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Inconsistency-Adaptive Dialogical Logic

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Abstract. Even when inconsistencies are present in our premise set, we can sensibly distinguish between good and bad arguments relying on these premises. In making this distinction, the inconsistency-adaptive approach of Batens strikes a particularly nice balance between inconsistency-tolerance and inferential strength. In this paper, we use the machinery of Batens' approach to extend the paraconsistent approach to dialogical logic as developed by Rahman and Carnielli. In bringing these frameworks closer together, we obtain a dynamic mechanism for the systematic study of dialogues in which two parties exchange arguments over a central claim, in the possible presence of inconsistencies.

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1. Introduction

In classical logic (**CL**) a contradiction trivializes any premise set. Yet many have argued that we regularly face inconsistencies in our argumentative practices [11,15,16,20,21], and we do not just stop arguing in the face of an inconsistency. Rather, we reason on despite the presence of inconsistencies, relying on the remaining information at hand. A wide variety of logics have been devised for representing such reasoning in the presence of inconsistencies.

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They are characterized by the invalidation of the *Ex Falso Sequitur Quodlibet* (EFSQ) principle, according to which any formula ψ is derivable from two formulas φ and $\neg\varphi$. These logics are called *paraconsistent* logics.

We are interested here in two particular frameworks in which a number of paraconsistent logics have been defined: dialogical logic and adaptive logic. Our aim is to bring both frameworks closer together by combining some of their key features into a new system. The framework of dialogical logic we refer to has its roots in the work of Lorenzen and Lorenz [18,19], and more recently the work of Rahman and his collaborators, see e.g. [12–14,17,22,25]. Dialogical logic is an alternative semantics that is neither model-theoretic nor proof-theoretic. Rather, the meaning of the logical connectives is given in terms of their use in argumentative practices. Dialogical logic is conceived as a game between a *Proponent* and an *Opponent*. The Proponent asserts an initial thesis, which the Opponent attacks. If the Proponent is able to defend her thesis irrespective of the attacks made by the Opponent, then the Proponent has a winning strategy in the dialogical game and her thesis is valid. We explain the basics of dialogical logic in Sect. 2.

In recent years, dialogical logic has taken a pluralist turn. In [23] for instance, Rahman and Carnielli present two paraconsistent dialogical logics obtained by modifying the manner in which players can attack negated statements in dialogues. The first of these logics, called $\mathbf{L-D}$, is only partially successful in avoiding the validity of EFSQ: inconsistencies between complex formulas still cause explosion in $\mathbf{L-D}$. The second paraconsistent dialogical logic, called $\mathbf{D}+$, fares better in this respect. However, both logics are rather weak in terms of inferential power. For instance, even in the absence of inconsistencies they invalidate applications of rules such as Disjunctive Syllogism (DS), Contraposition (CP), and Modus Tollens (MT).

As observed by Van Bendegem and Rahman [27,30] the unconditional invalidity of the likes of DS, CP, and MT is a high price to pay for tolerating inconsistencies. A remedy anticipated by Batens and Van Bendegem [7,30] is to incorporate techniques from the adaptive logics framework within the dialogical semantics. Adaptive logics are tools for modeling defeasible reasoning. We are particularly concerned with inconsistency-adaptive logics, which were originally developed for representing reasoning in the presence of inconsistencies [3]. Using methods stemming from the adaptive logics framework, we show how the paraconsistent dialogician can have her cake and eat it too by defining a system weak enough to successfully avoid logical explosion in the presence of inconsistencies, yet strong enough to validate applications of DS, DP, MT and the like in the absence of inconsistencies. The result is a system called *inconsistency-adaptive dialogical logic* (**IAD**).

In preparation of **IAD** we define a simple basic paraconsistent dialogical logic, **LLD**, inspired by Rahman and Carnielli's $\mathbf{L-D}$ (Sect. 3). Next, we strengthen **LLD** by allowing for conditional moves in dialogues, which are subject to further justification. The resulting inconsistency-adaptive dialogical logic, **IAD**, is defined and illustrated in Sect. 4. In Sect. 5, we compare **IAD** with existing approaches to paraconsistency in dialogical logic (Sect. 5.1) and in adaptive logic (Sect. 5.2). An important technical result is shown in Sect. 5.2, namely the correspondence—at least as far as finite premise sets are concerned—between **IAD** and the inconsistency-adaptive logic $\mathbf{CLuN^r}$ from [8–10]. We end the paper with some further illustrations of how adaptive logicians and dialogical logicians can mutually benefit from a closer collaboration (Sect. 6).

2. Dialogical Logic

In this section we provide a basic introduction to dialogical **CL**. We begin by presenting some basic definitions (Sect. 2.1). There are two kinds of rules for dialogical logic: particle rules and structural rules. The particle rules, which we give in Sect. 2.2, describe how players can attack and defend earlier moves, depending on the logical form of the formulas asserted in the dialogue. The structural rules, given in Sect. 2.3 for **CL**, subject plays in dialogical games to further conditions. In order to define a dialogical notion of validity, we also need the concept of a winning strategy in a dialogical game. This concept is defined in Sect. 2.4.[1]

2.1. Basic Definitions

Let $\mathcal{L}$ be a propositional language, defined as follows:

$\varphi ::= \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \neg\varphi$

Lower case letters $p, q, r, \ldots$ refer to atomic formulas in $\mathcal{L}$. We use lower case Greek letters $\varphi, \psi, \chi \ldots$ to refer to $\mathcal{L}$-formulas, and upper case Greek letters $\Gamma, \Sigma, \Delta, \ldots$ to refer to *finite* sets of $\mathcal{L}$-formulas.[2] To define the structural rules, we will make use of two labels, **P** and **O**, standing for the players of the games, the *Proponent* and the *Opponent* respectively. The identities of **P** and **O** are not relevant at the local level.[3] That is why to define the particle rules we will make use of player variables **X** and **Y** (with $\mathbf{X} \neq \mathbf{Y}$).

Dialogues are defined by means of two kinds of speech-acts: assertions and requests. We will use force symbols '!' for assertions and '?' for requests. A *move* is an expression of the form **X** - e where **X** is a player variable and e is either an assertion or a request.

We use $n := r_i$ and $m := r_j$ with $r_i, r_j \in \mathbb{N}^*$ for the utterance of the rank the players choose according to the rule [SR0] given in Sect. 2.3. Ranks are positive integers bounding the number of attacks and defences the players can perform in a play.

A *play* is a sequence of moves which complies with the game rules. In the dialogues we present, the initial thesis $\psi[\varphi_1, \ldots, \varphi_n]$ amounts to the claim

[1] Throughout Sect. 2 we rely on the definitions of [12], although we sometimes make some minor notational changes for the sake of uniformity. For a less dense introduction to dialogical **CL**, see [25].

[2] In Sect. 6.3 we show how countably infinite premise sets can also be dealt with within our framework.

[3] The identities of **P** and **O** will be defined by means of the structural rule [SR0] given in Sect. 2.3.

that there is a winning strategy for the conclusion ψ given the concession of $\varphi_1, \ldots, \varphi_n$. The premises $\varphi_1, \ldots, \varphi_n$ are referred to as the initial concessions. In case the premise set is empty, the initial thesis is simply ψ. The *dialogical game* for a claim $\psi[\varphi_1, \ldots, \varphi_n]$ (respectively ψ) is the set $\mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) of all the plays with $\psi[\varphi_1, \ldots, \varphi_n]$ (respectively ψ) as the initial thesis.[4]

For every move M in a given sequence $\mathcal{S}$ of moves, $p_{\mathcal{S}}(M)$ denotes the position of M in $\mathcal{S}$. Positions are counted starting with 0. We will also use a function F such that the intended interpretation of $F_{\mathcal{S}}(M) = [m', Z]$ is that in the sequence $\mathcal{S}$, the move M is an attack (if $Z = A$) or a defence (if $Z = D$) against the move of previous position m'.

2.2. Particle Rules

The particle rules for propositional dialogical logic are defined in the following table:

Assertion	Attack	Defence
$\mathbf{X}$ -$!\varphi \wedge \psi$	$\mathbf{Y}$ - $?\wedge_L$	$\mathbf{X}$ - $!\varphi$
	or	or
	$\mathbf{Y}$ - $?\wedge_R$	$\mathbf{X}$ - $!\psi$ respectively
$\mathbf{X}$ - $!\varphi \vee \psi$	$\mathbf{Y}$ - $?\vee$	$\mathbf{X}$ - $!\varphi$ or $\mathbf{X}$ - $!\psi$
$\mathbf{X}$ - $!\neg\varphi$	$\mathbf{Y}$ - $!\varphi$	$- - -$
		No Defence
$\mathbf{X}$ - $!\varphi \rightarrow \psi$	$\mathbf{Y}$ - $!\varphi$	$\mathbf{X}$ - $!\psi$
$\mathbf{X}$ - $!\psi[\varphi_1, \ldots, \varphi_n]$	$\mathbf{Y}$ - $!\varphi_1$	$\mathbf{X}$ - $!\psi$
	$\ldots$	
	$\mathbf{Y}$ - $!\varphi_n$	

Particle rules are abstract descriptions consisting of sequences of moves such that the first member of that sequence is an assertion, the second is an attack and the third is a defence against the attack (except in the case of negation, for which there is no possible defence). They are abstract because they are defined independently of any specific context of argumentation and independently of the players' identities. When a player asserts a conjunction, she (**X**) is committed to give a justification for both conjuncts. That is why the attacker (**Y**) is allowed to perform a request by means of which she chooses which conjunct (left or right) to defend. In the case of a disjunction it is the defender (**X**) who chooses. Indeed, an agent uttering a disjunction is committed to give a justification for (at least) one of the disjuncts. An attack may be a request or an assertion (in the case of the negation) or even a composite speech-act (in the case of the conditional or a formula of the form $\psi[\varphi_1, \ldots, \varphi_n]$).

[4] This notational device, suggested to us by an anonymous referee, has its roots in constructive type theory, where the use of hypotheses is indicated between brackets to the right of the claim. For more details on the relation of dialogical logic to constructive type theory, see [24]. Where $\Sigma = \{\varphi_1, \ldots, \varphi_n\}$, we will sometimes write $[\Sigma]$ instead of $[\varphi_1, \ldots, \varphi_n]$, for the sake of presentation.

Throughout the paper, we will denote the absence of possible defences with dashes, like in the table above.

2.3. Structural Rules

The structural rules we now define provide the global or structural level of semantics specifying the context of argumentation:

[SR0][Starting rule]. [(i) If the initial thesis is of the form $\psi[\varphi_1, \ldots, \varphi_n]$, then for any play $\mathcal{P} \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ we have:
(a) $p_{\mathcal{P}}(\mathbf{P}$ - $!\psi[\varphi_1, \ldots, \varphi_n])= 0$
(b) $p_{\mathcal{P}}(\mathbf{O}$ - $n := r_1)= 1$ and $p_{\mathcal{P}}(\mathbf{P}$ - $m := r_2)= 2$.
(ii) If the initial thesis is of the form ψ, then for any play $\mathcal{P} \in \mathcal{D}(\psi)$ we have:
(a') $p_{\mathcal{P}}(\mathbf{P}$ - $!\psi)= 0$
(b') $p_{\mathcal{P}}(\mathbf{O}$ - $n := r_1) = 1$ and $p_{\mathcal{P}}(\mathbf{P}$ - $m := r_2) = 2$.]

Clause (a) (respectively (a')) warrants that every play in $\mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) starts with $\mathbf{P}$ asserting the thesis $\psi[\varphi_1, \ldots, \varphi_n]$ (respectively ψ). In clause (b) (respectively (b')) the players choose their respective repetition ranks[5] among the positive integers. We recall that a rank is a positive integer bounding the number of attacks and defences which the players can perform in a play. For strategic reasons it is sufficient to consider the case in which the Opponent chooses rank 1 and the Proponent rank 2 (see Sect. 2.4).

[SR1][Classical development rule]. [For any move M in $\mathcal{P}$ such that $p_{\mathcal{P}}(M) > 2$ we have $F_{\mathcal{P}}(M) = [m', Z]$, where $Z \in \{A, D\}$ and $m' < p_{\mathcal{P}}(M)$. Let r be the repetition rank of player $\mathbf{X}$ and $\mathcal{P} \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) such that:
- the last member of $\mathcal{P}$ is a $\mathbf{Y}$-move,
- $M_0 \in \mathcal{P}$ is a $\mathbf{Y}$-move of position m_0,
- there are n moves $M_1, \ldots, M_n$ of player $\mathbf{X}$ in $\mathcal{P}$ such that $F_{\mathcal{P}}(M_1) = F_{\mathcal{P}}(M_2) = \cdots = F_{\mathcal{P}}(M_n) = [m_0, Z]$ with $Z \in \{A, D\}$.

Let N be an $\mathbf{X}$-move such that $F_{\mathcal{P}\frown N}(N) = [m_0, Z]$. Then $\mathcal{P} \frown N \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) if and only if $n < r$.[6]]

[SR1] ensures that after the repetition ranks have been chosen, every move either attacks or defends against a previous move made by the other player; players move alternately, and the number of attacks and defences they can perform in reaction to a same move is bounded by their repetition ranks.

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[5] A move M' performed by $\mathbf{X}$ in a dialogue is a *repetition* of a previous move M if (i) M and M' are two attacks performed by $\mathbf{X}$ against the same move N performed by $\mathbf{Y}$, or (ii) M and M' are two defences performed by $\mathbf{X}$ in response to the same attack N performed by $\mathbf{Y}$.
The ranks guarantee the finiteness of plays by limiting the repetitions allowed in a dialogue. We follow the rule formulated by Clerbout in [12, p. 788] in which the rank chosen by the players applies uniformly to the whole dialogue, and for defences as well as for attacks. By contrast, in Lorenz [18], players may choose different ranks for attacks and defences respectively.

