



Metalogical Decorations of Logical Diagrams

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Abstract. In recent years, a number of authors have started studying Aristotelian diagrams containing metalogical notions, such as tautology, contradiction, satisfiability, contingency, strong and weak interpretations of (sub)contrariety, etc. The present paper is a contribution to this line of research, and its main aims are both to extend and to deepen our understanding of metalogical diagrams. As for extensions, we not only study several metalogical decorations of larger and less widely known Aristotelian diagrams, but also consider metalogical decorations of another type of logical diagrams, viz. duality diagrams. At a more fundamental level, we present a unifying perspective which sheds new light on the connections between new and existing metalogical diagrams, as well as between object- and metalogical diagrams. Overall, the paper studies two types of logical diagrams (viz. Aristotelian and duality diagrams) and four kinds of metalogical decorations (viz. those based on the opposition, implication, Aristotelian and duality relations).

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1. Introduction

The Aristotelian square of oppositions is a diagram that displays four statements or notions, and certain logical relations holding between them. It has a very rich tradition, going back—together with the discipline of logic itself—to the works of Aristotle. Over the centuries, authors such as Avicenna, William of Sherwood, John Buridan, Boole and Frege have studied the square and other, larger Aristotelian diagrams [16, 17, 40, 48, 63, 67]. Since the beginning of

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the twenty-first century, logicians have started studying Aristotelian diagrams in a more systematic and mathematically precise way, which has led to significant theoretical advances, e.g. concerning the classification of the various types of Aristotelian diagrams [1, 59, 79, 81].

The overwhelming majority of Aristotelian diagrams that have been studied so far have *object-level* decorations: they visualize the Aristotelian relations holding between formulas coming from (the object language of) some given logical system (for example, propositional logic [55], modal logic [35], temporal logic [68], epistemic logic [50], deontic logic [56] and dynamic logic [22]), or between the natural language expressions of some given lexical field (for example, color terms [44], singular expressions [75] and subjective quantification [80]). In the last two or three years, however, authors such as Béziau and Seuren have also started investigating Aristotelian diagrams that are decorated with *meta-level* notions such as tautology, contradiction, satisfiability, contingency, strong and weak interpretations of (sub)contrariety, etc. [4, 5, 33, 72]. As will be explained later in this paper, this transition from object- to metalogical decorations of Aristotelian diagrams is quite important, since it sheds some interesting new light on the ways in which the Aristotelian relations are usually defined. It should also be emphasized that these metalogical diagrams are not only important from a strictly theoretical perspective, but are also relevant in more practically oriented contexts. For example, it has been argued that the metalogical square can be fruitfully used in teaching metalogic to certain groups of students [24].

In sum, then, metalogical diagrams are becoming an active topic of investigation in contemporary philosophical logic. The present paper is a contribution to this line of research, and its main aims are both to *extend* and to *deepen* our understanding of metalogical diagrams. We will now explain in some more detail how each of these aims will be achieved.

The paper offers extensions vis-à-vis previous work in at least three ways. First of all, we will study much *larger* metalogical diagrams: while previous work has focused on metalogical squares and hexagons, we will also study metalogical decorations for larger Aristotelian diagrams, such as octagons, dodecagons, and even three-dimensional polyhedra such as the rhombic dodecahedron. Secondly, we will also study metalogical decorations for *less widely known families* of Aristotelian diagrams. This applies not only to the larger diagrams mentioned in the first point (obviously, a rhombic dodecahedron is less familiar to most people than an ordinary square of oppositions), but also to the smaller diagrams, such as hexagons: previous work has focused exclusively on a single family of hexagons, viz. the Jacoby–Sesmat–Blanché hexagons, but in this paper we will also consider other, less widely known families of hexagons, such as the ‘Sherwood–Czezowski’ hexagons, the ‘unconnected-4’ hexagons and the ‘unconnected-8’ hexagons. Thirdly, we will show that metalogical decorations can be given not only for Aristotelian diagrams, but also for *other types* of logical diagrams, such as Hasse diagrams and, most importantly, duality diagrams.

There are also various ways in which the paper will deepen our understanding of metalogical diagrams. First of all, we will offer a more *precise* account of the relationship between object- and metalogical decorations of Aristotelian diagrams. Secondly, we will discuss the *connections* between the new metalogical diagrams presented in this paper and those that were previously studied. For example, we will show that the metalogical squares and hexagons studied by Béziau [4, 5], Diaconescu [33] and Seuren [72] can be seen as subdiagrams or specific instances of the diagrams presented here. Thirdly, because of these connections, the paper also offers a *unifying perspective* on metalogical diagrams. After all, if two seemingly unrelated metalogical diagrams can both be seen as specific instances of a single diagram, then they turn out to be intimately related to each other.

The paper is organized as follows. Sections 2 and 3 mainly prepare the ground for the analysis that will be presented in the later sections. Section 2 provides a series of increasingly more abstract ways of defining the Aristotelian relations, and discusses which of these definitions are best able to accommodate meta- as well as object-logical decorations of Aristotelian diagrams. Next, Sect. 3 provides some background on other types of logical relations, viz. the opposition, implication and duality relations. Sections 4–8 contain the core results of this paper: we present and study various metalogical decorations of various kinds of logical diagrams. Section 4 studies several Aristotelian diagrams that are decorated with the opposition relations, and discusses their connections with earlier work on metalogical decorations of Aristotelian diagrams. Next, Sect. 5 studies Aristotelian diagrams for the implication relations, and explores their connections with Aristotelian diagrams for the opposition relations (which were studied in Sect. 4) and for abstract ordering relations. Section 6 concludes our study of metalogical decorations for Aristotelian diagrams, by exploring some Aristotelian diagrams for the Aristotelian relations themselves, and for the duality relations. After this, we turn from metalogical Aristotelian diagrams to metalogical duality diagrams. In Sect. 7, we first study duality diagrams for the opposition and implication relations. Section 8 then explores some duality diagrams for the Aristotelian relations, and for the duality relations themselves. Finally, Sect. 9 summarizes the results that have been obtained in this paper, and highlights some general themes that we have touched upon along the way.

2. Defining the Aristotelian Relations

Before we start constructing new metalogical diagrams, it is important to clearly understand why a single type of diagrams—viz. Aristotelian diagrams—can be used for meta- as well as object-logical decorations. In other words, how is it possible for the Aristotelian relations to hold between metalogical statements as well as between object-logical formulas? The exact way in which the Aristotelian relations are defined turns out to be highly relevant in answering this question. In this section, we will introduce a series of increasingly more abstract ways of defining the Aristotelian relations, and discuss to what extent

these definitions are able to accommodate metalogical statements as well as object-logical formulas.

The oldest definition dates back to Aristotle himself, and has been used throughout the history of philosophical logic [62]. In contemporary work on Aristotelian diagrams, too, it is by far the most widely used definition [8, 9]. The formulation is entirely informal, and looks as follows:

Definition 2.1. Two statements φ and ψ are said to be

<i>contradictory</i>	iff	φ and ψ cannot be true together	and
		φ and ψ cannot be false together,	
<i>contrary</i>	iff	φ and ψ cannot be true together	and
		φ and ψ can be false together,	
<i>subcontrary</i>	iff	φ and ψ can be true together	and
		φ and ψ cannot be false together,	
<i>in subalternation</i>	iff	φ entails ψ	and
		ψ does not entail φ .	

Because of its informal nature, this definition can apply both to object- and to metalogical statements. To illustrate this, consider the ambiguity of the word ‘true’ in the definition of contrariety: this word can stand for ‘truth in a model’ (in case two object-logical formulas are said to be contrary), or for ‘absolute, informal truth’ (in case two metalogical statements are said to be contrary). A problem with Definition 2.1 is that it makes the Aristotelian relations entirely insensitive to the ‘background logic’. Which Aristotelian relation holds between two (object-logical) formulas partially depends on the logical system that is being assumed. The well-known issue of ‘existential import’ can be seen as an illustration of this problem: in classical syllogistics, there is a subalternation from $\forall x(Sx \rightarrow Px)$ to $\exists x(Sx \wedge Px)$, but in contemporary predicate logic, these formulas stand in no Aristotelian relation at all [30, Section 4]. In order to deal with this problem, a new and more precise way of defining the Aristotelian relations has been proposed [75, 79, 80]:

Definition 2.2. Let S be a logical system, which is assumed to have Boolean operators and a model-theoretic semantics \models . Two formulas $\varphi, \psi \in \mathcal{L}_S$ are said to be

<i>S-contradictory</i>	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \models \varphi \vee \psi,$
<i>S-contrary</i>	iff	$S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \varphi \vee \psi,$
<i>S-subcontrary</i>	iff	$S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \varphi \vee \psi,$
<i>in S-subalternation</i>	iff	$S \models \varphi \rightarrow \psi$	and	$S \not\models \psi \rightarrow \varphi.$

First of all, note that this definition stays very close to the original, more informal Definition 2.1. The condition that φ and ψ cannot be true together is formalized as $S \models \neg(\varphi \wedge \psi)$, i.e. S has no models in which φ and ψ are simultaneously true; similarly, the condition that φ and ψ cannot be false together is formalized as $S \models \varphi \vee \psi$, or equivalently, $S \models \neg(\neg\varphi \wedge \neg\psi)$, i.e. S has no models in which φ and ψ are simultaneously false. Furthermore, note that unlike Definition 2.1, Definition 2.2 is capable of dealing with the logic-sensitivity of the Aristotelian relations; for example, it is now possible to say

that two formulas are S_1 -contrary, but S_2 -contradictory, for logical systems S_1 and S_2 . The downside of explicitly specifying the logical system S , however, is that Definition 2.2 only applies to object-logical formulas: conditions such as $S \models \neg(\varphi \wedge \psi)$ and $S \models \varphi \rightarrow \psi$ are only meaningful for formulas φ and ψ from the object language \mathcal{L}_S .

A key insight of Definition 2.2 is that the Aristotelian relations are fully determined by the Boolean structure of the logical system S . This suggests a third and final way of defining these relations, which abstracts away from the concrete details of S , and only focuses on its Boolean structure:

Definition 2.3. Let $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \perp_{\mathbb{B}} \rangle$ be a Boolean algebra. Two elements $x, y \in B$ are said to be

- \mathbb{B} -contradictory iff $x \wedge_{\mathbb{B}} y = \perp_{\mathbb{B}}$ and $x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$,
- \mathbb{B} -contrary iff $x \wedge_{\mathbb{B}} y = \perp_{\mathbb{B}}$ and $x \vee_{\mathbb{B}} y \neq \top_{\mathbb{B}}$,
- \mathbb{B} -subcontrary iff $x \wedge_{\mathbb{B}} y \neq \perp_{\mathbb{B}}$ and $x \vee_{\mathbb{B}} y = \top_{\mathbb{B}}$,
- in \mathbb{B} -subalternation iff $x \wedge_{\mathbb{B}} y = x$ and $x \wedge_{\mathbb{B}} y \neq y$.

Because of its level of abstraction, this definition is fully general. First, note that it subsumes Definition 2.2 as a special case: if S is a logical system as specified in Definition 2.2 (i.e. having Boolean connectives), then its Lindenbaum–Tarski algebra $\mathbb{B}(S) := \{[\varphi] \mid \varphi \in \mathcal{L}_S\}$ is a Boolean algebra, and in the case of contrariety, for example, we have for all $\varphi, \psi \in \mathcal{L}_S$:¹

- φ and ψ are S -contrary iff $S \models \neg(\varphi \wedge \psi)$ and $S \not\models \varphi \vee \psi$
- iff $[\varphi] \wedge [\psi] = \perp$ and $[\varphi] \vee [\psi] \neq \top$
- iff $[\varphi]$ and $[\psi]$ are $\mathbb{B}(S)$ -contrary.

Another set of special cases of Definition 2.3 arises if we take the Boolean algebra \mathbb{B} to be the powerset $\wp(X)$ of some set X . In this case, two sets $X_1, X_2 \subseteq X$ are $\wp(X)$ -contrary iff $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 \neq X$ —in other words, iff X_1 and X_2 are disjoint but not exhaustive. If we take X to be the Lindenbaum–Tarski algebra $\mathbb{B}(S)$ of some (non-trivial) logical system S , we see how Aristotelian relations can hold between metalogical statements. For example, if we consider the sets $X_1 := \{\top\}, X_2 := \{\perp\} \subseteq \mathbb{B}(S)$, we see that X_1 and X_2 are $\wp(\mathbb{B}(S))$ -contrary, which means exactly that the metalogical properties of being an S -tautology and being an S -contradiction—or equivalently, the metalogical statements “ φ is an S -tautology” and “ φ is an S -contradiction”—are contrary to each other. Finally, it will also be interesting to consider the case $X := \mathbb{B}(S) \times \mathbb{B}(S)$, i.e. X is the set of all pairs of (equivalence classes of) formulas of the logical system S . Now we can define $X_1 := \{([\varphi], [\psi]) \mid \varphi$ and ψ are S -contrary $\}$ and $X_2 := \{([\varphi], [\psi]) \mid \varphi$ and ψ are S -subcontrary $\}$, and see that X_1 and X_2 are $\wp(\mathbb{B}(S) \times \mathbb{B}(S))$ -contrary, which means exactly that the metalogical relations of being S -contrary and being S -subcontrary—or equivalently, the metalogical statements “ φ and ψ are S -contrary” and “ φ and ψ are S -subcontrary”—are themselves contrary to each other.

¹ We make use here of the fact that the Aristotelian relations are only defined up to logical equivalence, i.e. if $\varphi \equiv_S \varphi'$ and $\psi \equiv_S \psi'$, then $R(\varphi, \psi)$ iff $R(\varphi', \psi')$, for all Aristotelian relations R . This can easily be derived from Definition 2.2.

Definition 2.3 thus explains why the Aristotelian relations can hold between meta- as well as object-logical statements: these relations are defined in a Boolean algebra, and this Boolean algebra can be situated at the object-logical level—in the the case of $\mathbb{B}(\mathbf{S})$ —, or at the metalogical level—in the cases of $\wp(\mathbb{B}(\mathbf{S}))$ and $\wp(\mathbb{B}(\mathbf{S}) \times \mathbb{B}(\mathbf{S}))$. In Sects. 4, 5 and 6 we will study metalogical decorations of Aristotelian diagrams that are based on both of the latter cases. We finish this section by summarizing the dialectical progression that has been developed:

- Definition 2.1 accommodates object- and metalogical statements, but fails to handle logic-sensitivity (at the object-level).
- Definition 2.2 is able to handle logic-sensitivity (at the object-level), but can only accommodate object-logical formulas.
- Definition 2.3 accommodates object- and metalogical statements, and is able to handle logic-sensitivity (at the object- and meta-level).

3. Opposition, Implication and Duality

In this section, we will introduce some other types of logical relations that are closely related to the Aristotelian relations, and that will play an important role later in the paper. Section 3.1 discusses the opposition and implication relations, which will be studied extensively in Sects. 4, 5 and 7. Next, Sect. 3.2 deals with the duality relations, which will figure prominently in Sects. 6 and 8.

3.1. The Opposition and Implication Relations

In [79] it is pointed out that the set of Aristotelian relations has a number of problematic, or at least peculiar, properties; for example, there is a clear conceptual asymmetry between the relations of contradiction, contrariety and subcontrariety on the one hand, and the relation of subalternation on the other. Furthermore, it is argued that these properties can best be explained by viewing the Aristotelian relations as being hybrid between two other types of relations, viz. the *opposition* and *implication* relations.

We will now define these two new types of relations. Recall that in Sect. 2, we showed that there are at least three distinct ways of defining the Aristotelian relations; in exactly the same way, there are also at least three ways of defining the opposition and implication relations. In this paper, however, we will only be dealing with these relations at the *object-logical* level (i.e. as holding between formulas of the language $\mathcal{L}_{\mathbf{S}}$ of some logical system \mathbf{S}), and thus there is no need for the full generality that was provided by Definition 2.3. Hence, we can define the opposition and implication relations in a way that is analogous to Definition 2.2 (if desired, the analogues of Definitions 2.1 and 2.3 could easily be given too).

