Log. Univers. 12 (2018), 141–206

1661-8297/18/010141-66, *published online* February 21, 2018
https://doi.org/10.1007/s11787-018-0186-7

Logica Universalis

Venn Diagram with Names of Individuals and Their Absence: A Non-classical Diagram Logic

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Abstract. Venn diagram system has been extended by introducing names of individuals and their absence. Absence gives a kind of negation of singular propositions. We have offered here a non-classical interpretation of this negation. Soundness and completeness of the present diagram system have been established with respect to this interpretation.

Mathematics Subject Classification. Primary 03B99; Secondary 00A66.

Keywords. Diagram logic, absence of individuals, completeness, non-classical semantics.

1. Introduction

The study of logic in terms of diagrams dates back to Euler (1707–1783, [8]), Venn (1834–1923, [21]) and Peirce (1839–1914, [14]). However, representation of a statement in linear form by a string of symbols has dominated the logic-culture. After a gap for about 100 years through the publications of Shin [15], Hammer [11], Barwise and Gerard Allwein [1] interest in diagrammatic studies in the context of logic has been renewed and regular conferences, workshops, schools and publications are taking place in recent decades.

In [3], diagrams of Shin and Hammer have been extended by incorporation of names of individuals. Parallelly, in Spider diagrams [9,12,13,17,18] existential and constant spiders have been introduced. This diagram system, indeed strives for greater expressibility. Further development can be viewed in [4,16,20]. But what is unique in [3] is the representation of absence of individuals. The introduction of absence explicitly was motivated from ancient Indian logical system [7] and an attempt to deal with the context of open universe diagrammatically [5] followed as a natural consequence. Since then some other studies have been carried out on depicting both the presence and absence of individuals in a diagram [2,6,19]. The most recent publications in this series [2,19] report a comparative empirical study as regards cognitive advantage between depiction of absence of an individual and not its explicit depiction in a diagram.

In the context of closed universe where it is assumed that the universe of discourse is pre-fixed which is usually represented by a rectangle, absence of an individual a in a region would be equivalent to its presence in the complementary region with respect to the given universe. This is the classical standard understanding and was considered in [3], though it was observed afterwards that the formalism requires some improvement. However, there may be another interpretation of the absence of an individual in a region and that is the theme of the present work. Under this interpretation, absence of an individual in the extension of a predicate does not necessarily imply the individual's presence in the complement of the extension with respect to the universe. For an initial discussion on this kind of interpretation we refer to [6]. However, in order to make this paper as much self contained as possible we discuss briefly about the motivation of such an interpretation. In the second interpretation, from the absence of an individual in a region r we cannot infer its presence in the complement of r though from the presence of an individual in r we can infer its absence in the complement of r. From the point of view of information theory, 'a is here in A' and 'a is not here in A' are both meaningful pieces of information. 'a is outside A' is a much stronger information than merely 'a is not here in A'. The agent who informs 'a is outside A' means to convey that he/she knows where 'a' is and that location is not A. While 'a is not in A' is a negative information, 'a is in $\bar{\mathrm{A}}$ (the complement of A)' is a positive information. By the depiction of the absence of a in the diagram we intend to reflect the negative information.

Besides, in the set theoretic context, there is the notion of recursively enumerable set A of which the inside elements are tractable while the complement is not. There is a program that will find the object a if $a \in \mathrm{A}$, but no program will be able to locate a if it is outside. Our present interpretation is more general than recursive enumerability. All these and other motivations have been discussed in [5,6]. We shall formally define this interpretation in Sect. 5 and establish soundness and completeness of our proposed diagram-system in the present paper. In defining present diagram system Shin's system Venn-II [15] will be mainly followed. But incorporation of individual and its absence within the diagram-system would require substantial modification in syntax, semantics and the procedure of proving the completeness. Additionally, the non-standard semantics renders significant difference from all other diagrammatic systems.

It is to be noted that depiction of absence is a way of representing negation in a diagram. There have been very few attempts to represent negation in diagram systems. This may be because traditionally, the collection of categorical propositions A, E, I, O are closed w.r.t negations and the use of emptiness (by shading) and non-emptiness (by cross) takes care of negated statements. The negation of an A statement is an O statement and vise versa. Similar is the relationship between E and I statements. The problem arises with the singular propositions and their negations (see [6] for a detailed discussion). The general propositions A, E, I, O have both contraries and contradictories but singular propositions have only contradictories. Thus depiction of absence in the present context is the depiction of negation of singular propositions and which, along with the present non-classical interpretation will require different assumptions and formal rules. These will be apparent in Sect. 3 with the introduction and elimination rules of the absence of individuals.

In spider diagram system a horizontal bar is used on top of a diagram to represent negation [9,17]. Thus negation of singular propositions could be depicted in it, of course, with classical interpretation. However, presence of an individual a and absence of an individual b simultaneously seem to be better represented in the present diagrammatic system from the angle of well-matchedness and iconicity [10].

This paper is organized in the following order:

Section 2: The diagrammatic Language,
Section 3: Rules of transformation,
Section 4: Some useful notions,
Section 5: Semantics of the diagram system and soundness theorem,
Section 6: Completeness theorem,
Section 7: Concluding remarks: differences of our system with the others,

Appendix: Proof of Lemma 5.5 required for soundness theorem.

We have tried to separate syntax from semantics with sufficient rigour. Usually in the diagram-logic literature syntax and semantics overlap which in our opinion misses the target namely establishing that diagrams themselves, without much explanation by words, can be the conveyor of information. The objective that manipulation of diagrams should be like manipulation of symbols also gets lost. We have deliberately tried to adopt the type of a standard logic text in the syntax-semantics divide. For this reason it may appear that there are often unnecessary steps in derivations. But this attempt has, we believe, rendered our system free from ambiguity (use of semantic notions in syntactic proofs) while distancing itself from the approach of abstract syntax [9,12,13,18].

2. The Diagrammatic Language

We intend to proceed as is done in logic texts, viz. first the alphabet and then the well formed formulas – in our case, primitive symbols which are basic diagrams and then more complex diagrams i.e. well formed diagrams.

2.1. Primitive Symbols

 : Rectangle; representing the universe,

 : Closed curve; representing monadic predicate,

 : Shading; representing emptiness,

x: Cross; representing non-emptiness,

a, b, c,...: Names of individuals (finitely many),
$\bar{a}$, $\bar{b}$, $\bar{c}$,...: Absence of individual named a, b, c,...,
A, B, C,...: Names for closed curve or labels (finitely many),
—: Line connecting crosses (x's) and rectangles,
- - -: Broken line connecting individuals (a's).

Definition 2.1 (*Diagrammatic objects*). The following items are diagrammatic objects:

(1) shading (),
(2) cross (x),
(3) names of individuals (a, b, c,...),
(4) names of individuals with bars ($\bar{a}$, $\bar{b}$, $\bar{c}$,...),
(5) sequence of crosses (x's) connected by — (line connecting crosses, in short lcc),
(6) sequence of individuals (a's) connected by - - - (broken line connecting individuals, in short lci) for each name 'a'.

Definition 2.2 (*Node*).

Each x in an lcc is called a x-node.
Each a in an lci is called an a-node.
x-nodes and a-nodes are simply called nodes if there is no confusion.

Definition 2.3 (*Blank closed curve*).

Any closed curve without any diagrammatic object is called a blank closed curve.

Definition 2.4 (*Diagram*).

A rectangle containing finitely many closed curves or diagrammatic objects or both are the building blocks of basic diagrams (vide Fig. 1). Besides, there may be a diagram which is a sequence of such basic diagrams joined by lines.

Of these diagrams some are taken to be well formed (see Sect. 2.2).

Definition 2.5 (*Region, basic region, minimal region*).

A basic region is the space enclosed by a rectangle or a closed curve. A basic region included by a closed curve A shall also be denoted by A whenever necessary.

The rectangle and the closed curves together divide the space within the rectangle into disjoint spaces. Each such space is called a minimal region.

A region is union of some minimal regions.

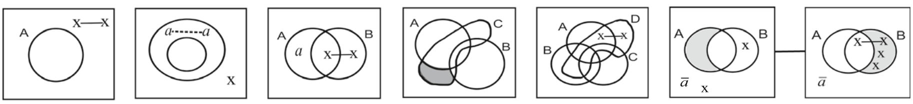


FIGURE 1. Diagrams

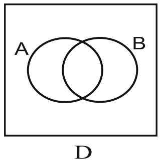


FIGURE 2.

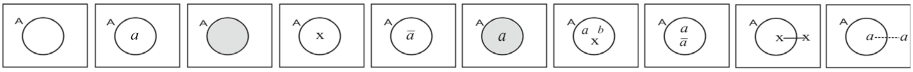


FIGURE 3. Type-I diagrams

Example 2.1. In Fig. 2, the diagram D has three basic regions A, B and the basic region enclosed by the rectangle. Four minimal regions (−A) · (− B), A · B, A · (− B) and (− A) · B. A region is the union of some of this minimal regions, for example, ((A · B) + (A · (− B))) is a region [+, · and − denote respectively the join, intersection and relative complement of regions: see also Sect. 2.3 for definitions].

2.2. Well-Formed Diagram (wfd)

Type-I

(1) A single blank closed curve within a rectangle with a single label attached with it is a Type-I diagram.
(2) A single closed curve having a label with one or more diagrammatic objects inscribed within it or outside it but within the rectangle is a wfd provided the following conditions i-iv are satisfied.
(i) Two nodes of an lcc or lci will not occur in the same minimal region.
(ii) Two single x's or a's or $\bar{a}$'s will not occur in the same minimal region.
(iii) If shading occurs in a minimal region it should cover the entire minimal region.
(iv) An individual 'a' will not occur in more than one minimal region.

Following are some examples (vide Fig. 3).

Type-II

A Type-II diagram is a diagram with more than one closed curves within rectangle that can be ordered in a sequence C_1, C_2, ..., C_n (say) such that the closed curve C_i divides all the minimal regions obtained from the closed curves C_1, C_2, ..., C_{i-1} in exactly two minimal regions and which satisfies the following conditions:

(i) The minimal regions so formed may have or may not have entries of diagrammatic objects.
(ii) The closed curve should not pass through the signs x, a, or $\bar{a}$ or labels, A, B,. . .
(iii) The diagram may contain lcc or lci with the restriction that two nodes of the same lcc or lci will not appear in the same minimal region.

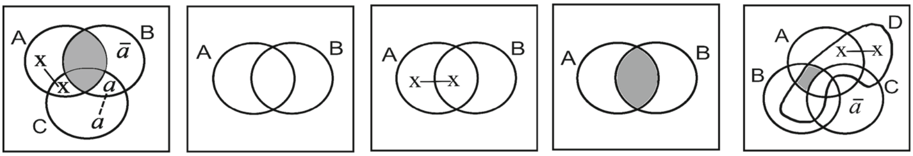


Figure 4. Type-II diagrams

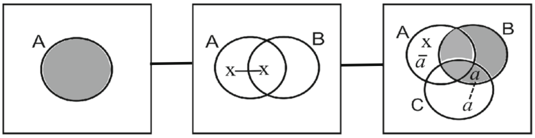


Figure 5. Type-III diagram

(iv) If there is shading in a minimal region it has to cover the entire minimal region.
(v) Two single x's or a's or $\bar{a}$'s will not occur in the same minimal region.
(vi) Labels should be attached to each closed curve and different labels for different closed curves within the same rectangle.
(vii) An individual 'a' should not occur in more than one minimal region.

Following are some examples (vide Fig. 4).

Note:

We follow here Venn's method of constructing overlapping closed curves. There are various other definitions of Venn diagrams. It is established that for any number $n > 1$ it is always possible to draw a Venn diagram satisfying the conditions stated in the definition of type-II diagrams.

Type-III

If D_1, D_2, ..., D_n, n $\geq$ 2 are of type-I or type-II diagrams then the diagram D' resulting from connecting them by straight lines (written as $D_1 - D_2 - \ldots - D_n$) is a wfd of Type-III. Each D_i is called a component of the diagram D' (vide Fig. 5).

Note: By a proper part of a type-III diagram D we shall mean a diagram obtained by dropping some of the components of D.

Henceforth by diagram we shall mean wfd.

Note: Let r be a region obtained by union of minimal regions $m_1, m_2, \ldots, m_n$. Then we say that r has a x-sequence or r has an a-sequence if and only if each m_i contains exactly one node of that sequence and none of the nodes fall in any minimal region outside r.

2.3. Counterpart Relation

The counterpart relation is a relation between the regions of any two diagrams defined as follows.

Two basic regions are counterparts if and only if they are regions enclosed by curves having the same label.

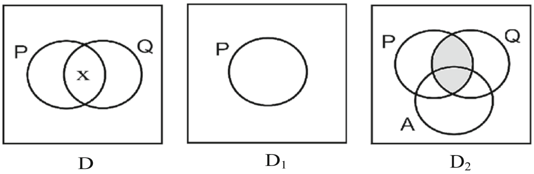


FIGURE 6. Example of counterpart relation between type-II diagrams

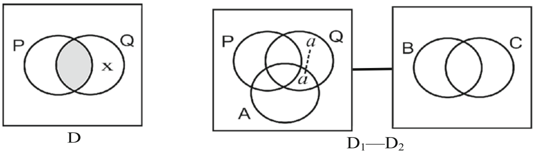


FIGURE 7. Example of counterpart relation between type-I/II and type-III diagram

If r and r' are regions of diagram D and s and s' are regions of diagram D', r is the counterpart of s and r' is the counterpart of s', then $r + r'$ is the counterpart of $s + s'$, $r \cdot r'$ is the counterpart of $s \cdot s'$ and $r - r'$ is the counterpart of $s - s'$, where +, · and − denote respectively the join, intersection and relative complement of regions. By $-$ r we shall denote the relative complement of a region r w.r.t the rectangle. It follows that if r is the counterpart of s then $-$ r is the counterpart of $-$ s.

Example 2.2. In Fig. 6, D_1 has the counterpart of the basic region P in D and D_2 has the counterpart of the basic region P and the basic region Q in D. The minimal region $P \cdot Q$ in D has a x and the counterpart of $P \cdot Q$ in D_2 has shading.

Example 2.3. In Fig. 7, the first component D_1 has the counterpart of the basic region P and counterpart of the basic region Q in D and the second component D_2 has no counterpart in D. The minimal region $P \cdot Q$ has shading in D and the counterpart of $P \cdot Q$ is blank in the first component of D_1. Again, the minimal region $Q - P$ has x in D and the counterpart of $Q - P$ has an a-sequence in the first component D_1.

Example 2.4. In Fig. 8, D_3 has the counterpart of the basic region A in D_1 and the counterpart of the basic region B in D_2. A in D_1 has a x-sequence but A in D_3 has shading. B in D_2 has a-sequence, but in D_3 the minimal region $A \cdot B$ has shading and there is no counterpart of $A \cdot B$.

D_4 has the counterpart of the basic region Q in D_1 but it has no counterpart in D_2. Q in D_1 has shading and Q in D_4 has a-sequence.

Definition 2.6 (*Identity*).

Two type-I/II diagrams are identical if and only if

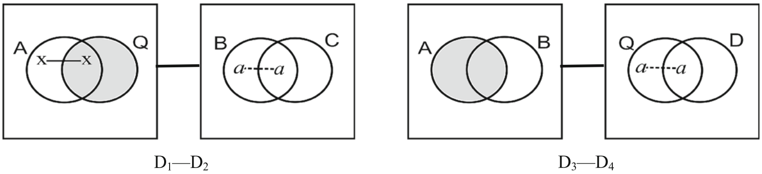


FIGURE 8. Example of counterpart relation between type-III diagrams

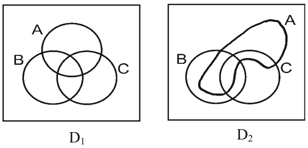


FIGURE 9. Example of identity

(i) the set of labels used in the two diagrams are same,
(ii) a region in one is shaded if and only if its counterpart in the other is shaded,
(iii) a region of one has a x-sequence if and only if its counterpart in the other has a x-sequence,
(iv)) a region of one has an a-sequence if and only if its counterpart in the other has an a-sequence,
(v) a region in one has $\bar{a}$ if and only if its counterpart in the other has $\bar{a}$.

[This definition is a modification of Hammer [11]].

Two type-III diagrams are identical if and only if each component of one is identical with some component of the other.

Example 2.5. In Fig. 9, D_1 and D_2 are drawn representation of same diagram.

2.4. Normal Form of type-I/II Diagrams

Let D (vide Fig. 10) be a type-II diagram having two closed curves. Let the minimal regions of D be m_1, m_2, m_3 and m_4.

Now m_1, m_2, m_3 and m_4 represent respectively the regions
A · (−B),
A · B,
(−A) · B and
(−A) · (−B).

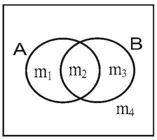


FIGURE 10.

These are all the possible combinations of the basic regions A, B and their complements −A, −B taken two together, i.e. for two curves we have $2^2 (= 4)$ minimal regions and $2^2 (= 4)$ combinations.

Suppose D has 3 curves A, B and C. Then all possible combinations of the basic regions A, B, C and their complements −A, −B, −C taken three together are

A · (−B) · C,
A · B · C,
(−A) · B · C,
(−A) · (−B) · C,
A · (−B) · (−C),
A · B · (−C),
(−A) · B · (−C),
(−A) · (−B) · (−C).

Definition 2.7 (*Normal form*).

Let a type-II diagram D have n curves $C_1, C_2, \ldots, C_n$ ($n \geq 2$). The diagram D is said to be in its normal form if it satisfies the following properties

(i) there are 2^n minimal regions in D,
(ii) all possible combinations of the basic regions $C_1, C_2, \ldots, C_n$ and their complements $-C_1, -C_2, \ldots, -C_n$ taken n together should be represented by these 2^n minimal regions.

Note: From the first two properties we can say that, a combination should be represented by exactly one minimal region.

It is clear that for any diagram with n closed curves there exists a unique Venn diagram which is in the normal form.

Example 2.6. Let us considered the following diagrams (vide Fig. 11).

These diagrams have the same number (= 3) of basic regions but varying number of minimal regions. It may be noted that only diagram D_1 is in the normal form. The diagram D_2 has 5 ($\neq 2^3$) minimal regions, i.e. it violates the first property of normal form. The diagram D_3 has 8 ($= 2^3$) minimal regions but the combination ((−A) · (−B) · C) is not represented, i.e. it violates the second property of normal form. Also the combination (A · B · (−C)) was represented by two minimal regions in the diagram D_3. The diagram D_4 has 10 ($\neq 2^3$) minimal regions, i.e. it violates the first property of normal form.

Note:

(i) A type-I diagram is in normal form by default.

