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\} = \left\{ \begin{matrix} 0^1 \\ 7^0 \\ 3^0 \end{matrix} \begin{bmatrix} +5 \\ +4 \end{bmatrix} [+9] \right\}$$

We get the new basic interval length $+5^0$ out of $L\{0^0 + (1 \times 12), 7^0\}$, i.e., $L\{0^1, 7^0\}$ and also a new length of the frame interval: $+5^0$ The traditional name for chords of that matrix is (minor) *sixth chord*. It characterizes the length of the frame interval with respect to a minor chord: $+9^0$. It is an elliptic and context-dependent name: (a) It is elliptic because the full name with respect to the *inner* structure of that chord would be *major-triad/fourth/major-sixth-chord*. The name with respect to its basic matrix would be simply *major-triad/fourth-chord*. (b) It is context-dependent because it could be understood as indicating a functional relation to another chord (3-tone-minor-chord in root position). But then it is not a name of the chord itself. It is a name of a relation to another chord. Then it is a name of a special case of *internal* harmony, usually called *fist inversion*. We explicate *fist inversion* by one application of I^u to a minor or major triad in root position.

But we can apply I^u to any chord within our logical space. I.e., that we can apply it to chords with more than three tones (e.g., seventh chords) and to tonally unpleasant chords as well:

$$I^u \left\{ \begin{matrix} 10^0 \\ 9^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +1 \\ +6 \\ +3 \end{bmatrix} \begin{bmatrix} +7 \\ +9 \end{bmatrix} [+10] \right\} = \left\{ \begin{matrix} 0^1 \\ 10^0 \\ 9^0 \\ 3^0 \end{matrix} \begin{bmatrix} +2 \\ +1 \\ +6 \end{bmatrix} \begin{bmatrix} +3 \\ +7 \end{bmatrix} [+9] \right\}$$

[10] We omit the superscript "1" of the basic matrices.

Applying I^u twice yields the second inversion of a minor triad in root position:

$$I^u I^u \left\{ \begin{matrix} 7^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\} = \left\{ \begin{matrix} 3^1 \\ 0^1 \\ 7^0 \end{matrix} \begin{bmatrix} +3 \\ +5 \end{bmatrix} [+8] \right\}$$

Three application of I^u reproduces the matrix of the triad and the basic number of each tone, but one octave higher:

$$I^u I^u I^u \left\{ \begin{matrix} 7^0 \\ 3^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\} = \left\{ \begin{matrix} 7^1 \\ 3^1 \\ 0^1 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\}.$$

It would be easy to characterize the corresponding chord inversion operator downwards I^d. The first application of this operator to a (tonal) triad would yield another second inversion of this chord.

7.2.2. Barré oerators β_{+i} and β_{-j}. Barré-operators have the form β_{+i} or the form β_{-j}, respectively. The index $+i$ indicates the number of steps of increasing the pitch of each tone of a argument chord. β_{+i} characterizes the move of a barreing one step "higher" (e.g., on a guitar). $\beta_{+3}C$ says, that the chord C is transformed into a chord with an identical matrix (isomorphic pattern of interval lengths) via increasing the pitch of each tone $t_i \in C$ by three steps on the chromatic scale. $\beta_{-j}C$ does the same thing in the opposite direction.

The general rules of Barré-operators are:

$$\beta_{+i} \left\{ \begin{matrix} t_{n+1} \\ t_n \\ \cdot \\ \cdot \\ \cdot \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ \vdots \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix} \right\} \Rightarrow \left\{ \begin{matrix} t_{n+1}+i \\ t_n+i \\ \cdot \\ \cdot \\ \cdot \\ t_2+i \\ t_1+i \end{matrix} \begin{bmatrix} \{t_{n+1}+i, t_n+i\} \\ \{t_n+i, t_{n-1}+i\} \\ \vdots \\ \{t_3+i, t_2+i\} \\ \{t_2+i, t_1+i\} \end{bmatrix} \right\}$$

$$\beta_{-j} \left\{ \begin{matrix} t_{n+1} \\ t_n \\ \cdot \\ \cdot \\ \cdot \\ t_2 \\ t_1 \end{matrix} \begin{bmatrix} \{t_{n+1}, t_n\} \\ \{t_n, t_{n-1}\} \\ \vdots \\ \{t_3, t_2\} \\ \{t_2, t_1\} \end{bmatrix} \right\} \Rightarrow \left\{ \begin{matrix} t_{n+1}-j \\ t_n-j \\ \cdot \\ \cdot \\ \cdot \\ t_2-j \\ t_1-j \end{matrix} \begin{bmatrix} \{t_{n+1}-j, t_n-j\} \\ \{t_n-j, t_{n-1}-j\} \\ \vdots \\ \{t_3-j, t_2-j\} \\ \{t_2-j, t_1-j\} \end{bmatrix} \right\}$$

Because of $\{t_k+i, t_l+i\} =_{IL} \{t_k+i, t_l+i\}$ and $\{t_k-j, t_l-j\} =_{IL} \{t_k+i, t_l+i\}$ for all k and l we get immediately that the chords C, $\beta_{+i}C$ and $\beta_{-j}C$ have isomorphic matrices.