Definition 3.1. Let \mathbf{S} be a logical system as in Definition 2.2. Two formulas $\varphi, \psi \in \mathcal{L}_{\mathbf{S}}$ are said to be

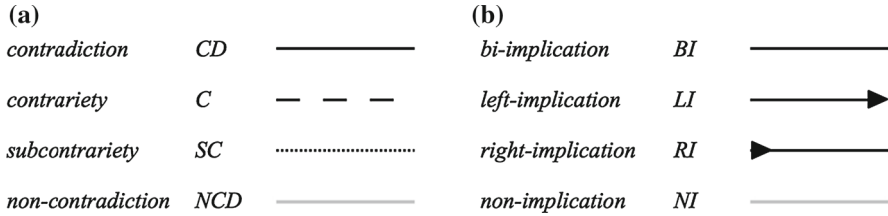


FIGURE 1. Code for visualizing **a** the opposition relations and **b** the implication relations

- S-contradictory* iff $S \models \neg(\varphi \wedge \psi)$ and $S \models \varphi \vee \psi$,
- S-contrary* iff $S \models \neg(\varphi \wedge \psi)$ and $S \not\models \varphi \vee \psi$,
- S-subcontrary* iff $S \not\models \neg(\varphi \wedge \psi)$ and $S \models \varphi \vee \psi$,
- S-non-contradictory* iff $S \not\models \neg(\varphi \wedge \psi)$ and $S \not\models \varphi \vee \psi$.

These relations are abbreviated as CD_S , C_S , SC_S and NCD_S , respectively. The set \mathcal{OG}_S consisting of these four relations is called the *opposition geometry* of S , i.e. $\mathcal{OG}_S := \{CD_S, C_S, SC_S, NCD_S\}$.

Definition 3.2. Let S be a logical system as in Definition 2.2. Two formulas $\varphi, \psi \in \mathcal{L}_S$ are said to be

- in S-bi-implication* iff $S \models \varphi \rightarrow \psi$ and $S \models \psi \rightarrow \varphi$,
- in S-left-implication* iff $S \models \varphi \rightarrow \psi$ and $S \not\models \psi \rightarrow \varphi$,
- in S-right-implication* iff $S \not\models \varphi \rightarrow \psi$ and $S \models \psi \rightarrow \varphi$,
- in S-non-implication* iff $S \not\models \varphi \rightarrow \psi$ and $S \not\models \psi \rightarrow \varphi$.

These relations are abbreviated as BI_S , LI_S , RI_S and NI_S , respectively. The set \mathcal{IG}_S consisting of these four relations is called the *implication geometry* of S , i.e. $\mathcal{IG}_S := \{BI_S, LI_S, RI_S, NI_S\}$.

The opposition and implication relations will be visualized using the code shown in Fig. 1. It should be noted that *S*-left-implication coincides with *S*-subalternation; consequently, the Aristotelian geometry $\mathcal{AG}_S := \{CD_S, C_S, SC_S, LI_S\}$ is *hybrid* between the opposition and implication geometries, i.e. $\mathcal{AG}_S \subseteq \mathcal{OG}_S \cup \mathcal{IG}_S$. Furthermore, the Aristotelian geometry is hybrid in an *informationally optimal* fashion. To explain this claim, [79] introduces a well-motivated information measure \mathfrak{m} that allows us to compare the information levels of any two opposition and/or implication relations. Next, note that any two formulas stand in exactly one opposition relation and in exactly one implication relation. Essentially, if the formulas happen to be *S*-contingent, it can be shown that one of these two relations is strictly more informative than the other (according to \mathfrak{m}) iff it is Aristotelian [79, Section 5].²

To illustrate this, consider the modal logic $S5$ and the $S5$ -contingent formulas $\Box p$, $\Diamond p$, $\Box \neg p$ and $\Diamond \neg p$. It is well-known that the Aristotelian relations holding between these formulas yield a classical Aristotelian square, which is shown in Fig. 2b. However, in exactly the same fashion, the opposition

² If neither of the two relations is strictly more informative than the other, then neither is Aristotelian, and thus the two formulas stand in no Aristotelian relation at all.

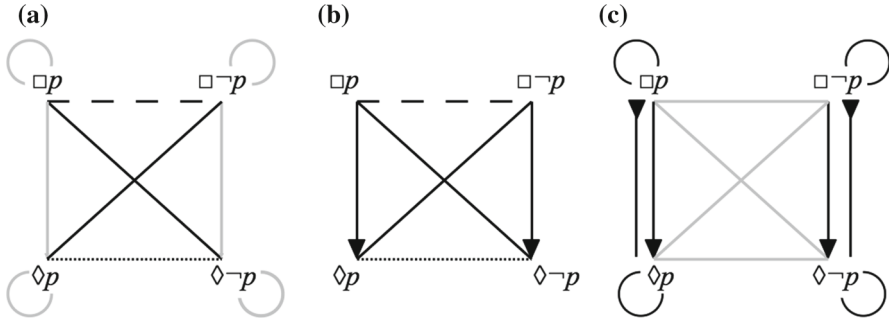


FIGURE 2. The **a** opposition, **b** Aristotelian and **c** implication square for $\{\Box p, \Box \neg p, \Diamond p, \Diamond \neg p\}$

and implication relations holding between these formulas yield opposition and implication squares, which are shown in Fig. 2a, c, respectively. Now consider some pairs of formulas:

- $\Box p$ and $\Box \neg p$: these stand in the opposition relation C_{S5} and the implication relation NI_{S5} ; according to **m**, C_{S5} is strictly more informative than NI_{S5} , and C_{S5} is an Aristotelian relation, while NI_{S5} is not.
- $\Diamond p$ and $\Box \neg p$: these stand in the opposition relation CD_{S5} and the implication relation NI_{S5} ; according to **m**, CD_{S5} is strictly more informative than NI_{S5} , and CD_{S5} is an Aristotelian relation, while NI_{S5} is not.
- $\Box p$ and $\Diamond p$: these stand in the opposition relation NCD_{S5} and the implication relation LI_{S5} ; according to **m**, LI_{S5} is strictly more informative than NCD_{S5} , and LI_{S5} is an Aristotelian relation, while NCD_{S5} is not.

The Aristotelian square in Fig. 2b is thus hybrid between the opposition and implication squares in Fig. 2a, c, respectively: every pair of formulas stands in an opposition relation and an implication relation, and the Aristotelian square visualizes exactly the more informative of these two relations.

The opposition and implication geometries were primarily introduced because of their relation to the Aristotelian geometry, but they also turn out to be very interesting in their own right. We now briefly mention some basic results from [79] that will be useful later in this paper.

Lemma 3.3. *For all formulas $\varphi, \psi \in \mathcal{L}_S$, the following hold:*

- | | | | | | | | |
|-----|------------------------|-----|----------------------------------|-----|-----------------------|-----|---------------------------------|
| 1a) | $CD_S(\varphi, \psi)$ | iff | $CD_S(\neg\varphi, \neg\psi)$, | 1b) | $BI_S(\varphi, \psi)$ | iff | $BI_S(\neg\varphi, \neg\psi)$, |
| 2a) | $C_S(\varphi, \psi)$ | iff | $SC_S(\neg\varphi, \neg\psi)$, | 2b) | $LI_S(\varphi, \psi)$ | iff | $RI_S(\neg\varphi, \neg\psi)$, |
| 3a) | $SC_S(\varphi, \psi)$ | iff | $C_S(\neg\varphi, \neg\psi)$, | 3b) | $RI_S(\varphi, \psi)$ | iff | $LI_S(\neg\varphi, \neg\psi)$, |
| 4a) | $NCD_S(\varphi, \psi)$ | iff | $NCD_S(\neg\varphi, \neg\psi)$, | 4b) | $NI_S(\varphi, \psi)$ | iff | $NI_S(\neg\varphi, \neg\psi)$. |

Lemma 3.4. *For all formulas $\varphi, \psi \in \mathcal{L}_S$, the following hold:*

1a)	$CD_S(\varphi, \psi)$	iff	$BI_S(\neg\varphi, \psi)$,	1b)	$CD_S(\varphi, \psi)$	iff	$BI_S(\varphi, \neg\psi)$,
2a)	$C_S(\varphi, \psi)$	iff	$RI_S(\neg\varphi, \psi)$,	2b)	$C_S(\varphi, \psi)$	iff	$LI_S(\varphi, \neg\psi)$,
3a)	$SC_S(\varphi, \psi)$	iff	$LI_S(\neg\varphi, \psi)$,	3b)	$SC_S(\varphi, \psi)$	iff	$RI_S(\varphi, \neg\psi)$,
4a)	$NCD_S(\varphi, \psi)$	iff	$NI_S(\neg\varphi, \psi)$,	4b)	$NCD_S(\varphi, \psi)$	iff	$NI_S(\varphi, \neg\psi)$,
5a)	$BI_S(\varphi, \psi)$	iff	$CD_S(\neg\varphi, \psi)$,	5b)	$BI_S(\varphi, \psi)$	iff	$CD_S(\varphi, \neg\psi)$,
6a)	$LI_S(\varphi, \psi)$	iff	$SC_S(\neg\varphi, \psi)$,	6b)	$LI_S(\varphi, \psi)$	iff	$C_S(\varphi, \neg\psi)$,
7a)	$RI_S(\varphi, \psi)$	iff	$C_S(\neg\varphi, \psi)$,	7b)	$RI_S(\varphi, \psi)$	iff	$SC_S(\varphi, \neg\psi)$,
8a)	$NI_S(\varphi, \psi)$	iff	$NCD_S(\neg\varphi, \psi)$,	8b)	$NI_S(\varphi, \psi)$	iff	$NCD_S(\varphi, \neg\psi)$.

Lemma 3.5. *For all formulas $\varphi, \psi \in \mathcal{L}_S$, the following hold:*

1. *there is exactly one $R \in \mathcal{OG}_S$ such that $R(\varphi, \psi)$,*
2. *there is exactly one $S \in \mathcal{IG}_S$ such that $S(\varphi, \psi)$.*

Lemma 3.5 essentially states that \mathcal{OG}_S and \mathcal{IG}_S are *partitions* of $\mathbb{B}(S) \times \mathbb{B}(S)$ (recall that $\mathbb{B}(S)$ denotes the Lindenbaum–Tarski algebra of S). In the contemporary study of Aristotelian diagrams, partitions play an important role, because of their intimate connection to *bitstring semantics* [25, 30, 82]. In this paper, this connection will not be explored in any detail; we will suffice by briefly describing the analogy between the partition-based bitstring semantics for classical propositional logic (CPL) and the partition-based bitstring semantics for the opposition and implication relations.

In CPL, the formulas p and q induce the partition $\{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$ of the class of all CPL-models (cf. the 4 rows in a truth table for a binary propositional connective); therefore, all Boolean combinations of formulas in this partition can be represented by bitstrings of length 4. Entirely analogously, the sets $\{([\varphi], [\psi]) \mid S \models \neg(\varphi \wedge \psi)\}$ and $\{([\varphi], [\psi]) \mid S \models \varphi \vee \psi\}$ induce the partition $\{CD_S, C_S, SC_S, NCD_S\} = \mathcal{OG}_S$ of $\mathbb{B}(S) \times \mathbb{B}(S)$; therefore, in this partition as well, all Boolean combinations of relations can be represented by bitstrings of length 4. Similar remarks apply to \mathcal{IG}_S .

3.2. The Duality Relations

The final set of logical relations that we will study consists of the well-known *duality relations*. Like the Aristotelian relations (but unlike the opposition and implication relations), the duality relations will be used in this paper at both the object- and the metalogical level, i.e. as holding between formulas/operators of the language \mathcal{L}_S of some logical system S as well as between metalogical statements/operators. They will therefore be defined in a fully general way (analogous to Definition 2.3 of the Aristotelian relations).

Definition 3.6. Consider Boolean algebras $\mathbb{A} = \langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, \neg_{\mathbb{A}}, \top_{\mathbb{A}}, \perp_{\mathbb{A}} \rangle$ and $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \perp_{\mathbb{B}} \rangle$. The duality relations between n -ary operators $O_1, O_2: \mathbb{A}^n \rightarrow \mathbb{B}$ are defined as follows:

- O_1 and O_2 are *identical* iff
 $\forall a_1, \dots, a_n \in A: O_1(a_1, \dots, a_n) = O_2(a_1, \dots, a_n)$,
- O_1 and O_2 are each other's *external negation* iff
 $\forall a_1, \dots, a_n \in A: O_1(a_1, \dots, a_n) = \neg_{\mathbb{B}} O_2(a_1, \dots, a_n)$,

- O_1 and O_2 are each other's *internal negation* iff
 $\forall a_1, \dots, a_n \in A: O_1(a_1, \dots, a_n) = O_2(\neg_{\mathbb{A}} a_1, \dots, \neg_{\mathbb{A}} a_n)$,
- O_1 and O_2 are each other's *dual* iff
 $\forall a_1, \dots, a_n \in A: O_1(a_1, \dots, a_n) = \neg_{\mathbb{B}} O_2(\neg_{\mathbb{A}} a_1, \dots, \neg_{\mathbb{A}} a_n)$.

These relations are abbreviated as ID, ENEG, INEG and DUAL, respectively. The set \mathcal{DG} consisting of these four relations is called the *duality geometry*, i.e. $\mathcal{DG} := \{\text{ID}, \text{ENEG}, \text{INEG}, \text{DUAL}\}$.

Although the duality relations are strictly speaking defined between operators $O_1, O_2: \mathbb{A}^n \rightarrow \mathbb{B}$, it will be convenient to also be able to apply them to the elements of \mathbb{B} directly: we will say that $O_1(a_1, \dots, a_n)$ and $O_2(a_1, \dots, a_n)$ stand in some duality relation R iff the operators O_1 and O_2 stand in R . Consider, for example, conjunction and disjunction in classical propositional logic (CPL): these can be viewed as binary operators $\wedge, \vee: \mathbb{B}(\text{CPL}) \times \mathbb{B}(\text{CPL}) \rightarrow \mathbb{B}(\text{CPL})$. It trivially holds for all $\varphi, \psi \in \mathcal{L}_{\text{CPL}}$ that $\varphi \wedge \psi \equiv_{\text{CPL}} \neg(\neg\varphi \vee \neg\psi)$, so by Definition 3.6 we have $\text{DUAL}(\wedge, \vee)$. Moving from operators to concrete formulas, we can also say that $\text{DUAL}(p \wedge q, p \vee q)$ and $\text{DUAL}(\neg p \wedge \neg q, \neg p \vee \neg q)$. In exactly the same way, it can be shown that $\text{INEG}(p \wedge q, \neg p \wedge \neg q)$, $\text{INEG}(p \vee q, \neg p \vee \neg q)$, $\text{ENEG}(p \wedge q, \neg p \vee \neg q)$ and $\text{ENEG}(p \vee q, \neg p \wedge \neg q)$. Using the visual code shown in Fig. 3a, we thus obtain the duality square shown in Fig. 3b.

When viewed as relations between elements of $\mathbb{B}(\mathcal{S})$, the duality relations ID and ENEG correspond exactly to the implication relation $BI_{\mathcal{S}}$ and the opposition relation $CD_{\mathcal{S}}$, respectively. Furthermore, looking at the duality square in Fig. 3b, it looks like DUAL corresponds to $LI_{\mathcal{S}}$, while INEG corresponds to both $C_{\mathcal{S}}$ and $SC_{\mathcal{S}}$. Observations such as these might explain why some authors [20, 58] have come close to straightforwardly identifying Aristotelian and duality squares, for example by using Aristotelian terminology to describe a duality square (or vice versa), or by viewing one as a generalization of the other.