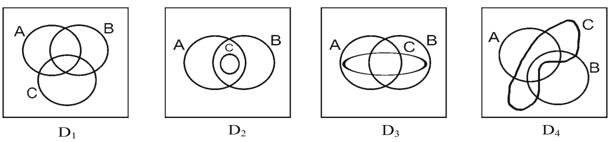


FIGURE 11. Example of normal form

(ii) A diagram is in normal form if and only if it is wfd.

So the formation rules are stated. Now, we pass on to the transformation rules as is done in the logic texts. Obviously, there are motivations behind introducing these rules but we refrain from stating them now. We would like to see that the intentions would be visible from the diagrams themselves. Otherwise the purpose of diagrammatic representation and diagram logic would be lost.

3. Rules of Transformation

3.1. Introduction Rules (for Closed Curves, $\bar{a}$ and x)

3.1.1. For Closed Curves. Let D be a type-I/II diagram. It may be transformed into a diagram by introducing a closed curve in D, obeying the restrictions 1–4 below:

(1) the newly introduced curve should divide all the minimal regions in D, into exactly two parts.
[It is to be noted that if a minimal region is shaded then it is divided into two minimal regions where both of them are shaded (vide Fig. 12))].

(2) If there is an x/a/$\bar{a}$ in a minimal region m of D, the new curve, say B, should be drawn in such a way that neither B passes through x/a/$\bar{a}$ nor the existing x/a/$\bar{a}$ appear in the intersection of the minimal region and the new curve i.e. m $\cdot$ B.
Examples of wrong introduction of closed curve (vide Figs. 13 and 14).

(3) (a) If a minimal region m included in D has a node of a x-sequence and a new closed curve, say B, is introduced in D, then x should be added in the region m $\cdot$ B and a line should connect the new x with one of the end x's of the existing x-sequence.

Example 3.1. Let us demonstrate introduction of the closed curve to B to the diagram D (vide Fig. 15).

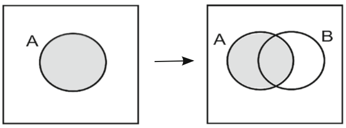


FIGURE 12.

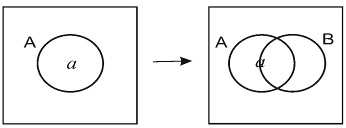


FIGURE 13.

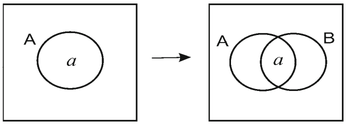


Figure 14.

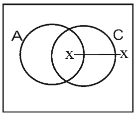


Figure 15.

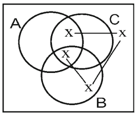


Figure 16.

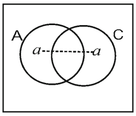


Figure 17.

When we introduce a new curve to B to D (vide Fig. 15), the minimal regions A · C and −(A + C) are divided in minimal regions ((A · C) − B) and (A · C · B) and (B − (A + C)) and −(A + C + B) respectively. We then add x in the regions (A · C · B) and (B −(A + C)) and connect them with one of the end x's of the existing x-sequence in the minimal regions ((A · C) − B) and −(A + C + B) to get the following diagram (vide Fig. 16).

(b) If a minimal region m included in D has a node of an *a*-sequence and a new closed curve, say B, is introduced in D, then *a* should be added in the region m · B and a broken line should connect the new *a* with one of the end *a*'s of the existing *a*-sequence.

Example 3.2. Let us demonstrate introduction of the closed curve B to the diagram D (vide Fig. 17).

When we introduce a new curve to B to D (vide Fig. 17), the minimal regions A − C and C − A are divided in minimal regions ((A · B)− C) and ((A − (B + C)) and ((C · B)− A) and ((C − (A + B)) respectively. We then add *a* in the regions ((A · B)− C) and ((C · B)− A) and connect them with one of the end *a*'s of the existing *a*-sequence in the minimal regions ((A − (B + C)) and ((C − (A + B)) to get the following diagram (vide Fig. 18).

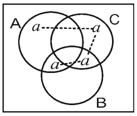


Figure 18.

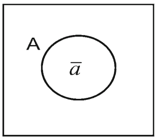


Figure 19.

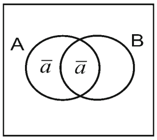


Figure 20.

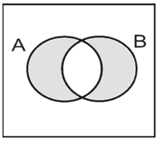


Figure 21.

(4) If a minimal region m included in D has $\bar{a}$ and a new closed curve, say B, is introduced in D, then $\bar{a}$ should be added in the region m · B.

Example 3.3. Let us demonstrate introduction of the closed curve to B to the diagram D (vide Fig. 19).

When we introduce a new curve to B to D, the basic region A is divided in minimal regions (A · B) and (A − B). We add $\bar{a}$ in the region (A · B) to get the following diagram (vide Fig. 20).

3.1.2. For $\bar{a}$. Let D be a type-I/II diagram, It may be transformed into a diagram by introducing $\bar{a}$ in a minimal region m of D if

(i) m is shaded or
(ii) a portion (may be whole) of − m, the complementary region of m has a-sequence.

Example 3.4. In diagram D (vide Fig. 21), the minimal regions A − B and B − A are shaded.

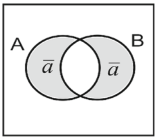


Figure 22.

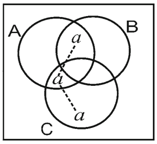


Figure 23.

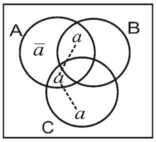


Figure 24.

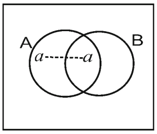


Figure 25.

Now using the above rule we introduce $\bar{a}$ in the minimal regions A − B and B − A (vide Fig. 22).

Example 3.5. In diagram D (vide Fig. 23), the region (((A · B) − C) + ((A · C) − B) + (C − (A + B)) has a-sequence.

Now using the above rule we can introduce $\bar{a}$ in any of the minimal regions (A − (B + C)),(B − (A + C)),((B · C) − A), (A · B · C), −(A + B + C). Without loss of generality we can introduce $\bar{a}$ for example in the minimal region (A − (B + C)) (vide Fig. 24).

3.1.3. For x. Let D be a type-I/II diagram.

(i) This may be transformed into a diagram by introducing a x-sequence in some region r of D if there is an a-sequence in r.
(ii) This may be transformed into a diagram by introducing a x-sequence in D such that each minimal region of D has a node of this x-sequence.

Example 3.6. We can introduce x-sequence in D (vide Fig. 25) in the region ((A − B) + (A · B)) and get the following diagram (vide Fig. 26).

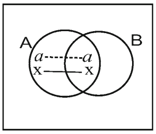


Figure 26.

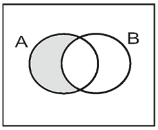


Figure 27.

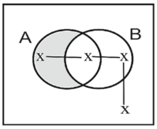


Figure 28.

Example 3.7. We can introduce x-sequence in D (vide Fig. 27) such that all the minimal regions, i.e. A − B, A · B, B − A, −(A + B) has a node of this x-sequence and get the following diagram (vide Fig. 28).

For a type-III diagram while using introduction rule we can introduce closed curves or diagrammatic objects in any one or more components.
Note:

(1) That a x-sequence may be introduced spreading over all the minimal regions is due to the fact that in the interpretation we take the universe to be non-empty. Here we differ from Shin [15] and Hammer [11].
(2) There are no rules to introduce constants. We will see that this restriction is basic for non-classical interpretation of the diagrammatic language (cf. conclusion).

3.2. Extension Rule (for lcc, lci and Components)

3.2.1. For lcc and lci. Let D be a type-I/II diagram containing a x-sequence or an *a*-sequence in some region r, then it may be transformed into a diagram by introducing x or *a* in a minimal region m (which is outside of r) connecting it with the existing x or *a*-sequence with a line or a broken line respectively.

Example 3.8. We can introduce x in D (vide Fig. 29) in any of the minimal regions (A −(B + C)), ((A · C) − B), (A · B · C), (C − (A + B)), −(A + B + C), say we introduce x in (A · B · C). Then this x is connected with any end of the existing x-sequence with a line. Thus we get either the diagram D_1 or the diagram D_2 but not the diagram D_3 (vide Fig. 30) [It is to be noted that D_3 is a valid spider diagram [12]].

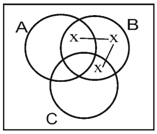


Figure 29.

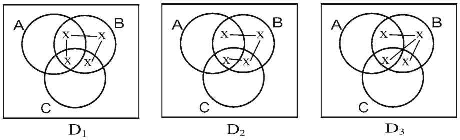


Figure 30.

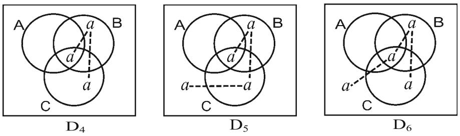


Figure 31.

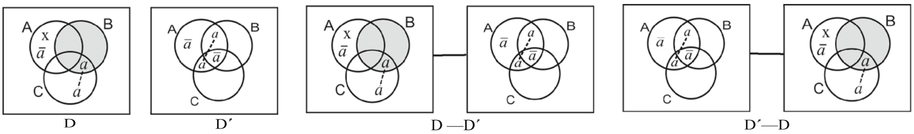


Figure 32.

Similarly from the diagram D_4, we get the diagrams D_5 and D_6 (vide Fig. 31).

3.2.2. For Components. A diagram D may be transformed into a diagram $D^{'}-D$ or $D-D^{'}$ by connecting any diagram $D^{'}$ to D.

Example 3.9. We may transform D into either the diagram $D-D^{'}$ or the diagram $D^{'}-D$ (vide Fig. 32) w.r.t any diagram $D^{'}$.

3.3. Elimination Rules (for lcc, lci, Shading, $\bar{a}$ and Closed Curves)

3.3.1. For lcc and lci. (i) A type-I/II diagram D may be transformed into a diagram by eliminating entire sequence of nodes of x's or a's.

Example 3.10. We can eliminate the entire x-sequence from D (vide Fig. 33) to get the following diagram (vide Fig. 34).

Also we can eliminate the entire a-sequence from D to get the following diagram (vide Fig. 35).

We can eliminate both x-sequence and a-sequence from D to get the following diagram (vide Fig. 36).

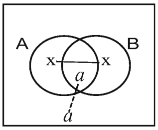


Figure 33.

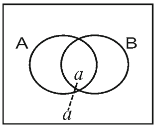


Figure 34.

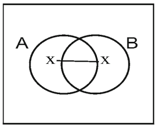


Figure 35.

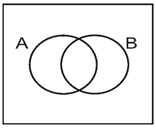


Figure 36.

(ii) If a type-I/II diagram contains a sequence of x's or a's with more than one node and some nodes fall in shaded region, then it may be transformed into a diagram by eliminating those nodes in the shaded region and preserving the remaining nodes in a chain.

Example 3.11. We can eliminate the node of the x-sequence from D (vide Fig. 37) that falls in the shaded region to get the following diagram (vide Fig. 38).

Also we can eliminate the node of the a-sequence from D that falls in the shaded region to get the following diagram (vide Fig. 39).

We can eliminate the nodes of the both of the sequences from D that fall in the shaded region to get the following diagram (vide Fig. 40).

(iii) If a type-I/II diagram contains a sequence of a's with more than one node and one node falls in a minimal region containing $\bar{a}$ then it may be transformed into a diagram by eliminating that node and preserving the remaining nodes in a chain.

Example 3.12. We can eliminate the node of the a-sequence from D (vide Fig. 41) that fall in the region containing $\bar{a}$ to get the following diagram (vide Fig. 42).

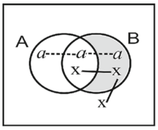


FIGURE 37.

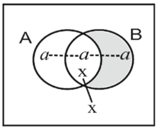


FIGURE 38.

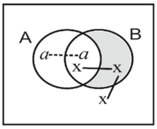


FIGURE 39.

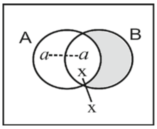


FIGURE 40.

3.3.2. For Shading. A type-I/II diagram D may be transformed into a diagram by eliminating shading from any minimal region of D.

Example 3.13. We can eliminate the shading from D to get the diagram D_1 (vide Fig. 43).

3.3.3. For $\bar{a}$. A type-I/II diagram D may be transformed into a diagram by eliminating $\bar{a}$ from any minimal region of D.

Example 3.14. We can eliminate $\bar{a}$ from D to get the diagram D_1 (vide Fig. 44).

3.3.4. For Closed Curve. For a type-I diagram closed curve elimination is not allowed since then the diagram obtained will not be well-formed.

A type-II diagram D may be transformed into a diagram, say $D^{'}$, by eliminating a closed curve, say B, provided the following conditions hold:

(1) For all regions r, if $B \cdot r$ is shaded then $r - B$ is also shaded and vise-versa.
(2) For all regions r, if $B \cdot r$ has $\bar{a}$ then $r - B$ also has $\bar{a}$ and vise-versa.

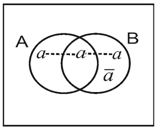


Figure 41.

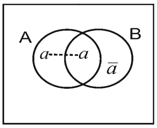


Figure 42.

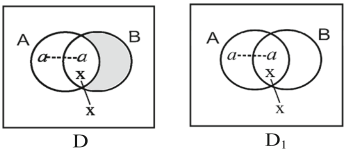


Figure 43.

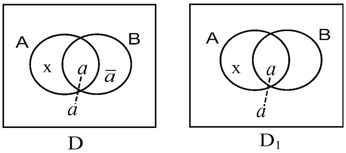


Figure 44.

If such conditions are satisfied then D' is obtained through following steps:

(i) B is eliminated. Let the resulting diagram (not necessarily wfd) be D_1.

(ii) A diagram D_2 is obtained thus − if D_1 is in normal form, D_1 is D_2. If not, then we reduce D_1 to its normal form D_2. [This reduction is always possible] (cf. Sect. 2.4)

(iii) All the diagrammatic objects of D_1 are transferred into the respective counterpart regions in D_2.

(iv) If more than one node of a x-sequence/a-sequence fall within the same minimal region of the diagram D_2 then only one such node is to be retained.

The resulting diagram is D'.

Example 3.15. The minimal regions $A - B$, $B \cdot A$, $C \cdot A$ and $A - C$ are all shaded. Thus we can eliminate the curve B or the curve C or both of the curves from the diagram D (vide Fig. 45) to get the well-formed diagrams D_1, D_2, D_3 respectively (vide Fig. 46).

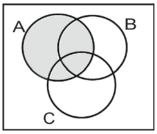


FIGURE 45.

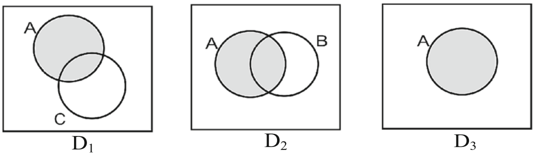


FIGURE 46.

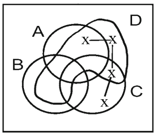


FIGURE 47.

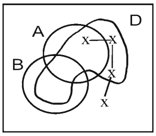


FIGURE 48.

But we cannot eliminate curve A as neither B − A nor C − A is shaded.

Example 3.16. From the diagram D in Fig. 47) if we eliminate the closed curve C then we get the diagram D_1 (vide Fig. 48).

Now the diagram D_1 is not in normal form as it has 10 ($\neq 2^3$) minimal regions, so we reduce D_1 to the diagram D_2 (vide Fig. 49).

All the diagrammatic objects of D_1 are transferred into the respective counterpart regions in D_2 (vide Fig. 50).

Finally by erasing the extra node from the minimal region D − B − A, we get the wfd diagram $D^{'}$ (vide Fig. 51).

Example 3.17. In Fig. 52a the minimal regions A − B, B · A have $\bar{a}$. Thus we can eliminate the curve B from the diagram to get the well-formed diagrams $D^{'}$ (vide Fig. 53). But we cannot eliminate curve A as there is no $\bar{a}$ in B − A.

On the other hand B cannot be eliminated from diagram vide Fig. 52b.

For type-III diagram while using elimination rule we can eliminate closed curves or diagrammatic objects in any one or more components.

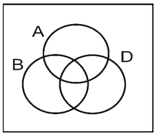


FIGURE 49.

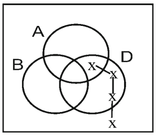


FIGURE 50.

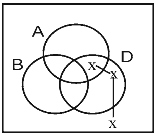
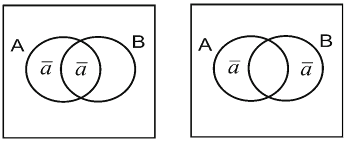


FIGURE 51.



(a) **(b)**

FIGURE 52.

3.4. Unification Rule

3.4.1. For Type-I or Type-II Diagram. Diagrams D_1 and D_2 may be transformed into a diagram by uniting them in one diagram, obeying the following steps.

(1) All the closed curves of D_2 of which there were no counterparts in D_1, are introduced in D_1 to obtain the diagram D.
(2) All the diagrammatic objects of D_2 are drawn in the respective counterpart regions of the diagram D, obeying the rule of introduction of closed curve.

[i.e. if a region r in D_2 has a x and if c(r), the counterpart region of r in D, has the minimal regions m_1, m_2, …, m_n within it due to the introduction of curves, then each of the m_i has a node of the above mentioned x-sequence with a line connecting them. Similarly for a-sequence. If a minimal region m in D_2 has shading or $\bar{a}$, then c(m), the counterpart region of m in D, is divided into two parts due to introduction of closed curve and thus each part has shading or $\bar{a}$ respectively].

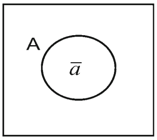


Figure 53.

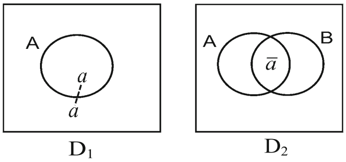


Figure 54.

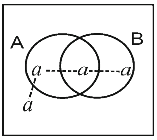


Figure 55.

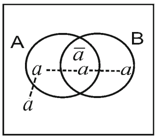


Figure 56.

The resulting diagram is called the Unification of two diagrams and is denoted as Uni(D_1, D_2).

One can observe that the operator Uni is commutative. Also the operator Uni is associative. So, the process of unification can be extended to n number of diagrams i.e. Uni(D_1, D_2, ..., D_n).

Example 3.18. Let us take the diagrams D_1 and D_2 (vide Fig. 54). Now we first introduce the curve B in the diagram D_1 to obtain the diagram D (vide Fig. 55).

Then we draw all the diagrammatic objects of D_2 (i.e. $\bar{a}$) in the respective counterpart region of D to obtain the diagram Uni(D_1, D_2) (vide Fig. 56).