[6] "$\mathcal{P} \frown N$" denotes the extension of $\mathcal{P}$ with N.

[SR2][Formal rule]. [The sequence $\mathcal{S}$ is a play only if the following condition is fulfilled: if $N = \mathbf{P}$ - $!\psi$ is a member of $\mathcal{S}$, for any atomic sentence ψ, then there is a move $M = \mathbf{O}$ - $!\psi$ in $\mathcal{S}$ such that $p_{\mathcal{S}}(M) < p_{\mathcal{S}}(N)$.]

This rule means that **P** can assert an atomic sentence ψ only if **O** previously asserted the same atomic sentence ψ. To define the last structural rule, we need the following definition:

[D1][X-terminal]. [Let $\mathcal{P}$ be a play in $\mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) the last member of which is an **X**-move. If there is no **Y**-move N such that $\mathcal{P}^\frown N \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) then $\mathcal{P}$ is said to be **X**-terminal.]

[SR3][Winning rule for plays]. [Player **X** wins a play $\mathcal{P} \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) if and only if $\mathcal{P}$ is **X**-terminal.]

According to [SR3], **X** wins a play if it is **Y**'s turn to play and no move is available to **Y**.

2.4. Strategy and Validity

The rules of the game do not say anything about validity or how to play. Dialogical validity is grasped at the strategic level. The thesis of **P** is valid if and only if **P** has a winning strategy according to the following definition:

[D2][Strategy]. [A strategy of a player **X** in $\mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) is a function s_x which assigns a legal **X**-move to every non terminal play $\mathcal{P} \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) the last member of which is a **Y**-move.

A **X**-strategy is winning if it leads to **X**'s win no matter how **Y** plays.]

On the basis of the definition of winning strategy, we can define the notion of consequence for dialogical **CL**; that is, a dialogical logic respecting [SR0]–[SR3], the so-called **CL**-rules:

[D3][CL-consequence]. [$\Sigma \vdash_{\mathbf{CL}} \varphi$ (respectively $\vdash_{\mathbf{CL}} \varphi$) iff according to the **CL**-rules, there is a **P**-winning strategy for the thesis $\varphi[\Sigma]$ (respectively φ).]

Before we turn to an example, one more comment is in order. Clerbout [12, p. 791] showed that there is a **P**-winning strategy when **O** chooses rank 1 if and only if there is a **P**-winning strategy for any other choice of **O**. Moreover, if **O** chooses rank 1 then if there is a **P**-winning strategy, **P** never has to choose a rank higher than 2 in order to win a dialogue. In some cases, **P** needs rank 2 because she needs the concession of both conjuncts of a conjunction asserted by **O** in order to win. By contrast, if there is an **O**-winning strategy, then **O** can win already by picking rank 1. (This is linked to the fact that **O** always chooses her rank first and that she does not play under the formal restriction.) For these reasons, we will use ranks 1 and 2 for **O** and **P** respectively for all illustrations in Sects. 2 and 3.

Let us now show that there is a **P**-winning strategy for $q[p, \neg p]$, an instance of (EFSQ). In the following table, representing a dialogue, the numbers of moves are indicated in the outer columns. The numbers in the inner columns indicate which moves are attacked. The **O**-moves are in the **O**-column and the

P-moves in the **P**-column. For reasons of transparency, we will systematically omit the force symbol '!' in our examples.

	O			P	
				$q[p, \neg p]$	0
1	$n := 1$			$m := 2$	2
3.1	p	0			
3.2	$\neg p$				
	$---$		3.2	p	4

Explanation We start the dialogue in with [SR0]. In move 3, **O** attacks **P**'s initial thesis. **P** cannot defend herself against this attack because she is not allowed to assert q, an atomic sentence, if **O** did not concede q before (see [SR2]). The only move available to **P** is to counter-attack the move 3.2, which is what **P** does in move 4. After this move, there is no more possible move for **O**, and **P** wins the dialogue.

There is also a **P**-winning strategy for $q[p \vee q, \neg p]$, an instance of DS:

	O			P	
				$q[p \vee q, \neg p]$	0
1	$n := 1$			$m := 2$	2
3.1	$p \vee q$	0		(q)	(6)
3.2	$\neg p$				
5	p (q)		3.1	$?\vee$	4
	$---$		3.2	p	6

Explanation To see that **P** has a winning strategy for this inference, we need to take into account two possible plays. Either **O** answers **P**'s attack in move 4 by asserting p in move 5, or **O** answers **P**'s attack in move 4 by asserting q in move 5. In the first case, **P** next attacks the formula $\neg p$ asserted by **O** in move 3.2, which she does in move 6. In the second case (indicated in between brackets), **P** replies to **O**'s attack in move 3 by asserting q in move 6. In both cases, **P** wins the play, so there is a **P**-winning strategy for the inference from $p \vee q$ and $\neg p$ to q.

3. Paraconsistent Dialogical Logic

In the introduction, we argued that there are good reasons for defining argumentative contexts in which the apparition of inconsistencies does not systematically cause explosion. Non-explosive logics are usually called paraconsistent logics. To get a paraconsistent dialogical logic, it is sufficient to add a restriction with respect to the use of the particle rule for negation by **P**. The idea is that **P** should not be allowed to attack a negation if **O** did not attack the same negated sentence before.[7] Such a modification occurs at the structural level. Indeed, the local meaning of negation is still the same, what is changed is the application of that rule in the particular context of paraconsistent dialogical logic. The other structural rules remain the same.

[SR4.1][Negation rule]. [The sequence $\mathcal{S}$ is a play only if the following condition is fulfilled: If there is a move $N_1 =$**P** - $!\psi$ in $\mathcal{S}$ such that:
1. $p_{\mathcal{S}}(N_1) = n_1$
2. $F_{\mathcal{S}}(N_1) = [m_1, A]$ and
3. $m_1 = p_{\mathcal{S}}(M_1)$ such that $M_1 =$**O** - $!\neg\psi$.

Then, there is a move $M_2 =$**O** - $!\psi$ in $\mathcal{S}$ such that:
1. $p_{\mathcal{S}}(M_2) = m_2$ and $m_2 < n_1$,
2. $F_{\mathcal{S}}(M_2) = [n_2, A]$ and
3. $n_2 = p_{\mathcal{S}}(N_2)$ such that $N_2 =$**P** - $!\neg\psi$.]

[SR4.1] ensures that **P** is allowed to attack a sentence of the form $\neg\varphi$ if and only if **O** has already attacked the same sentence $\neg\varphi$ before. Consequently there is no **P**-winning strategy for $q[p, \neg p]$, as is shown in the following dialogue:

	O			P	
				$q[p, \neg p]$	0
1	$n := 1$			$m := 2$	2
3.1	p	0			
3.2	$\neg p$				

Explanation With the standard rules, **P** won with an attack on the negation asserted by **O** in move 3.2. This move is not available to **P** anymore because according to [SR4.1] she is not allowed to attack an assertion of $\neg p$ if **O** did not attack the same formula before. So, the last move is an **O**-move and **P** cannot do anything. **O** wins.

A further consequence of [SR4.1] is that **P** has no winning strategy for $q[p \vee q, \neg p]$ either:

	O			P	
				$q[p \vee q, \neg p]$	0
1	$n := 1$			$m := 2$	2
3.1	$p \vee q$	0			
3.2	$\neg p$				
5	p		3.1	$?\vee$	4

[7] The idea is due to [23] although our approach is somewhat different. Our **LLD** (Lower Limit Dialogic) is different from Rahman and Carnielli's **L-D** (Literal Dialogues). In the latter, **P** is not allowed to challenge a negative literal (i.e., the negation of an atomic formula) if **O** has not previously challenged an occurrence of the same negative literal. **P** *is* allowed to challenge a *complex* negative formula even if **O** has not previously challenged any occurrence of the same complex negative formula. As a result, both systems have a slightly different set of theorems. See Sect. 5.1 for more details.

Explanation According to [SR4.1], **P** is not allowed to attack the negation asserted by **O** in move 3.2 and **O** wins. Notice that only one play is relevant to see that there is a winning strategy for **O**. In the dialogue for the same inference in **CL** (see Sect. 2.4) we showed that no matter the choices made by **O**, **P** would win. Here, choosing q in defence to move 4 would not have been optimal because it would have offered **P** the means to win.[8]

For reasons which will be clear in the next section, we call the dialogical logic enriched by means of [SR4.1] *Lower Limit Dialogic* (**LLD**).

[D4][LLD-consequence]. [$\Sigma \vdash_{\mathbf{LLD}} \varphi$ (respectively $\vdash_{\mathbf{LLD}} \varphi$) iff according to the **LLD**-rules there is a **P**-winning strategy for the thesis $\varphi[\Sigma]$ (respectively φ).]

In view of the **LLD**-validity of Modus Ponens (MP), the deduction theorem holds for **LLD**. That is, if $\Sigma \vdash_{\mathbf{LLD}} \varphi$, then $\vdash_{\mathbf{LLD}} \bigwedge \Sigma \rightarrow \varphi$. In Sect. 5.2, we show that, for finite premise sets, **LLD**-consequence corresponds to **CLuN**-consequence (see Theorem 1). The latter logic is a well-known paraconsistent logic devised by Batens.

As it stands, **LLD** is very weak. Indeed, while it invalidates EFSQ, it invalidates DS, Modus Tollens (MT), Contraposition (CP), Double Negation Elimination (DNE), Double Negation Introduction (DNI), as well as all other schemes of the following list, even if no inconsistency is involved:

$$p, \neg p \nvdash_{\mathbf{LLD}} q \tag{1}$$
$$p, \neg p \vee q \nvdash_{\mathbf{LLD}} q \tag{2}$$
$$\neg\neg p \nvdash_{\mathbf{LLD}} p \tag{3}$$
$$\neg(p \wedge q) \nvdash_{\mathbf{LLD}} \neg p \vee \neg q \tag{4}$$
$$\neg(p \vee q) \nvdash_{\mathbf{LLD}} \neg p \wedge \neg q \tag{5}$$
$$\neg(p \rightarrow q) \nvdash_{\mathbf{LLD}} p \wedge \neg q \tag{6}$$
$$p \nvdash_{\mathbf{LLD}} \neg\neg p \tag{7}$$
$$\neg p \vee \neg q \nvdash_{\mathbf{LLD}} \neg(p \wedge q) \tag{8}$$
$$\neg p \wedge \neg q \nvdash_{\mathbf{LLD}} \neg(p \vee q) \tag{9}$$
$$p \wedge \neg q \nvdash_{\mathbf{LLD}} \neg(p \rightarrow q) \tag{10}$$
$$p \rightarrow q \nvdash_{\mathbf{LLD}} \neg q \rightarrow \neg p \tag{11}$$
$$\neg q \rightarrow \neg p \nvdash_{\mathbf{LLD}} p \rightarrow q \tag{12}$$
$$p \rightarrow q, \neg q \nvdash_{\mathbf{LLD}} \neg p \tag{13}$$

Van Bendegem [30] argues that the restriction added in [SR4.1] is too strong, and that the dialogical approach to paraconsistency should be defined in a more flexible way. Following this line of thought, we propose a more

[8] Strictly speaking, since **P** chose rank 2, she is allowed to attack the disjunction in move 3.1 once more. Clearly, that wouldn't change much, since **O** can simply defend by stating p once more. The attack and defence would be identical to the ones already performed in moves 4 and 5. For reasons of space and transparency, we opted to leave such redundant moves out of the dialogue, and will continue doing so in the remainder of this paper.

dynamical approach in the next section, relying on the tools of inconsistency-adaptive logic.

4. Inconsistency-Adaptive Dialogical Logic

4.1. Reliable Formulas

In the possible presence of inconsistencies the question of whether or not a given inference rule is valid is not a simple yes/no question. In **CL** all of DS, MT, CP, DNE, and DNI are valid, and the result is a logic in which any premise set containing some inconsistency trivializes the consequence set. In **LLD** these rules are invalid, but the result is a logic that is too weak to capture how we actually reason. We need to find a middle way if we want to end up with a logic that is neither too weak nor too strong.

One way to end up in this Goldilocks zone of inconsistency-tolerant logics is to make the application of some rules of inference dependent on the behavior of the formulas to which we apply them. Consider, for instance, the following two applications of DS:

$$p, \neg p \vee q \vdash q \tag{14}$$

$$p, \neg p, \neg p \vee q \vdash q \tag{15}$$

In (14) the premises p and $\neg p \vee q$ behave consistently. In (15) the premises p and $\neg p$ do not. If our standard of deduction is **CL**, but we nonetheless want to tolerate inconsistency, then whenever the formulas to which a classically valid rule is applied behave consistently, no trouble arises. Problems occur only when inconsistencies are present, for then applying the rule in question may lead to triviality. For instance, allowing all applications of DS in a logic which also validates the Addition rule (from φ to infer $\varphi \vee \psi$) will immediately give rise to explosion whenever an inconsistency is present.[9] Thus, whether or not an application of DS is 'safe' depends to a large extent on how the premises of the argument behave. This suggests that we treat a rule like DS in a more dynamic way: an application of DS is safe whenever the formulas to which we apply the rule behave consistently, as in (14). An application of DS is unsafe whenever the formulas to which we apply the rule behave inconsistently, as in (15).

In order for a rule like DS to be safely applicable, however, it is not sufficient to demand that the formulas to which we apply it behave consistently. Consider the following case:

$$p, q, \neg p \vee \neg q, \neg p \vee r \vdash r \tag{16}$$

Here, we rely on p to infer r via DS. The formula p behaves consistently in the sense that there is no **P**-winning strategy for the conjunction $p \wedge \neg p$ given the concession of the premises. But in view of the premises of (16) it follows that either p behaves inconsistently or q does, since the disjunction $(p \wedge \neg p) \vee (q \wedge \neg q)$

[9] Given the formulas φ and $\neg\varphi$ we can infer $\varphi \vee \psi$ via Addition. Next, we can infer ψ from $\neg\varphi$ and $\varphi \vee \psi$ by DS. This works for any random formula ψ.

is an **LLD**-consequence of the premises. Given this knowledge, it seems too risky to go ahead and apply DS to $\neg p \vee r$ and r.