However, it has also been argued extensively that although \mathcal{AG} and \mathcal{DG} sometimes yield similar-looking diagrams, they are two conceptually independent sets of logical relations, and should thus not be confused with each other [29, 54, 77, 84]. For example, it is easy to show that all duality relations are symmetric, i.e. for all $R \in \mathcal{DG}$ and for all operators O_1, O_2 , it holds that

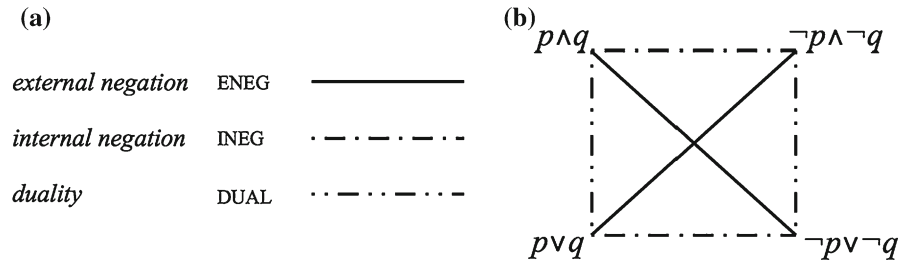


FIGURE 3. **a** Code for visualizing the duality relations; **b** duality square for $\{p \wedge q, \neg p \wedge \neg q, p \vee q, \neg p \vee \neg q\}$

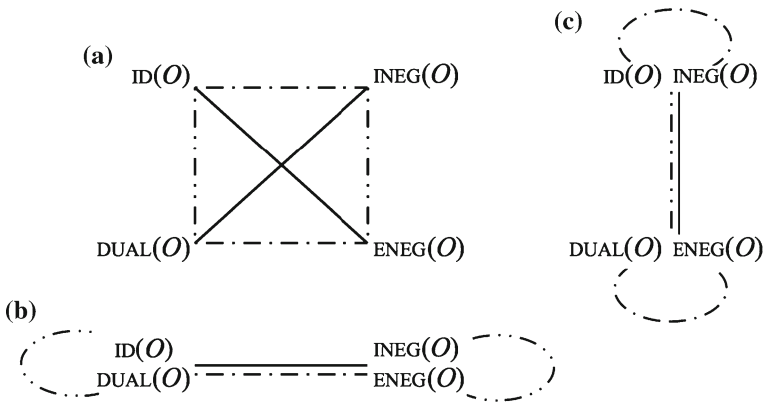


FIGURE 4. **a** Non-degenerated duality square, **b** degenerated duality diagram for operators that are their own dual, **c** degenerated duality diagram for operators that are their own internal negation

$R(O_1, O_2)$ iff $R(O_2, O_1)$; this clearly shows that DUAL cannot be identified with LI_5 , since the latter is asymmetric. Another important difference is that unlike the Aristotelian relations, the duality relations are *functional*, i.e. for all $R \in \mathcal{DG}$ and for all operators O_1 , there exists a unique operator O_2 such that $R(O_1, O_2)$.³ The duality relation R can thus be seen as a function, and we will often write $O_2 = R(O_1)$. For example, since $DUAL(\wedge, \vee)$, we can also say that $DUAL(\wedge) = \vee$. Since the duality relations are symmetric, the corresponding functions are idempotent, i.e. $R(R(O_1)) = O_1$ for all $R \in \mathcal{DG}$ and operators O_1 ;⁴ in other words: $R \circ R = ID$. More generally, it is well-known that when its elements are viewed as functions, \mathcal{DG} forms a *Klein four group* [36, 53, 65, 66, 83], which has the following Cayley table:

\circ	ID	ENEG	INEG	DUAL
ID	ID	ENEG	INEG	DUAL
ENEG	ENEG	ID	DUAL	INEG
INEG	INEG	DUAL	ID	ENEG
DUAL	DUAL	INEG	ENEG	ID

This Cayley table and the duality square in Fig. 4a might suggest that $ID(O)$, $ENEG(O)$, $INEG(O)$ and $DUAL(O)$ are pairwise distinct for all operators O . However, there are also exist operators O that are their own dual, i.e. $DUAL(O) = O = ID(O)$; it then follows that also $ENEG(O) = INEG(O)$, and thus the duality square in Fig. 4a degenerates into the horizontal du-

³ For example, the formula $p \wedge q$ is contrary to many formulas (e.g. to $\neg p$, to $\neg q$, to $\neg p \wedge \neg q$, etc.), but it has only one internal negation (viz. $\neg p \wedge \neg q$).

⁴ Proof: $R(O_1) = O_2 \Leftrightarrow R(O_1, O_2) \Leftrightarrow R(O_2, O_1) \Leftrightarrow R(O_2) = O_1$, and thus $R(R(O_1)) = R(O_2) = O_1$.

ality diagram in Fig. 4b. Viewing the elements of \mathcal{DG} as relations, we thus find that $\text{ID} \cap \text{DUAL} \neq \emptyset \neq \text{ENEG} \cap \text{INEG}$. A typical example of an operator that is its own dual is negation $\neg: \mathbb{B}(S) \rightarrow \mathbb{B}(S)$, since $\neg\varphi \equiv_S \neg\neg\neg\varphi$ for all $\varphi \in \mathcal{L}_S$. Completely analogously, there also exist operators O that are their own internal negation, i.e. $\text{INEG}(O) = O = \text{ID}(O)$; it then follows that also $\text{ENEG}(O) = \text{DUAL}(O)$, and thus the duality square in Fig. 4a degenerates into the vertical duality diagram in Fig. 4c. Again viewing the elements of \mathcal{DG} as relations, we find that $\text{ID} \cap \text{INEG} \neq \emptyset \neq \text{ENEG} \cap \text{DUAL}$. A typical example of an operator that is its own internal negation is the biconditional $\leftrightarrow: \mathbb{B}(\text{CPL}) \times \mathbb{B}(\text{CPL}) \rightarrow \mathbb{B}(\text{CPL})$, since $\varphi \leftrightarrow \psi \equiv_{\text{CPL}} \neg\varphi \leftrightarrow \neg\psi$ for all $\varphi, \psi \in \mathcal{L}_{\text{CPL}}$.⁵

The basic perspective on duality that has been described so far can be extended and generalized in various ways. For example, duality patterns for composed operators are studied in [21]. Another generalization, which will turn out to be very useful in Sect. 8, is the so-called *generalized Post duality* [41,51]. Recall that according to Definition 3.6, the INEG-relation involves negating *all* the operator’s arguments. Although the most canonical examples of duality indeed obey this requirement—e.g. the internal negation of $p \wedge q$ is $\neg p \wedge \neg q$; see Fig. 3b—, there are also important examples in which internal negation is applied to only one of the operator’s arguments, such as generalized quantifiers (on the relational perspective), subject negation, and the public announcement operator [22,29,37,45,47]. Generalized Post duality accommodates these examples, by ‘splitting’ the INEG-relation into n independent relations $\text{INEG}_1, \dots, \text{INEG}_n$; as a consequence, the DUAL-relation is also split into n independent relations $\text{DUAL}_1, \dots, \text{DUAL}_n$.

Definition 3.7. Consider Boolean algebras $\mathbb{A} = \langle A, \wedge_{\mathbb{A}}, \vee_{\mathbb{A}}, \neg_{\mathbb{A}}, \top_{\mathbb{A}}, \perp_{\mathbb{A}} \rangle$ and $\mathbb{B} = \langle B, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}}, \neg_{\mathbb{B}}, \top_{\mathbb{B}}, \perp_{\mathbb{B}} \rangle$. The generalized Post duality relations between n -ary operators $O_1, O_2: \mathbb{A}^n \rightarrow \mathbb{B}$ are defined as follows:

- ID and ENEG are defined as before (Definition 3.6),
- for $1 \leq i \leq n$, we define $\text{INEG}_i(O_1, O_2) :\Leftrightarrow \forall a_1, \dots, a_n \in A: O_1(a_1, \dots, a_n) = O_2(a_1, \dots, a_{i-1}, \neg_{\mathbb{A}} a_i, a_{i+1}, \dots, a_n)$,
- for $1 \leq i \leq n$, we define $\text{DUAL}_i(O_1, O_2) :\Leftrightarrow \forall a_1, \dots, a_n \in A: O_1(a_1, \dots, a_n) = \neg_{\mathbb{B}} O_2(a_1, \dots, a_{i-1}, \neg_{\mathbb{A}} a_i, a_{i+1}, \dots, a_n)$.

From the perspective of generalized Post duality, an n -ary operator O thus possesses $n + 1$ independent negation positions, viz. 1 external negation and n internal negations. Consequently, if $n > 1$, the generalized Post duality behavior of such an n -ary operator cannot be visualized by means of a simple square diagram, but rather requires an $(n + 1)$ -dimensional hypercube. For the binary operator of conjunction, for instance, we have $\text{INEG}_1(p \wedge q, \neg p \wedge q)$, $\text{DUAL}_1(p \wedge q, p \vee \neg q)$, $\text{INEG}_2(p \wedge q, p \wedge \neg q)$, $\text{DUAL}_2(p \wedge q, \neg p \vee q)$, etc.; all these facts can be visualized by means of a three-dimensional generalized Post duality

⁵ After having dealt with operators that coincide with their own dual or their own internal negation, one might wonder whether there are also operators that coincide with their own external negation. However, it is easy to show that if an operator $O: \mathbb{A}^n \rightarrow \mathbb{B}$ satisfies $\text{ENEG}(O) = O$, then \mathbb{B} has to be the trivial Boolean algebra in which $\perp_{\mathbb{B}} = \top_{\mathbb{B}}$.

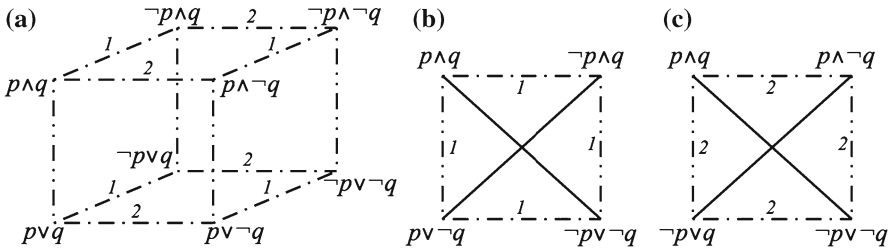


FIGURE 5. **a** Generalized Post duality cube, and **b**, **c** two generalized Post duality squares

cube, which is shown in Fig. 5a.⁶ Note that $INEG_1 \circ INEG_2$ and $ENEG \circ INEG_1 \circ INEG_2$ are the classical $INEG$ - and $DUAL$ -relations, respectively. Furthermore, it can be shown group-theoretically that the generalized Post duality cube in Fig. 5a contains exactly 14 duality squares as subdiagrams [21], two of which are shown in Fig. 5b, c.

Finally, it should be noted that there are binary operators O such that $INEG_1(O) = INEG_2(O)$ (in this case, it also holds that O is its own ‘classical’ internal negation, since $INEG(O) = (INEG_1 \circ INEG_2)(O) = (INEG_1 \circ INEG_1)(O) = ID(O) = O$). For example, for the operator $O: \mathbb{B}(S5) \times \mathbb{B}(S5) \rightarrow \mathbb{B}(S5): (\varphi, \psi) \mapsto \Box(\varphi \leftrightarrow \psi)$ it holds that $INEG_1(O) = INEG_2(O)$, since $\Box(\neg\varphi \leftrightarrow \psi) \equiv_{S5} \Box(\varphi \leftrightarrow \neg\psi)$.⁷ In such cases, the generalized Post duality cube degenerates into a generalized Post duality square, which is shown in Fig. 6a. If we ignore the classical $INEG$ - and $DUAL$ -relations, this square can be decomposed into a square for $INEG_1/DUAL_1$ and one for $INEG_2/DUAL_2$, which are shown in Fig. 6b, c, respectively.

4. Aristotelian Diagrams for the Opposition Relations

We have now arrived at the core sections of the paper, which will be devoted to constructing and studying various metalogical decorations of various kinds of logical diagrams. In this section, we will study Aristotelian diagrams for the

⁶ Note that in case of generalized duality diagrams, we attach subscripts to the diagram’s $INEG$ - and $DUAL$ -edges to indicate which $INEG_i$ or $DUAL_i$ -relation they represent. If an $INEG$ - or $DUAL$ -edge does not have any subscript, it still represents the classical $INEG$ - or $DUAL$ -relation, respectively. Finally, note that in the cube in Fig. 5a only $INEG_1$, $INEG_2$, and $DUAL$ have been visualized, for reasons of visual simplicity ($ENEG$ corresponds to the long diagonals of the cube, $INEG$ corresponds to the diagonals of the top and bottom faces of the cube, $DUAL_1$ corresponds to the diagonals of the front and back faces of the cube, and $DUAL_2$ corresponds to the diagonals of the left and right faces of the cube).

⁷ One might wonder why we do not stick to the simpler example of the biconditional (\leftrightarrow), since it also holds that $INEG_1(\leftrightarrow) = INEG_2(\leftrightarrow)$. However, for the biconditional we even have $INEG_1(\leftrightarrow) = INEG_2(\leftrightarrow) = ENEG(\leftrightarrow)$, and thus its generalized Post duality cube does not simply degenerate into a square, but even further, into a binary duality diagram resembling the one shown in Fig. 4c.

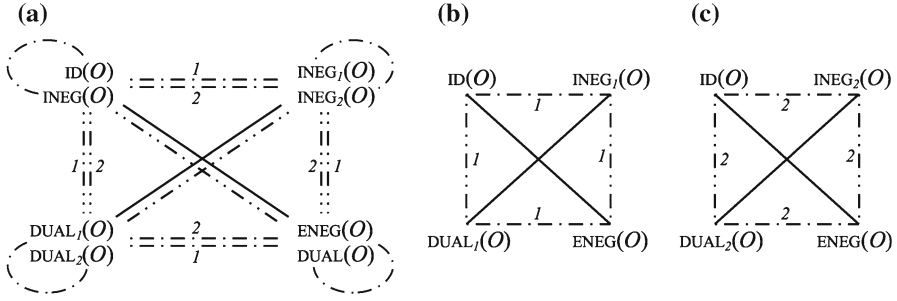


FIGURE 6. **a** Degenerated generalized Post duality cube for binary operators whose $INEG_1$ and $INEG_2$ coincide, and its decomposition into two generalized Post duality squares for **b** $INEG_1/DUAL_1$ and **c** $INEG_2/DUAL_2$

opposition geometry (\mathcal{OG}).⁸ Section 4.1 introduces an Aristotelian rhombic dodecahedron for \mathcal{OG} . Sections 4.2–4.5 study a number of interesting subdiagrams of this rhombic dodecahedron, and discuss their connections with earlier work on metalogical decorations of Aristotelian diagrams.

4.1. An Aristotelian Rhombic Dodecahedron for the Opposition Relations

It is well-known that if we take the Boolean closure of the CPL-fragment $\{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$, we get a Boolean algebra of 16 formulas, viz. the binary connectives applied to the propositional atoms p and q . This Boolean algebra can be visualized by means of an ordinary (two-dimensional) Hasse diagram. It was shown in [85] that this Hasse diagram can also be visualized as a three-dimensional polyhedron, viz. a rhombic dodecahedron (RDH). In [55, 78], however, it is shown that (a variant of) the RDH can also be used to visualize the Aristotelian relations holding in this Boolean algebra. Furthermore, the Hasse RDH and the Aristotelian RDH turn out to be intimately related to each other (this was already suggested in [74]; the mathematical details are worked out in [28]).

Given the close analogy between the CPL-fragment $\{p \wedge q, p \wedge \neg q, \neg p \wedge q, \neg p \wedge \neg q\}$ and the opposition geometry $\mathcal{OG} = \{CD, C, SC, NCD\}$ that was described at the end of Sect. 3.1 (in particular, see Lemma 3.5), it should not be surprising that very similar results can be obtained for \mathcal{OG} . We begin by considering the Boolean closure of \mathcal{OG} , i.e. $\wp^\cup(\mathcal{OG}) := \{\bigcup \mathcal{X} \mid \mathcal{X} \subseteq \mathcal{OG}\}$. This is a Boolean algebra with 16 elements; its bottom element is the empty relation over $\mathbb{B}(S)$, and its top element is $CD \cup C \cup SC \cup NCD = \mathbb{B}(S) \times \mathbb{B}(S)$, i.e. the universal relation over $\mathbb{B}(S)$. Figure 7 shows a (two-dimensional) Hasse diagram for $\wp(\mathcal{OG})$. Following Zellweger’s [85] suggestion, this Hasse diagram

⁸ Throughout this section (and the next ones), we will usually omit reference to the logical system S , and thus simply write \mathcal{OG} instead of \mathcal{OG}_S , CD instead of CD_S , etc.