Additionally, we observe that one application of β_{+1} on triads of the form $\left\{ \begin{matrix} 7^0 \\ 4^0 \\ 0^0 \end{matrix} \begin{bmatrix} +4 \\ +3 \end{bmatrix} [+7] \right\}$ coincides with the application of the operator sequence $--_2$:

$$\beta_{+1}\left\{\begin{matrix}7^0\\4^0\\0^0\end{matrix}\begin{bmatrix}+3\\+4\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}=\left\{\begin{matrix}8^0\\5^0\\1^0\end{matrix}\begin{bmatrix}+3\\+4\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}$$

$$-_2\left\{\begin{matrix}7^0\\4^0\\0^0\end{matrix}\begin{bmatrix}+3\\+4\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}=\left\{\begin{matrix}8^0\\4^0\\1^0\end{matrix}\begin{bmatrix}+4\\+3\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}$$

$$-\left\{\begin{matrix}8^0\\4^0\\1^0\end{matrix}\begin{bmatrix}+4\\+3\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}=\left\{\begin{matrix}8^0\\5^0\\1^0\end{matrix}\begin{bmatrix}+3\\+4\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}$$

This demonstrates that the iteration of operators acting only with respect to interval lengths can show stable arithmetical effects. The 12-fold application of β_{+1} is identical with the threefold application of I^u:

$$\underbrace{\beta_{+1}\ldots\beta_{+1}}_{12-times}\left\{\begin{matrix}7^0\\4^0\\0^0\end{matrix}\begin{bmatrix}+3\\+4\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}=I^uI^uI^u\left\{\begin{matrix}7^0\\4^0\\0^0\end{matrix}\begin{bmatrix}+3\\+4\end{bmatrix}\begin{bmatrix}+7\end{bmatrix}\right\}$$

8. Some Applications and Perspectives

Applications of our logic of chords can be of interest for several disciplines and practices: logic itself, philosophy, music theory, computer science, cognitive science, linguistics, etc. Let us mention some candidates:

(1) It should be possible to embed the entire language of classical propositional logic (propositional constants/variables, negation and binary connectives) isomorphically into a well-defined sublanguage of our molecular logic of chords. Following this line we are able to represent any tautology and any contradiction formally and audibly.

(2) An interesting task would be to find precise characterizations of special cases of internal harmony corresponding to major-minor tonality, dodecaphony, serial music etc.

(3) If a formal theory of rhythm is available, it can be combined with the logic of chords to a more general theory of music. It is also promising to find an axiomatization.

(4) Chord sequences are interesting cases in the context of Wittgenstein's concept of *family resemblances*. Chords resemble one another regarding tones, regarding basic numbers of tones, intervals, interval lengths of the same grade or of different grades, the basic numbers of interval lengths, more complex parts of matrices etc.

(5) Our vocabulary allows a precise distinction between the inner structure of chords, internal and external aspects of harmony. A nice example is the famous *Neapolitan chord*. The name itself has a cultural context: associated with the Neapolitan School. If we analyze compositions, we will find that *Neapolitan chord* is not a proper name, because there are a lot of different chords with a different number of tones and very different matrices. What is common to them is that they are all elements of the set of major

chords. The internal harmony can be described by using another chord as reference point (usually as tonic) and apply a unary operator to one of its X-dominants (usually called subdominant). But if we speak about its function using formulations like "preparing the dominant" or "functions as a subdominant without being a subdominant", then this belongs to an external (metaphorical) kind of speaking or indicates conceptual confusion.

(6) If one puts a Neapolitan chord into the "wrong" position of a cadence, then it produces something which is called "violation of expectation" in cognitive science. On the one hand the logic of chords can be used to make this concept much more precise. On the other hand iterated applications of chord operators yield expected and unexpected continuations of sequences of chords in a systematic way. If such sequences are used in experiments in neuroscience, the obtained data can be interpreted in the context of formally controlled experimental settings.

(7) We can use chord operators in the context of analyzing pieces of music. Using chord operators is an alternative way in describing internal harmonic relations.

(8) An interesting topic in computer science is the automatic recognition of patterns. The logic of chords offers an alternative tool to have access to these patterns.

(9) Looking at chord operators and matrices of chords could also be of interest for composers. With respect to the inner form of chords there is an alternative to all-interval twelve-ton rows: *all-interval-length chords*. All-interval-length chords are structures which contain each of the interval lengths $+1^0, +2^0, \ldots, +10^0, +11^0$ at least once. *Perfect* all-interval-length chords are structures there we have each of the mentioned interval lengths exactly once. A nice example is the matrix

$$\left\{ \begin{bmatrix} +6^0 \\ +4^0 \\ +5^0 \\ +2^0 \\ +1^0 \end{bmatrix} \begin{bmatrix} +10^0 \\ +9^0 \\ +7^0 \\ +3^0 \end{bmatrix} \begin{bmatrix} +3^1 \\ +11^0 \\ +8^0 \end{bmatrix} \begin{bmatrix} +5^1 \\ +0^1 \end{bmatrix} \begin{bmatrix} +6^1 \end{bmatrix} \right\}.$$

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