In order to make a more refined distinction between safe and unsafe applications of rules of inference, we will rely on an idea that stems from Batens' adaptive logics framework.[10] First, we define a set of formulas called the *set of abnormalities* (Ω) which contains all inconsistencies in our formal language:

$$\Omega =_{\text{df}} \{\varphi \wedge \neg\varphi \mid \varphi \in \mathcal{L}\}$$

Where Θ is a finite subset of Ω, let $Dab(\Theta)$ abbreviate the classical disjunction of the members of Θ ('Dab' is short for 'disjunction of abnormalities'). If Θ is a singleton containing only one member, say φ, then $Dab(\Theta) = \varphi \wedge \neg\varphi$. Next, we will say that it is 'safe' to apply a rule like DS whenever the formulas to which the rule is applied behave reliably with respect to the premises:

[D5] **[Reliability]** [Let $\varphi[\Sigma]$ be the thesis of the Proponent. A formula ψ behaves *reliably* with respect to Σ iff there is no formula $Dab(\Theta)$ such that:

(i) $\psi \wedge \neg\psi \in \Theta$, and
(ii) $\Sigma \vdash_{\mathbf{LLD}} Dab(\Theta)$, and
(iii) $\Sigma \nvdash_{\mathbf{LLD}} Dab(\Theta \backslash \{\psi \wedge \neg\psi\})$.]

If Σ is empty, then no disjunction of abnormalities is an **LLD**-consequence of Σ, so all formulas behave reliably with respect to the empty set. In view of (ii) and (iii), $\psi \wedge \neg\psi$ is indispensable in order to have a winning strategy for $Dab(\Theta)$ given the concession of the formulas in Σ.

Let (14) be the thesis of the Proponent. Applying Definition [D5] it is clear that p behaves reliably with respect to the premise set $\{p, \neg p \vee q\}$, as no disjunction of abnormalities is an **LLD**-consequence of this set. In (15), p no longer behaves reliably in view of the premise set $\{p, \neg p, \neg p \vee q\}$, since the formula $Dab(\{p \wedge \neg p\})$ meets conditions (i)–(iii) of Definition [D5]. Likewise, in (16) the formula $Dab(\{p \wedge \neg p, q \wedge \neg q\})$ meets conditions (i)–(iii), so—as desired—p does not behave reliably in view of the premise set $\{p, q, \neg p \vee \neg q, \neg p \vee r\}$.

The following case illustrates why condition (iii) of Definition [D5] is necessary:

$$p, \neg p \vee q, r \wedge \neg r \vdash q \tag{17}$$

In view of the premises of (17), we want DS to be applicable to p and $\neg p \vee q$, since we can safely take these formulas to behave consistently in the absence of further knowledge. For this application of DS, it does not matter that another formula, r, behaves inconsistently. But notice that the disjunction $Dab(\{p \wedge \neg p, r \wedge \neg r\})$ is an **LLD**-consequence of $\{p, \neg p \vee q, r \wedge \neg r\}$, and that this disjunction of abnormalities meets conditions (i) and (ii) of Definition [D5]. It is condition (iii) that prevents this disjunction from causing p to behave unreliably: the disjunct $p \wedge \neg p$ is not indispensable to $Dab(\{p \wedge \neg p, r \wedge \neg r\})$ since $p, \neg p \vee q, r \wedge \neg r \vdash_{\mathbf{LLD}} r \wedge \neg r$.

As we mentioned earlier on, the idea of applying **LLD**-invalid rules of inference in a more dynamic way, making use of the notion of reliability, stems

[10] See e.g. [6] for a general introduction to adaptive logics. For the adaptive treatment of inconsistency, see e.g. [8,10].

from the adaptive logics framework.[11] In what follows, we will implement this idea in a formally precise way within the framework of dialogical logic, resulting in an inconsistency-adaptive dialogical logic, **IAD**.

4.2. Inconsistency-Adaptive Dialogues

In classical dialogical logic, the Proponent can always attack a negated formula affirmed by the Opponent, provided that it is her turn to play and that she respects the formal restriction rule. We saw how this led to a logic in which inconsistencies trivialize the consequence set, and remedied the problem by restricting the application of the particle rule for negation by a new structural rule, the negative formula rule. This restriction, however, is very strong. It leaves the Proponent with very few means to attack negated statements, resulting in a logic that is, like many paraconsistent logics, rather weak in terms of inferential power.

Here, we present a new approach in the form of a middle way in which the Proponent is allowed to attack negated formulas affirmed by the Opponent, provided that she commits herself to the reliable behavior of the formula in question. In order to implement this provision in the dialogical setting, we will formulate a new rule, the *inconsistency-adaptive negation rule*, which extends the negative formula rule [SR4.1] from Sect. 3. Before we can define this rule, however, we need to make two preparatory changes to our framework.

First, we need a way to represent assumptions in dialogues. This we do by introducing an extra column for each player, called the *condition column*. If in some move M a player makes the assumption that a formula behaves reliably, we will add this assumption to M's condition column.

Second, the assertion that a formula does or does not behave reliably involves claims about **LLD**-consequence, as can be seen from Definition [D5]. In order to be able to settle such claims *within* an **IAD**-dialogue, we introduce the distinction between our main dialogue and its subdialogue(s). In the *main dialogue* the players play according to the rules of **IAD**. In justifying or challenging the reliability of formulas, one of the players may open a *subdialogue* in which a thesis is settled by playing according to the rules of **LLD**.[12]

Let us make things a bit more precise. In **IAD**, *moves* are sequences of the general form $\mathbf{X} - e - C - d$, where $\mathbf{X}$ and e are as before, where C is the condition corresponding to the move in question, and where d is either the main dialogue—in which case we write d_1—or a subdialogue—in which case we write $d_{1.i}$ for the i-th subdialogue. We generalize the starting rule as follows in order to incorporate these changes:

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[11] In fact, the name 'Lower Limit Dialogic' refers to a terminological convention used by adaptive logicians. Adaptive logics strengthen their so-called lower limit logic in the same way that **IAD** strengthens **LLD**.

[12] The first change is inspired by the proof theory of the adaptive logics framework, where a new column is added to proofs in order to keep track of conditional moves. The second change is inspired by [26], where the authors make a similar distinction between the upper section of a dialogue and its subdialogue(s).

[SR0.1][Starting rule for IAD]. [(i) If the initial thesis is of the form $\psi[\varphi_1, \ldots, \varphi_n]$, then for any play $\mathcal{P} \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ we have:
(a) $p_{\mathcal{P}}(\mathbf{P}$ - $!\psi[\varphi_1, \ldots, \varphi_n]$ - $\emptyset$ - $d_1)= 0$
(b) $p_{\mathcal{P}}(\mathbf{O}$ - $n := r_1$ - $\emptyset$ - $d_1)= 1$ and $p_{\mathcal{P}}(\mathbf{P}$ - $m := r_2$ - $\emptyset$ - $d_1)= 2$.
(ii) If the initial thesis is of the form ψ, then for any play $\mathcal{P} \in \mathcal{D}(\psi)$ we have:
(a′) $p_{\mathcal{P}}(\mathbf{P}$ - $!\psi$ - $\emptyset$ - $d_1)= 0$
(b′) $p_{\mathcal{P}}(\mathbf{O}$ - $n := r_1$ - $\emptyset$ - $d_1)= 1$ and $p_{\mathcal{P}}(\mathbf{P}$ - $m := r_2$ - $\emptyset$ - $d_1)= 2$.]

The classical development rule [SR1] is likewise generalized to conditional moves by replacing moves of the form $\mathbf{X}{-}e$ with moves of the form $\mathbf{X}{-}e{-}C{-}d$. At the start of the dialogue, the condition set is empty, since no assumptions were made yet. We are now ready to define the inconsistency-adaptive negation rule for **IAD**:

[SR4.2] **[IAD negation rule]** [The sequence $\mathcal{S}$ is a play only if the following condition is fulfilled: If there is a move $N = \mathbf{P}{-}!\psi - C - d$ in the sequence $\mathcal{S}$ such that:

1. $p_{\mathcal{S}}(N) = n$
2. $F_{\mathcal{S}}(N) = [m, A]$ and
3. $m = p_{\mathcal{S}}(M)$ such that $M = \mathbf{O}{-}!\neg\psi - \emptyset - d$

Then one of the following two conditions holds:

1. N is performed by **P** in accordance with the **LLD** negation rule [SR4.1], or
2. $N = \mathbf{P}! - \psi - \mathfrak{R}^{\Sigma}_{\psi} - d$, where $\mathfrak{R}^{\Sigma}_{\psi}$ abbreviates that ψ behaves reliably in view of the premise set Σ.]

If **O** already attacked the same statement before, then the **IAD** negation rule allows **P** to attack a negated statement in line with [SR4.1]. If not, then **P** may attack this statement under the assumption that it behaves reliably. The latter attack is made by a *conditional move*: in attacking a statement $\neg\psi$, **P** adds $\mathfrak{R}^{\Sigma}_{\psi}$ to the condition, thereby committing herself to the reliability of ψ in view of the premise set Σ.

Before we show how the Opponent can attack conditional moves, let us already illustrate the use of the **IAD** negation rule. Reconsider our dialogue for the application of DS to the premise set $\Sigma = \{p \vee q, \neg p\}$:[13]

	O				P		
d_1							
					$q[p \vee q, \neg p]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3.1	$p \vee q$	$\emptyset$	0				
3.2	$\neg p$	$\emptyset$					
5	p	$\emptyset$		3.1	$?\vee$	$\emptyset$	4
	$---$			3.2	p	$\mathfrak{R}^{\Sigma}_{p}$	6

[13] For reasons to be explained shortly, we require **O** to pick rank 2 in the inconsistency-adaptive setting.

Explanation We saw that in **LLD** the Proponent has no means to attack the premise $\neg p$, which ultimately causes her to lose the dialogue. The **IAD** negation rule, however, allows the Proponent to attack this premise on the condition that the negated formula p behaves reliably in view of Σ (move 6 in the dialogue).

Of course, once we tolerate the presence of inconsistencies we know that formulas will not always behave reliably in view of a given premise set. So the Opponent should be allowed to attack the assumption that a formula behaves reliably. This she can do by claiming that the abnormality corresponding to the formula in question is part of a *Dab*-formula that is a consequence of the premise set. But from Definition [D5] it is clear that more is required in order for this formula to behave unreliably: the *Dab*-formula provided by the Opponent should be an **LLD**-consequence of the premise set (condition (ii) of [D5]), and it should contain the abnormality corresponding to the formula under attack as an indispensable part (condition (iii) of [D5]). So the Proponent will be able to counter-attack in one of two ways, corresponding to conditions (ii) and (iii) of [D5]. The dynamics of this process is caught by the following rule:

Particle rule for the reliability operator $\mathfrak{R}$		
Assertion	Attack	Defence
$\mathbf{X}$–$!\varphi$– $\mathfrak{R}^{\Sigma}_{\varphi} - d_1$	$\mathbf{Y}$–$?\mathfrak{R}^{\Sigma}_{\varphi} Dab(\Theta) - \emptyset - d_1$ (where $\varphi \wedge \neg\varphi \in \Theta$)	$\mathbf{X}$–$!\mathfrak{F}_{\Sigma}(Dab(\Theta)) - \emptyset - d_1$
		Or $\mathbf{X}$ counter-attacks $\mathbf{X}$–$!\mathfrak{I}_{\Sigma}(Dab(\Theta\backslash\{\varphi \wedge \neg\varphi\})) - \emptyset - d_1$ (where $Dab(\Theta)\backslash\{\varphi \wedge \neg\varphi\} \neq \emptyset$)

Player **Y** attacks the conditional statement of **X** by providing a formula $Dab(\Theta)$ including the formula $\varphi \wedge \neg\varphi$ as one of its disjuncts. Player **X** replies either by claiming that it is false that the *Dab*-formula in question is an **LLD**-consequence of the premise set, or by claiming that the abnormality $\varphi \wedge \neg\varphi$ is not indispensable to the *Dab*-formula given by **Y**. In the first case, she uses the *failure operator* $\mathfrak{F}$. In the second case, she uses the *indispensability operator* $\mathfrak{I}$. In both cases, **X**'s claim concerns defendability under the concession of the members of Σ in **LLD**. If she uses the $\mathfrak{F}$-operator, she claims that the *Dab*-formula given by **Y** is not an **LLD**-consequence of the premises (condition (ii) of [D5] again). If she uses the $\mathfrak{I}$-operator, she claims that $Dab(\Theta\backslash\{\varphi \wedge \neg\varphi\})$ is an **LLD**-consequence of the premises (condition (iii) of [D5] again). In order to settle claims about **LLD**-defendability *within* an **IAD**-dialogue, we introduced the notion of a subdialogue earlier in this section. When **X** defends herself using either the $\mathfrak{F}$-operator or the $\mathfrak{I}$-operator, a subdialogue may open in the continuation of the dialogue. We consider the $\mathfrak{F}$-operator first.