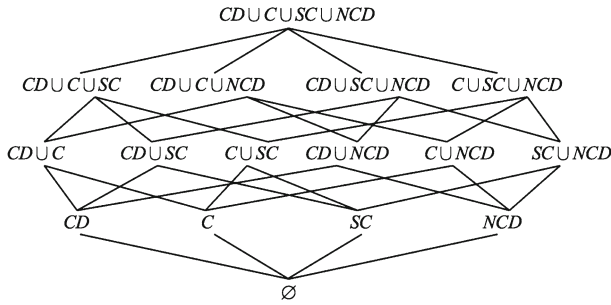


FIGURE 7. Two-dimensional Hasse diagram for $\wp^U(\mathcal{OG})$

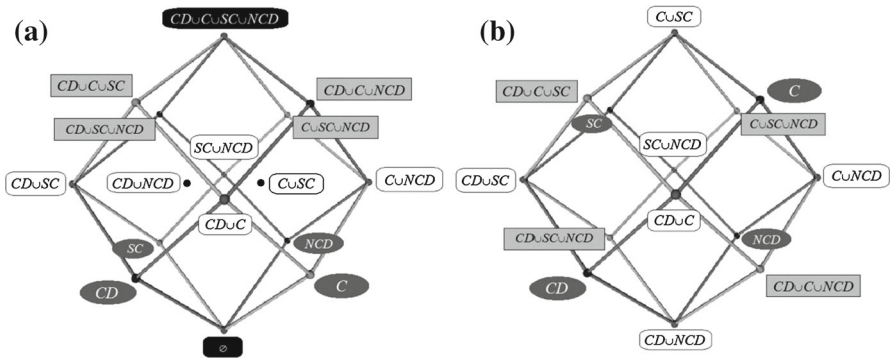


FIGURE 8. **a** Hasse and **b** Aristotelian rhombic dodecahedron for $\wp^U(\mathcal{OG})$

can also be visualized as a rhombic dodecahedron; this Hasse RDH for $\wp^U(\mathcal{OG})$ is shown in Fig. 8a. The corresponding Aristotelian RDH is shown in Fig. 8b.⁹

The elements in the Hasse RDH and the Aristotelian RDH are not formulas of some logical system \mathbb{S} —i.e. elements of $\mathbb{B}(\mathbb{S})$ —, but rather binary relations defined over that system—i.e. elements of $\wp(\mathbb{B}(\mathbb{S}) \times \mathbb{B}(\mathbb{S}))$ —; both of these RDHs are thus *metalogical* diagrams. In particular, the Aristotelian RDH for \mathcal{OG} in Fig. 8b is by far the largest and most complex Aristotelian diagram with a metalogical decoration that has been studied thus far.

The internal structure of the Aristotelian RDH for \mathcal{OG} can best be described in terms of its subdiagrams. For example, it is well-known that every Aristotelian RDH—regardless of its decoration—contains exactly six strong

⁹ Note that the Aristotelian RDH in Fig. 8b contains only 14 relations: like almost all Aristotelian diagrams in the literature, it does not contain $\wp^U(\mathcal{OG})$'s bottom element \emptyset and top element $\mathbb{B}(\mathbb{S}) \times \mathbb{B}(\mathbb{S})$. (It has been suggested that these elements are actually not *absent* from Aristotelian diagrams, but should rather be thought of as *coinciding* in the diagrams' centers of symmetry [69, 74]; recently, it has been shown that this suggestion is essentially correct, and can be derived from a general mathematical account of the relationship between Hasse diagrams and Aristotelian diagrams [28].)

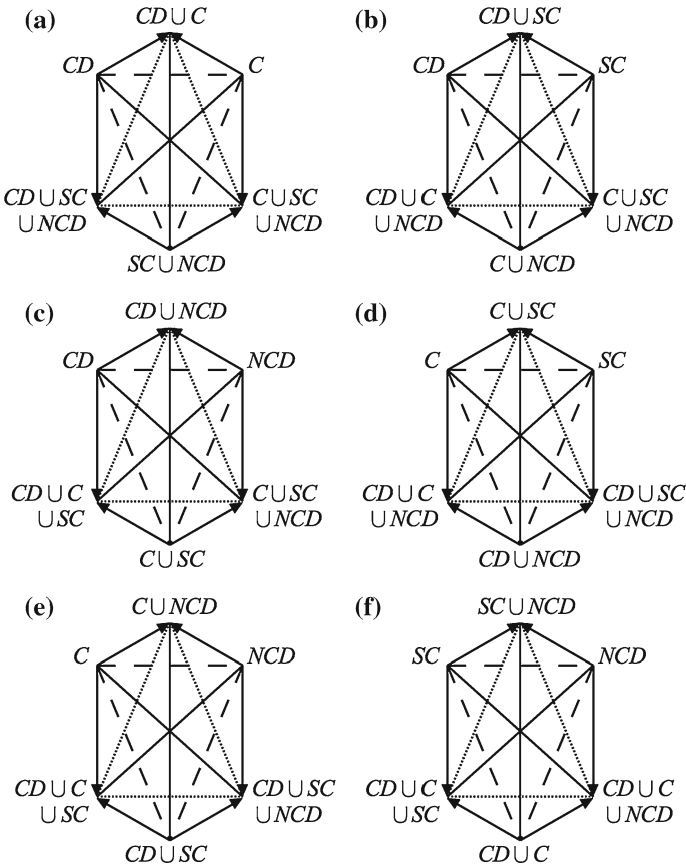


FIGURE 9. The six strong Jacoby–Sesmat–Blanché hexagons that are subdiagrams of the Aristotelian RDH for $\mathcal{O}\mathcal{G}$

Jacoby–Sesmat–Blanché hexagons [55,69,74,80].¹⁰ Figure 9 shows what these hexagons look like in the concrete case of the Aristotelian RDH for $\mathcal{O}\mathcal{G}$. In ongoing research [32,81], we are working on an exhaustive typology of *all* subdiagrams that can be found inside an Aristotelian RDH (again, regardless of its decoration). All these results can then straightforwardly be applied to the specific case of the Aristotelian RDH for $\mathcal{O}\mathcal{G}$. In the next few subsections, however, we will not strive for such exhaustiveness, but rather focus on some

¹⁰ The Jacoby–Sesmat–Blanché (JSB) hexagons are so-called because they were first studied in the 1950s by Jacoby [43], Sesmat [70] and Blanché [10–13] (back then not with metalogical decorations, of course). The distinction between strong and weak JSB hexagons was first introduced by Pellissier [64]. Using the terminology of Definition 2.3, a JSB hexagon is said to be *strong* iff the join of its three contrary elements is the top element of the Boolean algebra in which it is defined; it is said to be *weak* otherwise. For example, the JSB hexagon in Fig. 9a is strong, because its contrary elements are CD , C and $SC \cup NCD$, and their join $CD \cup C \cup (SC \cup NCD)$ is indeed the top element of $\wp^{\cup}(\mathcal{O}\mathcal{G})$.

specific subdiagrams of this Aristotelian RDH that turn out to be particularly interesting.

4.2. Béziau’s Hexagon for Tautology and Related Metalogical Notions

One of the subdiagrams of the Aristotelian RDH for \mathcal{OG} that turns out to be particularly interesting is the JSB hexagon shown in Fig. 10a. This diagram does not occur in the list of six given in Fig. 9, because it is a *weak* JSB hexagon; after all, the join of its contrary elements is $C \cup SC \cup NCD$, which is not the top element of $\wp^{\cup}(\mathcal{OG})$. Furthermore, this hexagon can also be reformulated using metalogical statements about formulas $\varphi, \psi \in \mathcal{L}_S$, as shown in Fig. 10b.

Let’s now see what happens if we fill in the same formula φ twice in these metalogical statements, as is shown in Fig. 11a. In terms of relations, this means that we are intersecting each relation $R \in \wp^{\cup}(\mathcal{OG})$ with the identity relation $\Delta := \{([\varphi], [\varphi]) \mid \varphi \in \mathcal{L}_S\}$ on $\mathbb{B}(S)$, and are thus no longer working in the Boolean algebra $\wp^{\cup}(\mathcal{OG})$, but rather in the Boolean algebra $\wp^{\cup}(\{R \cap \Delta \mid$

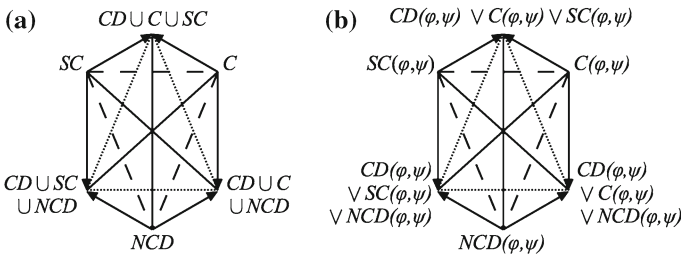


FIGURE 10. A weak Jacoby–Sesmat–Blanché hexagon inside the Aristotelian RDH for $\wp(\mathcal{OG})$, shown in terms of **a** relations and **b** statements

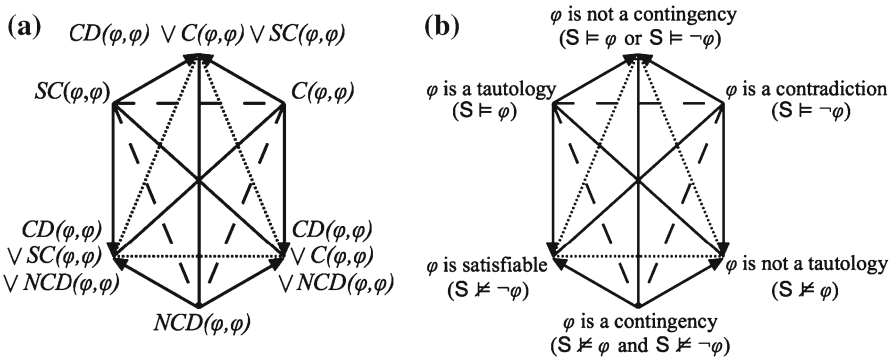


FIGURE 11. **a** A strong Jacoby–Sesmat–Blanché hexagon for statements of the form $R(\varphi, \varphi)$ (with $R \in \wp^{\cup}(\mathcal{OG})$), and **b** its reformulation using more familiar terminology

$R \in \mathcal{OG}\}) = \{(\bigcup \mathcal{X}) \cap \Delta \mid \mathcal{X} \subseteq \mathcal{OG}\}$. Assuming that the underlying logical system S is consistent,¹¹ we get:

$$\begin{aligned} CD(\varphi, \varphi) &\text{ iff } S \models \neg(\varphi \wedge \varphi) \text{ and } S \models \varphi \vee \varphi \text{ iff impossible,} \\ C(\varphi, \varphi) &\text{ iff } S \models \neg(\varphi \wedge \varphi) \text{ and } S \not\models \varphi \vee \varphi \text{ iff } \varphi \text{ is a contradiction,} \\ SC(\varphi, \varphi) &\text{ iff } S \not\models \neg(\varphi \wedge \varphi) \text{ and } S \models \varphi \vee \varphi \text{ iff } \varphi \text{ is a tautology,} \\ NCD(\varphi, \varphi) &\text{ iff } S \not\models \neg(\varphi \wedge \varphi) \text{ and } S \not\models \varphi \vee \varphi \text{ iff } \varphi \text{ is a contingency.} \end{aligned}$$

The impossibility of $CD(\varphi, \varphi)$ —i.e. the fact that $CD \cap \Delta = \emptyset$ —has an important consequence for the type of Aristotelian diagram that we are dealing with: even though the original JSB hexagon in Fig. 10 is a *weak* one, the new JSB hexagon in Fig. 11 is *strong*. After all, although $C \cup SC \cup NCD$ is not the top element of $\wp^\cup(\mathcal{OG})$, the fact that $CD \cap \Delta = \emptyset$ entails that $(C \cap \Delta) \cup (SC \cap \Delta) \cup (NCD \cap \Delta) = (C \cup SC \cup NCD) \cap \Delta$ is indeed the top element of $\wp^\cup(\{R \cap \Delta \mid R \in \mathcal{OG}\})$. Furthermore, we also find that

$$\begin{aligned} (\varphi, \varphi) \in CD \cup SC \cup NCD &\text{ iff } SC(\varphi, \varphi) \text{ or } NCD(\varphi, \varphi) \\ &\text{ iff } \varphi \text{ is a tautology or } \varphi \text{ is a contingency} \\ &\text{ iff } \varphi \text{ is satisfiable,} \\ (\varphi, \varphi) \in CD \cup C \cup NCD &\text{ iff } C(\varphi, \varphi) \text{ or } NCD(\varphi, \varphi) \\ &\text{ iff } \varphi \text{ is a contradiction or } \varphi \text{ is a contingency} \\ &\text{ iff } \varphi \text{ is not a tautology,} \\ (\varphi, \varphi) \in CD \cup C \cup SC &\text{ iff } C(\varphi, \varphi) \text{ or } SC(\varphi, \varphi) \\ &\text{ iff } \varphi \text{ is a contradiction or } \varphi \text{ is a tautology} \\ &\text{ iff } \varphi \text{ is not a contingency.} \end{aligned}$$

By filling in the same formula twice, the six relations in the original JSB hexagon in Fig. 10 thus turn out to correspond to some well-known metalogical notions, such as being a tautology, being satisfiable, being a contradiction and being contingent. At this point, it might be objected that two of the six relations do not correspond to a ‘primitive’ metalogical notion: $(CD \cup C \cup NCD) \cap \Delta$ and $(CD \cup C \cup SC) \cap \Delta$ can only be ‘negatively described’, as *not* being a tautology and *not* being a contingency, respectively. However, this discrepancy in lexicalization is entirely to be expected, since it is perfectly in line with previous, empirical work on natural language decorations of the classical square and strong JSB hexagon. Linguists have found exactly the same discrepancy in various closed lexical fields, such as the quantifiers and the temporal adverbs.¹² The correspondence between $\wp^\cup(\{R \cap \Delta \mid R \in \mathcal{OG}\})$ and the lexical field of tautology and related metalogical notions shows that this partial lexicalization pattern arises not only in natural languages, but even in metalogical jargon.

More importantly, however, because of this correspondence, the JSB hexagon in Fig. 11a can be reformulated as the more familiar JSB hexagon

¹¹ We will return to this assumption at the end of this subsection.

¹² Using the familiar A/I/E/O abbreviations for the elements of the square, and writing U and Y for the hexagon’s uppermost and lowermost elements, the non-lexicalized elements are exactly O and U. A pragmatic (neo-Gricean) explanation for the non-lexicalization of the O-element in the square has been developed in [38,39], and later extended to also account for the non-lexicalization of the U-element in the JSB hexagon [46,73].

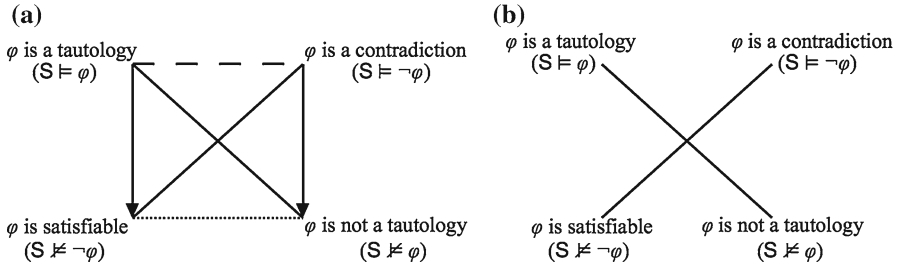


FIGURE 12. **a** A classical Aristotelian square for metalogical notions, on the assumption that S be consistent, **b** a degenerated Aristotelian ‘cross’ for the same metalogical notions, without the assumption that S be consistent

in Fig. 11b. This metalogical hexagon was first studied by Béziau in [4, Paragraph 3.2.4] and [5], and later also by Diaconescu [33]. What we have shown here is that this hexagon can be seen as (a special instance of) a subdiagram of the Aristotelian RDH for $\mathcal{O}\mathcal{G}$ that was introduced in the previous subsection.

Finally, it should be emphasized that the construction of Aristotelian diagrams for metalogical notions such as tautology and satisfiability crucially depends on the assumption that the underlying logical system S be *consistent*. To illustrate this, we will ignore (non-)contingency, and thus focus on the Aristotelian square in Fig. 12a (which can be seen as a subdiagram of the JSB hexagon in Fig. 11). The contrariety, the subcontrariety and the two subalternations in this square only hold if S is assumed to be consistent. For example, if S is not consistent, i.e. if there are no S -models, then we simultaneously have $S \models \varphi$ and $S \models \neg\varphi$, and thus lose the contrariety between φ being a tautology and φ being a contradiction. Similar remarks apply to the other Aristotelian relations in the square (except for the two contradictions). In sum, then, without the assumption that S be consistent, the metalogical square in Fig. 12a ‘degenerates’ into the Aristotelian ‘cross’ in Fig. 12b.