3.4.2. For Type-III Diagram with Type-I or Type-II Diagram. If $D_1 - D_2$ is a type-III diagram and D_3 be a type-I/II diagram then they may be transformed into the type-III diagram Uni(D_1, D_3)− Uni(D_2, D_3), by unification rule (vide Fig. 57).

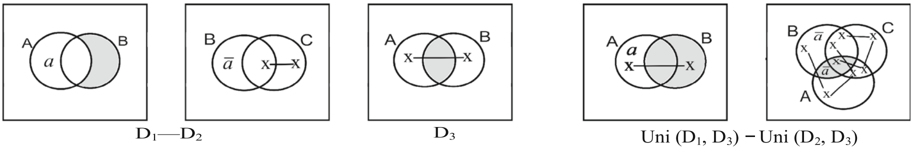


Figure 57.

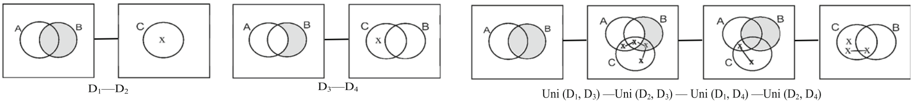


Figure 58.

3.4.3. For Type-III Diagram with Type-III Diagram. If $D_1 - D_2$ and $D_3 - D_4$ be two type-III diagrams then they may be transformed into the type-III diagram Uni(D_1, D_3)– Uni(D_2, D_3)– Uni(D_1, D_4)– Uni(D_2, D_4), by unification rule (vide Fig. 58).

Clearly the above rule can be extended to the unification of type-III diagrams with any number n_1, n_2 of components.

3.5. Rule of Splitting Sequences

Let D be a type-I/II diagram containing a x-sequence or an a-sequence in a region r. Let $m_1, m_2, \ldots, m_n$ be all the minimal regions contained in r.

Then D may be transformed into a type-III diagram $D_1 - D_2 - \cdots - D_n$ such that

(i) each D_i has the counterparts of all the basic regions of D,
(ii) each D_i has only one x or an a in the counterpart regions of m_i ($1 \leq i \leq n$),
(iii) any other diagrammatic object in any region r of D will be present in the counterparts c(r) in each D_i.

Example 3.19. From the diagram D using the rule of splitting sequences we get the diagram D_1 (vide Fig. 59).

Similarly, if we have the diagram D_2, using the rule of splitting sequences we get the diagram D_3 (vide Fig. 60).

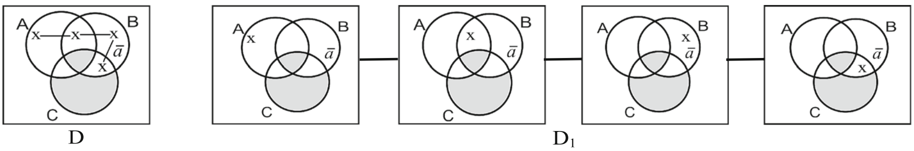


Figure 59.

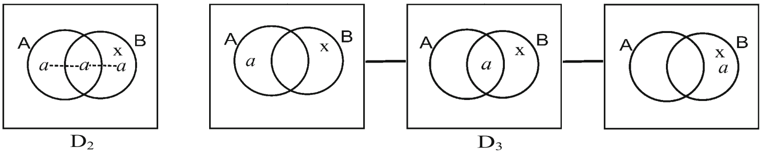


FIGURE 60.

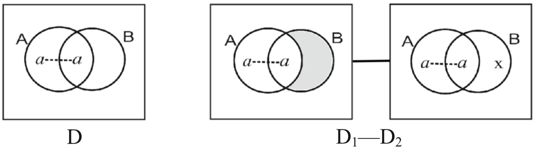


FIGURE 61.

3.6. Rule of Excluded middle

(a) If D is a type-I/II diagram such that there is a minimal region m containing no diagrammatic object then it may be transformed into a type-III diagram $D_1 - D_2$ such that
(i) both D_1 and D_2 have the counterparts of all the basic regions of D,
(ii) the counterpart of m in D_1 is shaded and in D_2 has a x and
(iii) all the other diagrammatic objects remain the same in D_1 and D_2 as in D.

Example 3.20. From the diagram D using the rule of excluded middle we get the diagram $D_1 - D_2$ (vide Fig. 61).

(b) If D is a diagram such that there is a minimal region m containing either x or m has no diagrammatic object and the region $-$ m does not contain an a-sequence, then it may be transformed into a type-III diagram $D_1 - D_2$ such that
(i) both D_1 and D_2 have the counterparts of all the basic regions of D,
(ii) the counterpart of m in D_1 has an a and in D_2 has $\bar{a}$ [if m has x then the x also remains in m in both D_1 and D_2] and
(iii) all the other diagrammatic objects remain the same in D_1 and D_2 as in D.

Example 3.21. The minimal region C − A − B of D does not contain any diagrammatic object. Then applying the rule of excluded middle to the diagram

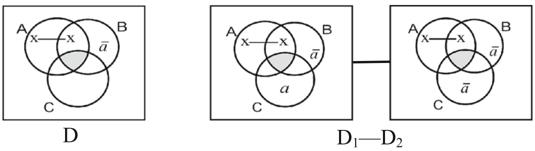


FIGURE 62.

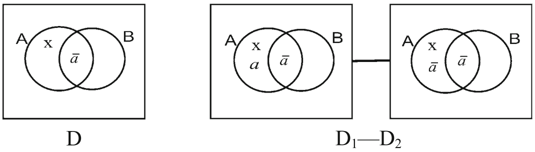


FIGURE 63.

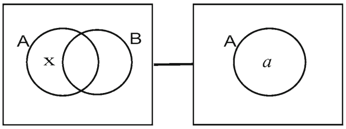


FIGURE 64.

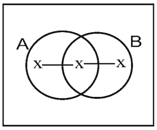


FIGURE 65.

D we get the diagram $D_1 - D_2$ where the minimal region c(C − A − B) has a in D_1 and $\bar{a}$ in D_2 (vide Fig. 62).

Example 3.22. The minimal region A − B of D contain x. Then applying the rule of excluded middle to the diagram D we get the diagram $D_1 - D_2$ where the minimal region c(A − B) has a in D_1 and $\bar{a}$ in D_2 (vide Fig. 63).

For a type-III diagram excluded middle rule can be used for one or more components.

3.7. Rule of Construction

A wfd $D_1 - D_2 - \cdots - D_n$ may be transformed into a diagram D if each of D_1, D_2, ..., D_n can be transformed into D by some of the previously mentioned rules.

Example 3.23. Let us consider the following diagram $D_1 - D_2$ (vide Fig. 64).

Now from the diagram D_1 we can obtain the following diagram D by using the rule of introduction for lcc (vide Fig. 65).

Again by using the rule of introduction of x we can obtain the following diagram D' from the diagram D_2 (vide Fig. 66).

Applying the elimination rule for lci we can obtain the following diagram D'' (vide Fig. 67).

Applying the introduction of closed curve rule we can obtain the following diagram D''' (vide Fig. 68).

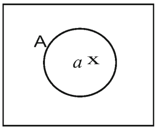


FIGURE 66.

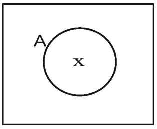


FIGURE 67.

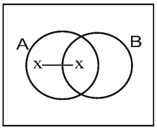


FIGURE 68.

And finally applying the rule of introduction for lcc we can obtain the diagram D from the diagram $D^{'''}$. So we can obtain the diagram D from both the diagrams D_1 and D_2 and thus we can obtain the diagram D from $D_1 - D_2$.

Definition 3.1 (*ρ-equivalence*).

Let D and $D^{'}$ be two wfds. Then D $\rho D^{'}$ holds if and only if there is a sequence of diagrams $D_1(\equiv D)$, D_2, . . ., $D_n(\equiv D^{'})$ such that D_{i+1} is obtainable from D_i (i = 1, 2,. . . , n-1) by one of the previously mentioned rules.

Diagrams D and $D^{'}$ are said to be ρ-equivalent if and only if D $\rho D^{'}$ and $D^{'} \rho$ D.

Note that since the unification rule requires at least two diagrams for being applicable, in the definition of ρ this rule does not apply.

3.8. Inconsistency Rules

Definition 3.2 (*Inconsistent diagram*).

(1) A type-I/II diagram D is said to be an Inconsistent diagram if either of the following conditions hold.
(i) There is a minimal region m in D such that it has both shading and a x.
(ii) There is a minimal region m in D such that it has both shading and an *a*.
(iii) There is a minimal region m in D such that it has both $\bar{a}$ and an *a*.
(iv) Every minimal region in D is shaded.
(2) A type-III diagram D is said to be an Inconsistent diagram if all the components of D are inconsistent in the sense (1).

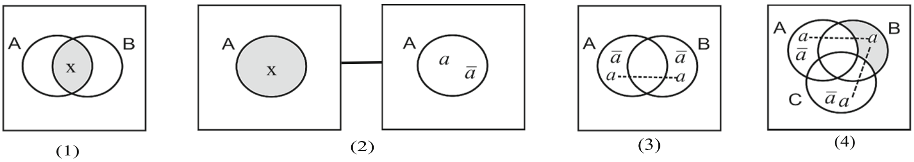


FIGURE 69.

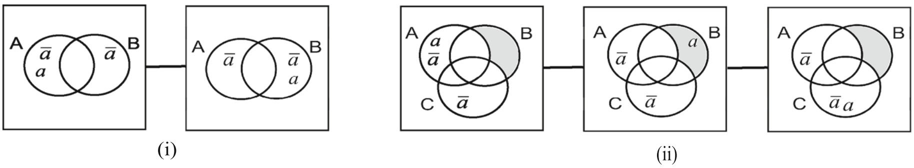


FIGURE 70.

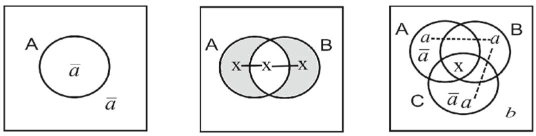


FIGURE 71.

(3) Any diagram D which is ρ-equivalent to an inconsistent diagram of kinds (1) and (2) is an inconsistent diagram.
(4) No other diagram is inconsistent.

Example 3.24. Here are few examples of inconsistent diagrams.

The diagrams in Fig. 69(3), (4) are ρ-equivalent to the inconsistent diagrams D_1 (vide Fig. 70(i)) and D_2 (vide Fig. 70(ii)) respectively.

Definition 3.3 (*Consistent diagram*).

(1) If a type-I/II diagram D is not inconsistent then D is consistent.
(2) A type-III diagram D is said to be a consistent diagram if and only if at least one of the components of D is consistent.

Example 3.25. Here are few examples of consistent diagrams (Fig. 71).

Inconsistency Rules

(i) An inconsistent diagram may be transformed into any diagram.

Example 3.26. From the inconsistent diagram D using the inconsistency rule we can get any diagram, for example we can get the diagram D_1 (vide Fig. 72).

(ii) A type-III diagram may be transformed into a diagram by dropping an inconsistent component from it.

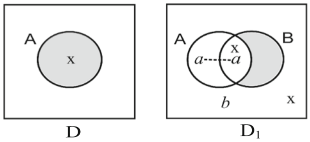


FIGURE 72.

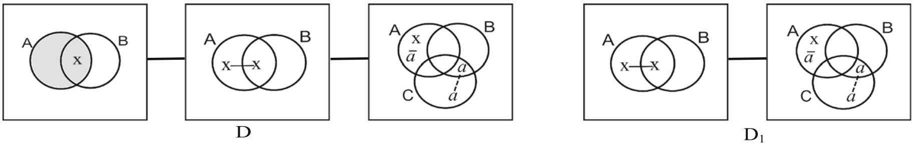


FIGURE 73.

Example 3.27. From the type-III diagram D using the inconsistency rule we can drop the inconsistent component and get the diagram D_1 (vide Fig. 73).

Definition 3.4 (*Provability or syntactic consequence*).
A diagram D is provable from a non-empty, finite set Δ of diagrams ($\Delta \vdash$ D) if and only if there is a sequence of diagrams $D_1, D_2, \ldots, D_n (\equiv D)$ such that each diagram is either a member of Δ or is obtainable from earlier diagrams in the sequence by one of the rules of transformation.

So, any rule may be written as $D \vdash D^{'}$ where $D^{'}$ is obtained from D by any of the rules.

If $D_1 \rho D_2$ then $D_1 \vdash D_2$ holds but not the converse.

Diagrams D_1 and D_2 are syntactically equivalent if and only if $D_1 \vdash D_2$ and $D_2 \vdash D_1$.

4. Some Useful Notions

We shall now define some notions that would be used in the sequel, the respective usages will be mentioned at the end of the section.

Definition 4.1 (*Tautology*).

(1) A type-I/II diagram D is said to be a Tautology if D has no diagrammatic objects.
(2) Any diagram which is syntactically equivalent to a Tautology of kind (1) is a Tautology.

If a diagram D is a tautology we write $\vdash$D.

It is to be noted that Tautology is taken to be a syntactic notion in this work.

Proposition 4.2. *If $\vdash D$ then for any $D^{'}$, $D^{'} \vdash D$ holds.*

Proof. Let $\vdash$D hold. Then either (i) D is a type-I/II tautology with no diagrammatic object or (ii) D is syntactically equivalent to a tautology.

Case-(i):
D is a type-I/II tautology with no diagrammatic object.
(a) Let D' be a type-I/II diagram.
$D' \vdash$D holds by steps (i) to (iii).

(i) Elimination of diagrammatic objects in D'.
(ii) Introduction of closed curves in D' which are in D and not in D'.
(iii) Elimination of closed curves from D' which are not in D.
(b) Let D' be a type-III diagram where $D' \equiv D_1' - D_2' - D_3' - \ldots - D_n'$.

Then $D_i' \vdash$ D, for all D_i' $(1 \leq i \leq$ n), since all D_i''s are type-I/II diagrams and for any type-I/II diagrams D_i', $D_i' \vdash$ D holds [already proved in (a)].

Thus $D' \vdash$ D, by the rule of construction.

Case-(ii):
D is syntactically equivalent to a type-I/II tautology D_t.

Then $D_t \vdash$ D and D $\vdash D_t$. Now for any diagram D', $D' \vdash D_t$ holds (by case-i). Thus $D' \vdash$ D, by transitivity. □

Definition 4.3 (*Tautologous minimal region*).

A minimal region m in a diagram D is called a tautologous region if and only if any of the following holds:

(i) D is a type-I/II diagram and m has no diagrammatic object.
(ii) D is a type-III diagram with D_1 and D_2 as components, m occurs both in D_1 and D_2 and m has shading in one and a part of x-sequence in the other.[D is said to be tautologous at m w.r.t x].
(iii) D is a type-III diagram with D_1 and D_2 as components, m occurs both in D_1 and D_2 and m has $\bar{a}$ in one and a part of a-sequence in the other.[D is said to be tautologous at m w.r.t a].
(iv) D is a type-III diagram with D_1 and D_2 as components, m occurs both in D_1 and D_2 and m has $\bar{a}$ in one and a part of x-sequence in the other.[D is said to be tautologous at m w.r.t (x, $\bar{a}$)].

[Henceforth, we shall use only the term tautologous at m for all the three kinds stated above].

Example 4.1. The tautologous region of D (vide Fig. 74) are (A − B) and (B − A).

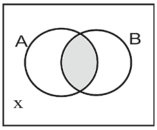


FIGURE 74.

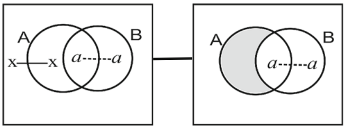


FIGURE 75.

Example 4.2. In Fig. 75, D is tautologous at (A − B) w.r.t x.

Definition 4.4. D_{max}

For a diagram D we define a diagram D_{max} as follows:

Case-i:

Let D be a type-I or type-II consistent diagram.

(1) If D has no diagrammatic object then D_{max} is obtained by introducing a x-sequence such that each minimal region of D has a node of this x-sequence.

(2) D has only one diagrammatic object:

(a) If D has a x-sequence in a region then D_{max} is the diagram obtained by all possible extensions of x-sequence put together.

(b) If D has a shading in a minimal region then D_{max} is the diagram obtained by (i) introducing $\bar{a}$ in the shaded minimal region for all a, (ii) introducing a x-sequence such that each minimal region of D has a node of this sequence, (iii) erasing the node of the x-sequence introduced in (ii) which falls in the shaded region, (iv) putting together the diagrams obtained in (i), (ii), and (iii).

(c) If D has an a-sequence in a region then D_{max} is the diagram obtained by (i) introducing x- sequence in the same region where a-sequence is present and (ii) introducing $\bar{a}$ in all other minimal regions (where nodes of the a-sequence are not present). (iii) Putting together all possible extensions of a-sequence and x sequence.

(d) If D has a $\bar{a}$ in a minimal region then D_{max} is the diagram obtained by introducing x-sequence such that each minimal region of D has a node of this sequence.

(3) D has more than one diagrammatic objects:

(a) If D has a sequence (x-sequence or a-sequence) such that some nodes fall in the shaded region, then we first eliminate the nodes which fall in the shaded region. Let the newly obtained diagram be D_1. Then D_{max} is the unification of the max-diagrams for each diagrammatic object of D_1.

(b) If D has an a-sequence such that some nodes fall in the region containing $\bar{a}$, then we first eliminate the nodes which fall in the that region. Let the newly obtained diagram be D_1. Then D_{max} is the unification of the max-diagrams for each diagrammatic object of D_1.

(c) If D has no such sequence (as in the cases in (a) and (b)) then D_{max} is the unification of the max-diagrams for each diagrammatic object of D.

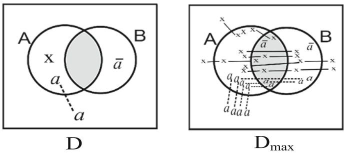


Figure 76.

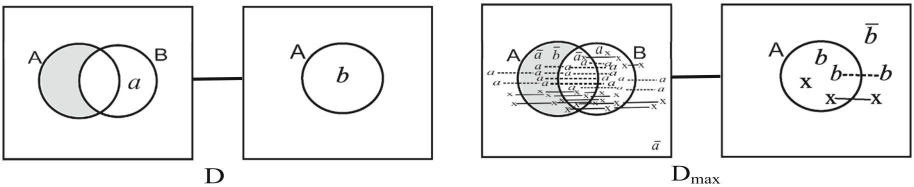


Figure 77.

[The idea of D_{max} has been taken from Shin and Hammer with necessary modification because of the introduction of diagrammatic objects a and $\bar{a}$]. Note: Through this process D_{max} never becomes inconsistent unless D is itself inconsistent.

Example 4.3. If D has more than one diagrammatic objects then D_{max} (vide Fig. 76) is the unification of the max-diagrams for each diagrammatic object.