Particle rule for the failure operator $\mathfrak{F}$		
Assertion	Attack	Defence
$\mathbf{X}$–$!\mathfrak{F}_{\Sigma}\varphi - \emptyset - d_1$	$\mathbf{Y}$–$!\varphi[\Sigma] - \emptyset - d_{1.i}$	– – –
	$\mathbf{Y}$ opens a subdialogue $d_{1.i}$	No defence

In attacking **X**'s statement of the form $!\mathfrak{F}_\Sigma\varphi$, **Y** opens a new subdialogue in which she defends the thesis $\varphi[\Sigma]$. In line with the starting rule, we start a subdialogue with the empty condition set. Given the set-up of **IAD**-dialogues, it will always be the Opponent who attacks expressions of the form $!\mathfrak{F}\varphi$, and who defends the thesis $\varphi[\Sigma]$. Consequently, it is now the Opponent who should play under the formal restriction. Indeed—following [26], where the $\mathfrak{F}$-operator was first introduced—we require that there is a switch in formal play whenever the $\mathfrak{F}$-operator is attacked. To implement the switch at the structural level, we need to replace [SR2] with the following new rules [SR2.1] and [SR.2.2] which regulate the formal restriction:[14]

[SR2.1] [Formal restriction for IAD] [If **X** plays under formal restriction, then the sequence $\mathcal{S}$ is a play only if the following condition is fulfilled: if $N = \mathbf{X}\text{–}!\psi - C_j - d$ is a member of $\mathcal{S}$, for any atomic sentence ψ, then there is a move $M = \mathbf{Y}\text{–}!\psi - C_i - d$ in $\mathcal{S}$ such that $p_\mathcal{S}(M) < p_\mathcal{S}(N)$.]

[SR2.2] [Application of the formal restriction rule in IAD] [The application of the formal restriction is regulated by the following conditions:

1. In the main dialogue d_1, if $\mathbf{X} = \mathbf{P}$, then **X** plays under the formal restriction.
2. If **X** opens a subdialogue $d_{1.i}$, then **X** plays under the formal restriction.]

In **CL**-dialogues and **LLD**-dialogues it is generally sufficient for **O** to choose rank 1 (see Sect. 2.4). However, in **IAD**-dialogues a switch in formal restriction may occur when a new subdialogue is opened in accordance with [SR2.1] and [SR2.2], and it is possible that **O** plays under the formal restriction in this subdialogue. Because of this, it might now be the case that there is an **O**-winning strategy beginning with **O** choosing rank 2, but not with **O** choosing rank 1. For this reason, we let both players pick rank 2 in the illustrations in this section.

Since the thesis defended in the subdialogue is a claim about **LLD**-consequence, we play according to the rules of **LLD** in the subdialogue, as ensured by the following structural rule:

[SR4.3] [Application of the negation rules] [In the main dialogue d_1, **P** attacks negations in accordance with the inconsistency-adaptive negation rule [SR4.2]. In a subdialogue $d_{1.i}$ the player who plays under the formal restriction attacks negations in accordance with the **LLD** negation rule [SR4.1].]

Let us now reconsider our example we started above. Suppose **O** attacks the condition attached to move 6 in the dialogue in line with the particle rule for the $\mathfrak{R}$-operator, giving the *Dab*-formula $p \wedge \neg p$. A good strategy for **P** is to claim that it is false that this *Dab*-formula is an **LLD**-consequence of the

[14] Our approach differs from [26], where the switch is defined at the local level, i.e. inside the particle rule. Strictly speaking the identity of the player who plays under the formal restriction is not related to the local semantic level, so we defined the regulation of this matter in a separate structural rule.

premise set, using the $\mathfrak{F}$-operator. The continuation of the dialogue looks as follows:

	O				P		
d_1							
					$q[p \vee q, \neg p]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3.1	$p \vee q$	$\emptyset$	0				
3.2	$\neg p$	$\emptyset$					
5	p	$\emptyset$		3.1	$?\vee$	$\emptyset$	4
	$- - -$			3.2	p	$\mathfrak{R}_p^{\Sigma}$	6
7	$?\mathfrak{R}_p^{\Sigma}(p \wedge \neg p)$	$\emptyset$	6		$\mathfrak{F}_{\Sigma}(p \wedge \neg p)$	$\emptyset$	8
$d_{1.1}$							
9	$p \wedge \neg p[p \vee q, \neg p]$	$\emptyset$	8		$- - -$		
13	$p \wedge \neg p$	$\emptyset$		9	$p \vee q$	$\emptyset$	10.1
				9	$\neg p$	$\emptyset$	10.2
11	$?\vee$	$\emptyset$	10.1		q	$\emptyset$	12
				13	$?\wedge_L$	$\emptyset$	14

Explanation Using the particle rule for the $\mathfrak{R}$-operator, **O** attacks the condition of move 6, claiming that p does not behave reliably in view of the *Dab*-formula $p \wedge \neg p$. **P** defends herself by claiming that this *Dab*-formula is not an **LLD**-consequence of Σ, using the $\mathfrak{F}$-operator (move 8). Next, **O** attacks **P**'s move in line with the particle rule for the $\mathfrak{F}$-operator: She opens a new subdialogue $d_{1.1}$, claiming that she is able to defend $p \wedge \neg p$ given the concessions of Σ by **P**. **P** challenges this assertion by conceding the formulas in Σ (moves 10.1 and 10.2). The dialogue continues as usual, but with **O** playing under the formal restriction (so **O** cannot introduce new atomic formulas in the dialogue). As a consequence of this switch in formal restriction, **O** cannot answer to **P**'s attack to give the left conjunct of the conjunction (move 14) she asserted in move 13 (the conjunct in question is p, an atomic formula not yet asserted by **P** within $d_{1.1}$). For the same reason, she cannot attack the formula $\neg p$ asserted by **P** in move 10.2. Consequently, she loses the subdialogue. In general:

[SR3.1] **[Winning rule for subdialogues]** [A subdialogue $d_{1.i}$ is won by **X** if it is **Y**'s turn and there are no more moves available to **Y**. If **X** wins the subdialogue, we return to the main dialogue d_1 in which it is (still) **Y**'s turn.]

In line with [SR3.1], we return to our main dialogue d_1 after move 14, where it is **O**'s turn to make the next move. However, in the main dialogue too **O** has run out of options. Consequently, **P** wins the dialogue in view of [SR3], and the inference from $p \vee q$ and $\neg p$ to q is **IAD**-valid.[15]

[D6] **[IAD-consequence]** [$\Sigma \vdash_{\mathbf{IAD}} \varphi$ (respectively $\vdash_{\mathbf{IAD}} \varphi$) iff according to the **IAD**-rules there is a **P**-winning strategy for the thesis $\varphi[\Sigma]$ (respectively φ).]

[15] Remember that we present only the "best" strategies available. So if **P** wins the dialogue, she has a winning strategy and the inference under dispute is valid.

By definition [D6]:

$$p \vee q, \neg p \vdash_{\mathbf{IAD}} q$$

So far we have only defined and illustrated the use of the operators $\mathfrak{R}$ and $\mathfrak{F}$. Let us next turn to the $\mathfrak{I}$-operator, the use of which is regulated by the following rule:

Particle rule for the indispensability operator $\mathfrak{I}$		
Assertion	Attack	Defence
$\mathbf{X}$–$!\mathfrak{I}_\Sigma\varphi - \emptyset - d_i$	$\mathbf{Y}$–$?\mathfrak{I}_\Sigma\varphi - \emptyset - d_i$	$\mathbf{X}$–$!\varphi[\Sigma] - \emptyset - d_{i.j}$ $\mathbf{X}$ opens a subdialogue $d_{i.j}$

We explicate the use of this rule by means of an example. Let $\Sigma = \{p, \neg q \to \neg p, r \wedge \neg r\}$. The Proponent claims that $q[\Sigma]$:[16]

	O				P		
d_1							
					$q[\Sigma]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3.1	p	$\emptyset$	0		q	$\emptyset$	24
3.2	$\neg q \to \neg p$	$\emptyset$					
3.3	$r \wedge \neg r$	$\emptyset$					
5	$\neg p$			3.2	$\neg q$	$\emptyset$	4
	$---$			5	p	$\mathfrak{R}_p^\Sigma$	6
7	$?\mathfrak{R}_p^\Sigma((p \wedge \neg p) \vee (r \wedge \neg r))$	$\emptyset$	6				
				7	$\mathfrak{I}_\Sigma(r \wedge \neg r)$	$\emptyset$	8
9	$?\mathfrak{I}_\Sigma(r \wedge \neg r)$	$\emptyset$	8				
$d_{1.1}$							
					$r \wedge \neg r[\Sigma]$	$\emptyset$	10
11.1	p	$\emptyset$	10		$r \wedge \neg r$	$\emptyset$	16
11.2	$\neg q \to \neg p$	$\emptyset$					
11.3	$r \wedge \neg r$	$\emptyset$					
13	r	$\emptyset$		11.3	$?\wedge_L$	$\emptyset$	12
15	$\neg r$	$\emptyset$		11.3	$?\wedge_R$	$\emptyset$	14
17	$?\wedge_L$	$\emptyset$		16	r	$\emptyset$	18
19	$?\wedge_R$	$\emptyset$	16		$\neg r$	$\emptyset$	20
21	r	$\emptyset$	20		$---$		
	$---$			15	r	$\emptyset$	22
d_1							
23	q	$\emptyset$	4		$---$		

[16] We write '$q[\Sigma]$' instead of '$q[p, \neg q \to \neg p, r \wedge \neg r]$' for reasons of presentation.

Explanation In this dialogue, **O** attacks the conditional move 6 by providing the *Dab*-formula $(p \wedge \neg p) \vee (r \wedge \neg r)$ in move 7.[17] The particle rule for the $\mathfrak{R}$-operator leaves **P** with two possible replies. Either she can claim that it is false that this *Dab*-formula is an **LLD**-consequence of Σ, using the $\mathfrak{F}$-operator; or she can claim that the abnormality $p \wedge \neg p$ is not indispensable to the *Dab*-formula given by **O**, using the $\mathfrak{I}$-operator. Note that if **P** were to opt for the first reply, she would lose the ensuing subdialogue, since the *Dab*-formula given by **O** *is* an **LLD**-consequence of Σ: one of its disjuncts is a premise. Therefore, strategically, it is better for **P** to go into counter-attack and use the $\mathfrak{I}$-operator (move 8).

Next, **O** attacks **P** to justify her claim that the abnormality $p \wedge \neg p$ is not indispensable to the disjunction $(p \wedge \neg p) \vee (r \wedge \neg r)$. In defence to this attack—and in line with the particle rule for the $\mathfrak{I}$-operator – **P** has to show that $r \wedge \neg r$ is an **LLD**-consequence of Σ. To this end, a new subdialogue opens in which **P** defends $r \wedge \neg r$ given the concession of $p, \neg q \rightarrow \neg p, r \wedge \neg r$ (move 10).[18]

In subdialogue $d_{1.1}$, there is no switch as to which player plays under the formal restriction. As in the main dialogue, it is **P** who defends the initial thesis and carries the burden of proof. Therefore it is **P** who plays under the formal restriction. (As we saw, the situation is different in subdialogues triggered by the $\mathfrak{F}$-operator, where it is **O** who defends the initial thesis and, consequently, plays under the formal restriction.)

Clearly, the thesis $r \wedge \neg r[p, \neg q \rightarrow \neg p, r \wedge \neg r]$ is **LLD**-valid, hence **P** wins the subdialogue (move 22). This means that after move 22 we return to our main dialogue, where it is **O**'s turn to play. She does so by attacking the formula $\neg q$ in move 4. In doing so, however, she introduces the new atomic formula q (move 23), which allows **P** to use this atom as well, and to reply to **O**'s first attack in d_1 (move 24). After this defence by **P**, no more moves are available to **O**, hence **P** wins the dialogue. By Definition [D6]:

$$p, \neg q \rightarrow \neg p, r \wedge \neg r \vdash_{\mathbf{IAD}} q$$

4.3. Some more illustrations

We end Sect. 4 with two more examples which further illustrate the use of the **IAD**-rules. First, we illustrate that there is no **P**-winning strategy for $q[p \wedge \neg p]$ with the **IAD**-rules:

[17] Which disjunctions of abnormalities **O** introduces when attacking conditional **P**-moves in **IAD**-dialogues, is a strategic matter on our account. We leave it open whether and how our analysis can benefit from a more systematic study of the heuristics involved in constructing the 'right' disjunctions of abnormalities when attacking conditional moves.

[18] Attacks and their respective defences usually appear on the same line in a dialogue. For reasons of presentation, we opted to break this habit whenever a player defends herself against an attack by means of the $\mathfrak{I}$-operator (as in move 10 above).

	O				P		
d_1							
					$q[p \wedge \neg p]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3	$p \wedge \neg p$	$\emptyset$	0			$\emptyset$	
5	p	$\emptyset$		3	$?\wedge_1$	$\emptyset$	4
7	$\neg p$	$\emptyset$		3	$?\wedge_2$	$\emptyset$	6
	$- - -$			7	p	$\mathfrak{R}_p^\Sigma$	8
9	$?\mathfrak{R}_p^\Sigma(p \wedge \neg p)$	$\emptyset$	7		$\mathfrak{F}_\Sigma(p \wedge \neg p)$	$\emptyset$	10
$d_{1.1}$							
11	$p \wedge \neg p[p \wedge \neg p]$	$\emptyset$	10		$- - -$	$\emptyset$	
13	$p \wedge \neg p$	$\emptyset$		11	$p \wedge \neg p$	$\emptyset$	12
17	p	$\emptyset$		13	$?\wedge_1$	$\emptyset$	14
15	$?\wedge_1$	$\emptyset$	12		p	$\emptyset$	16
19	$\neg p$	$\emptyset$		13	$?\wedge_2$	$\emptyset$	18
	$- - -$	$\emptyset$		19	p	$\emptyset$	20
21	$?\wedge_2$	$\emptyset$	12		$\neg p$	$\emptyset$	22
23	p	$\emptyset$	22		$- - -$		

Explanation To attack the assertion of the form $\neg p$ (move 7), **P** has to perform a conditional move assuming that p behaves consistently (move 8). Next, **O** attacks the condition (move 9) and **P** answers making use of the $\mathfrak{F}$-operator in accordance with the rule for the $\mathfrak{R}$-operator (move 10). Notice that a counter-attack with the $\mathfrak{I}$-operator is not allowed (there is no disjunct to delete from the *Dab*-formula). When **O** attacks the $\mathfrak{F}$-operator, she opens a subdialogue $d_{1.1}$ in which she has to play under formal restriction (in accordance with [SR2.1] and [SR2.2]). Since the thesis $p \wedge \neg p[p \wedge \neg p]$ is clearly **LLD**-valid, she has a winning strategy for it. Recall that the move 23 is allowed because **P** already attacked $\neg p$ before (move 20). **O** wins the subdialogue. Moreover, no more moves are available to **P** d_1. So **P** has no winning strategy for $q[p \wedge \neg p]$ and this thesis is **IAD**-invalid.