4.3. Aristotelian Hexagons for Strong and Weak (Sub)contrariety

Throughout the history of philosophical logic, the relations of contrariety and subcontrariety have been defined in two related, but subtly different ways. The resulting notions can be called *strong* and *weak* (sub)contrariety. Working in a logical system S ,¹³ the definitions look as follows:

¹³ Recall that in Sect. 2, we distinguished at least three ways of defining the Aristotelian relations, viz. Definitions 2.1, 2.2 and 2.3. The distinction between strong and weak (sub)contrariety can be made in each of these approaches. We choose to work in the line of Definition 2.2 here, because the distinction between strong and weak (sub)contrariety only plays a role at the object-logical level. (At the metalogical level, we exclusively work with the strong notions of (sub)contrariety throughout this paper.)

Definition 4.1. Two formulas $\varphi, \psi \in \mathcal{L}_S$ are said to be

<i>strongly S-contrary</i>	iff $S \models \neg(\varphi \wedge \psi)$	and	$S \not\models \varphi \vee \psi,$
<i>weakly S-contrary</i>	iff $S \models \neg(\varphi \wedge \psi),$		
<i>strongly S-subcontrary</i>	iff $S \not\models \neg(\varphi \wedge \psi)$	and	$S \models \varphi \vee \psi,$
<i>weakly S-subcontrary</i>	iff		$S \models \varphi \vee \psi.$

The strong notions of (sub)contrariety are defined in terms of a \models -condition and a $\not\models$ -condition; the corresponding weak notions keep the former, but discard the latter. Note that the notions of (sub)contrariety defined in Sects. 2 and 3 are the strong ones (also see Footnote 13). Other uses of the strong notions can be found in [62, 74, 79]. In contrast, the weak notions of (sub)contrariety are used in [14, 57, 71]. In recent years, the distinction itself has become the topic of some discussion. For example, Humberstone [42] links the strong and weak notions of (sub)contrariety to “traditionalist” and “modernist” approaches to logic, while Demey and Smessaert [27] show that the relation between the two notions can be understood in terms of Gricean pragmatics. Finally, and most relevant for our current purposes, Béziau [4, Paragraph 4.1.2] has used the strong and weak notions of (sub)contrariety to define a metalogical decoration for an Aristotelian hexagon.

Because of their definition, the strong notions of contrariety and subcontrariety trivially belong to \mathcal{OG} (recall Definition 3.1). However, the corresponding weak notions can also be expressed in terms of opposition relations:

$$\begin{aligned}
 (\varphi, \psi) \in CD \cup C & \text{ iff } CD(\varphi, \psi) \text{ or } C(\varphi, \psi) \\
 & \text{ iff } (S \models \neg(\varphi \wedge \psi) \text{ and } S \models \varphi \vee \psi) \text{ or} \\
 & \quad (S \models \neg(\varphi \wedge \psi) \text{ and } S \not\models \varphi \vee \psi) \\
 & \text{ iff } S \models \neg(\varphi \wedge \psi) \\
 & \text{ iff } \varphi \text{ and } \psi \text{ are weakly S-contrary,}
 \end{aligned}$$

and analogously, $(\varphi, \psi) \in CD \cup SC$ iff φ and ψ are weakly S-subcontrary. Using ‘s’- and ‘w’-subscripts for ‘strong’ and ‘weak’, respectively, we thus have:

$$\begin{aligned}
 C_s &= C, & SC_s &= SC, \\
 C_w &= CD \cup C, & SC_w &= CD \cup SC.
 \end{aligned}$$

The strong and weak notions of (sub)contrariety thus all belong to $\wp^U(\mathcal{OG})$, and hence occur in (subdiagrams of) the Aristotelian RDH that was constructed in Sect. 4.1. Consider, for example, the strong JSB hexagon that was already shown in Fig. 9a, but is repeated here (modulo some rotations and reflections) as Fig. 13a, and reformulated using the strong/weak terminology as Fig. 13b. This hexagon visualizes the Aristotelian relations holding between the notions of contradiction, strong contrariety and weak contrariety (and their negations).¹⁴ It can also be used to shed some new light on the interesting question *why* an otherwise well-regulated piece of scientific jargon such as the term *contrary* can come to be ambiguous.

¹⁴ Completely analogously, one can of course also construct a strong JSB hexagon for strong/weak subcontrariety instead of strong/weak contrariety; see Fig. 9b.

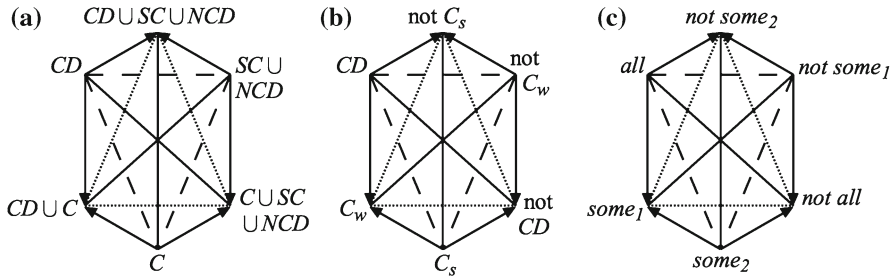


FIGURE 13. **a** A strong JSB hexagon with elements of $\wp^U(OG)$, **b** its reformulation in terms of weak and strong contrariety, and **c** an analogous strong JSB hexagon for the unilateral and bilateral interpretations of the natural language quantifier *some*

The point is that the term *contrary* is ambiguous in a highly systematic fashion: it has a weak interpretation (which is compatible with *contradiction*) and a strong interpretation (which is incompatible with *contradiction*). A strong JSB hexagon such as the one shown in Fig. 13c has been used in [73, p. 624] to explain why the natural language expression *some* is ambiguous between a *unilateral* interpretation *some*₁ (which is compatible with *all*, i.e. *at least one*) and a *bilateral* interpretation *some*₂ (which is incompatible with *all*, i.e. *some but not all*).¹⁵ The precise linguistic-cognitive details of this explanation need not concern us here, but given the striking analogy between *all/some*₁/*some*₂ and *CD/C_w/C_s*, it should not be surprising if a broadly similar account also applies to the latter.

4.4. An Aristotelian Octagon for Strong and Weak (Sub)contrariety

In the previous subsection, we constructed a strong JSB hexagon for strong and weak contrariety (see Figs. 9a, 13), and another strong JSB hexagon for strong and weak subcontrariety (see Fig. 9b and Footnote 14). So far, however, we have not yet considered the *interaction* between these two (pairs of) notions. The key insight in studying this interaction is that strong contrariety and strong subcontrariety are themselves contrary to each other.¹⁶ After all, as was already said in Sect. 2, it holds that $C \cap SC = \emptyset$ and $C \cup SC \neq \mathbb{B}(S) \times$

¹⁵ The distinction between unilateral and bilateral interpretations can also be made for other quantifiers, such as *many* and *few* [80, p. 484ff.], and even for richer, non-closed lexical fields [73, p. 640ff.]. For example, by replacing *all* and *some* by resp. *human* and *animal* in Fig. 13c, one can explain the ambiguity of the word *animal*, which has a biological interpretation (which is compatible with *human*) as well as a more ‘everyday life’ interpretation (which is incompatible with *human*). Extensive linguistic research has shown that systematic ambiguities such as these show up across a wide range of natural languages.

¹⁶ To re-emphasize a point that was already made earlier (see Footnote 13): note that in saying that strong contrariety and strong subcontrariety are themselves *contrary*, we are using the notion of *contrariety* (in italics) at the metalogical level, and are thus making use of the strong notion of contrariety. To put it more explicitly: strong contrariety and strong subcontrariety are themselves *strongly* contrary.

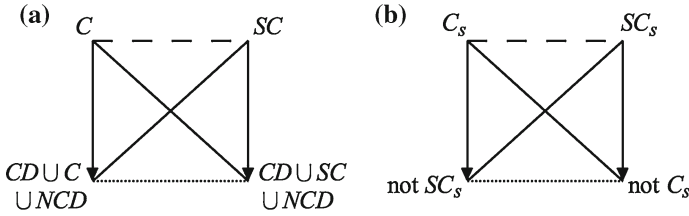


FIGURE 14. **a** Aristotelian square with elements of $\wp^U(\mathcal{OG})$, **b** its reformulation in terms of strong (sub)contrariety

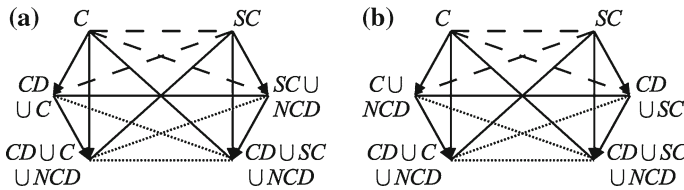


FIGURE 15. Two Sherwood–Czeżowski hexagons that result from adding **a** weak contrariety ($CD \cup C$) and **b** weak subcontrariety ($CD \cup SC$) to the square in Fig. 14

$\mathbb{B}(S)$, i.e. $C \cap SC$ is the bottom element, but $C \cup SC$ is not the top element of the Boolean algebra $\wp^U(\mathcal{OG})$. Consequently, strong contrariety and strong subcontrariety can be used to construct a classical Aristotelian square, which is shown in Fig. 14.

In order to integrate the weak notions of (sub)contrariety into this square, we need to ‘decompose’ its subalternations. First of all, recalling that weak contrariety is $C_w = CD \cup C$, we can add C_w to the square by decomposing its left subalternation into $C \rightarrow CD \cup C$ and $CD \cup C \rightarrow CD \cup C \cup NCD$. In order to keep the resulting diagram closed under contradiction, we also add the $\wp^U(\mathcal{OG})$ -complement of $CD \cup C$, viz. $SC \cup NCD$, by decomposing the square’s right subalternation into $SC \rightarrow SC \cup NCD$ and $SC \cup NCD \rightarrow CD \cup SC \cup NCD$. The resulting Aristotelian diagram is a Sherwood–Czeżowski hexagon, which is shown in Fig. 15a.¹⁷ Secondly, recalling that weak subcontrariety is $SC_w = CD \cup SC$, we can add weak subcontrariety in an entirely analogous fashion, and will thereby obtain a second Sherwood–Czeżowski hexagon, which is shown in Fig. 15b.

These two Sherwood–Czeżowski hexagons can be combined into a single Aristotelian diagram, viz. a Buridan octagon, which is shown in Fig. 16a.¹⁸

¹⁷ The Sherwood–Czeżowski hexagons are so-called because they were long thought to have first been studied in the 1950s by Czeżowski [19], but it has recently been argued that they were already used by the thirteenth-century logician William of Sherwood [48, 49].

¹⁸ The Buridan octagons are so-called because they were first studied by the fourteenth-century logician John Buridan [40, 67]. Buridan octagons that can be embedded as Aristotelian subdiagrams inside an Aristotelian RDH, have recently also been called ‘rubicubes’, based on the cube-like shape with two rhombic faces of this embedding

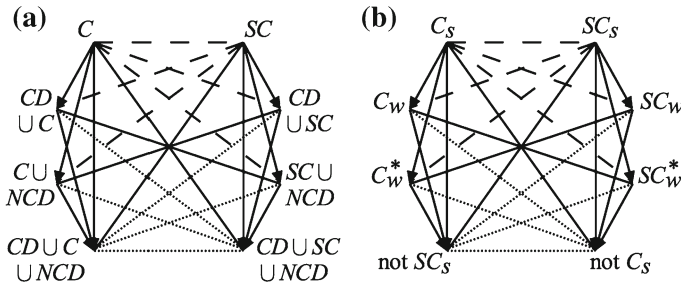


FIGURE 16. **a** Buridan octagon with elements of $\wp^{\cup}(\mathcal{O}\mathcal{G})$, **b** reformulation in terms of strong/weak (sub)contrariety

Note that the four relations in the ‘middle’ of this octagon (i.e. those that do not occur in the square in Fig. 14) do not stand in any Aristotelian relation at all (except for the obvious contradictions between $CD \cup C$ and $SC \cup NCD$, and between $C \cup NCD$ and $CD \cup SC$). Consider, for example, the relations $CD \cup C$ and $C \cup NCD$:

- since $(CD \cup C) \cap (C \cup NCD) = C \neq \emptyset$, it follows that $CD \cup C$ and $C \cup NCD$ are neither contradictory nor contrary,
- since $(CD \cup C) \cup (C \cup NCD) = CD \cup C \cup NCD \neq \mathbb{B}(S) \times \mathbb{B}(S)$, it follows that $CD \cup C$ and $C \cup NCD$ are not subcontrary,
- since $(CD \cup C) \cap (C \cup NCD) = C \neq CD \cup C$, it follows that there is no subalternation from $CD \cup C$ to $C \cup NCD$,
- since $(C \cup NCD) \cap (CD \cup C) = C \neq C \cup NCD$, it follows that there is no subalternation from $C \cup NCD$ to $CD \cup C$.

Recalling that the Buridan octagon in Fig. 16a is a subdiagram of the Aristotelian RDH for $\wp^{\cup}(\mathcal{O}\mathcal{G})$ that was described in Sect. 4.1, it should be noted that this Buridan octagon consists of exactly those relations $R \in \wp^{\cup}(\mathcal{O}\mathcal{G})$ that contain either C or SC but not both, i.e. such that $C \subseteq R$ iff $SC \not\subseteq R$. We can also construct the ‘complement’ of this Buridan octagon, which contains exactly those relations $R \in \wp^{\cup}(\mathcal{O}\mathcal{G})$ such that $C \subseteq R$ iff $SC \subseteq R$. It is well-known that the complement of a Buridan octagon is itself a strong JSB hexagon [78, 81],¹⁹ and hence, this complement is a strong JSB hexagon inside the Aristotelian RDH for $\wp^{\cup}(\mathcal{O}\mathcal{G})$, viz. the one shown in Fig. 9c.

Finally, and perhaps more importantly for our current purposes, the Buridan octagon in Fig. 16a also suggests that there is not just *one*, but actually *two* ways in which the strong notion of (sub)contrariety can be weakened.

Footnote 18 continued

[30, 32, 78, 80, 81]. The term ‘rhombicube’ was introduced recently, and derives from the fact that if we consider a Buridan octagon as an Aristotelian subdiagram embedded inside the Aristotelian RDH, it has a cube-like shape with two rhombic faces.

¹⁹ There are 8 relations $R \in \wp^{\cup}(\mathcal{O}\mathcal{G})$ such that $C \subseteq R$ iff $SC \subseteq R$, but two of them are $\wp^{\cup}(\mathcal{O}\mathcal{G})$ ’s bottom element \emptyset (for which it holds that $C \not\subseteq \emptyset$ and $SC \not\subseteq \emptyset$) and its top element $\mathbb{B}(S) \times \mathbb{B}(S)$ (for which it holds that $C \subseteq \mathbb{B}(S) \times \mathbb{B}(S)$ and $SC \subseteq \mathbb{B}(S) \times \mathbb{B}(S)$), and these do not occur inside any Aristotelian diagram (recall Footnote 9).

Recall that the strong notion of contrariety ($C_s = C$) is defined in terms of a \models -condition and a $\not\models$ -condition. As was discussed in the previous subsection, several authors have proposed a weaker version of contrariety by dropping the $\not\models$ -condition, thereby obtaining $C_w = CD \cup C$. The Buridan octagon suggests, however, that there is also an alternative notion of weak contrariety, viz. $C_w^* = C \cup NCD$. It is easy to see that this alternative notion of weak contrariety is ‘dual’ to the original one, in the sense that it can be obtained from strong contrariety by dropping the \models -condition instead of the $\not\models$ -condition. Looking again at Fig. 16a, we cannot distinguish between the two weak notions of contrariety, since $CD \cup C$ and $C \cup NCD$ stand in exactly the same Aristotelian relations to the relations of the original square (i.e. C , SC , $CD \cup C \cup NCD$ and $CD \cup SC \cup NCD$):

- both are entailed by strong contrariety (i.e. C),
- both entail the absence of strong subcontrariety (i.e. $CD \cup C \cup NCD$),
- both are contrary to strong subcontrariety (i.e. SC),
- and finally, both are subcontrary to the absence of strong contrariety (i.e. $CD \cup SC \cup NCD$).

Similar remarks can be made, of course, for subcontrariety: in addition to strong subcontrariety ($SC_s = SC$) and the original notion of weak subcontrariety ($SC_w = CD \cup SC$), we also get a new notion of weak subcontrariety ($SC_w^* = SC \cup NCD$) that is ‘dual’ to the original one. Using this new terminology, the relations in the Buridan octagon in Fig. 16a can be reformulated more evocatively as in Fig. 16b.