Case-ii:

Let D be a type-III consistent diagram, say $D_1 - D_2 - \cdots - D_n$ such that none of the D_i is inconsistent. Then D_{max} is $D_{1max} - D_{2max} - \cdots - D_{nmax}$.

If there are some inconsistent components then D_{max} is obtained by droping the inconsistent components and then applying the above procedure.

Example 4.4. If D is a type-III diagram $D_1 - D_2$ then D_{max} is the diagram $D_{1max} - D_{2max}$ (vide Fig. 77).

Proposition 4.5. *For any consistent wfd D we have $D \vdash D_{max}$ and $D_{max} \vdash D$, that is any consistent diagram D is syntactically equivalent to D_{max}.*

Proof. Let D be a type-I/II diagram.

Case-i:

D has no diagrammatic object.

Then D_{max} is obtained by introducing a x-sequence such that each minimal region of D_{max} has a node of this x-sequence.

$\therefore D \vdash D_{max}$ [by rule of introduction of x- sequence].

Conversely, by using the rule of elimination of diagrammatic objects we get $D_{max} \vdash D$.

Case-ii:

D has only one diagrammatic object.

Sub case-(a):

D has a x-sequence in a region.

Then D_{max} is the diagram obtained by all possible extensions of x-sequence put together.

Let D_1, D_2, ..., D_n be the extensions of the x-sequence.

So, D_{max} = Uni(D, D_1, D_2, ..., D_n).

Now D $\vdash$ D_i, i= 1, 2,..., n [by the rule of extension of x-sequence].

$\therefore$ D $\vdash$ D_{max} [by the rule of unification].

Conversely, by using the rule of elimination of diagrammatic objects we get D_{max} $\vdash$ D.

Sub case-(b):

D has shading in a minimal region.

Then D_{max} is the diagram obtained by (i) introducing $\bar{a}$ in the shaded minimal region for all a, (ii) introducing a x-sequence such that each minimal region of D has a node of this sequence, (iii) erasing the node of the x-sequence introduced in (ii) which falls in the shaded region, (iv) putting together the diagrams obtained in (i), (ii), and (iii).

$\therefore$ D $\vdash$ D_1 [by the rule of introduction of $\bar{a}$].

$\therefore$ D_1 $\vdash$ D_2 [by the rule of introduction of x-sequence].

$\therefore$ D_2 $\vdash$ D_3 [by the rule of elemination of part of x-sequence].

Now D_{max} is the diagram Uni (D_1, D_2, D_3).

So, D $\vdash$ D_{max} by the rule of unification.

Conversely, by using the rule of elimination of diagrammatic objects we get D_{max} $\vdash$ D.

Sub case-(c):

D has an a- sequence in a region.

Then D_{max} is the diagram obtained by introducing x-sequence in the same region where a-sequence is present and $\bar{a}$ in all other minimal regions (where nodes of the a-sequence are not present). Also all possible extensions of a and x are put together in D_{max}(for all constants a).

Now using the rule of introduction of x-sequence we get D $\vdash$ D_1 (say).

Then using the rule of introduction of $\bar{a}$ we get D_1 $\vdash$ D_2 (say).

D_2 has a x-sequence in the same region of D where there is an a-sequence and $\bar{a}$ in all other minimal regions.

By D_{2i}, i, i = 1, 2,..., n we denote all possible extensions of the x-sequence and the a-sequence in D_2.

So, D_2 $\vdash$ D_{2i}.

Hence, D_2 $\vdash$ Uni(D_2, D_{21}, D_{22}, ..., $D_{2(n-1)}$, D_{2n}) $\equiv$ D_{max}.

Hence,D $\vdash$ D_{max}.

By using the rule of elimination of diagrammatic objects we get D_{max} $\vdash$ D.

Sub case-(d):

D has a $\bar{a}$ in a minimal region.

Then D_{max} is the diagram obtained by introducing x-sequence such that each minimal region of D has a node of this sequence.

$\therefore$ D $\vdash$ D_{max} [by introduction of x-sequence].

By using the rule of elimination of diagrammatic objects we get $D_{max} \vdash D$.

Case-iii:

D has more than one diagrammatic object.

D $\vdash$ D_1 (elimination rule if required).

$D_1 \vdash D_{max}$, follows from the unification rule.

$\therefore$ D $\vdash$ D_{max} [by transitivity]

Conversely, by using the rule of elimination of diagrammatic objects we get $D_{max} \vdash D$.

Let D be a type-III diagram.

Without loss of generality we can assume that D is the diagram $D_1 - D_2$. Then D_{max} is the diagram $D_{1max} - D_{2max}$.

Now by using the above arguments we get $D_1 \vdash D_{1max}$.

$\therefore D_1 \vdash D_{1max} - D_{2max}$ [by the rule of introduction of components].

Similarly, $D_2 \vdash D_{1max} - D_{2max}$.

$\therefore$ D $\vdash$ D_{max} [by the rule of construction].

Again by using the rule of elimination of diagrammatic objects we get $D_{1max} \vdash D_1$.

$\therefore D_{1max} \vdash D_1 - D_2$ [by the rule of introduction of components].

Similarly, $D_{2max} \vdash D_1 - D_2$.

$\therefore D_{max} \vdash$ D [by the rule of construction].

Thus D $\vdash$ D_{max} and $D_{max} \vdash$ D. □

The notions Tautologous minimal region and D_{max} have been used in completeness proof.

5. Semantics

Definition 5.1 (*Model*).

We define a model to be a triple (U, I, h), where

(i U is a non-empty set,

(ii) I is a function assigning subsets of U to all regions of all wfds such that

1. I(r) = U, whenever r is a basic region enclosed by a rectangle,
2. I(r) = I(s), whenever r and s are two basic regions that are labeled by the same label,
3. if r and s are regions of a diagram D, then $I(r + s) = I(r) \cup I(s)$,
4. if r and s are two regions of a diagram D, then $I(r \cdot s) = I(r) \cap I(s)$,
5. if r and s are regions of a diagram D, then $I(r - s) = I(r) \backslash I(s)$,

(iii) h is a partial function assigning objects h(a) of U to the names of individuals a. Since h is a partial function h(a) may not be defined for some a.

It follows from 5 and (2.3) that $I(-r) = U \backslash I(r)$.

Definition 5.2 (*True in a model*).

Let $M = (U, I, h)$ be a model. We say that a type-I/II diagram D is True in M (denoted by $M \Vdash D$) if and only if the following conditions are satisfied.

(i) If r is shaded then $I(r) = \phi$ (null set).
(ii) If r ($\equiv m_1 + m_2 + \cdots + m_n$) has x-sequence then $I(r) \neq \phi$ [$I(m_1) \neq \phi$ or $I(m_2) \neq \phi$ or …or $I(m_n) \neq \phi$] (or being inclusive).
(iii) If r ($\equiv m_1 + m_2 + \cdots + m_n$) has a then $h(a)$ is an element of the universe and $h(a) \in I(r)$ [$h(a) \in I(m_1)$ $\mathfrak{or}$ $h(a) \in I(m_2)$ $\mathfrak{or} \ldots \mathfrak{or}$ $h(a) \in I(m_n)$] ($\mathfrak{or}$ being exclusive).
(iv) If $\bar{a}$ is in r then either $h(a)$ does not exist in U or if $h(a)$ exists in U then $h(a) \notin I(r)$ i.e. $h(a) \in U \backslash I(r)$.

If D is a type-III diagram and let $D \equiv D_1 - D_2 - \cdots - D_n$. Then $M \Vdash D$ if and only if $M \Vdash D_i$ for at least one of the components D_i.

Definition 5.3 (*Logical consequence*).

Let Δ be a set of diagrams and D be a diagram. We say that D is a Logical Consequence of Δ and write $\Delta \models D$ if and only if D is true in every model in which every member of Δ is true.

A rule is sound if and only if for every instance $D \vdash D^{'}$ of the rule, $D \models D^{'}$ holds.

Soundness

Theorem 5.4 (Soundness Theorem) *For any set $\Delta \cup \{D\}$ of diagrams, if $\Delta \vdash D$ then $\Delta \models D$.*

Proof. We prove the theorem by using Lemmas 5.5 and 5.6.

Lemma 5.5. *All the transformation rules are sound.*

The proof is added in the Appendix.

Lemma 5.6. *A tautology is true in all interpretations.*

Proof.
(1) Suppose D is a type-I/II tautology, then there is no diagrammatic object in D. Then vacuously D is true in any model.
(2) Any diagram which is syntactically equivalent to a Tautology is a Tautology. □

Let us consider a diagram D which is syntactically equivalent to a Tautology D_1.

i.e. $D \vdash D_1$ and $D_1 \vdash D$ where D_1 is a type-I/II tautology.

But D_1 is a tautology, so D_1 is true in every model. Since $D_1 \vdash D$, we get because of soundness of the rules $D_1 \models D$. Thus D is true in every model. □

Proof of Soundness Theorem

Let $\Delta \vdash$ D.

Then there is a sequence of diagrams D_1, D_2, …, D_n ($\equiv$ D) such that each diagram is either a member of Δ or is obtainable from earlier diagrams in the sequence by one of the rules of transformation.

Since all the transformation rules are sound, $\Delta \models$ D is established by induction on the length of the sequence D_1, D_2, …, D_n.

Note: In soundness theorem Δ need not be finite.

6. Completeness

The ultimate objective of this section is to establish completeness of the diagram system developed with respect to the semantics as stated in Sect. 5, i.e. $\Delta \models$ D implies $\Delta \vdash$ D where Δ is a finite collection of diagrams and D is a single diagram.

Before proceeding to the completeness theorem proper, we prove an important result (Theorem 6.1) which will be used in the completeness proof.

Theorem 6.1. *If D is a consistent type-I/II diagram then D is true in some model.*

Proof. Case-1: D is a Tautology. Then D is true in every model (see Lemma 5.6).

Case-2: D is a consistent diagram other than tautology.

A model for the diagram shall be obtained through the following steps:

1. Enumerate all the minimal regions of diagram D.
2. Assign null set(ϕ) to the minimal regions with shading.
3. Mark the minimal regions with $\bar{a}$ but not falling in category-2. Similarly mark for other symbols $\bar{b}$, $\bar{c}$, …
4. Consider the minimal regions with a node of a-sequence. Take the least numbered minimal region that do not fall within categories 2 and 3. Assign an element h(a) to that region and to no other region h(a) is to be assigned. Similarly for the other constants.
5. Consider the x-sequence and mark the minimal regions with a node of x-sequence. Assign elements x_1, x_2, … (all distinct) to the minimal regions not falling in category-2.
6. Assign distinct new objects y_1, y_2, … to all other minimal regions that are yet unassigned (there may be some minimal regions in category-3, category-4 and minimal regions with no diagrammatic objects).
7. Take a basic region. It is split into minimal regions. Take the union of the assignments to all such minimal regions. This is the set-assignment to the basic region taken. Similarly, assign sets all the basic regions (including the rectangle).

□

Note: The above procedure gives one model of an arbitrary diagram, but there may be other models also of this same diagram.

Example 6.1. Let us consider the consistent diagram D (vide Fig. 78).

A model for the diagram D shall be obtained through the following steps:

1. We first enumerate all the minimal regions of diagram D (vide Fig. 79) viz. 1–8.
2. We assign null set(ϕ) to the minimal regions with shading. i.e. I(1) = ϕ and I(5) = ϕ.
3. We Mark the minimal region viz.3 with $\bar{a}$ but not falling in category-2.
4. Consider the minimal regions viz. 2 and 6 with a node of a-sequence. We take the least numbered minimal region that do not fall within categories 2 and 3. Assign an element h(a) to that region.i.e. h(a) $\in$ I(2).

Similarly, for b we have h(b) $\in$ I(7).

5. Consider the x-sequence and mark the minimal regions viz. 1,8,7 and 3 with a node of x-sequence. Assign elements x_1, x_2, ... (all distinct) to the minimal regions not falling in category-2 i.e. $x_1 \in$ I(3), $x_2 \in$ I(7) and $x_3 \in$ I(8).
6. We assign distinct new objects y_1, y_2, ... to all other minimal regions that are yet unassigned i.e. $y_1 \in$ I(4) and $y_2 \in$ I(6).
7. We take a basic region A and split it into minimal regions. Take the union of the assignments to all such minimal regions. Thus set-assignment to the basic region A is I(A)= I(1) $\cup$ I(2) $\cup$ I(5) $\cup$ I(6). i.e. I(A)= {h(a), y_2}.
   Similarly, I(B)= I(2) $\cup$ I(3) $\cup$ I(4) $\cup$ I(5) = {h(a), x_1, y_1}.
   I(C) = I(6) $\cup$ I(5) $\cup$ I(4) $\cup$ I(7) = {y_2, y_1, x_2, h(b)}.
   And for the region enclosed by rectangle we have I(U) = I(1) $\cup$ I(2) $\cup$ I(3) $\cup$ I(4) $\cup$ I(5) $\cup$ I(6) $\cup$ I(7) $\cup$ I(8) = {h(a), h(b), x_1, x_2, x_3, y_1, y_2}.

We shall first establish a special case of completeness theorem.

Lemma 6.2. *If* $\models D$ *then* $\vdash D$ *holds.*

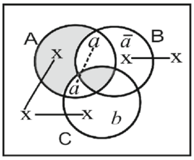


FIGURE 78.

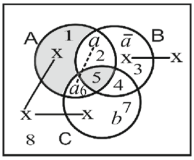


FIGURE 79.

Proof. Case-1:

Let D be a type-I/II diagram and $\models$ D holds i.e. D is true in every model. We want to show that either- (i) D does not have any diagrammatic object in any region r or (ii) D has an lcc spread over the whole rectangle.

If not, then either shades or lci or $\bar{a}$ or lcc (which is not spread over the whole rectangle) shall occur in a region r.

If shading occurs in a region r, we can construct an model M = (U, I, h) such that I(r) $\neq \phi$, so that D is not true in M, a contradiction to our assumption.

Similarly for a lcc or lci or $\bar{a}$ we can construct a model M such that D is not true in M, contradicting our assumption.

Thus either D does not have any diagrammatic object in any region r or D has an lcc spread over the whole rectangle.

If D does not have any diagrammatic object in any region r then $\vdash$ D holds (by definition of $\vdash$ D).

If D has an lcc spread over the whole rectangle, we can get a $D^{'}$ with no diagrammatic object just by elimination of the lcc.

Conversely, from that particular $D^{'}$ we get D by the rule of introduction of x-sequence [see note below].

Case-2:

Let D be a type-III diagram such that $D \equiv D_1 - D_2 - \cdots - D_n$ and $\models$ D holds.

Then we can show that either

(i) there is a component D_i such that $\models D_i$ holds.

or (ii) D is syntactically equivalent to a type-III diagram $D^{'}$ such that for at least one diagrammatic object in D there exists a tautologous pair ($D_i^{'}$, $D_j^{'}$) in $D^{'}$ and having no other diagrammatic object in $D_i^{'}$ and $D_j^{'}$.

If not, then we can construct a model M such that D is not true in M, contradicting our assumption [see demonstration below, Example 6.4].

In both the cases we now show that $\vdash$ D holds.

Case-1:

Suppose, there is a component D_i in D such that $\models D_i$ holds. Thus $\vdash D_i$ holds (already proved in case-1).

So, $D \vdash D_i$ holds [by Proposition 4.2].

Also, $D_i \vdash D$ holds [by the rule of extension of components].

Thus D is syntactically equivalent to D_i.

Hence, $\vdash$ D holds by the definition of tautology.

Case-2:

Suppose, D is syntactically equivalent to a type-III diagram $D^{'}$ such that for at least one diagrammatic object in D there exists a tautologous pair ($D_i^{'}$, $D_j^{'}$) in $D^{'}$ and having no other diagrammatic object in $D_i^{'}$ and $D_j^{'}$.

Let us consider a type-I/II diagram D'' such that—(i) D'' has all the basic regions of D'_i and D'_j and (ii) there is no diagrammatic object in D''.

$\therefore D'' \vdash D'_i - D'_j$ (by rule of excluded middle).
$\therefore D'' \vdash D'$ (by rule of extension of component).
$\therefore D'' \vdash D$ (by transitivity as $D' \vdash D$).

Again, $\vdash D''$ holds because D'' is a type-I/II diagram and there is no diagrammatic object in D''.

Thus $D \vdash D''$ [by Proposition 4.2].

Thus D is syntactically equivalent to D'' and $\vdash D$ holds by the definition of tautology. □

Note: Suppose, D have an lcc spread over the whole rectangle. Let D' be any type-I/II diagram such that D' does not have any diagrammatic object. Then D is syntactically equivalent to D'.

Then, $D \vdash D'$ holds by steps (i) to (iii).

(i) Elimination of diagrammatic objects in D.
(ii) Introduction of closed curves in D which are in D' and not in D.
(iii) Elimination of closed curves from D which are not in D'.

Again, $D' \vdash D$ holds by steps (i) to (iii).

(i) Introduction of x-sequence in D'.
(ii) Introduction of closed curves in D' which are in D and not in D'.
(iii) Elimination of closed curves from D' which are not in D.

Thus D is syntactically equivalent to D' and $\vdash D$ holds by transitivity ($\vdash D'$ holds because D' is a type-I/II diagram and there is no diagrammatic object in D').

Example 6.2. A type-III diagram D is interpreted by inclusive or of the components.

So, one possibility for $\models D$ is that one component is a tautology (i.e. without any diagrammatic objects) e.g. the Fig. 80.

The other possibility is that two of the components together are true in all interpretations and thus making this pair a tautologous one, e.g. Fig. 81.

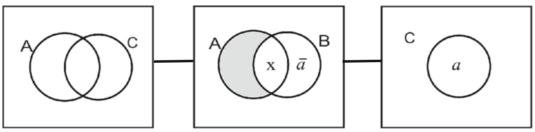


FIGURE 80.

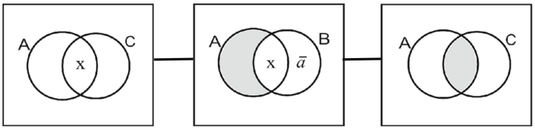


FIGURE 81.

Here the 1st and 3rd components are tautologous. Although the 2nd and 3rd components are also tautologous this pair can not be considered since one of them, the second one, has other diagrammatic objects.

We now proceed towards the schema of completeness theorem. A particular construction of a diagram D^+ from a diagram D will be required to develope this schema. So we define D^+ as below.

Definition 6.3. D^+

Let $D \models D'$. Where D is a type-I/II diagram we define D^+ as follows:

Case-1: Let D' be a type-I diagram. Then $D^+ \equiv D$.

Case-2: Let D' be a type-II diagram. Then D^+ is the diagram obtained by introducing closed curves to D that enclose basic regions of D' of which there are no counterparts in D.