The following example, $\neg q[\Sigma]$, with $\Sigma = \{\neg(p \wedge q), \neg\neg p\}$ illustrates how **P** can make various conditional moves within the same dialogue:

	O				P		
d_1							
					$\neg q[\Sigma]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3.1	$\neg(p \wedge q)$	$\emptyset$	0		$\neg q$	$\emptyset$	4
3.2	$\neg\neg p$	$\emptyset$					
5	q	$\emptyset$	4		$---$		
	$---$			3.2	$\neg p$	$\mathfrak{R}^{\Sigma}_{\neg p}$	6
7	$?\mathfrak{R}^{\Sigma}_{\neg p}(\neg p \wedge \neg\neg p)$	$\emptyset$	6		$\mathfrak{F}_{\Sigma}(\neg p \wedge \neg\neg p)$	$\emptyset$	8
$d_{1.1}$							
9	$\neg p \wedge \neg\neg p[\Sigma]$	$\emptyset$	8		$---$		
11	$\neg p \wedge \neg\neg p$	$\emptyset$		9	$\neg(p \wedge q)$	$\emptyset$	10.1
					$\neg\neg p$	$\emptyset$	10.2
13	$\neg p$	$\emptyset$		11	$?\wedge_L$	$\emptyset$	12
	$---$			13	p	$\emptyset$	14
d_1							
15	p	$\emptyset$	6		$---$		
	$---$			3.1	$p \wedge q$	$\mathfrak{R}^{\Sigma}_{p\wedge q}$	16
17	$?\mathfrak{R}^{\Sigma}_{p\wedge q}((p \wedge q)$ $\wedge\neg(p \wedge q))$	$\emptyset$	16		$\mathfrak{F}_{\Sigma}((p \wedge q)$ $\wedge\neg(p \wedge q))$	$\emptyset$	18
$d_{1.2}$							
19	$(p \wedge q) \wedge \neg(p \wedge q)[\Sigma]$	$\emptyset$	18		$---$		
21	$(p \wedge q) \wedge \neg(p \wedge q)$	$\emptyset$		19	$\neg(p \wedge q)$	$\emptyset$	20.1
					$\neg\neg p$	$\emptyset$	20.2
23	$p \wedge q$	$\emptyset$		21	$?\wedge_L$	$\emptyset$	22
				23	$?\wedge_L$	$\emptyset$	24
25	$?\wedge_L$	$\emptyset$	16		p	$\emptyset$	26

Explanation Here, **P** performs a first conditional move (move 6), assuming that $\neg p$ behaves consistently, which is immediately attacked by **O** (move 7). **P** answers making use of the $\mathfrak{F}$-operator (move 8) and **O** opens a subdialogue while attacking that statement (move 9). **P** wins the subdialogue (move 14) because she played the last move and **O** cannot attack the negations asserted by **P** (move 10) since, in the subdialogue, **O** plays under the formal restriction in accordance with [SR2.2]. Next, we go back to the main dialogue d_1 where it is **O**'s turn to play. The dialogue continues. In move 16 **P** makes a new conditional attack, asserting $p \wedge q$ and attacking **O**'s move 3.1. **O** attacks the condition (move 17) and **P** answers making use of the $\mathfrak{F}$-operator. A new subdialogue $d_{1.2}$ opens, in which **O** again plays under formal restriction and loses (move 24). We return to the main dialogue d_1 where it is again **O**'s turn to play. She attacks the move 16 by asking the left conjunct (move 25). **P** answers. At this stage no more moves are available to **O**, and **P** wins the dialogue. Note that **O** might have asked the right conjunct instead of the left one. But that would not have changed anything: **O** already had conceded both of the conjuncts (moves 5 and 15). Note also that, once she asked for the left conjunct, **O** can no longer ask for the right conjunct, since she chose rank 2 and since she already attacked move 16 twice (moves 17 and 25).

Note that **O** could have played very differently in the above dialogue by picking different *Dab*-formulas when attacking **P**'s conditional moves. For instance, she could have picked the *Dab*-formula $(\neg p \wedge \neg\neg p) \vee ((p \wedge q) \wedge \neg(p \wedge q))$ in any of moves 7 or 17. However, this would not have given her a winning strategy in any of the ensuing subdialogues, since $(\neg p \wedge \neg\neg p) \vee ((p \wedge q) \wedge \neg(p \wedge q))$ is not an **LLD**-consequence of Σ. We leave the verification of these details to the reader.

5. Related Work

5.1. IAD and dialogical logic

The negation rule [SR4.1] of **IAD** is inspired by the rules defined by Rahman & Carnielli [23]. It is worth noting that in [23] two different paraconsistent dialogical logics are defined, namely **L-D** (Literal Dialogues) and **D+** (Paraconsistent Positive Dialogues). The relevant difference between both is that the negation rule of the former is restricted to negated atomic formulas (negative literal rule) while the negation rule of the latter is formulated in a general way and does not contain any such restriction.[19] In **IAD**, the negation rules [SR4.1] and [SR4.2] are applied indifferently to atomic and complex formulas.

In **L-D** EFSQ is invalid only in case the formula which behaves inconsistently is atomic:[20]

$$\nvdash (p \wedge \neg p) \rightarrow q \tag{18}$$

$$\vdash ((p \vee p) \wedge \neg(p \vee p)) \rightarrow q \tag{19}$$

By contrast, both are invalid in **D+**. However, as a consequence of the generalisation of the negation rule, DNE ($\neg\neg p \rightarrow p$) is not valid in **D+** even if we play with the classical development rule. Moreover, DS (as well as CPOS and other principles) is invalidated even if no inconsistency occurs in the dialogue. As observed by Rahman and Van Bendegem [27,30], the invalidity of DS is a high price to pay for going paraconsistent. A more flexible alternative is desired. Here inconsistency-adaptive logic and paraconsistent dialogical logic meet.

In order to solve this difficulty, Rahman & Van Bendegem define a notion of adaptive validity in a dialogical framework. Their analysis is rather different from **IAD**. Indeed, **IAD** involves new rules by means of which conditional moves are implemented. By contrast, Rahman & Van Bendegem do not define any new rules for dialogues. Rather, they come up with a new definition of validity based on an analysis at the strategic level. In their approach, a formula is *valid*

[19] Another difference is that negation in **D+** is defined in conditional form, i.e. each formula φ has its corresponding constant $\bot_\varphi$ such that $\neg\varphi \Leftrightarrow \varphi \rightarrow \bot_\varphi$. It it thus possible to distinguish between $\bot_p$ and $\bot_q$ for example, the first holding for inconsistencies related to p, the second for the inconsistencies related to q. For more details, we refer to [23, Sec. 5.1].

[20] In its treatment of (18) and (19), **L-D** behaves exactly like Arruda's system **V1** from [1] (see also [2, Sec. 7], where this system is called $\mathbf{PI^v}$).

by adaptation if it is valid according to the standard definition of validity and is *free of paraconsistent redundancies*:

[D7][Free of Paraconsistent Redundancies]. [A formula is said to be free of paraconsistent redundancies iff **P** wins under the standard structural rules and she can win under the following conditions:

1. She can attack every **O**-formula at least once (e.g. with one of the possible attacks on a conjunction) in any of the possible **O**-variants (not necessarily in the same **O**-variant).[21]
2. She can defend herself at least once (e.g. with one of the possible defences of a disjunction) against all attacks in any of the possible **O**-variants (not necessarily in the same **O**-variant).
3. She can use at least one occurrence of any atomic **O**-formula in any of the possible **O**-variants (not necessarily the same **O**-variant).]

However, this is not sufficient to validate DS $(((p \vee q) \wedge \neg p) \rightarrow q)$ for which another device is required. Let us explain the point on the basis of the following dialogue as developed by Rahman & Van Bendegem [27, p. 303].

	O			P	
				$((p \vee q) \wedge \neg p) \rightarrow q$	0
1	$n := 1$			$m := 2$	2
3	$(p \vee q) \wedge \neg p$	0		q	12
5	$p \vee q$		3	$?\wedge_L$	4
7	$\neg p$		3	$?\wedge_R$	6
9	p		5	$?\vee$	8
	$- - -$		7	p	10
	O			**P**	
11	q			$?\vee_{[q]}$	[8]

In the dialogue above, **P** wins in move 10 according to the rules of standard dialogical logic. However, **P** does not answer the challenge of move 3 and thus the formula does not comply with the definition of validity by adaptation. Therefore, we add in a subdialogue the possibility for **P** to ask $?\vee_{[q]}$ to enable him to show she can answer the challenge in at least one **O**-variant. Now, **P** can answer the attack of move 3 in move 12 and the formula is claimed to be valid by adaptation.[22]

As a result, the following holds in Rahman & Van Bendegem's dialogical logic:

$$\vdash ((p \vee q) \wedge \neg p) \rightarrow q \tag{20}$$

$$\vdash (((p \vee q) \wedge \neg p) \wedge p) \rightarrow q \tag{21}$$

$$\nvdash ((p \vee q) \wedge \neg p) \wedge p) \rightarrow r \tag{22}$$

$$\nvdash p \rightarrow (\neg p \rightarrow q) \tag{23}$$

[21] An **O**-variant is a possible development of a dialogue triggered by an **O**-choice.

[22] For the sake of clarity, we made some minor changes to the formulation offered in [27].

$$\nvdash (p \wedge \neg p) \rightarrow q \tag{24}$$

The difference between (21) and (22) is a consequence of the additional device we add for DS as in the dialogue above. Indeed, in a dialogue for (21) **P** can answer q in at least one **O**-variant, while in a dialogue for (22) she cannot answer r in any **O**-variant. **LLD** and **IAD** behave as follows with respect to (20)–(24):

$$\nvdash_{\mathbf{LLD}} ((p \vee q) \wedge \neg p) \rightarrow q \tag{25}$$

$$\nvdash_{\mathbf{LLD}} (((p \vee q) \wedge \neg p) \wedge p) \rightarrow q \tag{26}$$

$$\nvdash_{\mathbf{LLD}} (((p \vee q) \wedge \neg p) \wedge p) \rightarrow r \tag{27}$$

$$\nvdash_{\mathbf{LLD}} p \rightarrow (\neg p \rightarrow q) \tag{28}$$

$$\nvdash_{\mathbf{LLD}} (p \wedge \neg p) \rightarrow q \tag{29}$$

$$\vdash_{\mathbf{IAD}} ((p \vee q) \wedge \neg p) \rightarrow q \tag{30}$$

$$\vdash_{\mathbf{IAD}} (((p \vee q) \wedge \neg p) \wedge p) \rightarrow q \tag{31}$$

$$\vdash_{\mathbf{IAD}} (((p \vee q) \wedge \neg p) \wedge p) \rightarrow r \tag{32}$$

$$\vdash_{\mathbf{IAD}} p \rightarrow (\neg p \rightarrow q) \tag{33}$$

$$\vdash_{\mathbf{IAD}} (p \wedge \neg p) \rightarrow q \tag{34}$$

One might wonder why (30)–(34) hold. The reason is that φ is an **IAD**-theorem iff φ is a theorem of **CL**. In an **IAD**-dialogue the Proponent is allowed to conditionally attack negated statements. In order to successfully attack such conditional statements, the Opponent must give a *Dab*-formula which is an **LLD**-consequence of the premise set. But if the premise set is empty, no *Dab*-formula whatsoever is an **LLD**-consequence of it. Therefore the Proponent will have a successful defence ready: using the failure operator she can show that any *Dab*-formula given by the Opponent is not an **LLD**-consequence of the empty premise set. Note, however, that $p \vee q, \neg p, p \nvdash_{\mathbf{IAD}} q$ as well as $p \vee q, \neg p, p \nvdash_{\mathbf{IAD}} r$, and $p, \neg p \nvdash_{\mathbf{IAD}} q$. This is in line with the adaptive approach. Adaptive logicians are interested in logical consequence rather than theoremhood. Rahman & Van Bendegem do not define a consequence relation, but such a development might be considered.

5.2. IAD and adaptive logic

In this section we show that logical consequence in **LLD** corresponds to logical consequence in Batens' paraconsistent logic **CLuN** (Sect. 5.2.1), and that logical consequence in **IAD** corresponds to logical consequence in the inconsistency-adaptive logic $\mathbf{CLuN^r}$ (Sect. 5.2.2).[23]

5.2.1. LLD and CLuN. We first outline the result that **LLD**-consequence and **CLuN**-consequence are identical (Theorem 1), and next turn to the non-monotonic extensions (Theorem 2). Our results are restricted to finite premise

[23] The propositional fragment of **CLuN** was first introduced in [2] under the name **PI**. See e.g. [4,9] for more details on **CLuN** and $\mathbf{CLuN^r}$.

sets.[24] Our proof outlines rely on existing results established by dialogicians and adaptive logicians. For the details, we refer to these results where necessary.