4.5. Correcting Béziau’s Hexagon for Strong and Weak (Sub)contrariety

In the previous two subsections, we have constructed Aristotelian diagrams for the strong and weak notions of contrariety, and their interaction. In each of these diagrams, the relation of contradiction played an important role, in the sense that CD is a subset of exactly half of the relations at its vertices (for example, in Buridan octagon in Fig. 16, it holds that $CD \subseteq C_w, SC_w$, not C_s , not C_s). However, none of these diagrams contains CD ‘by itself’. Béziau [4, Paragraph 4.1.2] has studied the interaction between CD and strong/weak (sub)contrariety), and claims that this interaction can be visualized by means of a JSB hexagon [4, Figure 40], which is shown here as Fig. 17a. Note that Béziau [4, p. 30] explicitly defines the notions of strong (sub)contrariety and weak (sub)contrariety, and his definition corresponds exactly to our Definition 4.1. However, Béziau does not state explicitly how he understands the relation labeled as ‘not CD ’ in his hexagon; we will return to this missing definition very soon (see Footnote 21).

Upon closer examination, Béziau’s hexagon turns out to be only partially correct; more specifically, we find:

- the three contrarities and the six subalternations are correct,
- the three contradictions are incorrect,
 - for example, C_s and SC_w are said to be contradictory, but since $C_s \cap SC_w = C \cap (CD \cup SC) = \emptyset$ and $C_s \cup SC_w = C \cup (CD \cup SC) = CD \cup C \cup SC \neq \mathbb{B}(S) \times \mathbb{B}(S)$, these relations are contrary, rather than contradictory,

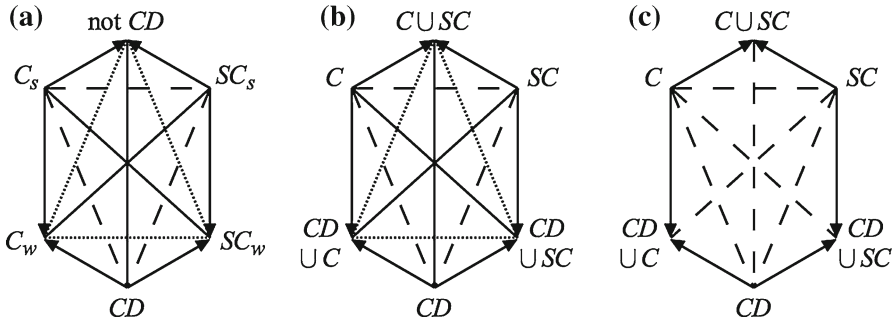


FIGURE 17. **a** Béziau’s partially correct hexagon, **b** a plausible reformulation in terms of elements of $\wp^U(\mathcal{O}\mathcal{G})$, and **c** the corrected version

- the three subcontrarities are incorrect, for example, C_w and SC_w are said to be subcontrary, but since $C_w \cup SC_w = (CD \cup C) \cup (CD \cup SC) = CD \cup C \cup SC \neq \mathbb{B}(S) \times \mathbb{B}(S)$, these relations are *not* subcontrary; in fact, it can be shown that they do not stand in any Aristotelian relation at all.

The contradictions in Béziau’s hexagon in Fig. 17a, b should thus be replaced with contrarities, and similarly, the subcontrarities should be deleted (and not replaced with any Aristotelian relation at all). Figure 17c shows the Aristotelian diagram that is the result of making these corrections to Béziau’s original hexagon. It should be emphasized that this corrected hexagon is highly exotic, since it does not obey the nearly universally accepted principle that Aristotelian diagrams should consist of pairs of contradictory formulas.²⁰

Based on the observations made above, a uniform explanation can be given for the errors in the hexagon in Fig. 17a, b: Béziau seems to have ignored the relation NCD . After all, the incorrect contradictions and subcontrarities are based on the assumption that the universal relation over $\mathbb{B}(S)$ is $CD \cup C \cup SC$, whereas actually it is $CD \cup C \cup SC \cup NCD$.²¹ Putting it in less abstract terms, Béziau seems to have ignored the fact that there exist pairs

²⁰ The systematic study of such non-standard Aristotelian diagrams is still in its infancy; some preliminary results can be found in [60, 61].

²¹ Whether this explanation also applies to the contradiction between CD and ‘not CD ’, depends on how the latter relation is interpreted (recall that Béziau does not explicitly say what he means by ‘not CD ’). If ‘not CD ’ is interpreted as $C \cup SC \cup NCD$, then CD and ‘not CD ’ are indeed contradictory to each other, and this relation in Béziau’s hexagon in Fig. 17a is correct after all. However, given the uniform explanation for the two other contradictions and the three subcontrarities, it seems far more likely that Béziau means ‘not CD ’ to be interpreted as $C \cup SC$ —this interpretation is also assumed in Fig. 17b—and in that case, it is incorrect to say that CD and ‘not CD ’ are contradictory to each other (since $CD \cup (C \cup SC) \neq \mathbb{B}(S) \times \mathbb{B}(S)$).

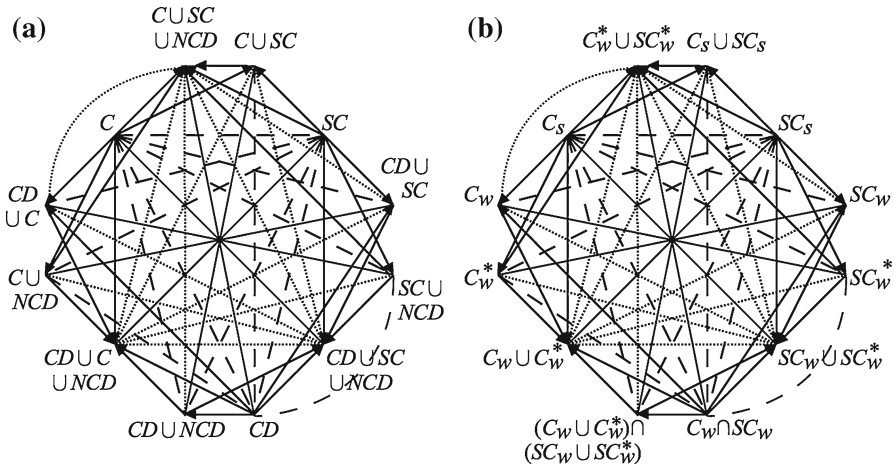


FIGURE 18. **a** The complement-closure of Béziau’s hexagon, **b** its reformulation in terms of strong/weak/weak* (sub)contrariety

of formulas—e.g. the pairs (p, q) and (p, p) in CPL—that are not contradictory, not contrary, and not subcontrary to each other (all we can say is that $NCD_{CPL}(p, q)$ and $NCD_{CPL}(p, p)$).²²

A correct Aristotelian diagram for the interaction between CD and strong/weak subcontrariety can be obtained by considering the *complement-closure* of Béziau’s hexagon, i.e. the Aristotelian diagram that is obtained by adding the complements of all relations that occur in the original diagram.²³ The resulting diagram is a dodecagon, which is shown in Fig. 18. This dodecagon contains all the other Aristotelian diagrams shown in Sects. 4.3–4.5 as subdiagrams. Comparing this dodecagon to the Aristotelian RDH that was described in Sect. 4.1, we see that it lacks only two relations, namely precisely those that were originally forgotten by Béziau: NCD and its contradictory, $CD \cup C \cup SC$.

One can also adopt an alternative perspective on the relationship between Béziau’s hexagon in Fig. 17a and the Aristotelian RDH for \mathcal{OG} in Fig. 8b. Recall that $\wp^{\cup}(\mathcal{OG})$ consists of 16 relations, which are of the form $\bigcup \mathcal{X}$, for $\mathcal{X} \subseteq \mathcal{OG}$. Now, if $NCD = \emptyset$, then these 16 relations collapse pairwise into 8 relations; for example, $CD \cup NCD = CD \cup \emptyset = CD$. The table below describes all these collapses:

²² Using Pellissier’s [64] terminology, yet another reformulation might look as follows: Béziau takes the hexagon in Fig. 17 to be a *strong* JSB hexagon, but actually it is a *weak* JSB hexagon. However, this reformulation has to be taken with a grain of salt: Béziau’s hexagon is incorrect in its contradictions and subcontrarieties, so it is not a proper JSB hexagon to begin with, and thus the question whether it is a strong or a weak one does not even arise, strictly speaking.

²³ For most Aristotelian diagrams, this operation does not make much sense, since they are already closed under negation, and are thus their own complement-closure.

RDH		collapse	collapse		RDH
CD	\rightarrow	CD	$C \cup SC$	\leftarrow	$C \cup SC \cup NCD$
$CD \cup NCD$	\nearrow			\nwarrow	$C \cup SC$
C	\rightarrow	C	$CD \cup SC$	\leftarrow	$CD \cup SC \cup NCD$
$C \cup NCD$	\nearrow			\nwarrow	$CD \cup SC$
SC	\rightarrow	SC	$CD \cup C$	\leftarrow	$CD \cup C \cup NCD$
$SC \cup NCD$	\nearrow			\nwarrow	$CD \cup C$
NCD	\rightarrow	$[\emptyset]$	$[CD \cup C \cup SC]$	\leftarrow	$CD \cup C \cup SC$
$[\emptyset]$	\nearrow			\nwarrow	$[CD \cup C \cup SC \cup NCD]$

In this table, square brackets indicate that a relation is the top- or bottom element in its respective Boolean algebra, and is thus not included in Aristotelian diagrams (recall Footnote 9). We thus see that the $16 - 2 = 14$ relations of the Aristotelian RDH for \mathcal{OG} pairwise collapse into the $\frac{16}{2} - 2 = 6$ relations of Béziau’s hexagon.²⁴ In other words, Béziau’s hexagon is correct after all (and plays a role analogous to our RDH), if only we are prepared to make the assumption that $NCD = \emptyset$. This leads to a more charitable interpretation of Béziau’s hexagon, since the only substantial criticism that can now be made is that he should have stated more explicitly that he was working under the assumption that $NCD = \emptyset$. The problem with this interpretation, however, is that the assumption that $NCD = \emptyset$ is itself extremely unrealistic. For example, in any consistent logical system S , if φ is S -contingent, it holds that $NCD_S(\varphi, \varphi)$, and thus definitely $NCD_S \neq \emptyset$. In all reasonable logical systems, it will thus be the case that $NCD \neq \emptyset$, which brings us back to the original conclusion that Béziau’s hexagon in Fig. 17 is partially incorrect, and should be replaced by the dodecagon in Fig. 18 or even the full RDH in Fig. 8b.

We will finish this section with a more general remark. In recent years, Béziau has studied at least two metalogical decorations of Aristotelian diagrams, viz. a JSB hexagon for tautology, satisfiability and related notions—see Fig. 11b—and a JSB hexagon for the interaction between contradiction and strong and weak notions of (sub)contrariety—see Fig. 17a. Until now, these metalogical hexagons appeared to be two ‘independent’ Aristotelian diagrams. In this section, however, we have shown that these hexagons are intimately related to each other, since both of them can be seen as (special instances of) subdiagrams of one and the same metalogical diagram, viz. the Aristotelian RDH for \mathcal{OG} that was described in Sect. 4.1.

5. Aristotelian Diagrams for the Implication Relations

In this section we continue our study of metalogical decorations for Aristotelian diagrams, by considering various Aristotelian diagrams for the implication geometry (\mathcal{IG}). Section 5.1 shows how all the diagrams for \mathcal{OG} that were

²⁴ Furthermore, note that on the assumption that $NCD = \emptyset$, Béziau’s hexagon is a *strong* JSB hexagon. It is still conceptually very different, however, from the six strong JSB hexagons inside the Aristotelian RDH for \mathcal{OG} that were listed in Fig. 9, since the latter six do not depend on the assumption that $NCD = \emptyset$.

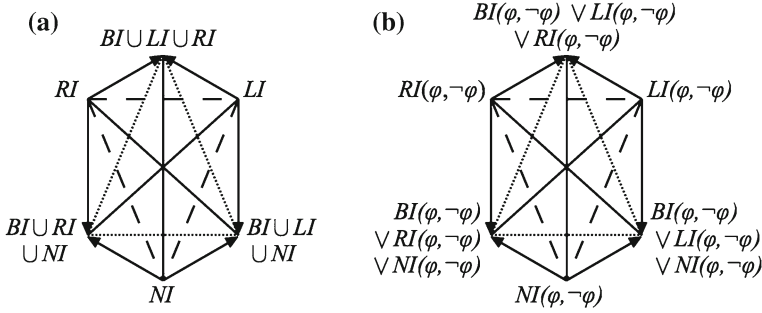


FIGURE 19. **a** A weak Jacoby–Sesmat–Blanché hexagon decorated with elements of $\wp^{\cup}(\mathcal{IG})$, **b** the result of filling in $(\varphi, \neg\varphi)$

studied in Sect. 4 can systematically be turned into diagrams for \mathcal{IG} . Next, Sect. 5.2 explores the connection between Aristotelian diagrams for \mathcal{IG} and Aristotelian diagrams for abstract ordering relations.

5.1. From Opposition to Implication Decorations of Aristotelian Diagrams

We have argued above that since \mathcal{OG} is a partition of $\mathbb{B}(\mathcal{S}) \times \mathbb{B}(\mathcal{S})$ —i.e. every pair of formulas stands in exactly one opposition relation—the elements of $\wp^{\cup}(\mathcal{OG})$ can be used to decorate an Aristotelian RDH. Furthermore, all Aristotelian diagrams that were discussed in Sect. 4 can be seen as subdiagrams of this Aristotelian RDH. Now, on the one hand, \mathcal{OG} and \mathcal{IG} are closely related to each other (recall Lemma 3.4), and on the other hand \mathcal{IG} is also a partition of $\mathbb{B}(\mathcal{S}) \times \mathbb{B}(\mathcal{S})$ —i.e. every pair of formulas stands in exactly one implication relation (recall Lemma 3.5). Consequently, it should come as no surprise that the elements of $\wp^{\cup}(\mathcal{IG}) = \{\cup \mathcal{X} \mid \mathcal{X} \subseteq \mathcal{IG}\}$ can also be used to decorate an Aristotelian RDH, and furthermore, all the insights and Aristotelian diagrams for $\wp^{\cup}(\mathcal{OG})$ that were discussed in Sect. 4 can straightforwardly be transposed to $\wp^{\cup}(\mathcal{IG})$.

We will now briefly describe some of these Aristotelian diagrams for \mathcal{IG} . First of all, in analogy to the weak JSB hexagon of opposition relations that was shown in Fig. 10a, we will consider the weak JSB hexagon of implication relations that is shown in Fig. 19a. In the light of Lemma 3.4, however, we will not fill in (φ, φ) in these relations, but rather $(\varphi, \neg\varphi)$; the resulting metalogical JSB hexagon is shown in Fig. 19b.²⁵ In terms of relations, this means that we are intersecting each relation $R \in \wp^{\cup}(\mathcal{IG})$ with the relation $\nabla := \{([\varphi], [\neg\varphi]) \mid \varphi \in \mathcal{L}_{\mathcal{S}}\}$ on $\mathbb{B}(\mathcal{S})$, and are thus no longer working in the Boolean algebra $\wp^{\cup}(\mathcal{IG})$, but rather in the Boolean algebra $\wp^{\cup}(\{R \cap \nabla \mid R \in \mathcal{IG}\}) = \{(\cup \mathcal{X}) \cap$

²⁵ It should be clear that we could also choose to systematically fill in $(\neg\varphi, \varphi)$ instead of $(\varphi, \neg\varphi)$: by Lemma 3.4 the resulting JSB hexagon would be identical to the one shown in Fig. 19b, modulo symmetry over the hexagon’s vertical axis—for example, $RI(\neg\varphi, \varphi)$ iff $LI(\varphi, \neg\varphi)$.

$\nabla \mid \mathcal{X} \subseteq \mathcal{IG}$.²⁶ Assuming that the underlying logical system S is consistent, we find that

$(\varphi, \neg\varphi) \in BI$	iff	impossible,
$(\varphi, \neg\varphi) \in LI$	iff	φ is a contradiction,
$(\varphi, \neg\varphi) \in RI$	iff	φ is a tautology,
$(\varphi, \neg\varphi) \in NI$	iff	φ is a contingency,
$(\varphi, \neg\varphi) \in BI \cup RI \cup NI$	iff	φ is satisfiable,
$(\varphi, \neg\varphi) \in BI \cup LI \cup NI$	iff	φ is not a tautology,
$(\varphi, \neg\varphi) \in BI \cup LI \cup RI$	iff	φ is not a contingency.