Case-3: Let D' be a type-III diagram where $D' \equiv D'_1 - D'_2 - \ldots - D'_m$ and there is at least one component D'_i of D' such that $D \models D'_i$ holds ($1 \leq i \leq m$). Then D^+ is the diagram obtained by introducing closed curves to D that enclose the basic regions of D'_i of which there are no counterparts in D [The diagram D^+ is obtained w.r.t component D'_i such that $D \models D'_i$. There can be more than one components D'_i and D'_j such that $D \models D'_i$ and $D \models D'_j$. So there can be two different diagrams D^+ w.r.t D'_i and D'_j].

Example 6.3. Let us consider the two diagrams D and D' ($\equiv D'_1 - D'_2$) (vide Fig. 82).

Here $D \models D'_1$. Then D^+ (vide Fig. 83) is the diagram obtained by introducing closed curves to D that enclose the basic regions of D'_1 of which there are no counterparts in D.

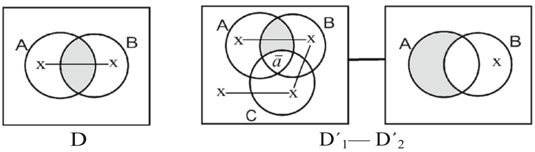


FIGURE 82.

Case-4: Let $D^{'}$ be a type-III diagram where $D^{'} \equiv D^{'}_1 - D^{'}_2 - \ldots - D^{'}_m$ and there is no component in $D^{'}$ such that $D \models D^{'}_i$ holds ($1 \leq i \leq m$) then D^+ is the diagram obtained by introducing closed curves to D that enclose all the basic regions of $D^{'}_i$ ($1 \leq i \leq m$) of which there are no counterparts in D.

Example 6.4. Let us consider the two diagrams D and $D^{'}$ ($\equiv D^{'}_1 - D^{'}_2$) (vide Fig. 84).

Here $D \models D^{'}$. There is no component in $D^{'}$ such that $D \models D^{'}_i$ holds (i = 1,2). Then D^+ (vide Fig. 85) is the diagram obtained by introducing closed curves to D that enclose all the basic regions of $D^{'}_i$ (i = 1, 2) of which there are no counterparts in D.

For the proof of Completeness we shall adopt the following strategy.

First, we prove completeness for single-premise i.e. if $D \models D^{'}$ then $D \vdash D^{'}$ where D and $D^{'}$ are single diagrams.

This consists of sub cases, the complete picture of which is shown in Fig. 86. Broadly there are the following divisions,

(i) either of D or $D^{'}$ is tautology, (ii) either of D or $D^{'}$ is inconsistent, (iii) both of D and $D^{'}$ are consistent type-I/II diagrams other than tautology,

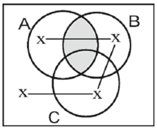


FIGURE 83.

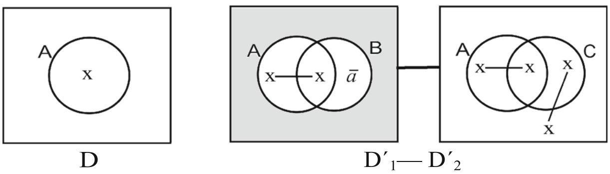


FIGURE 84.

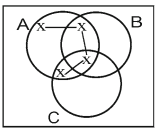


FIGURE 85.

(iv) D is type-I/II diagram and D' is type-III diagram other than tautology and (v) D is type-III diagram and D' is any diagram other than tautology.

To prove (iii) we have obtained necessary conditions on diagrams D and D' that hold when $D \models D'$, given that D has counterparts of all the regions in D'. Then we extend D to D_{max} and show that $D_{max} \vdash D'$. Since $D \vdash D_{max}$ we get $D \vdash D'$.

If D does not have counterparts of some of the basic regions of D' then we extend D to D^+. For definition of D^+ given $D \models D'$, see def 6.2. D^+ is an extension of D by introducing all closed curves of D' which are not in D. Since $D^+ \models D$ by elemination rule, we have $D^+ \models D'$ and by the previous sub case we get $D^+ \vdash D'$ and hence $D \vdash D'$.

Then the remaining cases when either of the diagrams D or D' is of type-III are dealt with.

Finally the completeness theorem is obtained by employing unification rule on the diagrams in the premise set when it contains more than one diagram.

Schema of Completeness Theorem

Figure 86.

Theorem 6.4. *If any one of the diagrams D or $D^{'}$ is a tautology and $D \models D^{'}$ hold then $D \vdash D^{'}$ holds.*

Proof. We can prove the theorem by using the following two lemmas (Lemmas 6.5 and 6.6).

Lemma 6.5. *If $D \models D^{'}$ and $D^{'}$ is a tautology then $D \vdash D^{'}$ holds.*

Proof. $\mathrm{D}^{'}$ is a tautology i.e. $\vdash \mathrm{D}^{'}$. Thus $\mathrm{D} \vdash \mathrm{D}^{'}$ holds for any D (Proposition 4.2). □

Lemma 6.6. *If D is a tautology and $D \models D^{'}$ then $D^{'}$ is a tautology and $D \vdash D^{'}$.*

Proof. Let D be a tautology and $\mathrm{D} \models \mathrm{D}^{'}$.

Since D is tautology, D is true in every model (cf. Lemma 5.6). So, $\mathrm{D}^{'}$ is true in every model.

Thus $\mathrm{D}^{'}$ is a tautology [by Lemma 6.2].

Since $\mathrm{D}^{'}$ is a tautology thus for any D, $\mathrm{D} \vdash \mathrm{D}^{'}$ (Proposition 4.2). □

Thus, when any one of the diagram D or $\mathrm{D}^{'}$ is a tautology and $\mathrm{D} \models \mathrm{D}^{'}$ holds, then $\mathrm{D} \vdash \mathrm{D}^{'}$ holds. □

Theorem 6.7. *If any one of the diagram D or $D^{'}$ is inconsistent and $D \models D^{'}$ holds then $D \vdash D^{'}$ holds.*

Proof. We can prove the theorem by using the following two lemmas (Lemmas 6.8 and 6.9).

Lemma 6.8. *If $D^{'}$ is inconsistent and $D \models D^{'}$ then D is inconsistent and $D \vdash D^{'}$.*

Proof. Let $\mathrm{D}^{'}$ be inconsistent and $\mathrm{D} \models \mathrm{D}^{'}$.

If possible let D be consistent.

Then D is true in some model (Theorem 6.1). But since $\mathrm{D}^{'}$ is false in every model,we face a contradiction to $\mathrm{D} \models \mathrm{D}^{'}$.

Thus D is inconsistent.

Thus for any $\mathrm{D}^{'}$, $\mathrm{D} \vdash \mathrm{D}^{'}$ holds by inconsistency rule. □

Lemma 6.9. *If $D \models D^{'}$ and D is inconsistent then $D \vdash D^{'}$.*

Proof. D is inconsistent implies $\mathrm{D} \vdash \mathrm{D}^{'}$. This holds because of inconsistency rule. □

Thus, when any one of the diagram D or $\mathrm{D}^{'}$ is inconsistent and $\mathrm{D} \models \mathrm{D}^{'}$ holds, then $\mathrm{D} \vdash \mathrm{D}^{'}$ holds. □

Lemma 6.10 gives necessary conditions that should hold for the diagram D given some conditions for $\mathrm{D}^{'}$ and such that $\mathrm{D}^{'}$ is a semantic consequence of D. This lemma will be used in Theorem 6.11 in which completeness theorem will be established in a special case.

Lemma 6.10. *Let D and $D^{'}$ be both type-I/II consistent diagrams other than tautology and $D \models D^{'}$ and for every region r of $D^{'}$ there is a counterpart region c(r) in D then the following hold:*

(1) If m is a minimal region in $D^{'}$ such that m is shaded then c(m) in D is shaded.
(2) If r is a region in $D^{'}$ such that r has a x-sequence then
(i) c(r) can contain any diagrammatic object if r spreads over the whole rectangle.
(ii) c(r) has a part of a x-sequence or an a-sequence or c(− r) has shading if r is a proper part of rectangle.
(3) If r is a region in $D^{'}$ such that r has an a-sequence then c(r) has a part of an a-sequence.
(4) If m is a minimal region in $D^{'}$ such that m has $\bar{a}$ then
either c(−m) in D has a part of the a-sequence,
or c(m) is shaded,
or c(m) has $\bar{a}$.

Proof. Let $D^{'}$ contain only one diagrammatic object.

Proof of (1): Let m be a minimal region in $D^{'}$ such that m is shaded.

Now c(m) should be shaded.

For if not, it may be blank, may contain part of a x-sequence, part of an a-sequence or $\bar{a}$.

In any case we can find a model M such that D is true in M and I(c(m))$\neq \phi$.

So there is an model M such that $M \Vdash D$ but $M \nVdash D^{'}$.

This contradicts $D \models D^{'}$.

Proof of (2): Let r be a region in $D^{'}$ such that r has a x-sequence.

Case-(i):

r is spread over the whole rectangle. Since there is only one diagrammatic object in this case a x-sequence and it is spread over the whole rectangle, $D^{'}$ is a tautology. So, $D \models D^{'}$ for any D. Thus c(r) can have any diagrammatic object.

Case-(ii):

r is a proper part of rectangle then c(r) has a part of x-sequence or an a-sequence or c(− r) has shading.

If not, c(r) is either shaded or blank or contain $\bar{a}$ and c(− r) can have any diagrammatic object other than shading.

In any case we can find a model M such that D is true in M and I(c(r))$= \phi$.

So there is an model M such that $M \Vdash D$ but $M \nVdash D^{'}$.

This contradicts $D \models D^{'}$.

Proof of (3): Let r be a region in $D^{'}$ such that r has an a-sequence.

Then c(r) has a part of an a-sequence.

If not, c(r) is either shaded or blank or contains a x-sequence or $\bar{a}$.

Then we can find a model M such that D is true in M and h(a)$\notin$ I(c(r)).

So there is an model M such that M $\Vdash$ D but M $\nVdash$ D$'$.

This contradicts D $\models$ D$'$.

Proof of (4): Let m be a minimal region in D$'$ such that m has $\bar{a}$.

Then either c($-$ m) has part of a-sequence or c(m) has shading or has $\bar{a}$.

If not then c($-$ m) has no a-sequence and c(m) has no shading or $\bar{a}$.

Then we can find a model M such that D is true in M and h(a)$\in$ I(c(m)).

So there is an model M such that M $\Vdash$ D but M $\nVdash$ D$'$.

This contradicts D $\models$ D$'$.

Let D$'$ contain more than one diagrammatic object.

Now we can say that D$'$ is the diagram Uni(D$'_1$, D$'_2$, …, D$'_n$) where each D$'_i$ contains only one diagrammatic object of D$'$.

Since D $\models$ D$'$ thus D $\models$ D$'_i$, for all i [since Uni(D$'_1$, D$'_2$, …, D$'_n$) is the conjunction of D$'_1$, D$'_2$, …, D$'_n$].

Now the conditions 1–4 already hold for each D$'_i$.

Thus these will also hold for D$'$. □

Theorem 6.11. *If D and D$'$ are both type-I/II consistent diagrams other than tautology such that for every region r of D$'$ there is a counterpart region c(r) in D and D $\models$ D$'$ then D $\vdash$ D$'$.*

Proof. We first consider the case when D$'$ contains only one diagrammatic object.

We know that D $\vdash$ D_{max} (Proposition 4.5).

Since for every region in D$'$ there exists an counterpart region in D, then all the conditions of Lemma 6.10 holds.

(1) Let in D$'$ there be only one minimal region m which is shaded, then c(m) in D is shaded (by Lemma 6.10).

Since we did not eliminate any shading in the construction of D_{max}, c(m) in D_{max} is also shaded.

Thus D_{max} $\vdash$ D$'$ [by possible application of rule of elimination other than this particular shading].

(2) Let in D$'$ there be only one x-sequence in a region r.

Case-(i):

If r spread over the rectangle then D$'$ is a tautology.

D_{max} $\vdash$ D$'$ [Proposition 4.2].

Case-(ii):

If r is a proper part of the rectangle then either c(r) has a part of x-sequence or an a-sequence or c($-$ r) has shading (Lemma 6.10).

If c(r) has a part of x-sequence or an a-sequence then c(r) in D_{max} has all possible extensions of x-sequence or a-sequence.

Thus D_{max} $\vdash$ D$'$ [by possible application of rule of elimination other than this particular sequence].

If c(− r) has shading then the region c(r) + c(− r) and the region c(r) both have x-sequences in D_{max} as per construction of D_{max}.

Thus $D_{max} \vdash D^{'}$ [by possible application of rule of elimination other than this particular sequence].

(3) If r is a region in $D^{'}$ such that m has an a-sequence, then c(r) has a part of an a-sequence.

Then c(r) in D_{max} has all possible extension of x-sequence and a-sequence and also $\bar{a}$.

Thus $D_{max} \vdash D^{'}$ [by possible application of rule of elimination other than this particular sequence].

(4) Let there be only one $\bar{a}$ in $D^{'}$ such that m has $\bar{a}$ then

either c(− m) in D has a part of a-sequence,

or c(m) is shaded,

or c(m) has $\bar{a}$.

If c(− m) in D has a part of a-sequence then c(m) in D_{max} has $\bar{a}$.

Thus $D_{max} \vdash D^{'}$ [by possible application of rule of elimination other than this $\bar{a}$].

If c(m) in D has shading then c(m) in D_{max} has shading and $\bar{a}$.

Thus $D_{max} \vdash D^{'}$ [by possible application of rule of elimination other than this $\bar{a}$].

If c(m) in D has $\bar{a}$ then c(m) in D_{max} has $\bar{a}$.

Thus $D_{max} \vdash D^{'}$ [by possible application of rule of elimination other than this $\bar{a}$].

Thus in any case $D_{max} \vdash D^{'}$.

So, because $D \vdash D_{max}$ we get $D \vdash D^{'}$ [by transitivity].

Let us consider the case when $D^{'}$ contain more than one diagrammatic object.

We can say that $D^{'}$ is the diagram Uni($D^{'}_1$, $D^{'}_2$, ..., $D^{'}_n$) where each $D^{'}_i$ contains only one diagrammatic object of $D^{'}$.

Now the conditions 1–4 already hold for each $D^{'}_i$.

Thus $D \vdash D^{'}_i$ (already proved in the previous case).

$\therefore$ $D \vdash D^{'}$ [since Uni($D^{'}_1$, $D^{'}_2$, ..., $D^{'}_n$) is the conjunction of $D^{'}_1$, $D^{'}_2$, ..., $D^{'}_n$]. □

We now consider the general case when the existence of counterpart condition is dropped.

Theorem 6.12. *If D and $D^{'}$ are both type-I/II consistent diagrams other than tautology and $D \models D^{'}$ then $D \vdash D^{'}$.*

Proof. We have $D \vdash D^{+}$ [by the rule of introduction of closed curves of $D^{'}$ for which there are no counterparts in D].

Again $D^{+} \vdash D$ [by the rule of elimination of closed curves].

Thus $D^{+} \models D$ [by soundness].

$\therefore \mathrm{D}^{+} \models \mathrm{D}^{'}$ [by transitivity, since $\mathrm{D} \models \mathrm{D}^{'}$ by assumption].
Now for every region in $\mathrm{D}^{'}$ there exists counterpart region in D^{+} and both of them are type-I/II diagrams. Also, $\mathrm{D}^{+} \models \mathrm{D}^{'}$.
$\therefore \mathrm{D}^{+} \vdash \mathrm{D}^{'}$ [by Theorem 6.11].
$\therefore \mathrm{D} \vdash \mathrm{D}^{'}$ [by transitivity].

□

We now take $\mathrm{D}^{'}$ as a type-III diagram. First we put some restrictions on D and $\mathrm{D}^{'}$ and finally the general case will be established. Lemma 6.13 is similar to Lemma 6.10 giving necessary conditions that holds when certain similar restrictions are imposed on D and $\mathrm{D}^{'}$.

Lemma 6.13. *Let*

(i) *$D \models D^{'}$ where $D^{'} \equiv D_1^{'} - D_2^{'} - \cdots - D_n^{'}$,*
(ii) *$D \models D^{''}$ do not hold for any proper part $D^{''}$ of $D^{'}$ and*
(iii) *for every region r in $D^{'}$ there exist a counterpart region c(r) in D.*

Then the conditions 1–4 hold:

(1) *If m is a minimal region of $D^{'}$ and m is shaded then exactly one of the following holds.*
(i) *c(m) is shaded.*
(ii) *c(m) contains no diagrammatic object and $D^{'}$ is tautologous at m w.r.t x or tautologous at m w.r.t some a.*
(iii) *c(m) contains $\bar{a}$ and $D^{'}$ is tautologous at m w.r.t x or w.r.t some $b(\neq a)$.*
(iv) *c(m) contains one node of a (non-degenerate) x-sequence or a-sequence.*
(2) *If r is a region in $D^{'}$ such that r has a x-sequence then one of the following holds.*
(i) *c(r) has a part of x-sequence or an a-sequence.*
(ii) *A part of c(r) has no diagrammatic object.*
(iii) *A part of c(r) contains $\bar{a}$.*
(3) *If r is a region in $D^{'}$ such that r has an a-sequence then exactly one of the following holds.*
(i) *c(r) has a part of an a-sequence.*
(ii) *There exists a minimal region m in r such that c(m) contains no diagrammatic object and $D^{'}$ is tautologous at m w.r.t a.*
(iii) *There exists a minimal region m in r and c(m) contains a part of x-sequence and $D^{'}$ is tautologous at m w.r.t a.*
(4) *If m is a minimal region of $D^{'}$ and m has $\bar{a}$ then exactly one of the following holds.*
(i) *Either part of c(− m) in D has a-sequence,*
or c(m) is shaded,
or c(m) has $\bar{a}$.
(ii) *c(m) contains no diagrammatic object.*
(iii) *c(m) contains one nodes of a (non-degenerate) x-sequence or a-sequence.*

(iv) c(m) contains a part of x and D' *is tautologous at m w.r.t a.*

The proof of (1) is by showing that if none of the conditions (i)– (iv) holds then $D \not\models D'$. Similarly (2), (3) and (4) are also established. However, we are giving some demonstrations of the cases that will make things transparent.

Example 6.5. Let us consider the two diagrams D and $D' (\equiv D'_1 - D'_2 - D'_3)$ (vide Fig. 87).

Here $D \models D'$, but there is no component in D' such that $D \models D'_i$ holds $(i = 1, 2, 3)$.