Theorem 1. $\Sigma \vdash_{\mathbf{LLD}} \psi$ *iff* $\models_{\mathbf{CLuN}} \psi$.

Proof outline. We make use of two results from the literature. First, in [9] Diderik Batens and Joke Meheus show that a Smullyan-style tableau method for **CLuN** is obtained from Smullyan's tableau method for **CL** simply by removing the following rule for signed formulas:[25]

$$\text{If } \mathsf{T}\neg\varphi, \text{ then } \mathsf{F}\varphi \tag{$\mathsf{T}\neg$}$$

Second, in [12] Nicolas Clerbout gives a procedure for transforming an atomically closed Smullyan tableau for a thesis φ into an extensive form[26] of a winning **P**-strategy in a game $\mathcal{D}(\varphi)$, and vice versa. Our proof strategy is to apply the procedure from [12] first to the tableau method for **CLuN** from [9] ($\Leftarrow$), and next to extensive forms of winning **P**-strategies in **LLD**-games ($\Rightarrow$).

$\Leftarrow$ In order to apply Clerbout's procedure to Batens and Meheus' tableau method for **CLuN**, we add the following rule for tableaux in **CLuN**:

$$\begin{array}{r}\text{If } \mathsf{T}\neg\varphi \text{ then } \mathsf{F}\varphi \text{ only if, on the same branch,}\\ (\mathsf{F}\neg) \text{ was already applied to } \mathsf{F}\neg\varphi\end{array} \tag{$\mathsf{T}\neg*$}$$

($\mathsf{T}\neg*$) is redundant in the tableau method for **CLuN**, as we can only apply it to formulas appearing on a branch that is already closed (since it contains two formulas $\mathsf{T}\neg\varphi$ and $\mathsf{F}\neg\varphi$). We need the addition of this rule, however, to ensure that complete closed branches are *atomically* closed, i.e. to ensure that whenever two signed formulas $\mathsf{T}\neg\varphi$ and $\mathsf{F}\neg\varphi$ occur on some branch, then in completing the tableau we take care that there is an *atomic* formula ψ such that $\mathsf{T}\neg\psi$ and $\mathsf{F}\neg\psi$ occur on the same branch. ($\mathsf{T}\neg*$) takes care that if no more rules are applicable in a **CLuN**-tableau and the tableau closes, then the tableau closes atomically. Consequently, we can use this tableau as an input to Clerbout's procedure (an atomically closed **CLuN**-tableau for φ is also an atomically closed **CL**-tableau for φ). The result is an extensive form of a winning **P**-strategy s_p in a game $\mathcal{D}(\varphi)$ for **CL**. In order to show that s_p is a winning **P**-strategy in $\mathcal{D}(\varphi)$ for **LLD**, all we need to show is that it respects the additional structural rule [SR4.1].

In Clerbout's procedure, T-signed (resp. F-signed) formulas in tableaux correspond to **O**-moves (resp. **P**-moves) in dialogues. The application of ($\mathsf{T}\neg$) to some formula $\mathsf{T}\neg\varphi$ in a tableau corresponds to a **P**-attack on some **O**-move labeled $\neg\varphi$ in a dialogue. Analogously, the application of ($\mathsf{F}\neg$) to some formula

[24] Since compactness holds for **CLuN**, Theorem 1 readily generalizes to cases in which Σ is infinite. Since compactness fails for $\mathbf{CLuN^r}$, this generalization cannot be carried out for Theorem 2.

[25] See [28] for the details on Smullyan-style tableaux. We are assuming that the equivalence operator is not primitive in our language. Note that in [9], **CLuN** is called '**P**',

[26] The *extensive form* of a dialogical game is the representation of the game in the form of a tree, and the extensive form of a strategy of player **X** is the tree representation of **X**'s strategy. See [12, Definitions 2–4] for the full definitions.

$\mathsf{F}\neg\varphi$ in a tableau corresponds to an **O**-attack on some **P**-move labeled $\neg\varphi$ in a dialogue.

Since in a (modified) **CLuN**-tableau the only rule applicable to formulas of the form $\mathsf{T}\neg\varphi$ is $(\mathsf{T}\neg *)$, and since this rule is only applicable in case $(\mathsf{F}\neg)$ was already applied to $\mathsf{F}\neg\varphi$ on the same branch, it follows by construction that in the extensive form resulting after applying the procedure from [12] there cannot be a **P**-attack on an **O**-move labeled $\neg\varphi$ *unless* **O** already attacked the same formula before. Hence [SR4.1] is respected, and the strategy s_p provided by the procedure is a winning **P**-strategy in $\mathcal{D}(\varphi)$ for **LLD**.

$\Rightarrow$ Given an extensive form of a winning **P**-strategy s_p in a game $\mathcal{D}(\varphi)$ for **LLD**, we can use this strategy as an input to the procedure from [12] (a winning **P**-strategy for φ in **LLD** is (a fortiori) a winning **P**-strategy for φ in **CL**). The result is an atomically closed **CL**-tableau for φ.

Suppose that, in the resulting tableau $\mathcal{T}$, a formula $\mathsf{F}\varphi$ occurs on a branch θ as a result of applying $(\mathsf{T}\neg)$ to some formula $\mathsf{T}\neg\varphi$. Then in the extensive form of s_p this corresponds to a **P**-move n at which **P** attacks a formula $\neg\varphi$. Since s_p is an **LLD**-strategy, the moves in its extensive form respect [SR4.1], and there must be an **O**-move $m < n$ at which **O** attacks the formula $\neg\varphi$ uttered by **P**. But then, by the construction from [12], if θ contains an application of $(\mathsf{T}\neg)$ to $\mathsf{T}\neg\varphi$, then higher up on the branch it will contain an application of $(\mathsf{F}\neg)$ to $\mathsf{F}\neg\varphi$. So all applications of $(\mathsf{T}\neg)$ in $\mathcal{T}$ are in fact applications of the weaker rule $(\mathsf{T}\neg *)$, and $\mathcal{T}$ is an atomically closed **CLuN**-tableau for φ.

So far, we have shown that a formula φ is a **CLuN**-theorem iff φ is an **LLD**-theorem. In view of the validity of MP in **LLD** and **CLuN**, the deduction theorem holds for these logics. Hence, our result readily generalizes: for all finite Σ, $\Sigma \vdash_{\mathbf{LLD}} \psi$ iff $\models_{\mathbf{CLuN}} \psi$. □

5.2.2. IAD and CLuN$^{\mathbf{r}}$. In showing the correspondence between **IAD** and **CLuN$^{\mathbf{r}}$** we will make use of the tableau method for **CLuN$^{\mathbf{r}}$** from [9], restricted to the propositional level.[27] To obtain a **CLuN$^{\mathbf{r}}$**-tableau for $\Sigma \models \psi$, first make a **CLuN**-tableau for this inference (see Sect. 5.2.1; there is no need for using the **CLuN**-redundant rule $(\mathsf{T}\neg *)$). Second, add the following rule (which, like $(\mathsf{T}\neg *)$, is redundant in **CLuN**):

$$\frac{T\neg\varphi}{T\varphi \mid F\varphi} \qquad (\mathsf{T}\neg **)$$

Next, label formulas that stem from the premises: prefix all premises in the tableau (signed formulas of the form $\mathsf{T}\varphi$ which appear at the top node) with a '•', and do the same for all signed formulas appearing at nodes which result from applying a rule to a formula labeled with a '•'. Given a branch θ, $\overline{\theta}$ denotes the set of formulas on θ which are labeled with a '•'.

A branch θ of the resulting tableau $\mathcal{T}$ verifies a formula $\varphi \wedge \neg\varphi$ iff $\mathsf{T}\varphi, \mathsf{T}\neg\varphi \in \theta$. Let $Ab(\theta) = \{\varphi \mid \mathsf{T}\varphi, \mathsf{T}\neg\varphi \in \theta\}$. Where $\Delta \subset \Omega$, θ verifies a formula $Dab(\Delta)$ iff θ verifies at least one member of Δ. Let $\Theta(\Sigma)$ be the set of disjunctions of the form $(\varphi_1 \wedge \neg\varphi_1) \vee \cdots \vee (\varphi_n \wedge \neg\varphi_n)$ verified

[27] In [9], **CLuN$^{\mathbf{r}}$** is referred to as '**P$^{\mathbf{r}}$**'.

by all $\overline{\theta} \in \mathcal{T}$. $U(\Sigma) = \{\varphi \mid \varphi \wedge \neg\varphi \in \Delta$ for some Δ such that $Dab(\Delta) \in \Theta(\Sigma)$ and there is no $\Delta' \subset \Delta$ such that $Dab(\Delta') \in \Theta(\Sigma)\}$.

As usual a branch *closes* as soon as it contains two nodes $\mathsf{T}\varphi$ and $\mathsf{F}\varphi$. Otherwise it remains *open*. A branch θ of a finished tableau for $\Sigma \models \varphi$ is *marked* iff it is open and $Ab(\theta) \nsubseteq U(\Sigma)$. A $\mathbf{CLuN^r}$-tableau for $\Sigma \models \varphi$ *closes* iff all of its branches are either closed or marked. For more details on constructing tableaux for $\mathbf{CLuN^r}$, see [9, Sec. 5].

Lemma 1. *φ behaves reliably in view of Σ iff $\varphi \notin U(\Sigma)$.*

Proof. Step 1. Let $\mathcal{T}$ be a complete **CLuN**-tableau for $\Sigma \models \psi$. $Dab(\Delta) \in \Theta(\Sigma)$ iff $Dab(\Delta)$ is verified by all $\overline{\theta} \in \mathcal{T}$ iff the **CLuN**-tableau for $\Sigma \models Dab(\Delta)$ closes iff (by [9, Theorem 1]) $\Sigma \models_{\mathbf{CLuN}} Dab(\Delta)$ iff (by Theorem 1) $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$.

Step 2. $\Rightarrow$ Suppose $\varphi \in U(\Sigma)$. By the definition of $U(\Sigma)$, there is a Δ such that (i) $\varphi \wedge \neg\varphi \in \Delta$, (ii) $Dab(\Delta) \in \Theta(\Sigma)$, and (iii) there is no $\Delta' \subset \Delta$ such that $Dab(\Delta') \in \Theta(\Sigma)$. By step 1, there is a Δ such that (i) $\varphi \wedge \neg\varphi \in \Delta$, (ii) $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$, and (iii) there is no $\Delta' \subset \Delta$ such that $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta')$. Since $\Delta \backslash \{\varphi \wedge \neg\varphi\} \subset \Delta$, it follows by [D5] that φ does not behave reliably in view of Σ.

$\Leftarrow$ Suppose φ does not behave reliably in view of Σ. By [D5], there is a $Dab(\Delta)$ such that (i) $\varphi \wedge \neg\varphi \in \Delta$, (ii) $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$, and (iii) $\Sigma \nvdash_{\mathbf{LLD}} Dab(\Delta \backslash \{\varphi \wedge \neg\varphi\})$. Let $\Delta' \subseteq \Delta$ be the smallest subset of Δ such that $\Sigma \vdash_{\mathbf{LLD}} \Delta'$. By (iii), it follows that $\varphi \wedge \neg\varphi \in \Delta'$. By our construction, there is no $\Delta'' \subset \Delta'$ such that $\Sigma \vdash_{\mathbf{LLD}} \Delta''$. By step 1, (i) $\varphi \wedge \neg\varphi \in \Delta'$, (ii) $Dab(\Delta') \in \Theta(\Sigma)$, and (iii) there is no $\Delta'' \subset \Delta'$ such that $Dab(\Delta'') \in \Theta(\Sigma)$. By the definition of $U(\Sigma)$, $\varphi \in U(\Sigma)$. □

Lemma 2. *For finite premise sets Σ, if $\Sigma \vdash_{\mathbf{IAD}} \psi$ then $\Sigma \models_{\mathbf{CLuN^r}} \psi$.*

Proof outline. Suppose $\Sigma \not\models_{\mathbf{CLuN^r}} \psi$. (i) If $\Sigma \not\models_{\mathbf{CL}} \psi$, then clearly $\Sigma \nvdash_{\mathbf{IAD}} \psi$, for **IAD** and **CL** share the same set of particle rules, and the structural rules of **IAD** impose further restrictions on **P**-moves in **CL**-dialogues (in particular on the application of the particle rule for negation). So if **P** has no winning strategy for $\psi[\Sigma]$ in **CL**, then **P** has no winning strategy for $\psi[\Sigma]$ in **IAD**.

(ii) Suppose $\Sigma \models_{\mathbf{CL}} \psi$. Consider an open and unmarked branch θ of a completed $\mathbf{CLuN^r}$-tableau $\mathcal{T}$ for this inference (since $\Sigma \not\models_{\mathbf{CLuN^r}} \psi$, there is at least one).

(i) Locate the top node n of the first (highest) L-application of $(\mathsf{T}\neg **)$ on θ.[28]

(ii) Let $\mathsf{T}\neg\varphi$ be the formula occurring at n. Consider n's child node m at which the formula $\mathsf{F}\varphi$ occurs as a result of the R-application of $(\mathsf{T}\neg **)$ at n. Let λ be the rightmost branch in $\mathcal{T}$ which contains m.

[28] An *L-application* or left-application of $(\mathsf{T}\neg **)$ to a formula $\mathsf{T}\neg\varphi$ on a branch θ of a tableau is an application of this rule resulting in θ's containing both a node labeled $\mathsf{T}\neg\varphi$ and a node labeled $\mathsf{T}\varphi$. Analogously, an *R-application* or right-application of $(\mathsf{T}\neg **)$ on a branch θ of a tableau is an application of this rule resulting in θ's containing both a node labeled $\mathsf{T}\neg\varphi$ and a node labeled $\mathsf{F}\varphi$.