The metalogical hexagon in Fig. 19b can thus be reformulated as the hexagon for tautology and related metalogical notions that was already shown in Fig. 11b. This well-known hexagon [4, 5, 33] can thus not only be seen as (a special instance) of a subdiagram of the Aristotelian RDH for \mathcal{OG} (as was shown in Sect. 4.2), but also as (a special instance) of a subdiagram of the Aristotelian RDH for \mathcal{IG} .

Another insight that can fruitfully be transposed from \mathcal{OG} to \mathcal{IG} is the distinction between strong and weak opposition relations. The notion of left-implication (LI) that was defined in Definition 3.2 can be called *strong* (notation: LI_s), since it is defined in terms of a \models - and a $\not\models$ -condition. This definition can be weakened in two ways that are ‘dual’ to each other: LI_w drops the $\not\models$ -condition, while LI_w^* drops the \models -condition.²⁷ Obviously, an analogous distinction can be made for right-implication. All these relations are elements of $\wp^\cup(\mathcal{IG})$:

$$\begin{array}{ll}
 LI_s = & LI, & RI_s = & RI, \\
 LI_w = & BI \cup LI, & RI_w = & BI \cup RI, \\
 LI_w^* = & LI \cup NI, & RI_w^* = & RI \cup NI.
 \end{array}$$

Together, these strong and weak notions of left- and right-implication can be used to decorate a Buridan octagon, which is shown in Fig. 20.

5.2. Aristotelian Diagrams for Ordering Relations

In this subsection we will explore the relationship between the Aristotelian RDH for $\wp^\cup(\mathcal{IG})$ and Aristotelian diagrams for abstract ordering relations. The latter were among those that have sparked a renewed interest in Aristotelian diagrams since the middle of the twentieth century [13], and have recently also been studied by Béziau [6]. Let’s start by briefly recalling the definitions of partial and total ordering relations:

Definition 5.1. A binary relation \leq on a set S is said to be a *partial order* iff it satisfies the following conditions:

²⁶ Recall that in Sect. 4.2, every relation $R \in \wp^\cup(\mathcal{OG})$ was intersected with the identity relation $\Delta = \{([\varphi], [\varphi]) \mid \varphi \in \mathcal{L}_S\}$. It is easy to see that $\nabla = CD \in \mathcal{OG}$ and $\Delta = BI \in \mathcal{IG}$. Furthermore, in [79] it is shown that CD and BI are the most informative relations of \mathcal{OG} and \mathcal{IG} , respectively. The operations described here and in Sect. 4.2 are thus entirely analogous, since both of them involve intersecting the relations of one geometry with the most informative relation of the other geometry.

²⁷ The notion of *entailment* in contemporary logic is typically assumed to be reflexive ($\varphi \models \varphi$), and thus does not correspond to LI_s , but rather to LI_w .

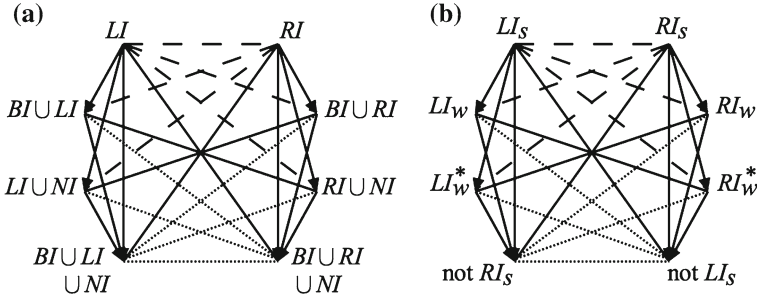


FIGURE 20. **a** Buridan octagon with elements of $\varphi^U(\mathcal{IG})$, **b** reformulation in terms of strong/weak left- and right-implication

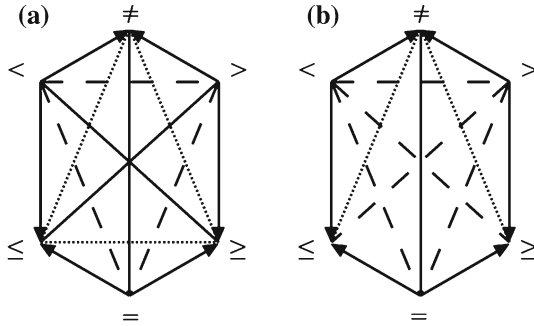


FIGURE 21. Aristotelian hexagon for **a** a total order, and **b** a partial order

- for all $x \in S: x \leq x$, (reflexivity)
- for all $x, y, z \in S: \text{if } x \leq y \text{ and } y \leq z, \text{ then } x \leq z$, (transitivity)
- for all $x, y \in S: \text{if } x \leq y \text{ and } y \leq x, \text{ then } x = y$. (antisymmetry)

Furthermore, \leq is said to be a *total order* iff it satisfies all the conditions mentioned above, and additionally also the following one:²⁸

- for all $x, y \in S: x \leq y \text{ or } y \leq x$. (totality)

Finally, if \leq is a partial or a total order, then the corresponding *strict* relation $<$ is defined as follows: $x < y :\Leftrightarrow (x \leq y \text{ and } x \neq y)$.²⁹

Consider a total order \leq on a set S . It follows straightforwardly from Definition 5.1 that the relations $\leq, \geq, <, >, =, \neq \in \varphi(S \times S)$ can be used to decorate a JSB hexagon, which is shown in Fig. 21a. For example, $<$ and $>$ are

²⁸ Note that if the totality axiom is added, the reflexivity axiom becomes redundant, since every relation that satisfies totality can easily be shown to also satisfy reflexivity.

²⁹ In case \leq is a total order, the corresponding strict relation can also be defined as follows: $x < y :\Leftrightarrow \text{not}(x \geq y)$. However, this alternative definition cannot be used if \leq is only a partial order: if \leq is a total order, then $\text{not}(x \geq y)$ is equivalent to $(x \leq y \text{ and } x \neq y)$, but if \leq is only a partial order, then these two statements are *not* equivalent to each other.

contrary to each other, since there are no $x, y \in S$ such that simultaneously $x < y$ and $x > y$, but there are $x, y \in S$ such that neither $x < y$ nor $x > y$ (for example, take $(x, y) := (x, x)$, for any $x \in S$). Furthermore, the JSB hexagon for total orders is *strong*, since the union of the contrary relations $<$, $>$ and $=$ is the universal relation $S \times S$ (i.e. the top element of the Boolean algebra $\wp(S \times S)$): it follows straightforwardly from the totality condition that for all $x, y \in S$: $x < y$ or $x > y$ or $x = y$. The fact that a total order can be used to decorate a strong JSB hexagon has long been known—in fact, it was already known in the 1950s by Sesmat [70] and Blanché [11–13], two of the authors after which the JSB hexagon was originally named (recall Footnote 10).

If the relation \leq is not assumed to be a total order, but merely a partial order, the Aristotelian hexagon in Fig. 21a undergoes some changes:

- two of the three contradictions change into contrarities, for example, $<$ and \geq are contraries, since $< \cap \geq = \emptyset$ (i.e. there are no $x, y \in S$ such that simultaneously $x < y$ and $x \geq y$), and $< \cup \geq \neq S \times S$ (i.e. there are $x, y \in S$ such that neither $x < y$ nor $x \geq y$)
- one of the three subcontrarities is lost (and not replaced with any Aristotelian relation at all), viz. \leq and \geq are no longer subcontrary to each other, since $\leq \cup \geq \neq S \times S$ (i.e. there are $x, y \in S$ such that neither $x \leq y$ nor $x \geq y$),
- the three contrarities and the six subalternations remain unchanged.

The resulting Aristotelian hexagon is shown in Fig. 21b, and was first studied in [60, Figure 59]. We would now like to point out a striking analogy between two conceptual processes that have been described in this section and the previous one:

- in Sect. 4.5: correcting Béziau’s hexagon, i.e. from Fig. 17a, b to 17c.
- just now: moving from a total order to a partial order, i.e. from Fig. 21a to 21b,

These two cases arise because of essentially one and the same reason. As was argued in Sect. 4.5, the first case shows that one cannot ignore the opposition relation of non-contradiction, i.e. one cannot assume (like Béziau seemed to do) that $NCD = \emptyset$. Completely analogously, the second case shows that in going from complete to partial orders (and thus dropping the totality axiom), one cannot ignore the relation of incomparability $\# := \{(x, y) \in S \times S \mid \text{not}(x \leq y) \text{ and } \text{not}(x \geq y)\} = \{(x, y) \in S \times S \mid \text{not}(x = y) \text{ and } \text{not}(x < y) \text{ and } \text{not}(x > y)\}$, i.e. one cannot assume that $\# = \emptyset$. From this perspective, it is particularly ironic that Béziau explicitly states that his hexagon for contradiction and strong and weak (sub)contrariety in Fig. 17a is “very similar to the ones presented by Sesmat and Blanché relating $<$, $>$, $=$, \leq , \geq , \neq ” [4, p. 30], and thus compares it to the hexagon for *total* orders in Fig. 21a, although it is actually much more similar to the hexagon for *partial* orders in Fig. 21b.

The Boolean closure of the hexagon for partial orders in Fig. 21b is a Boolean algebra consisting of 16 elements, which—after leaving out the empty and universal relations (recall Footnote 9)—can be used to decorate

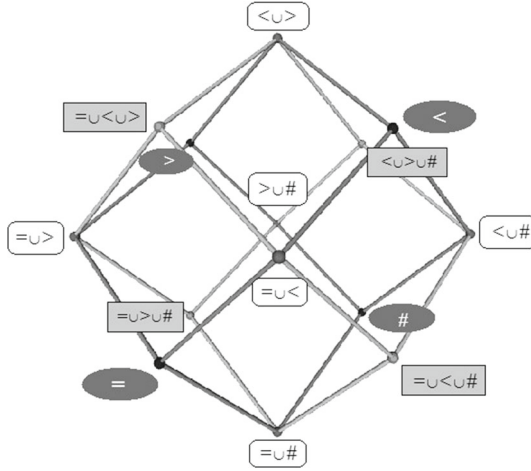


FIGURE 22. Aristotelian rhombic dodecahedron for partial orderings

an Aristotelian RDH. (A geometrical variant of this RDH can already be found in [60, Figure 59].) The atoms of this Boolean algebra are the relations $=$, $<$, $>$ and $\#$, and thus the Boolean algebra itself can be described as $\wp^{\cup}(\{=, <, >, \#\}) = \{\bigcup \mathcal{X} \mid \mathcal{X} \subseteq \{=, <, >, \#\}\}$; see Fig. 22.³⁰ All relations in the hexagon for partial orders in Fig. 21b (and its Boolean closure) can thus be written as $\bigcup \mathcal{X}$ for some $\mathcal{X} \subseteq \{=, <, >, \#\}$; for example, the relations \leq , \geq and \neq correspond to $= \cup <$, $= \cup >$ and $< \cup > \cup \#$, respectively. As a consequence, the hexagon for partial orders in Fig. 21b can be reformulated as in Fig. 23a.³¹

The Aristotelian hexagon in Fig. 23a can be seen as a subdiagram of the Aristotelian RDH for partial orders that was shown in Fig. 22. Needless to say, the latter also has many other subdiagrams, which can be exhaustively studied.

³⁰ Note that we can move back from partial to total orderings by imposing the totality axiom, which corresponds to assuming that $\# = \emptyset$. This assumption entails that the 16 relations of $\wp^{\cup}(\{=, <, >, \#\})$ collapse pairwise into 8 relations (this is analogous to the pairwise collapse of $\wp^{\cup}(\mathcal{OG})$ that is described toward the end of Sect. 4.5, where it is assumed that $NCD = \emptyset$). Ignoring the empty and universal relation, we thus find that the $16 - 2 = 14$ relations of the RDH for partial orders in Fig. 22 pairwise collapse into the $\frac{16}{2} - 2 = 6$ relations of the strong JSB hexagon for linear orders in Fig. 21a.

³¹ We are now also in a position to explain why the hexagons in Figs. 17c and 21b do not entirely consist of the same Aristotelian relations, despite the fundamental underlying analogy that was described earlier (ignoring NCD vs. ignoring $\#$). The differences between both diagrams arise from the fact that the hexagon’s top element in Fig. 17c is $C \cup SC$, whereas in Fig. 21b it is $< \cup > \cup \#$. To obtain a fully perfect analogy, we would need to put $C \cup SC \cup NCD$ at the top of the hexagon in Fig. 17c, or, alternatively, $< \cup >$ at the top of the hexagon in Fig. 21b. However, the first option is not the most plausible interpretation of Béziau’s use of ‘not CD ’ (recall Footnote 21), while the second one does not match our intuitive understanding of \neq (take, for example, the partial order of set inclusion, and consider the sets $A = \{1, 2\}$ and $B = \{2, 3\}$: even though neither $A \subset B$ nor $A \supset B$, we still want to be able to say that $A \neq B$).

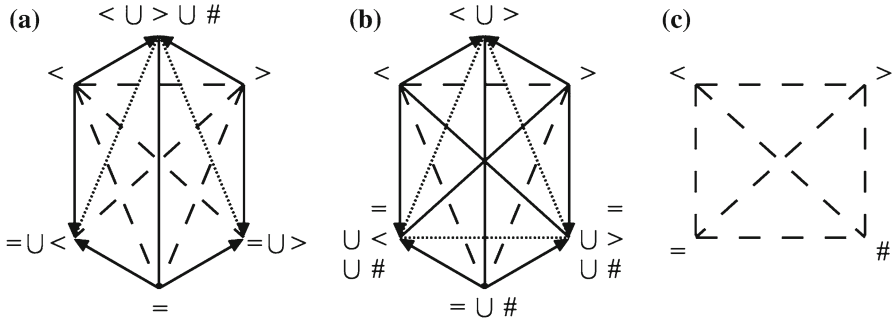


FIGURE 23. Three Aristotelian subdiagrams of the Aristotelian RDH for partial orders shown in Fig. 22

For now, we will mention only two such subdiagrams: the strong JSB hexagon in Fig. 23b and the ‘contrariety square’ in Fig. 23c, which are reformulations in terms of $\wp^U(\{=, <, >, \#\})$ of Aristotelian diagrams for partial orders that were originally studied by Béziau [6, Figures 3 and 9].

Concrete instances of the Aristotelian RDH for partial orders in Fig. 22 (or any of its subdiagrams) can be obtained by ‘filling in’ a concrete partial order. A well-known example of a partial ordering relation is *set inclusion*, and thus we immediately obtain an Aristotelian RDH for set inclusion [76]. Another example, which is more relevant for our current purposes, is based on the fact that S-entailment is a partial order over the Lindenbaum-Tarski algebra $\mathbb{B}(S)$ of a given logical system S , and hence, we immediately obtain an Aristotelian RDH for entailment. This diagram is exactly the RDH for the implication geometry \mathcal{IG} that was described in Sect. 5.1; in particular, the abstract relations $=, <, >$ and $\#$ are instantiated exactly as the implication relations BI, LI, RI and NI , respectively, and hence $\wp^U(\{=, <, >, \#\}) = \wp^U(\{BI, LI, RI, NI\}) = \wp^U(\mathcal{IG})$. This shows that the Aristotelian RDH for \mathcal{IG} is not only related to the RDH for \mathcal{OG} , as was shown in Sect. 5.1, but is also a specific instance of the RDH for partial orders.

6. Aristotelian Diagrams for Aristotelian and Duality Relations

In this section we conclude our study of metalogical decorations for Aristotelian diagrams. Section 6.1 studies some Aristotelian diagrams that are decorated with the Aristotelian relations themselves, while Sect. 6.2 studies some Aristotelian diagrams for the duality relations.

6.1. Aristotelian Diagrams for the Aristotelian Relations

Since the Aristotelian geometry is hybrid between the opposition and implication geometries ($\mathcal{AG} \subseteq \mathcal{OG} \cup \mathcal{IG}$), many of the diagrams that were studied in Sects. 4 and 5 can be viewed not only as Aristotelian diagrams for \mathcal{OG} or \mathcal{IG} , but also as Aristotelian diagrams for \mathcal{AG} itself. For example, since contrariety and subcontrariety are not only opposition relations, but also Aristotelian relations, the decoration of the Buridan octagon for strong and weak

(sub)contrariety in Fig. 16 can be seen as consisting of opposition relations, but equally well as consisting of Aristotelian relations. Still, the Aristotelian diagrams in Sects. 4 and 5 do not fully do justice to the hybrid nature of \mathcal{AG} , since none of them has a decoration that simultaneously contains opposition *and* implication relations. In this subsection, we will therefore focus exactly on Aristotelian diagrams with decorations containing both opposition and implication relations.