Let us consider the component D'_1, in D'_1 the minimal region ((− A) · (− B)) has $\bar{a}$, the minimal region (A · (− B)) has shading and D' is tautologous at (A · (− B)) w.r.t x, the minimal region (A · B) has $\bar{a}$ and the minimal region ((− A) · B) has *a*. Now, in diagram D, c((− A) · (− B)) contains no diagrammatic object, c(A · (− B)) has $\bar{a}$, c(A · B) has shading and c((− A) · B) has *a*.

Let us consider the component D'_2, in D'_2 the minimal region ((− A) · (− B)) has shading and D' is tautologous at ((− A) · (− B)) w.r.t x, the minimal region ((A · (− B)) · (B · (− A))) has x-sequence and the minimal region (A · B) has $\bar{a}$. Now, in diagram D, c((− A) · (− B)) contains no diagrammatic object, c((A · (− B)) · (B · (− A))) has a part of *a*-sequence and c(A · B) has shading.

Let us consider the component D'_3, in D'_3 the minimal region ((− A) · (− B)) has x, the minimal region (A · (− B)) has $\bar{a}$ the minimal region (A · B) has shading and the minimal region ((B · (− A)) · ((− A) · (− B))) has x-sequence. Now, in diagram D, c((− A) · (− B)) contains no diagrammatic object, c((A · (− B))) has $\bar{a}$, c(A · B) has shading and c((B · (− A)) · ((− A) · (− B))) has a part of *a*-sequence.

So, all the conditions of Lemma 6.13 are verified.

Theorem 6.14. *If D is a type-I/II diagram and* D' *is a type-III diagram and both of them are consistent diagrams other than tautologies such that for every region r of* D' *there is a counterpart region c(r) in D and* $D \models D'$ *then* $D \vdash D'$ *(where* $D' \equiv D'_1 - D'_2 - \cdots - D'_n$*).*

Proof. Case-1:

There is at least one component in D', say D'_i, such that $D \models D'_i$ holds.

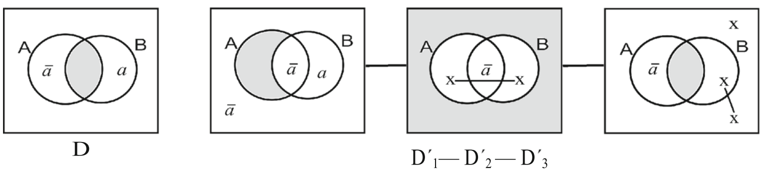


Figure 87.

For every region in $D_i^{'}$ there exists an counterpart region in D and D $\models D_i^{'}$ holds.

$\therefore D \vdash D_i^{'}$ [by Theorem 6.11].

Now $D_i^{'} \vdash D_1^{'} - D_2^{'} - \cdots - D_i^{'} - \cdots - D_n^{'}$ [by the rule of introduction of components].

$\therefore D \vdash D_1^{'} - D_2^{'} - \cdots - D_n^{'}$ [by transitivity].

That is $D \vdash D^{'}$.

Case-2:

$D \models D^{''}$ does not hold for any proper part $D^{''}$of $D^{'}$.

We split the cases into two sub cases.

Sub case-(i):

There is no $\bar{a}$ in D.

In this case D contains at least one non-degenerate x-sequence or at least one non-degenerate a-sequence or at least one blank region.

We now follow the following steps.

$D \vdash D_1$ say, (by all possible application of rule of splitting sequence).

Then we obtain $D_1 \vdash D_2$ say, (by possible multiple application of rule of excluded middle-(a) till there is no region left without a diagrammatic object).

Then we obtain $D_2 \vdash D^{\Box}$ say, (by possible multiple application of rule of excluded middle-(b) that is adding a or $\bar{a}$ in the minimal regions having a x till each of the regions having x have an a or $\bar{a}$).

So $D^{\Box}$ is a type-III diagram $D_1^{\Box} - D_2^{\Box} - \cdots - D_n^{\Box}$ where each component has only conjunctive information and there is no region without any diagrammatic objects.

Again, $D \vdash D^{\Box}$ (by transitivity).

Then $D^{\Box}_{max}$ is the diagram $D^{\Box}_{1max} - D^{\Box}_{2max} - \cdots - D^{\Box}_{nmax}$.

Now $D_1^{\Box} \vdash D^{\Box}_{1max}$ (Proposition 4.5).

$D_1^{\Box} \vdash D^{\Box}_{1max} - D^{\Box}_{2max} - \cdots - D^{\Box}_{nmax}$. (by extension of components).

Similarly for $D_2^{\Box}$ etc.

Hence by rule of construction we have $D^{\Box} \vdash D^{\Box}_{max}$.

Thus $D \vdash D^{\Box}_{max}$ (by transitivity).

Since for every region in $D^{'}$ there exists a counterpart region in D, then all the conditions of Lemma 6.13 hold.

(1) If m is a minimal region of $D^{'}$ and m is shaded then exactly one of the following holds

(i) c(m) is shaded.

(ii) c(m) contains no diagrammatic object and $D^{'}$ is tautologous at m w.r.t x or tautologous at m w.r.t some a.

(iii) c(m) contains $\bar{a}$ and $D^{'}$ is tautologous at m w.r.t x or w.r.t some $b(\neq a)$.

(iv) c(m) contains one node of a(non-degenerate) x-sequence or a-sequence.

In case (i), c(m) is shaded and hence c(m) in each $D^{\Box}_{imax}$ for all i this region is also shaded, as we did not eliminate any shading.

In case (ii), c(m) contains no diagrammatic object and $D^{'}$ is tautologous at m w.r.t x. Hence the region m in some of the components $D^{'}_1$, $D^{'}_2$, ..., $D^{'}_k$ has shading and m in the remaining components $D^{'}_{k+1}$, $D^{'}_{k+2}$, ..., $D^{'}_n$ has a x.

But $D^{\Box}$ is obtained by rule of excluded middle, so c(m) has shading in some of the components of $D^{\Box}$ and c(m) has x in some of the components of $D^{\Box}$. Thus $D^{\Box}_{max}$ also has the same diagrammatic objects in the respective counterpart regions.

c(m) contains no diagrammatic object and $D^{'}$ is tautologous at m w.r.t some a. Hence the region m in some of the components $D^{'}_1$, $D^{'}_2$, ..., $D^{'}_k$ has shading and m in the remaining components $D^{'}_{k+1}$, $D^{'}_{k+2}$, ..., $D^{'}_n$ has a a or $\bar{a}$.

This case can be dealt with similarly as above.

Case (iii) is not applicable here since D has no $\bar{a}$.

In case (iv), c(m) contains one node of a (non-degenerate) x-sequence or a-sequence. But $D^{\Box}$ is obtained by rule of splitting sequences followed by rule of excluded middle.

So c(m) in $D^{\Box}_{1max}$, $D^{\Box}_{2max}$, ..., $D^{\Box}_{kmax}$ has shading and c(m)in $D^{\Box}_{k+1max}$, $D^{\Box}_{k+2max}$, ..., $D^{\Box}_{nmax}$ has a x.

Hence we see that in $D^{\Box}_{max}$ the diagram $D^{'}$ is embedded. So we get $D^{\Box}_{max} \vdash D^{'}$ [by suitable application of rule of elimination and rule of construction].

Similarly, for any other diagrammatic object in $D^{'}$ we can show that $D^{\Box}_{max} \vdash D^{'}$.

Thus $D \vdash D^{'}$ (by transitivity) (vide Example 6.6).

Sub case-(ii):

There is $\bar{a}$ in some minimal regions in D.

Let there be $\bar{a}$ in m_1, m_2, ..., m_n of D.

Corresponding to each minimal region m_i we construct $D_{T_{m_i}}$ (i=1, 2,...,n) such that

- (i) $D_{T_{m_i}}$ is a type-III diagram $D_{1T_{m_i}} - D_{2T_{m_i}}$.
- (ii) $D_{1T_{m_i}}$, $D_{2T_{m_i}}$, for all $i = 1, 2, \ldots, n$ have the same basic regions as D.
- (iii) $c(m_i)$ has shading in one of the components and x in the other component.
- (iv) $D_{T_{m_i}}$ has no other diagrammatic objects.

Now we unite D with $D_{T_{m_1}}$, $D_{T_{m_2}}$, ..., $D_{T_{m_n}}$.

$\therefore D \vdash$ Uni(D, $D_{T_{m_1}}$, $D_{T_{m_2}}$, ..., $D_{T_{m_n}}$)(by rule of unification).

Now, Uni(D, $D_{T_{m_1}}$, $D_{T_{m_2}}$, ..., $D_{T_{m_n}}$) $\vdash D_1$ say,(by all possible applications of rule of splitting sequence).

$D_1 \vdash D_2$ say, (by possible multiple applications of rule of excluded middle-(a) till there is no region without diagrammatic object left).

$D_2 \vdash D^{\Box}$ say,(by possible multiple applications of rule of excluded middle-(b) till each of the regions having x have an a or $\bar{a}$).

So $D^{\Box}$ is a type-III diagram $D^{\Box}_1 - D^{\Box}_2 - \cdots - D^{\Box}_n$ where each component has only conjunctive information and there is no region without any diagrammatic objects.

Then $D^{\Box}_{max}$ is the diagram $D^{\Box}_{1max} - D^{\Box}_{2max} - \cdots - D^{\Box}_{nmax}$.

Now $D^{\Box}_1 \vdash D^{\Box}_{1max}$ (Proposition 4.5).

$D^{\Box}_1 \vdash D^{\Box}_{1max} - D^{\Box}_{2max} - \cdots - D^{\Box}_{nmax}$ (by extension of components).

Similarly for $D^{\Box}_2$ etc.

Hence by rule of construction we have $D^{\Box} \vdash D^{\Box}_{max}$.

Thus $D \vdash D^{\Box}_{max}$ (by transitivity).

We can show that $D^{\Box}_{max} \vdash D'$, by using the similar methods used in sub case-(i).

Thus $D \vdash D'$ (by transitivity) (vide Example 6.7). □

Example 6.6. Let us consider the following two diagram D and D' (vide Fig. 88).

Here, $D \models D'$. But, there is no component in D' for which $D \models D'_i$ (i=1,2) holds.

We use the rule of excluded middle. The resultant diagram is $D^{\Box}$ (vide Fig. 89).

Now from $D^{\Box}$ we construct $D^{\Box}_{max}$ (vide Fig. 90).

we can get the diagram D'_1 from $D^{\Box}_{1max}$ by using the elimination rule. And from $D^{\Box}_{2max}$ we get the diagram D'_2.

$\therefore D^{\Box}_{1max} \vdash D'$ [by rule of extension of components].

Similarly, $D^{\Box}_{2max} \vdash D'$.

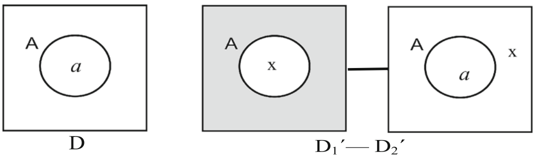


Figure 88.

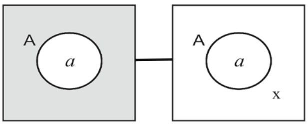


Figure 89.

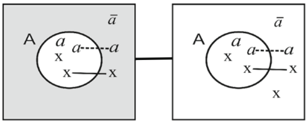


Figure 90.

Thus, $D^{\Box}_{max} \vdash D^{'}$ [by rule of construction].
$\therefore D \vdash D^{'}$.

Example 6.7. Let us consider the following two diagram D and $D^{'}$ (vide Fig. 91).

Here, $D \models D^{'}$. But, there is no component in $D^{'}$ for which $D \models D^{'}_{i}$ $(i = 1, 2, 3, 4)$ holds.

Now we first unite the diagram D with a type-III diagram $D_{T_{m_1}}$ (vide Fig. 92).

The resultant diagram is Uni(D, $D_{T_{m_1}}$) (vide Fig. 93).

Now we unite the diagram D with a type-III diagram $D_{T_{m_2}}$ (vide Fig. 94).

The resultant diagram is Uni(D, $D_{T_{m_1}}$, $D_{T_{m_2}}$) (vide Fig. 95).

We use the rule of splitting sequence. The resultant diagram is D_1 (vide Fig. 96).

We use the rule of excluded middle-(a) till there is no region without diagrammatic object left. The resultant diagram is D_2 (vide Fig. 97).

Then we use the rule of excluded middle-(b) till all the regions having x has an a or $\bar{a}$. The resultant diagram is $D^{\Box}$ (vide Fig. 98).

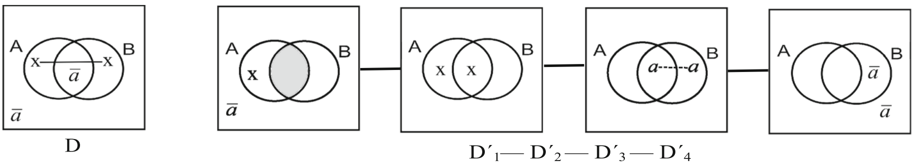


Figure 91.

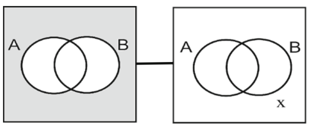


Figure 92.

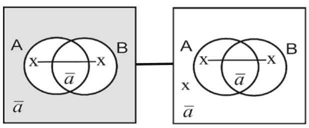


Figure 93.

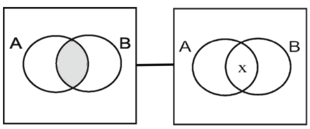


Figure 94.

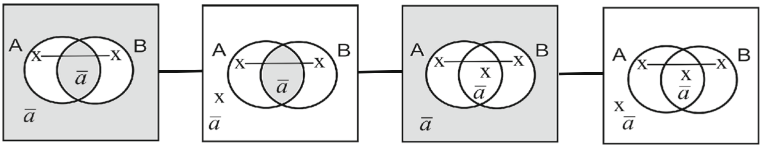


Figure 95.

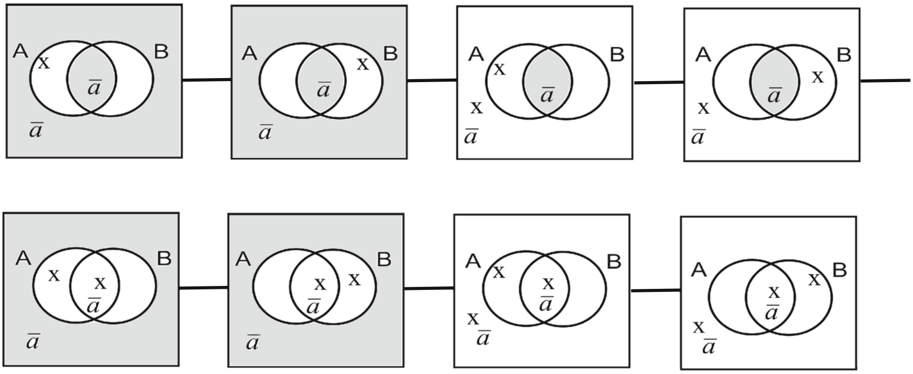


Figure 96.

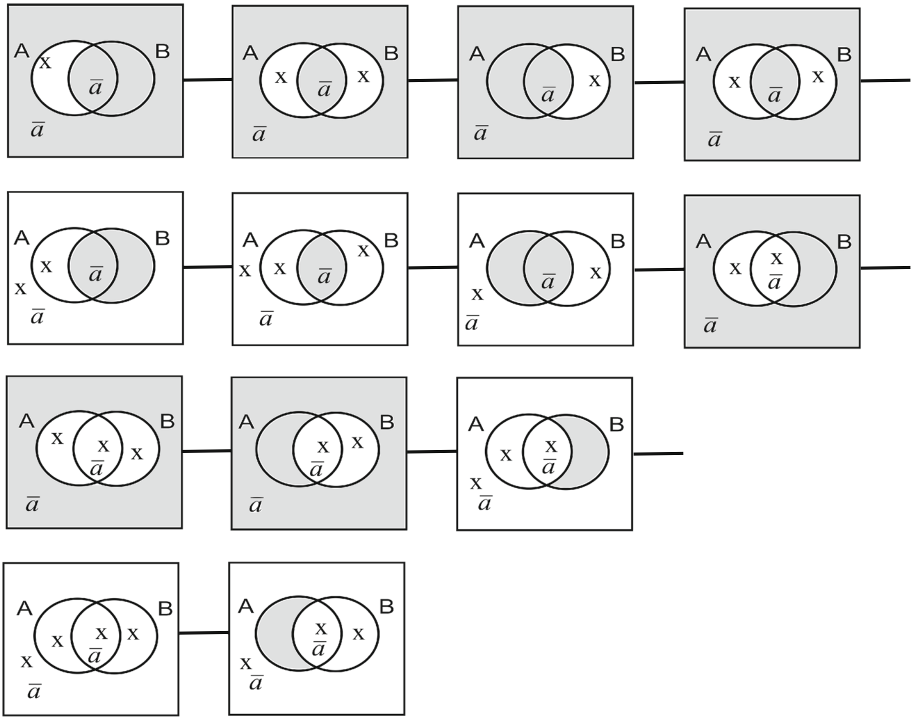


Figure 97.