We show that the following holds:

(a) At step (i) λ always contains at least one L-application of $(\mathsf{T}\neg{**})$.
(b) λ always exists, and is a closed branch.

Ad. (a). Suppose that, at step (i), λ contains no L-applications of $(\mathsf{T}\neg{**})$. Then all nodes on λ are either starting nodes, nodes generated via **CLuN**-rules, or nodes generated via R-applications of $(\mathsf{T}\neg{**})$. Note that all R-applications of $(\mathsf{T}\neg{**})$ are just instances of the **CL**-rule $(\mathsf{T}\neg)$. Therefore, all nodes on λ are either starting nodes or nodes generated via **CL**-rules. Since $\mathcal{T}$ is complete, λ then corresponds to some branch in the **CL**-tableau for $\Sigma \models \psi$. But then λ is closed, since by our supposition $\Sigma \models_{\mathbf{CL}} \psi$.

Ad. (b). By (a) we know that there is always an L-application of $(\mathsf{T}\neg{**})$ on λ at step (i), with top node n. Therefore we will always be able to move to the right at step (ii), i.e. to move to the rightmost branch containing the child m of the R-application of $(\mathsf{T}\neg{**})$ at n. Since **CLuN**-trees are finitely generated and Σ is finite, there always is such a rightmost branch. Moreover, it is easily checked that λ contains only applications of **CL**-valid rules, and since $\mathcal{T}$ is complete and $\Sigma \models_{\mathbf{CL}} \psi$, λ is closed. In fact, λ is identical to some branch μ in the completed **CL**-tableau $\mathcal{T}'$ for $\Sigma \models \psi$.

Using the procedure from [12], we next transform $\mathcal{T}'$ into an extensive form tree for $\Sigma \vdash_{\mathbf{CL}} \psi$. In this extensive form, μ corresponds to a **CL**-dialogue d for $\psi[\Sigma]$. We modify d into an **IAD**-dialogue d':

(1) For each **P**-move n in d: if n is the result of an application of the particle rule for negation to some **O**-assertion $\neg\varphi$, add the condition $\{\mathfrak{R}^{\Sigma}_{\varphi}\}$ to n, unless **O** already challenged the **P**-assertion $\neg\varphi$ at some move $m < n$.
(2) By our construction, each conditional **P**-challenge corresponds to an application of $(\mathsf{T}\neg)$ in μ. Locate the first (highest) application of $(\mathsf{T}\neg)$ in μ. By our construction again, this application corresponds to the first (highest) R-application of $(\mathsf{T}\neg{**})$ to a formula $\mathsf{T}\neg\varphi$ occurring at a node n in θ. Since θ contains the L-application of $(\mathsf{T}\neg{**})$ to $\mathsf{T}\neg\varphi$, $\varphi \in Ab(\theta)$. Since θ is unmarked, $\varphi \in U(\Sigma)$. Hence there is a Δ such that $\varphi\wedge\neg\varphi \in \Delta$, $Dab(\Delta) \in \Theta(\Sigma)$, and there is no $\Delta' \subset \Delta$ such that $Dab(\Delta') \in \Theta(\Sigma)$.

   By our construction, the first (highest) application of $(\mathsf{T}\neg)$ in μ corresponds to a **P**-move in d challenging an **O**-assertion of the form $\neg\varphi$. Suppose that d contains an **O**-move $m < n$ challenging the **P**-assertion $\neg\varphi$. Then m would correspond to an application of $(\mathsf{F}\neg)$ to $\mathsf{F}\neg\varphi$ in λ and in θ. But then θ would be closed, since $\mathsf{T}\neg\varphi \in \theta$. Contradiction. Hence, d contains no **O**-move $m < n$ challenging $\neg\varphi$. Consequently, we added the condition $\mathfrak{R}^{\Sigma}_{\varphi}$ to n in step (1). Now let m be the last move in d. Add a new move $m+1$ of the form $\mathbf{O}-?\mathfrak{R}(Dab(\Delta)) - \emptyset - d$, challenging the conditional move $\mathbf{P}-!\varphi - \{\mathfrak{R}^{\Sigma}_{\varphi}\} - d$.

Re-label all d-moves as d'-moves. We show that **O** has a winning strategy in the continuation of d'. In accordance with the particle rule for the $\mathfrak{R}$-operator, **P** can now play using either the $\mathfrak{I}$-operator or the $\mathfrak{F}$-operator in move $m+2$. It does not matter for our purposes which option **P** picks. What matters is that, whichever alternative **P** chooses, **O** has a winning strategy for the ensuing subdialogue. Since $\varphi \in U(\Sigma)$, φ does not behave reliably in view of Σ (by Lemma 1). If **P** defends using the $\mathfrak{F}$-operator, she will lose the ensuing sub-dialogue since $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$. If she counter-attacks using the $\mathfrak{I}$-operator, she will lose the ensuing sub-dialogue since $Dab(\Delta)$ is minimal.

It is safely left to the reader to verify that the completion of d' in line with **O**'s winning strategy does not violate any of the **IAD**-rules. By [D6], $\Sigma \nvdash_{\mathbf{IAD}} \psi$. □

Lemma 3. *For finite premise sets Σ, if $\Sigma \models_{\mathbf{CLuN^r}} \psi$ then $\Sigma \vdash_{\mathbf{IAD}} \psi$.*

Proof outline. Suppose $\Sigma \nvdash_{\mathbf{IAD}} \psi$. (i) If $\Sigma \not\models_{\mathbf{CL}} \psi$, then $\Sigma \not\models_{\mathbf{CLuN^r}} \psi$, for the upper limit logic of **CLuN^r** is **CL** and adaptive logics are never stronger than their upper limit logic [6, Sec. 5].

(ii) Suppose $\Sigma \models_{\mathbf{CL}} \psi$. Since $\Sigma \nvdash_{\mathbf{IAD}} \psi$, and since whenever **P** has a winning strategy in **LLD**, **P** has a winning strategy in **IAD**, it follows that $\Sigma \nvdash_{\mathbf{LLD}} \psi$. We first show the following:

(†) There is a move m of the form $\mathbf{O}-!\neg\varphi$ such that:

(a) For all **P**-winning strategies s_p for $\psi[\Sigma]$ in **CL**, s_p assigns the move $\mathbf{P}-!\varphi$ to m;
(b) For all **O**-winning strategies s_o for $\psi[\Sigma]$ in **LLD**, s_o assigns m to a **P**-move $n < m$ and **P** cannot challenge m in view of [SR4.1]; and
(c) For all **O**-winning strategies s_o for $\psi[\Sigma]$ in **IAD**, there is a $\Delta \subset \Omega$ such that $\varphi \wedge \neg\varphi$ is indispensable to Δ and $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$, and s_o assigns the move $\mathbf{O}-?\mathfrak{R}^{\Sigma}_{\varphi}(Dab(\Delta)) - \emptyset - d$ to $\mathbf{P}-!\varphi - \{\mathfrak{R}^{\Sigma}_{\varphi}\} - d$.

Ad. (a) and (b). By our supposition, $\Sigma \models_{\mathbf{CL}} \psi$ while $\Sigma \nvdash_{\mathbf{LLD}} \psi$. Hence there is a **P**-winning strategy s_p for $\psi[\Sigma]$ in **CL** and there is an **O**-winning strategy s_o for $\psi[\Sigma]$ in **LLD**. This is possible only in view of [SR4.1], since this rule constitutes the only difference between **CL**-games and **LLD**-games. So there must be a move m of the form $\mathbf{O}-!\neg\varphi$ such that s_o assigns m to a **P**-move $n < m$, and such that in the case of **CL** s_p assigns the move $\mathbf{P}-!\varphi$ to m, while in the case of **LLD** **P** cannot challenge m in view of [SR4.1].

Ad. (c). In an **IAD**-game the moves m such that (a) and (b) hold for m are exactly those moves which **P** can challenge only conditionally. For **O** to have a winning strategy for $\psi[\Sigma]$ in **IAD** she must, for at least one move m such that (a) and (b) hold for m, challenge **P**'s conditional move by using the particle rule for the $\mathfrak{R}$-operator (this holds in view of [SR4.2]). In doing so, she must provide a formula $Dab(\Delta)$ such that (i) $\varphi \wedge \neg\varphi$ is indispensable to Δ and (ii) $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$. If (i), then **O** has a winning strategy in a sub-dialogue triggered by **P** using the $\mathfrak{I}$-operator. If (ii) then **O** has a winning strategy in a sub-dialogue triggered by **P**'s using the $\mathfrak{F}$-operator. Since there is a move m such that (a) and (b) hold for m, and since **O** indeed has a winning strategy for $\psi[\Sigma]$ in **IAD** (by our supposition), it follows that (c) holds for m.

Since $\Sigma \vdash_{\mathbf{CL}} \psi$, there is a **P**-winning strategy for $\psi[\Sigma]$ in **CL**. Equivalently (since Σ is finite and since the deduction theorem holds for **CL**), there is a **P**-winning strategy for $\bigwedge\Sigma \rightarrow \psi$ in **CL**. Call this strategy s_p. Using the algorithm from [12], transform the extensive form of s_p into a closed, complete **CL**-tableau $\mathcal{T}$ for $\bigwedge\Sigma \rightarrow \psi$. By the construction, move m from (†) corresponds to a node n in $\mathcal{T}$ labeled $\mathsf{T}\neg\varphi$. By (†(a)) and the construction, it follows that $\mathcal{T}$ contains an application of ($\mathsf{T}\neg$) to n, resulting in a node n' labeled $\mathsf{F}\varphi$.

We now make the following modifications to $\mathcal{T}$. First, remove the top node $\mathsf{F}\bigwedge\Sigma \to \psi$, label the new top node $\mathsf{T}\bigwedge\Sigma$ with a '•', and do the same for all signed formulas appearing at nodes which result from applying a rule to a formula labeled with a '•'. Next, for all branches θ on $\mathcal{T}$ which contain n and n': At $n'-1$, the predecessor of n' on θ, create a new branch such that $n'-1$ has two immediate successors n' and n''. n', as before, is labeled $\mathsf{F}\varphi$. The new node n'' is labeled $\mathsf{T}\varphi$. n'' (resp. n') can be seen as the result of an L-application (resp. an R-application) of ($\mathsf{T}\neg**$) to n. Complete the branch containing n'' using only ($\mathsf{T}\neg**$) and the tableau rules for **CLuN**.

We re-interpret all applications of ($\mathsf{T}\neg$) on branches θ containing n and n' in the original tableau $\mathcal{T}$ as R-applications of ($\mathsf{T}\neg**$), so that all branches resulting from our modifications to $\mathcal{T}$ are complete branches of the **CLuN$^{\mathbf{r}}$**-tableau for $\Sigma \models \psi$.

By Theorem 1 and our supposition, $\Sigma \not\models_{\mathbf{CLuN}} \psi$. By the construction, (†(b)), and the **CLuN**-redundancy of ($\mathsf{T}\neg**$), at least one of the newly obtained branches remains open. Call this branch ι. Suppose that ι contains the R-application of ($\mathsf{T}\neg**$) to n, and hence contains n as well as n'. Then by our construction ι is a branch of the original tableau $\mathcal{T}$. But $\mathcal{T}$ is closed, so the supposition is false and ι contains the L-application of ($\mathsf{T}\neg**$) to n. Consequently, ι contains n as well as n'', and $\varphi \in Ab(\iota)$.

By (†(c)) it follows that there is a $\Delta \subset \Omega$ such that $\varphi\wedge\neg\varphi$ is indispensable to Δ and $\Sigma \vdash_{\mathbf{LLD}} Dab(\Delta)$. By [D5] φ does not behave reliably in view of Σ. By Lemma 1, $\varphi \in U(\Sigma)$. Hence, since $\varphi \in Ab(\iota)$, ι remains unmarked.

In sum, we constructed a complete, open, and unmarked branch of the **CLuN$^{\mathbf{r}}$**-tableau for $\Sigma \models \psi$. Consequently, $\Sigma \not\models_{\mathbf{CLuN^r}} \psi$.

Theorem 2. *For finite premise sets* Σ, $\Sigma \vdash_{\mathbf{IAD}} \psi$ *iff* $\Sigma \models_{\mathbf{CLuN^r}} \psi$.

Proof. Immediate in view of Lemmas 2 and 3. □

6. Variation

In this section we discuss some easy-to-define variants of **IAD** making use of insights borrowed from dialogical logicians and adaptive logicians.

6.1. Intuitionistic inconsistency-adaptive dialogical logic

An intuitionistic version of **IAD** is obtained by substituting the classical development rule [SR1] for the intuitionistic development rule [SR1.1]:

[SR1.1][Intuitionistic development rule].

[For any move M in $\mathcal{P}$ such that $p_{\mathcal{P}}(M) > 2$ we have $F_{\mathcal{P}}(M) = [m', Z]$ where $Z \in \{A, D\}$ and $m' < p_{\mathcal{P}}(M)$. Let r be the repetition rank of player $\mathbf{X}$ and $\mathcal{P} \in \mathcal{D}(\psi[\varphi_1, \ldots, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) such that:

1. For every move $M = \mathbf{Y} - e - C - d$ in $\mathcal{P}$, with $p_{\mathcal{P}}(M) = m$ and such that e is a complex formula, let:

- $M_1, \ldots, M_n \in \mathcal{P}$ be the n moves of **X** such that $F_{\mathcal{P}}(M_1) = \cdots = F_{\mathcal{P}}(M_n) = [m, A]$,
- The sequence $\mathcal{P} \frown N$ be such that $N = X - e - C - d$ and $F_{\mathcal{P}\frown N}(N) = [m,\, A]$

Then we have $\mathcal{P} \frown N \in \mathcal{D}(\psi[\varphi_1,\, \ldots,\, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) if and only if $n < r$.

2. For every **Y**-move $M' \in \mathcal{P}$ such that $p_{\mathcal{P}}(M') = m'$ and $F_{\mathcal{P}}(M') = [k,\, A]$, let the sequence $\mathcal{P} \frown N'$ be such that $N' = \mathbf{X} - e - C - d$ and $F_{\mathcal{P}\frown N'}(N') = [m',\, D]$. We have $\mathcal{P} \frown N' \in \mathcal{D}(\psi[\varphi_1,\, \ldots,\, \varphi_n])$ (respectively $\mathcal{D}(\psi)$) if and only if the following conditions hold:
   - There is no $N'' \in \mathcal{P}$ such that $F_{\mathcal{P}}(N'') = [m',\, D]$,
   - For every **Y**-move $M'' \in \mathcal{P}$ such that $p_{\mathcal{P}}(M'') > m'$ we have: if $F_{\mathcal{P}}(M'') = [h,\, A]$, then there is an **X**-move $P \in \mathcal{P}$ such that $F_{\mathcal{P}}(P) = [p_{\mathcal{P}}(M''),\, D]$.]

Intuitively, this rule states that when it is **X**'s turn, she is allowed to attack any formula previously uttered by **Y** in accordance with the limit fixed by the chosen rank, or she can defend herself against the last non-answered attack.

Note that [SR1.1] applies to the entire dialogue, including its possible subdialogues. As a consequence, players defending a thesis $\varphi[\Sigma]$ in a subdialogue now have to show that this inference too is intuitionistically valid.

The difference between classical **IAD** and intuitionistic **IAD** is illustrated by the following dialogue:

	O				P		
d_1							
					$p[\neg\neg p]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3	$\neg\neg p$	$\emptyset$	0				
	$---$			3	$\neg p$	$\mathfrak{R}_p^{\Sigma}$	4
5	p	$\emptyset$	4		$---$		

Explanation In the above dialogue for intuitionistic **IAD**, **O** wins without even attacking the condition of **P**'s move 4. Note that clause 2 of [SR1.1] prohibits **P** from defending herself against **O**'s attack in move 3: she is only allowed to defend herself against the last unanswered **O**-attack, which is the one in move 5. Due to this restriction, **P** runs out of moves and loses the dialogue.

The situation is very different in classical **IAD**. Here, **P** *is* allowed to defend herself against **O**'s attack in move 3, by asserting p in a new move 6. In the continuation of the dialogue, **O** can still attack **P**'s conditional move 4 in line with the particle rule for the $\mathfrak{R}$-operator, but she will lose the ensuing subdialogue no matter which *Dab*-formula she proposes, since there are no *Dab*-formulas among the **LLD**-consequences of the premise set. Once the subdialogue ends, there are no more **O**-moves available in the main dialogue, so the inference is valid for classical **IAD**.

As a further illustration of the workings of intuitionistic **IAD**, note that there is no **P**-winning strategy for the thesis $q[p,\, \neg q \to \neg p,\, r \wedge \neg r]$ in the dialogue from Sect. 4.2. In intuitionistic **IAD** move 24 in this dialogue cannot be performed by **P**, since she can no longer defend herself against **O**'s attack in move 3. **P** runs out of moves, and **O** wins the dialogue.

Interestingly, intuitionistic **IAD** is not strictly weaker than classical **IAD**. For instance, the inference from the premise set $\{\neg q, r \vee q, (p \vee \neg p) \rightarrow (q \wedge \neg q)\}$ to the conclusion r is valid in intuitionistic **IAD**, but not in classical **IAD**. In an **IAD**-dialogue for this inference, **P** has a winning strategy only if she conditionally attacks the premise $\neg q$, assuming that q behaves reliably. Due to the invalidity of excluded middle in intuitionistic logic, the abnormality $q \wedge \neg q$ is an **LLD**-consequence of the premise set only if we play with the classical development rule [SR1], not if we play with the intuitionistic development rule [SR1.1]. Consequently, q does not behave reliably if we use [SR1], while it does if we use [SR1.1]. We leave it as an exercise to write out the dialogues for this inference in the respective **IAD**-variants.

6.2. A Simple Variant for a Simple Strategy

In certain argumentative contexts we may want to permit attacks on a slightly weaker condition than **IAD** does. For instance, instead of demanding that the abnormality corresponding to the condition of a **P**-move is not an indispensable part of some *Dab*-formula, we could implement the weaker demand that the abnormality in question is not a consequence of our premise set (independently of its being or not being part of some longer *Dab*-formula).

[D8][Simply ok formulas] [Let $\varphi[\Sigma]$ be the thesis of the Proponent. A formula ψ is *simply ok* with respect to Σ iff $\Sigma \nvdash_{\mathbf{LLD}} \psi \wedge \neg\psi$.]

If Σ is empty, then no disjunction of abnormalities is an **LLD**-consequence of Σ, so all formulas are simply ok with respect to the empty set. We implement this idea via a new operator, the $\mathfrak{S}$-operator. In the **IAD** negation rule [SR4.2], replace '$\mathfrak{R}^{\Sigma}_{\psi}$' with '$\mathfrak{S}^{\Sigma}_{\psi}$'. The latter expression abbreviates that ψ is simply ok with respect to Σ. The particle rule for the new operator is as follows:

Particle rule for the $\mathfrak{S}$-operator		
Assertion	Attack	Defence
$\mathbf{X}$–$!\varphi - \boldsymbol{\mathfrak{S}}^{\Sigma}_{\varphi} - d$	$\mathbf{Y}$–$?\boldsymbol{\mathfrak{S}}^{\Sigma}_{\varphi} - \emptyset - d$	$\mathbf{X}$–$!\mathfrak{F}_{\Sigma}(\varphi \wedge \neg\varphi) - \emptyset - d$

All the rest remains as before. Call the resulting logic **SIAD** or Simple Inconsistency-Adaptive Logic. **IAD** implements the reliability strategy of the adaptive logics framework, whereas **SIAD** implements the simple strategy. An advantage of **SIAD** over **IAD** is that it removes the element of choice whenever **X** attacks a conditional **Y**-move: as opposed to the particle rule for the $\mathfrak{R}$-operator, the particle rule for the $\mathfrak{S}$-operator leaves **X** with no alternatives for picking a *Dab*-formula in order to show that a formula asserted by **Y** behaves unreliably.

The main disadvantage of **SIAD** is that it is explosive whenever disjunctions of abnormalities cannot be reduced to one of their disjuncts. Let $\varphi_1, \ldots, \varphi_n \in \Omega$. Then if for some premise set Σ it holds that $\Sigma \vdash_{\mathbf{LLD}} \varphi_1 \vee$

$\dots \vee \varphi_n$ while $\Sigma \nvdash_{\mathbf{LLD}} \varphi_1, \dots, \Sigma \nvdash_{\mathbf{LLD}} \varphi_n$, then anything is a consequence of Σ. Let, for instance $\Sigma = \{\neg p, \neg q, p \vee q\}$:

$$\neg p, \neg q, p \vee q \vdash_{\mathbf{SIAD}} r \tag{35}$$

In order for **P** to win this inference in an **IAD**-dialogue or in an **SIAD**-dialogue, she needs to conditionally attack either the **O**-assertion $\neg p$ or the **O**-assertion $\neg q$. In the **IAD**-dialogue, the conditions corresponding to these moves are $\mathfrak{R}_p^\Sigma$ and $\mathfrak{R}_q^\Sigma$ respectively. None of p or q is reliable in view of the **LLD**-consequence $(p \wedge \neg p) \vee (q \wedge \neg q)$. As a result, **O** will have a winning strategy for the **IAD**-dialogue. In the **SIAD**-dialogue, the conditions corresponding to **P**'s attack are $\mathfrak{S}_p^\Sigma$ and $\mathfrak{S}_q^\Sigma$ respectively. Each of p and q is simply ok, since neither $p \wedge \neg p$ nor $q \wedge \neg q$ is an **LLD**-consequence. As a result, **P** will have a winning strategy for the **SIAD**-dialogue.

The simple strategy is just one of a number of alternative strategies that a player can adopt when conditionally attacking a statement by the other player—and, admittedly, not a recommendable one in contexts where disjunctions of abnormalities cannot be reduced to one of their disjuncts. Within the adaptive logics framework a number of other strategies are available for arguing in the possible presence of inconsistent information [5, 29]. The incorporation of these within the dialogical setting is left for future investigation.

6.3. Finite Dialogues for Infinite Premise Sets

In line with the pluralist turn recently taken by dialogicians, we define a way of dealing with infinite premise sets within the framework of dialogical logic. This need not go against the original motivation underlying dialogical logic, since in our proposal dialogical games are still finitary.

The only change needed is a slight addition to the particle rule for conditional assertions of the form $!\psi[\varphi_1, \dots, \varphi_n]$. We introduce the following particle rule for dealing with countably infinite premise sets Σ:

Assertion	Attack	Defence
$\mathbf{X}{-}!\psi[\Sigma] - \emptyset - d$ (where Σ is countably infinite)	$\mathbf{Y}{-}?\infty - \emptyset - d$	$\mathbf{X}{-}!\psi[\varphi_1, \dots, \varphi_n] - \emptyset - d$ (where $\varphi_1, \dots, \varphi_n \in \Sigma$)

The defensive move of **X** can then be attacked by **Y** in line with the particle rule for assertions of the form $!\psi[\varphi_1, \dots, \varphi_n]$ from Sect. 1.

The general idea is that whenever a player claims that a formula is a consequence of Σ, whether we are playing in the main dialogue or in a sub-dialogue, she must show that the formula in question is a consequence of some finite subset of Σ. As an illustration, consider the premise set $\Sigma = \{p \vee q, \neg q, (q \wedge \neg q) \vee (r_i \wedge \neg r_i), (q \wedge \neg q) \to (r_i \wedge \neg r_i) \mid i \in \mathbb{N}\}$, which we borrow from [7, Sec. 5]. We show that $\Sigma \vdash_{\mathbf{IAD}} p$.

	O				P		
d_1							
					$p[\Sigma]$	$\emptyset$	0
1	$n := 2$	$\emptyset$			$m := 2$	$\emptyset$	2
3	$?\infty$	$\emptyset$	2		$p[p \vee q, \neg q]$	$\emptyset$	4
5.1	$p \vee q$	$\emptyset$	4				
5.2	$\neg q$	$\emptyset$					
7	q	$\emptyset$	6	5.1	$?\vee$	$\emptyset$	6
				5.2	q	$\mathfrak{R}_q^\Sigma$	8
9	$?\mathfrak{R}_q^\Sigma((q \wedge \neg q) \vee (r_3 \wedge \neg r_3))$	$\emptyset$	8		$\mathfrak{I}_\Sigma(r_3 \wedge \neg r_3)$	$\emptyset$	10
11	$?\mathfrak{I}_\Sigma(r_3 \wedge \neg r_3)$	$\emptyset$	10				
$d_{1.1}$							
				11	$r_3 \wedge \neg r_3[\Sigma]$	$\emptyset$	12
13	$?\infty$	$\emptyset$	12		$r_3 \wedge \neg r_3$ $[(q \wedge \neg q) \vee (r_3 \wedge \neg r_3), (q \wedge \neg q) \rightarrow (r_3 \wedge \neg r_3)]$	$\emptyset$	14

Explanation The *Dab*-formula $(q\wedge\neg q)\vee(r_3\wedge\neg r_3)$ chosen by **O** at move 9 is just one of an infinite number of possible *Dab*-formulas containing $q \wedge \neg q$ between which **O** can choose (we chose 'r_3' arbitrarily). The formula in question is indeed a consequence of Σ, but the disjunct $q \wedge \neg q$ is dispensable. To show this, **P** uses the $\mathfrak{I}$-operator and picks a finite subset of Σ of which the shorter *Dab*-formula $r_3 \wedge \neg r_3$ is a consequence.

It is easily verified that **P** has a winning strategy in $d_{1.1}$, due to the **LLD**-validity of $(q \wedge \neg q) \vee (r_3 \wedge \neg r_3), (q \wedge \neg q) \rightarrow (r_3 \wedge \neg r_3) \vdash r_3 \wedge \neg r_3$. Since **P** wins the subdialogue, she will also win the main dialogue.

The strategy adopted by the Proponent in this dialogue readily generalizes. Whatever *Dab*-formula the Opponent picks, the Proponent always has an answer ready which gives her a winning strategy for the dialogue, provided that she picks a 'good' finite subset of Σ. Hence $\Sigma \vdash_{\mathbf{IAD}} p$.

7. Conclusion

Our main aim was to bring closer together the frameworks of dialogical logic and adaptive logic. This we achieved by combining some of the key features of both approaches, resulting in **IAD**, an inconsistency-adaptive dialogical logic. **IAD** is a defeasible extension of the paraconsistent dialogical logic **LLD**, which is in turn inspired by the work of Rahman and Carnielli [23]. The extension integrates elements from the adaptive approach within the framework of dialogical logic. The result is a non-explosive dialogical logic in which applications of the particle rule for negation are sensitive to the consistent behaviour of the premises. By interpreting logical validity in terms of a winning strategy in a two-player game, **IAD** gives a game-theoretic interpretation to its counterpart logic $\mathbf{CLuN^r}$ from the adaptive logics framework (see Sect. 5.2).

To illustrate the modularity of our approach, we provided a number of ways in which **IAD** can be adjusted for use in different contexts. For instance, if we replace the classical development rule with the intuitionist development rule, we readily obtain an intuitionistic variant of **IAD**. Likewise, we can replace the reliability operator of **IAD** with a different operator for modeling the simple strategy of adaptive logic. It remains to be seen whether other adaptive strategies can likewise be represented by similar operators within dialogical logic. Future work will also point out to what extent we can use the dialogical framework for explicating the reasoning steps underlying other adaptive consequence relations.

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