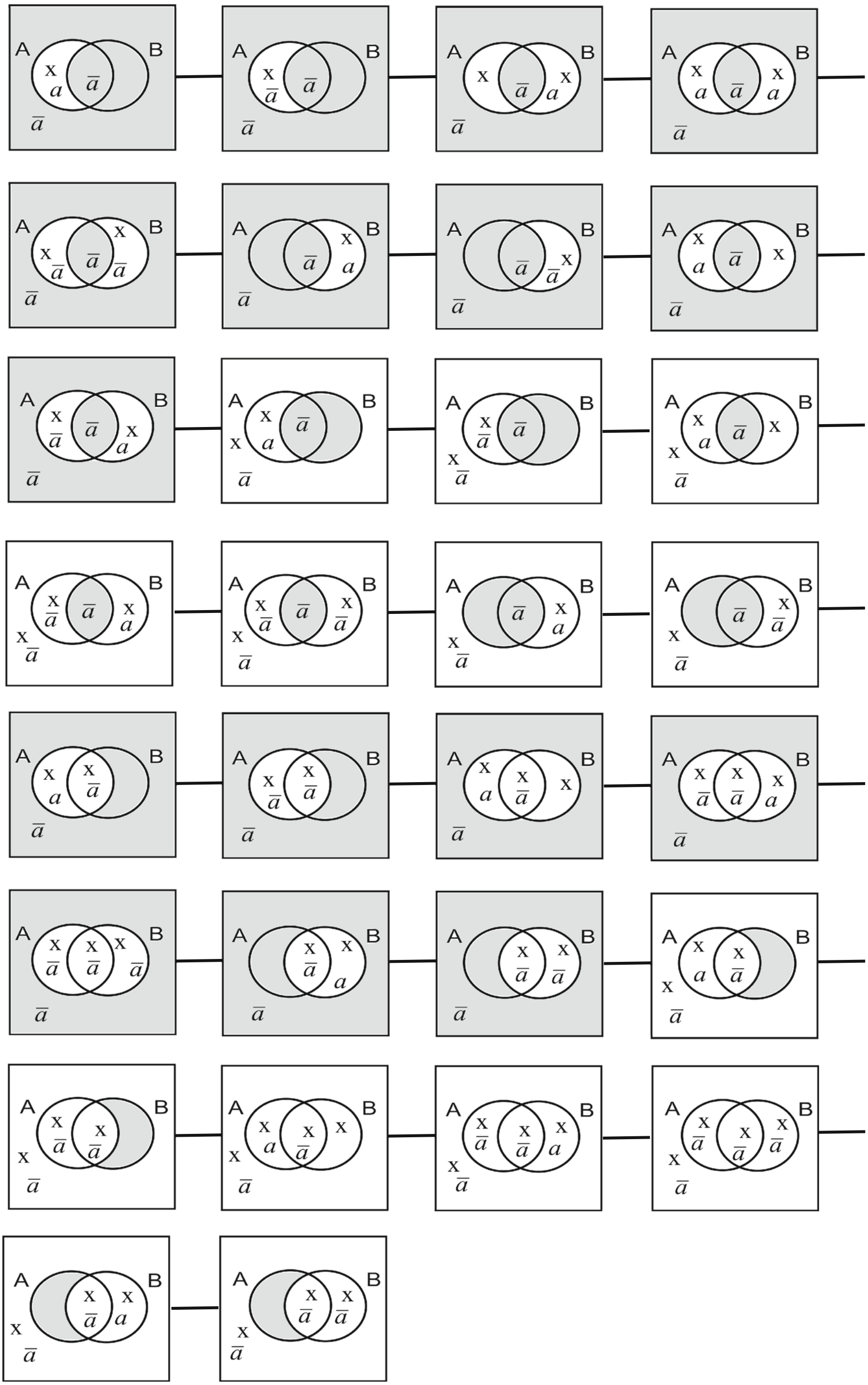


Figure 98.

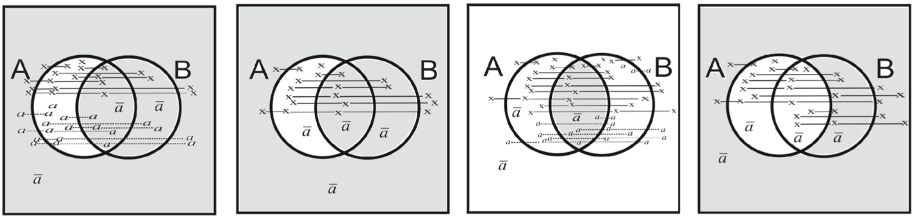


FIGURE 99.

Now from $D^{\square}$ we construct $D^{\square}_{max}$. In Fig. 99 we show a few components of $D^{\square}_{max}$ obtained from $D^{\square}_{1}$, $D^{\square}_{2}$, $D^{\square}_{13}$ and $D^{\square}_{18}$ respectively from the Fig. 98.

We can get the diagram $D^{'}_{1}$ from $D^{\square}_{1max}$ by using the elimination rule. And from $D^{\square}_{2max}$ we get the diagram $D^{'}_{4}$. Also, from $D^{\square}_{13max}$ and $D^{\square}_{18max}$ we get the diagram $D^{'}_{3}$ and $D^{'}_{2}$ respectively.

Similarly, from other components of $D^{\square}_{max}$ we get components of $D^{'}$.

$\therefore D^{\square}_{imax} \vdash D^{'}$ (i=1, 2,..., 30) [by rule of extension of components].

Thus, $D^{\square}_{max} \vdash D^{'}$ [by rule of construction].

$\therefore D \vdash D^{'}$.

Theorem 6.15. *If D is a type-I/II diagram and $D^{'}$ is a type-III diagram, both of them are consistent diagrams other than tautology and $D \models D^{'}$ then $D \vdash D^{'}$.*

Proof. Case-1:

There is at least one component in $D^{'}$, say $D^{'}_{1}$, such that $D \models D^{'}_{1}$ holds.

Now $D^{'}_{1}$ is a type-I/type-II diagram, such that $D^{'}_{1}$ may have some basic regions counterparts of which are not present in D. Then D^{+} is the diagram obtained by introducing closed curves to D such that these curves enclose basic regions of $D^{'}_{1}$ of which there are no counterparts present in D.

Thus $D \vdash D^{+}$ [by rule of introduction of closed curve].

Again $D^{+} \vdash D$ [by the rule of elimination of closed curves].

Thus $D^{+} \models D$ [by soundness].

$\therefore D^{+} \models D^{'}_{1}$ [by transitivity].

Now for every region in $D^{'}_{1}$ there exists a counterpart region in D^{+} and both of them are type-I/II diagrams and $D^{+} \models D^{'}_{1}$.

$\therefore D^{+} \vdash D^{'}_{1}$ [by Theorem 6.11].

$\therefore D \vdash D^{'}_{1}$ [by transitivity].

Now $D^{'}_{1} \vdash D^{'}_{1} - D^{'}_{2} - \cdots - D^{'}_{n}$ [by the rule of introduction of components].

$\therefore D \vdash D^{'}_{1} - D^{'}_{2} - \cdots - D^{'}_{n}$ [by transitivity].

$\therefore D \vdash D^{'}$.

Case-2:

$D \models D^{''}$ does not hold for any proper part $D^{''}$ of $D^{'}$.

D^{+} is the diagram obtained by introducing basic regions of $D^{'}_{1}$, $D^{'}_{2}$, ..., $D^{'}_{n}$ to D of which there were no counterparts present in D.

Thus $\mathrm{D} \vdash \mathrm{D}^{+}$ [by rule of introduction of closed curve].
Again $\mathrm{D}^{+} \vdash \mathrm{D}$ [by the rule of elimination of closed curves].
Thus $\mathrm{D}^{+} \models \mathrm{D}$ [by soundness].
$\therefore \mathrm{D}^{+} \models \mathrm{D}^{'}$ [by transitivity].
Now for every region in $\mathrm{D}^{'}$ there exists an counterpart region in D^{+} and $\mathrm{D}^{+} \models \mathrm{D}^{'}$.
$\therefore \mathrm{D}^{+} \vdash \mathrm{D}^{'}$ [by Theorem 6.14].
$\therefore \mathrm{D} \vdash \mathrm{D}^{'}$ [by transitivity].

□

Remark 6.16. Let

(i) $\mathrm{D} \models \mathrm{D}^{'}$ where $\mathrm{D}^{'} \equiv \mathrm{D}^{'}_1 - \mathrm{D}^{'}_2 - \cdots - \mathrm{D}^{'}_n - \cdots - \mathrm{D}^{'}_{n+m}$ and
(ii) $\mathrm{D} \models \mathrm{D}^{'}_1 - \mathrm{D}^{'}_2 - \cdots - \mathrm{D}^{'}_n$.

Then $\mathrm{D} \vdash \mathrm{D}^{'}_1 - \mathrm{D}^{'}_2 - \cdots - \mathrm{D}^{'}_{n+m}$.

Proof. Now from Theorem 6.15, we get $\mathrm{D} \vdash \mathrm{D}^{'}_1 - \mathrm{D}^{'}_2 - \cdots - \mathrm{D}^{'}_n$. So, by application of rule of extension of components we get $\mathrm{D} \vdash \mathrm{D}^{'}_1 - \mathrm{D}^{'}_2 - \cdots - \mathrm{D}^{'}_{n+m}$. □

Theorem 6.17. *If D is a type-III diagram and $D^{'}$ is any diagram,both of them are consistent diagrams other than tautology and $D \models D^{'}$ then $D \vdash D^{'}$.*

Proof. We take $\mathrm{D} \equiv \mathrm{D}_1 - \mathrm{D}_2 - \cdots - \mathrm{D}_n$.

Case-1:
Let $\mathrm{D}^{'}$ be a type-I/II diagram.
Since $\mathrm{D} \models \mathrm{D}^{'}$, $\mathrm{D}_i \models \mathrm{D}^{'}$ holds for all D_i $(1 \leq \mathrm{i} \leq \mathrm{n})$. Otherwise if say $\mathrm{D}_1 \not\models \mathrm{D}^{'}$, then there is a model M such that $\mathrm{M} \Vdash \mathrm{D}_1$ and $\mathrm{M} \nVdash \mathrm{D}^{'}$.
Then $\mathrm{M} \Vdash \mathrm{D}$ but $\mathrm{M} \nVdash \mathrm{D}^{'}$, a contradiction.
Now for any D_i, $\mathrm{D}_i \models \mathrm{D}^{'}$ holds and both of them are type-I/II diagrams.
$\therefore \mathrm{D}_i \vdash \mathrm{D}^{'}$ [by Theorem 6.12].
$\therefore \mathrm{D} \vdash \mathrm{D}^{'}$ [by rule of construction].

Case-2:
Let $\mathrm{D}^{'}$ be the type-III diagram $\mathrm{D}^{'}_1 - \mathrm{D}^{'}_2 - \cdots - \mathrm{D}^{'}_m$.
Since $\mathrm{D} \models \mathrm{D}^{'}$, $\mathrm{D}_i \models \mathrm{D}^{'}$ holds for all D_i $(1 \leq \mathrm{i} \leq \mathrm{n})$.
Now for any D_i, $\mathrm{D}_i \models \mathrm{D}^{'}$ holds and D_i is a type-I/II diagram and $\mathrm{D}^{'}$ is a type-III diagram.
$\therefore \mathrm{D}_i \vdash \mathrm{D}^{'}$ [by Theorem 6.15].
$\therefore \mathrm{D} \vdash \mathrm{D}^{'}$ [by rule of construction].

□

Theorem 6.18 (Completeness theorem). *For any finite set Δ of diagrams and any diagram D, if $\Delta \models D$ then $\Delta \vdash D$.*

Proof. Case-1:

Let $\Delta = \phi$. Then $\models$ D i.e. D is true in every model. Thus $\vdash$ D holds (by Lemma 6.2).

Case-2:

Let $\Delta \neq \phi$. Then we obtain a diagram $D^{'}$ from the diagram Δ by the rule of unification.

Thus $\Delta \vdash D^{'}$.

Since $D^{'}$ is obtained from Δ by the rule of unification, all the diagrammatic objects and closed curves of each diagram of Δ are included in $D^{'}$.

Thus $D^{'} \vdash D^*$ for all $D^* \in \Delta$ [by rule of elimination and construction (whenever necessary)].

Thus by soundness we have $D^{'} \models D^*$ for all $D^* \in \Delta$.

Since $\Delta \models D$, therefore $D^{'} \models D$.

Thus $D^{'} \vdash D$ (by Theorems 6.4 to 6.17).

$\therefore \Delta \vdash D$ (by transitivity). □

7. Concluding Remarks

We indicate below some of the basic differences of the diagram system presented in the paper with some other systems.

1. This system is non-classical in the following sense. From the Fig. 100a, b does not follow.

   Semantically it means that absence of a in A does not necessarily imply locating its presence in the complement $-$A. In the introduction the motivation behind this assumption has been discussed.

   Syntactically the above fact obtains because of there not being any introduction rule for constants [cf. Sect. 3.1.3, note]. In the classical context, for any constant a and closed curve A, the following diagram would be obtained (vide Fig. 101) as h(a) should belong to I(A) or in its complement. So, there should have been a rule of introduction of the constant a by taking an a-sequence spreading over the whole rectangle with one node in every minimal region (vide Fig. 102).

   Thus according to the classical position, it is required that if there is the name of an individual in the language, its referent must be present somewhere in the universe whereas in our case it is not so. Our system admits empty-terms.

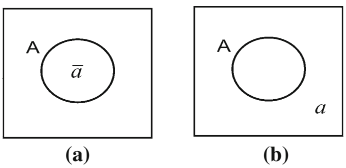


Figure 100.

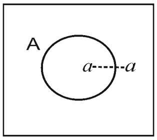


Figure 101.

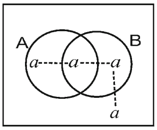


Figure 102.

2. In our diagram system there are additional primitive symbols (diagrammatic objects) viz. names of individuals a, their absence $\bar{a}$, line connecting a's (being interpreted by disjunctive $\mathfrak{or}$).
3. The blank rectangle is not a well-formed diagram in this system. We have followed the standard predicate logic system that requires at least one predicate symbol. It is to be noted that while in Shin [15, page no: 142, footnote] and Hammer the universe may be empty, ours is not.
4. Naturally, we have needed introduction and elimination rules for other diagrammatic objects. In fact there are introduction rules for closed curve (as in other systems) and additionally of $\bar{a}$ and x. Also we needed elimination rules for lci and $\bar{a}$ additionally.
5. Elimination rules for closed curved may be specially mentioned [c.f. Sect. 3.3.4]. In order to formulate this rule we have introduced the notion of normal form of a Venn diagram. For this, we had to make a little modification in the definition of Venn diagrams [c.f. Sect. 2.2].
6. Similarly, modifications and extensions of other rules have been required.
7. The notion of tautology is syntactic here. We have also introduced a notion called tautologous minimal region [c.f. Sect. 4, Definition 4.4].

Towards future directions of our work, we may mention the following:

(1) building diagram systems for standard classical logic. This system will be different from those of Shin and Hammer in that firstly there shall be representations of presence and absence of an individuals and hence our results will be different. Secondly this system will be obtained by incorporating a rule of introduction of individuals which is not present in the current system.
(2) We shall develope the diagram system with open universe i.e. there shall not be the bounding rectangle in the diagrams. A first attempt in this direction was made in [5]. Our studies would be to bridge the gaps in this effort and established completeness theorem rigorously following the technique of the present paper.

Acknowledgements

We acknowledge with gratitude some fruitful suggestions given by Gem Stapleton and Jim Burton in the course of development of this research and anonymous referee for pointing out some crucial points in the presentation. We acknowledge UGC UPE-II, School of Cognitive Science, Jadavpur University for funding this research.

Appendix A. All the Transformation Rules are Sound

The proof is obtained by dealing with all the transformation rules one by one.

(i) If $D \vdash D'$ by introduction of a new curve then $D \models D'$.

Proof. (a) Suppose a minimal region m in D is shaded. Let D be true in a model M. Thus I (m) = ϕ.

Let D' be obtained from D by introduction of a curve B. Then shading occurs in the minimal regions $m - B$ and $m \cdot B$ [as m is divided into two parts].

Now $I(m - B) \cup I(m \cdot B) = I((m - B) + (m \cdot B)) = I(m) = \phi$.

Thus $I(m - B) = \phi$ and $I(m \cdot B) = \phi$.

$\therefore M \Vdash D'$.

(b) Let a x-sequence occur in a region r in D, where $r = m_1 + m_2 + \cdots + m_k$.

Each of $m_1, m_2, \ldots, m_k$ contains one node of the x-sequence.

Let D be true in an arbitrary model M. Thus $\bigcup_{i=1}^{k} I(m_i) \neq \phi$.

Let D' be obtained from D by introduction of a curve B. Then we get new regions $m_i -$ B and $m_i \cdot$ B for each i. Because of the rule x is added in $m_i \cdot$ B for each i and these new x's are successively joined by line with one end of the existing x-sequence.

Now $I(m_i - B) \cup I(m_i \cdot B) = I((m_i - B) + (m_i \cdot B)) = I(m_i)$.

$\bigcup_{i=1}^{k} (I(m_i - B) \cup I(m_i \cdot B)) = \bigcup_{i=1}^{k} I(m_i) \neq \phi$.

$\therefore M \Vdash D'$.

(c) Let a a-sequence occur in a region r in D, where $r = m_1 + m_2 + \cdots + m_k$.

Each of $m_1, m_2, \ldots, m_k$ contains one node of the a-sequence.

Let D be true in an arbitrary model M. Let D' be obtained from D by introduction of a curve B. Then by using the similar method used in case-(b) we get $M \Vdash D'$.

(d) Suppose a minimal region m in D has $\bar{a}$.

Let D be true in an arbitrary model M.

Let D' be obtained from D by introduction of a curve B. Then there are $\bar{a}$ in the minimal regions $m - B$ and $m \cdot B$.

Case-1:

If $h(a)$ is not in the universe U. This implies,a does not occur in D.

So, a does not occur in D'?.

So M $\Vdash$ $D^{'}$.

Case-2:
If h(a) is in universe U, then h(a) $\notin$ I(m).
Now I(m − B) ∪ I(m · B) = I((m − B) + (m · B)) = I(m).
∴ h(a) $\notin$ (I(m − B) ∪ I(m · B)).
∴ h(a) $\notin$ I(m − B) and h(a) $\notin$ I(m · B).
∴ M $\Vdash$ $D^{'}$.
Since M is arbitrary. For every M, if M $\Vdash$ D then M $\Vdash$ $D^{'}$.
∴ D $\models$ $D^{'}$. □

(ii) If D $\vdash$ $D^{'}$ by introduction of $\bar{a}$ then D $\models$ $D^{'}$.

Proof. Case-(i):
Let m be a minimal region in D such that m is shaded.
Let M $\Vdash$ D, then I(m) = ϕ i.e. h(a) $\notin$ I(m) for any constant a.
$D^{'}$ is obtained by introducing $\bar{a}$ in c(m) [where c(m) is the counterpart region of m in $D^{'}$].
Now c(m) has shading and $\bar{a}$.
Since I(m) = ϕ therefore I(c(m)) = ϕ.
∴ h(a) $\notin$ I(c(m)) for any constant a.
∴ M $\Vdash$ $D^{'}$.
∴ D $\models$ $D^{'}$.

Case-(ii):
Let m be a minimal region in D such that a portion of − m, say r, has a-sequence.
Let M $\Vdash$ D, then h(a) ∈ I(r).
∴ h(a) $\notin$ I(m).
$D^{'}$ is obtained by introducing $\bar{a}$ in c(m) [where c(m) is the counterpart region of m in $D^{'}$].
Since h(a) $\notin$ I(m) therefore h(a) $\notin$ I(c(m))[as I(m) = I(c(m))].
∴ M $\Vdash$ $D^{'}$.
∴ D $\models$ $D^{'}$. □

(iii) If D $\vdash$ $D^{'}$ by introduction of x then D $\models$ $D^{'}$.

Proof. Case-(i):
Let r be a region of D containing a-sequence.
Let M $\Vdash$ D, then h(a) ∈ I(r).
$D^{'}$ is obtained by introducing x-sequence in c(r), where c(r) is the counterpart region of r in $D^{'}$.
Since I(r) = I(c(r)).
∴ h(a) ∈ I(c(r)). Thus I(c(r)) ≠ ϕ.
∴ M $\Vdash$ $D^{'}$.

$\therefore$ D $\models$ D$^{'}$.

Case-(ii):

Let D be any diagram.

D$^{'}$ is obtained by introducing a x-sequence in D such that each minimal region of D has a node of the x-sequence.

Then as universe is non-empty we get D $\models$ D$^{'}$. □

(iv) If D $\vdash$ D$^{'}$ by extension rule for lcc and lci then D $\models$ D$^{'}$.

Proof. Case-(i):

Let r be a region in D containing a x-sequence.

Let r be the region $m_1 + m_2 + \cdots + m_{n-1}$ where each m_i $[1 \leq i \leq n-1]$ has a node of the x-sequence.

Without loss of generality we may assume that the x-sequence has been extended to only one minimal region m_n.

Let M $\Vdash$ D.

Then I(r) $\neq \phi$.

$\therefore$ $I(m_1 + m_2 + \cdots + m_{n-1}) \neq \phi$.

$\therefore$ $I(m_1) \neq \phi$ or $I(m_2) \neq \phi$ or $\ldots$ or $I(m_{n-1}) \neq \phi$.

D$^{'}$ is obtained by introducing x to a minimal region $c(m_n)$ [where $c(m_i)$ is the counterpart region of m_i in D$^{'}$].

Since, $I(m_i) = I(c(m_i))$.

$\therefore$ $I(c(m_1)) \neq \phi$ or $I(c(m_2)) \neq \phi$ or$\ldots$or $I(c(m_{n-1})) \neq \phi$.

$\therefore$ $I(c(m_1) + c(m_2) + \cdots + c(m_{n-1})) \neq \phi$.

$\therefore$ $(I(c(m_1) + c(m_2) + \cdots + c(m_{n-1})) + I(c(m_n))) \neq \phi$.

$\therefore$ $I(c(m_1) + c(m_2) + \cdots + c(m_{n-1}) + c(m_n)) \neq \phi$.

Thus M $\Vdash$ D$^{'}$.

$\therefore$ D $\models$ D$^{'}$.

Case-(ii):

Let r be a region in D containing an a-sequence.

Let r be the region $m_1 + m_2 + \cdots + m_{n-1}$ where each m_i $[1 \leq i \leq n-1]$ has a node of the a-sequence.

Let M $\Vdash$ D. D$^{'}$ is obtained by introducing an a to a minimal region $c(m_n)$ [where $c(m_i)$ is the counterpart region of m_i in D$^{'}$]. Then by using the similar method used in case-(i) we get M $\Vdash$ D$^{'}$. □

(v) If D $\vdash$ D$^{'}$ by extension rule for components then D $\models$ D$^{'}$.

Proof. Let D$^{'}$ $\equiv$ D $- D_1 - D_2 - \cdots - D_n$.

If M $\Vdash$ D then M $\Vdash$ D$^{'}$(follows directly from Definition 5.2).

$\therefore$ D $\models$ D$^{'}$. □

(vi) If D $\vdash$ D$^{'}$ by elimination of entire sequence, shading and $\bar{a}$ then D $\models$ D$^{'}$.

Proof. Case-(a):

Let D contain only one diagrammatic object i.e. a x-sequence or an a-sequence or shading or $\bar{a}$.

If $D^{'}$ is obtained from D by elimination of entire sequence or shading or $\bar{a}$ then $D^{'}$ does not contain any diagrammatic object.

Then vacuously $D^{'}$ is true in any model.

Thus $D \models D^{'}$.

Case-(b):

Let D contain more than one diagrammatic object.

If $D^{'}$ is obtained from D by elimination of some diagrammatic objects then the other diagrammatic objects of D are present in $D^{'}$ in exactly the corresponding counterpart regions.

So if $M \Vdash D$ then $M \Vdash D^{'}$.

Thus $D \models D^{'}$. □

(vii) If $D \vdash D^{'}$ by elimination of part of sequence then $D \models D^{'}$.

Proof. Elimination of part of a x-sequence is permitted if the eliminated nodes fall within a shaded region r.

Case-(a):

Let $m_1, m_2, \ldots, m_k, m_{k+1}, \ldots, m_n$ be the minimal regions of D containing all the nodes of a x-sequence such that $m_1 + m_2 + \cdots + m_k \subseteq r$, $m_{k+1} + m_{k+2} + \cdots + m_n \nsubseteq r$ and let r be shaded.

Let $M \Vdash D$. Then $I(m_1 + m_2 + \cdots + m_k + m_{k+1} + m_{k+2} + \cdots + m_n)) \neq \phi$ and $I(r) = \phi$.

Since $I(r) = \phi$, thus $I(m_1 + m_2 + \cdots + m_k) = \phi$ and $I(m_{k+1} + m_{k+2} + \cdots + m_n) \neq \phi$.

Let $D^{'}$ be obtained by elimination of nodes of x-sequence from m_1, m_2, …, m_k.

Let $c(r)$, $c(m_1)$, $c(m_2)$,…, $c(m_n)$ be counterpart regions of r,m_1, m_2, …, m_n in $D^{'}$ respectively.

Then $c(r)$ is shaded and since $c(m_1)+c(m_2)+\cdots+ c(m_k) \subseteq c(r)$ thus $c(m_1)$, $c(m_2)$,…, $c(m_k)$ are shaded and $c(m_{k+1})$, $c(m_{k+2})$,…, $c(m_n)$ contains the nodes of x-sequence.

Now $I(m_i) = I(c(m_i))$.

$\therefore I(c(m_1)+c(m_2)+\cdots+ c(m_k)) = \phi$ and $I(c(m_{k+1})+c(m_{k+2})+\cdots+ c(m_n)) \neq \phi$.

$\therefore M \Vdash D^{'}$.

$\therefore D \models D^{'}$.

[see Example 3.11]

Elimination of part of an a-sequence is permitted if the eliminated nodes fall within a shaded region r.

Case-(b):

Let $m_1, m_2, \ldots, m_k, m_{k+1}, \ldots, m_n$ be the minimal regions of D containing all the nodes of an a-sequence such that $m_1 + m_2 + \cdots + m_k \subseteq r$, $m_{k+1} + m_{k+2} + \cdots + m_n \nsubseteq r$ and let r be shaded.

Let $M \Vdash D$. Then $h(a) \in I(m_1 + m_2 + \cdots + m_k + m_{k+1} + m_{k+2} + \cdots + m_n))$ and $I(r) = \phi$.

Since $I(r) = \phi$, thus $I(m_1 + m_2 + \cdots + m_k) = \phi$ and $h(a) \in I(m_{k+1} + m_{k+2} + \cdots + m_n)$.

Let D' be obtained by elimination of nodes of a-sequence from $m_1, m_2, \ldots, m_k$.

Let $c(r), c(m_1), c(m_2), \ldots, c(m_n)$ be counterpart regions of $r, m_1, m_2, \ldots, m_n$ in D' respectively.

Then $c(r)$ is shaded and since $c(m_1)+c(m_2)+\cdots+c(m_k) \subseteq c(r)$ thus $c(m_1), c(m_2), \ldots, c(m_k)$ are shaded and $c(m_{k+1}), c(m_{k+2}), \ldots, c(m_n)$ contains the nodes of a-sequence.

Now $I(m_i) = I(c(m_i))$.

$\therefore I(c(m_1)+c(m_2)+\cdots+c(m_k)) = \phi$ and $h(a) \in I(c(m_{k+1})+c(m_{k+2})+\cdots+c(m_n))$.

$\therefore M \Vdash D'$.

$\therefore D \models D'$.

Case-(c):

Let $m_1, m_2, \ldots, m_n$ be the minimal regions of D containing nodes of an a-sequence and some of them say, m_i has $\bar{a}$ also.

Let $M \Vdash D$. Then $h(a) \notin I(m_i)$ and $h(a) \in I(m_1)$ $\mathfrak{or}$ $h(a) \in I(m_2)$ $\mathfrak{or} \ldots \mathfrak{or}$ $h(a) \in I(m_{i-1})$ $\mathfrak{or}$ $h(a) \in I(m_{i+1})$ $\mathfrak{or} \ldots \mathfrak{or}$ $h(a) \in I(m_n)$.

Since $h(a) \notin I(m_i)$,

$\therefore h(a) \in I(m_1 + m_2 + \cdots + m_n)$.

Let D' be obtained by elimination of a from m_i.

Let $c(m_1), c(m_2), \ldots, c(m_n)$ be counterpart regions of $m_1, m_2, \ldots, m_n$ in D' respectively.

Then $c(m_i)$ has $\bar{a}$ and $c(m_1), c(m_2), \ldots, c(m_{i-1}), c(m_{i+1}), \ldots, c(m_n)$ contains the nodes of a-sequence.

Now $I(m_j) = I(c(m_j))$.

$\therefore h(a) \notin I(c(m_i))$ and $h(a) \in I(c(m_1) + c(m_2) + \cdots + c(m_{i-1}) + c(m_{i+1}) + \cdots + c(m_n))$.

$\therefore M \Vdash D'$.

$\therefore D \models D'$.

□

(viii) If $D \vdash D'$ by elimination of a closed curve then $D \models D'$.

Proof. Let D' be obtained from D by elimination of a closed curve. Then for every region r in D' there is a counterpart region $c(r)$ in D.

If possible let $D \not\models D'$, then there exists a model M such that $M \Vdash D$ but $M \nVdash D'$.

(i) Let r be a region in $D^{'}$ such that it has shading but $I(r) \neq \phi$.

Therefore, $I(c(r)) \neq \phi$.

But since any shading is not added in $D^{'}$, it must be the case that c(r) in D is shaded.

As $M \Vdash D$, $I(c(r)) = \phi$, a contradiction.

(ii) Let r be a region in $D^{'}$ such that it has x-sequence but $I(r) = \phi$.

Therefore, $I(c(r)) = \phi$.

But since we did not add any x-sequence in $D^{'}$, it must be the case that there exists a region $r^{'}$ in D such that $r^{'} \subseteq c(r)$and $r^{'}$ has a x-sequence.

As $M \Vdash D$, $I(r^{'}) \neq \phi$. Hence $I(c(r)) \neq \phi$, a contradiction.

(iii)Let r be a region in $D^{'}$ such that it has a-sequence but $h(a) \notin I(r)$.

Therefore, $h(a) \notin I(c(r))$.

But since we did not add any a-sequence in $D^{'}$, it must be the case that there exists a region $r^{'}$ in D such that $r^{'} \subseteq c(r)$ and $r^{'}$ has an a-sequence.

As $M \Vdash D$, $h(a) \in I(r^{'})$. Hence $h(a) \in I(c(r))$, a contradiction.

(iv) Let m be a minimal region in $D^{'}$ such that it has $\bar{a}$ but $h(a) \in I$ (m).

Therefore, $h(a) \in I(c(m))$ [c(m) is the region in D which is the union of two minimal regions separated by the elimination of closed curve].

But since we did not add any $\bar{a}$ in $D^{'}$, it must be the case that $\bar{a}$ is in the two minimal regions constituting c(m).

As $M \Vdash D$, $h(a) \notin I(c(m))$, a contradiction.

Thus for all M, if $M \Vdash D$ then $M \Vdash D^{'}$.

Hence $D \models D^{'}$. □

(ix) If $D \vdash D^{'}$ by inconsistency rule then $D \models D^{'}$.

Proof. Case-(a):

Suppose $D^{'}$ follows from D by the rule of inconsistency where D is of type-I/II.

$D \models D^{'}$ does not holds if and only if there exists a model M such that $M \Vdash D$ but $M \nVdash D^{'}$.

But since D is inconsistent, it follows from Definition 3.1 that it is false in every model. So there is no M such that $M \Vdash D$.

Thus $D \models D^{'}$ (vacuously).

Case-(b):

Let $D \equiv D_1 - D_2 - \cdots - D_n$, where one of the components, say D_i is an inconsistent diagram and $D^{'}$ is obtained by eliminating D_i.

Thus $D^{'} \equiv D_1 - D_2 - \cdots - D_{i-1} - D_{i+1} - \cdots - D_n$.

Let M be a model such that $M \Vdash D$.

$\therefore M \Vdash D_1 - D_2 - \cdots - D_n$ i.e. at least one of $D_1, D_2, \ldots, D_n$ is true in M.

But there is no M such that $M \Vdash D_i$.

$\therefore$ at least one of $D_1, D_2, \ldots, D_{i-1}, D_{i+1}, \ldots, D_n$ is true in M.

$\therefore M \Vdash D_1 - D_2 - \cdots - D_{i-1} - D_{i+1} - \cdots - D_n$.

$\therefore$ M $\Vdash$ D$'$.
$\therefore$ D $\models$ D$'$.

□

(x) If $\{D_1, D_2\} \vdash D'$ by unification rule then $\{D_1, D_2\} \models D'$.

Proof. Case-(a): For type-I/type-II diagram.

Let D_1 and D_2 be united in one diagram D'.

Let M $\Vdash$ D_1 and M $\Vdash$ D_2.

Then the counterparts of all the basic regions of D_1 and D_2 are present in D'.

Also all the diagrammatic objects of D_1 and D_2 are present in D' maintaining respective positions.

Thus M $\Vdash$ D'.

Case-(b): For type-III diagram with type-I/type-II diagram.

Let D_1 be a type-III diagram $D_{11} - D_{12}$ and D_2 be a type-II diagram.

D_1 and D_2 are united in one diagram D'.

Now D' is the diagram Uni(D_{11}, D_2) $-$ Uni(D_{12}, D_2).

Let M $\Vdash$ D_1 and M $\Vdash$ D_2.

Since M $\Vdash$ D_1 then either M $\Vdash$ D_{11} or M $\Vdash$ D_{12}.

Without loss of generality we can assume that M $\Vdash$ D_{11}.

Then because of the assumption that M $\Vdash$ D_2 and case-(a),M $\Vdash$ Uni(D_{11}, D_2).

So, M $\Vdash$ Uni(D_{11}, D_2) $-$ Uni(D_{12}, D_2).

Case-(c): For type-III diagram with type-III diagram.

Let $D_1 \equiv D_{11} - D_{12}$ and $D_2 \equiv D_{21} - D_{22}$ are united in one diagram D'.

Now D' is the diagram Uni(D_{11}, D_{21}) $-$ Uni(D_{12}, D_{21}) $-$ Uni(D_{11}, D_{22}) $-$ Uni(D_{12}, D_{22}).

Let M $\Vdash$ D_1 and M $\Vdash$ D_2.

Without loss of generality we can assume that M $\Vdash$ D_{11} and M $\Vdash$ D_{21}.

So, M $\Vdash$ Uni(D_{11}, D_{21}).

So,M $\Vdash$ D'. □

(xi) If D $\vdash$ D' by rule of splitting sequences then D $\models$ D'.

Proof. Let a region r in D have a x-sequence.

Let m_1, m_2, ..., m_n be the minimal regions within r, containing one node each of the x-sequence.

Let D' be obtained from D by the rule of splitting sequences.

$\therefore$ D' is the diagram $D'_1 - D'_2 - D'_3 - \ldots - D'_n$, where the counterpart of m_1, m_2, ..., m_n contains only one x.

Let M $\Vdash$ D.

$\therefore$ $I(m_1) \neq \phi$ or $I(m_2) \neq \phi$ or ... or $I(m_n) \neq \phi$.

$\therefore$ $I(c(m_1)) \neq \phi$ or $I(c(m_2)) \neq \phi$ or $\ldots$ or $I(c(m_n)) \neq \phi$ [where $c(m_1)$, $c(m_2),\ldots, c(m_n)$ are counterpart regions of $m_1, m_2, \ldots, m_n$ in D' respectively].
$\therefore$ $M \Vdash D'_1$ or $M \Vdash D'_2$ or $\ldots$ or $M \Vdash D'_n$.
$\therefore$ $M \Vdash D'$.
$\therefore$ $D \models D'$.
Let a region r in D have an a-sequence.

Let $m_1, m_2, \ldots, m_n$ be minimal regions within r, containing one node each of the a-sequence.

Let D' be obtained from D by the rule of splitting sequences.

$\therefore$ D' is the diagram $D'_1 - D'_2 - D'_3 - \ldots - D'_n$, where the counterparts of $m_1, m_2, \ldots, m_n$ contain only one a each.

Let $M \Vdash D$. Then by using the similar method used in the previous case we get $M \Vdash D'$.

$\therefore$ $D \models D'$. □

(xii) If $D \vdash D'$ by rule of excluded middle then $D \models D'$.

Proof. (a)Let m be a minimal region in D which has no diagrammatic objects.

Let D' be obtained from D by the rule of excluded middle. Thus D' is the diagram $D'_1 - D'_2$. We have two cases (i) and (ii).

(i) D'_1 and D'_2 have all the diagrammatic objects of D except that c(m) in D'_1 is shaded and c(m) in D'_2 has x [where c(m) is the counterpart of the region m].

Let $M \Vdash D$.
Then I(m) is either empty **or** non-empty (D' is tautologous at m w.r.t x).
Then $M \Vdash D'_1$ **or** $M \Vdash D'_2$.
$\therefore$ $M \Vdash D'$.
$\therefore$ $D \models D'$.

(ii) D'_1 and D'_2 have all the diagrammatic objects of D except that c(m) in D'_1 has a and c(m) in D'_2 has $\bar{a}$ [where c(m) is the counterpart of the region m].

Let $M \Vdash D$.
Then I(m) has either a **or** $\bar{a}$ (D' is tautologous at m w.r.t a) [for all a].
Then $M \Vdash D'_1$ **or** $M \Vdash D'_2$.
$\therefore$ $M \Vdash D'$.
$\therefore$ $D \models D'$.

(b) Let m be a minimal region in D which has x.

Let D' be obtained from D by the rule of excluded middle. Thus D' is the diagram $D'_1 - D'_2$, where D'_1 and D'_2 has all the diagrammatic objects of D except that c(m) in D'_1 has a and c(m) in D'_2 has $\bar{a}$ [where c(m) is the counterpart of the region m].

Let $M \Vdash D$, then $I(m) \neq \phi$.
Then I(m) has either a or $\bar{a}$ (tautologous at m w.r.t a) [for all a].

$\therefore$ M $\Vdash$ D$^{'}$.
$\therefore$ D $\models$ D$^{'}$.

□

(xiii) If D $\vdash$ D$^{'}$ by rule of construction then D $\models$ D$^{'}$.

Proof. Let D $\equiv$ D$_1$ − D$_2$ − $\cdots$ − D$_n$.
We obtain D$^{'}$ by rule of construction. So, D$_i$ $\vdash$ D$^{'}$ (by other rules) i=1,...,n.
Let M $\Vdash$ D.
So at least one component,say D$_i$, is true in M, that is M $\Vdash$ D$_i$.
So, M $\Vdash$ D$^{'}$.
$\therefore$ D $\models$ D$^{'}$. □

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Received: July 25, 2017.
Accepted: January 27, 2018.