The Aristotelian diagrams that were constructed in Sects. 4 and 5 are all based on the fact that \mathcal{OG} and \mathcal{IG} are *partitions* of $\mathbb{B}(\mathcal{S}) \times \mathbb{B}(\mathcal{S})$ (recall Lemma 3.5). It is well-known, however, that the Aristotelian geometry is *not* a partition—indeed, this was one of the main motivations for introducing \mathcal{OG} and \mathcal{IG} in [79]—, and thus we cannot straightforwardly apply the same technique as before. In particular, it does not make sense to introduce $\wp^\cup(\mathcal{AG}) := \{\bigcup \mathcal{X} \mid \mathcal{X} \subseteq \mathcal{AG}\}$, since the latter is not a Boolean algebra whose atoms are the Aristotelian relations.

Since the Aristotelian geometry is hybrid between the opposition and implication geometries ($\mathcal{AG} \subseteq \mathcal{OG} \cup \mathcal{IG}$), it makes sense to look at the meet (i.e. the coarsest common refinement) of the partitions \mathcal{OG} and \mathcal{IG} , which is defined as follows:

$$\mathcal{OG} \wedge \mathcal{IG} := \{R \cap S \mid R \in \mathcal{OG}, S \in \mathcal{IG}, R \cap S \neq \emptyset\}.$$

Since the partition $\mathcal{OG} \wedge \mathcal{IG}$ is a refinement of \mathcal{OG} as well as \mathcal{IG} , it follows that every Aristotelian relation can be seen as an element of the Boolean closure $\wp^\cup(\mathcal{OG} \wedge \mathcal{IG})$ [31]. Note, however, that there are $|\mathcal{OG}| \times |\mathcal{IG}| = 4 \times 4 = 16$ relations of the form $R \cap S$ for $R \in \mathcal{OG}$ and $S \in \mathcal{IG}$, and only one of them is irrelevant (it can easily be shown that $CD \cap BI = \emptyset$ if the underlying logical system \mathcal{S} is consistent). In other words, we have $|\mathcal{OG} \wedge \mathcal{IG}| = 15$, and thus $\wp^\cup(\mathcal{OG} \wedge \mathcal{IG})$ contains $2^{15} = 32768$ relations in total, which renders it practically infeasible to systematically study Aristotelian diagrams for \mathcal{AG} by viewing them as subdiagrams of some larger, Boolean closed Aristotelian diagram (in contrast to Sects. 4 and 5, where all Aristotelian diagrams for \mathcal{OG} and \mathcal{IG} were seen as subdiagrams of the Aristotelian RDH for \mathcal{OG} and the Aristotelian RDH for \mathcal{IG} , respectively).

We will therefore proceed in a more *local* fashion, and study some interesting Aristotelian diagrams for \mathcal{AG} without viewing them as subdiagrams of some larger, Boolean closed Aristotelian diagram. Furthermore, many of the Aristotelian diagrams that will be studied in this subsection do not hold in full generality, i.e. for *all* pairs of formulas (φ, ψ) , but only if we impose certain additional conditions on the formulas, such as satisfiability or contingency.³² For each diagram that is studied in this subsection, we will therefore explicitly state the additional conditions which it depends on.

³² Formally, this means that we intersect every relation in $\wp^\cup(\mathcal{OG} \wedge \mathcal{IG})$ with $\Delta_{A,B} := \{(\varphi, \psi) \mid \varphi \text{ satisfies condition } A, \psi \text{ satisfies condition } B\}$, and do not work with a partition of $\mathbb{B}(\mathcal{S}) \times \mathbb{B}(\mathcal{S})$, but rather of $(\mathbb{B}(\mathcal{S}) \times \mathbb{B}(\mathcal{S})) \cap \Delta_{A,B} = \Delta_{A,B}$.

The first Aristotelian diagram that we will study contains both weak opposition and weak implication relations. Löbner [52, p. 55] defines the following four relations:

- φ and ψ are compatible iff $S \not\models \neg(\varphi \wedge \psi)$,
- φ implies ψ iff $S \models \varphi \rightarrow \psi$,
- φ is contrary to ψ iff $S \models \neg(\varphi \wedge \psi)$,
- φ does not imply ψ iff $S \not\models \varphi \rightarrow \psi$.

These four relations correspond exactly to $SC \cup NCD$, $BI \cup LI$, $CD \cup C$ and $RI \cup NI$, respectively. Löbner thus exclusively works with *weak* opposition and implication relations: using notation from Sects. 4 and 5, his relations are SC_w^* , LI_w , C_w and RI_w^* , respectively.

Since \mathcal{OG} and \mathcal{IG} are partitions of $\mathbb{B}(S) \times \mathbb{B}(S)$ (recall Lemma 3.5), it is trivial to check that $CD \cup C$ and $SC \cup NCD$ are contradictory, and also that $BI \cup LI$ and $RI \cup NI$ are contradictory. If we do not impose any additional constraints, then these two contradictions are the only Aristotelian relations that obtain between Löbner’s four relations. For example, $CD \cup C$ and $BI \cup LI$ do not stand in any Aristotelian relation at all:

- $(p \wedge \neg p, p) \in CD \cup C$ and $(p \wedge \neg p, p) \in BI \cup LI$,
so $(CD \cup C) \cap (BI \cup LI) \neq \emptyset$,
so $CD \cup C$ and $BI \cup LI$ are neither contradictory nor contrary,
- $(p, q) \notin CD \cup C$ and $(p, q) \notin BI \cup LI$,
so $(CD \cup C) \cup (BI \cup LI) \neq \mathbb{B}(\text{CPL}) \times \mathbb{B}(\text{CPL})$,
so $CD \cup C$ and $BI \cup LI$ are not subcontrary,
- $(p, \neg p) \notin (CD \cup C) \cap (BI \cup LI)$ and $(p, \neg p) \in CD \cup C$,
so $(CD \cup C) \cap (BI \cup LI) \neq CD \cup C$,
so there is no subalternation from $CD \cup C$ to $BI \cup LI$,
- $(p, p) \notin (CD \cup C) \cap (BI \cup LI)$ and $(p, p) \in BI \cup LI$,
so $(CD \cup C) \cap (BI \cup LI) \neq BI \cup LI$,
so there is no subalternation from $BI \cup LI$ to $CD \cup C$.

Without imposing any additional constraints, Löbner’s relations thus constitute a ‘degenerated’ Aristotelian square (or ‘cross’), which is shown in Fig. 24a.

If we make the additional assumption that the relations’ first argument is satisfiable, however, the situation changes quite drastically. For example, based on this assumption, it can be shown that $CD \cup C$ and $BI \cup LI$ are

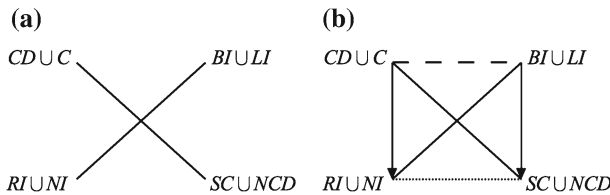


FIGURE 24. Löbner’s relations as decorations of **a** an Aristotelian cross (no constraints) and **b** a classical Aristotelian square (constraint: satisfiability of the first argument)

contrary to each other: it is still trivially the case that $(CD \cup C) \cup (BI \cup LI)$ is not the universal relation over $\mathbb{B}(S)$, and to see that $(CD \cup C) \cap (BI \cup LI)$ is empty, note that $(\varphi, \psi) \in (CD \cup C) \cap (BI \cup LI)$ entails that $S \models \neg(\varphi \wedge \psi)$ and $S \models \varphi \rightarrow \psi$, and thus $S \models \neg\varphi$, which contradicts our assumption that φ is satisfiable. In exactly the same way, it can be shown that $SC \cup NCD$ and $RI \cup NI$ are subcontrary to each other, and that there are subalternations from $BI \cup LI$ to $SC \cup NCD$ and from $CD \cup C$ to $RI \cup NI$. On the assumption that their first argument is satisfiable, Löbner’s relations thus end up constituting a classical Aristotelian square, which is shown in Fig. 24b. Although Löbner himself did not explicitly draw this metalogical square, he recognized that his four relations “constitute another Aristotelian square too, implication $[BI \cup LI]$ implying compatibility $[SC \cup NCD]$ and so on” [52, p. 55].

In Seuren [72, p. 11] we find another Aristotelian diagram that contains both opposition and implication relations. Seuren considers the same four relations as Löbner (although he calls them differently), and adds two more, viz. ‘strict compatibility’ and its complement. As indicated by Seuren himself [72, p. 3], the relation of strict compatibility is better known as *logical independence* or *unconnectedness* [79, Footnote 43],³³ and is defined as follows:

$$\begin{aligned} \varphi \text{ and } \psi \text{ are } \textit{unconnected} \quad \text{iff} \quad & S \not\models \neg(\varphi \wedge \psi) \text{ and } S \not\models \varphi \vee \psi \text{ and} \\ & S \not\models \varphi \rightarrow \psi \text{ and } S \not\models \psi \rightarrow \varphi. \end{aligned}$$

It should be clear that Seuren’s notion of strict compatibility can be represented as $NCD \cap NI$. Making use of Lemma 3.5, the complement of strict compatibility can be represented as follows:

$$\text{not}(NCD \cap NI) = \text{not}(NCD) \cup \text{not}(NI) = (CD \cup C \cup SC) \cup (BI \cup LI \cup RI).$$

Strict compatibility and its complement are (by definition) contradictory to each other. Furthermore, even without making any additional assumptions about the relations’ arguments, it can easily be shown that these two new relations enter into Aristotelian relations with Löbner’s original four; for example, $NCD \cap NI$ is contrary to $CD \cup C$ and to $BI \cup LI$, and stands in subalternation to $SC \cup NCD$ and to $RI \cup NI$. In total, Seuren’s relations thus constitute the Aristotelian hexagon shown in Fig. 25a; this type of hexagon is relatively unknown, but has recently been studied and called an ‘unconnected-4’ hexagon, since it contains exactly four pairs of vertices that do not stand in any Aristotelian relation at all [32, 78, 81].

If we do make the additional assumption that the relations’ first argument is satisfiable, then the cross constituted by Löbner’s four relations again turns into a classical Aristotelian square (recall Fig. 24), and the hexagon as a whole turns into a more familiar JSB hexagon, as shown in Fig. 25b, and originally in [72, Figure 4]. It should be noted that this is a *weak* JSB hexagon, since $(CD \cup C) \cup (BI \cup LI) \cup (NCD \cap NI) \neq \mathbb{B}(S) \times \mathbb{B}(S)$.³⁴

³³ See Footnote 2 of this paper for the relation between unconnectedness and the informativity account of the Aristotelian relations that is presented in [79].

³⁴ For example, $(p \vee \neg p, p) \in SC \cap RI$, and hence $(p \vee \neg p, p) \notin CD \cup C$ and $(p \vee \neg p, p) \notin BI \cup LI$ and $(p \vee \neg p, p) \notin NCD \cap NI$.

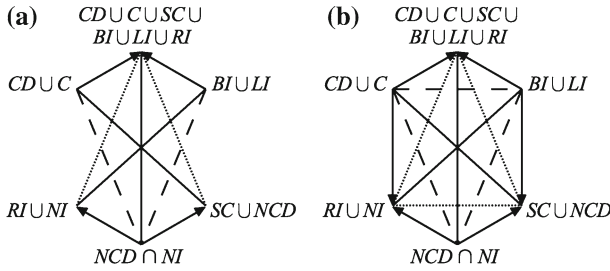


FIGURE 25. Seuren's relations as decorations of **a** an 'unconnected-4' hexagon (no constraints) and **b** a weak JSB hexagon (constraint: satisfiability of the first argument)

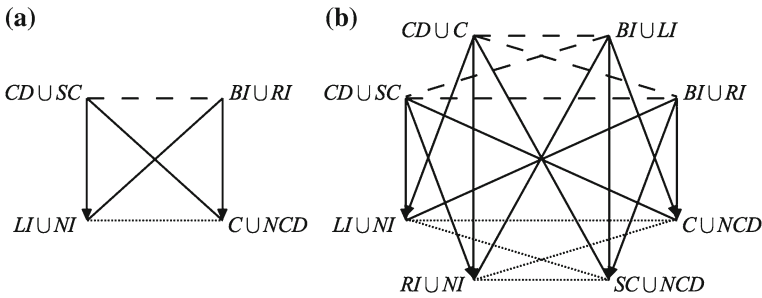


FIGURE 26. **a** Variant of Löbner's square (constraint: the first argument is not a tautology), **b** Aristotelian octagon (constraint: contingency of both arguments)

Löbner's squares in Fig. 24 are decorated with weak relations C_w and LI_w and the 'dually' weak relations SC_w^* and RI_w^* . A variant of Löbner's square can be obtained by changing which relations are 'ordinarily weak' and which ones are 'dually weak', i.e. by considering the relations $C_w^* = C \cup NCD$, $LI_w^* = LI \cup NI$, $SC_w = CD \cup SC$ and $RI_w = BI \cup RI$. If we do not make any additional assumptions, these four new relations again constitute a degenerated Aristotelian square, just like Löbner's four original ones (see Fig. 24a). However, if we make the additional assumption that the relations' first argument is not a tautology,³⁵ then they turn out to constitute a classical Aristotelian square, which is shown in Fig. 26a.

Löbner's original square in Fig. 24b and its variant in Fig. 26a can be combined with each other, thereby yielding an Aristotelian octagon. To ensure that both squares are classical (i.e. not degenerated), we have to assume that the relations' first argument is satisfiable and not a tautology, i.e. that it is

³⁵ Note the close connection between the conditions that are needed to turn the degenerated square into a classical square: for Löbner's original four relations, the condition is that the first argument is satisfiable (i.e. $S \not\models \neg\varphi$), for the four new relations, the condition is that the first argument is not a tautology (i.e. $S \not\models \varphi$).

contingent. If we additionally assume that the relations' second argument is also contingent, then the relations of the two squares enter into further Aristotelian relations with each other; for example, it can be shown that $CD \cup C$ (from Löbner's original square) and $BI \cup RI$ (from the variant to Löbner's square) are contrary to each other,³⁶ and also that $BI \cup LI$ (from Löbner's original square) and $CD \cup SC$ (from the variant) are contrary to each other.³⁷ The resulting metalogical octagon is shown in Fig. 26b.³⁸ This type of Aristotelian octagon has hitherto not been studied in any systematic way, but it can be found in [23, 37, 45, 47], where it receives an object-logical decoration consisting of categorical statements from syllogistics. In [31] it is shown that the octagon's object-logical (syllogistic) decoration is intimately related to the metalogical decoration that has been discussed here.

To finish this subsection, we will study two different ways to turn the weak JSB hexagon in Fig. 25b into a *strong* JSB hexagon. Note that the reason for the JSB hexagon in Fig. 25b being weak is that its contrariety triangle consists of $CD \cup C$, $BI \cup LI$ and $NCD \cap NI$, and thus lacks SC and RI (see Footnote 34); in order to obtain a strong JSB hexagon, we thus need to add these two relations to the hexagon's contrariety triangle. One possibility is to combine both relations into $SC \cup RI$, and add this to $NCD \cap NI$, which results in the Aristotelian hexagon in Fig. 27a. If we make the assumption that the relations' first argument is satisfiable and that their second argument is contingent, then this hexagon can be shown to be a strong JSB hexagon. An alternative possibility is to add SC to $CD \cup C$, and RI to $BI \cup LI$, which results in the Aristotelian hexagon in Fig. 27b. This second possibility is conceptually more elegant than the first one, since it better respects the distinction between the opposition and implication geometries.³⁹ Furthermore, if we make the assumption that both of the relations' arguments are contingent, then the diagram in Fig. 27b can be shown to be a strong JSB hexagon as well.

6.2. Aristotelian Diagrams for the Duality Relations

We now turn to Aristotelian diagrams for the duality relations ID, ENEG, INEG and DUAL, which were introduced in Sect. 3.2. Just like the Aristotelian relations (but unlike the opposition and implication relations), the duality relations do *not* constitute a partition of $\mathbb{B}(S) \times \mathbb{B}(S)$: they are neither jointly exhaustive

³⁶ Proof: it is trivial that $(CD \cup C) \cup (BI \cup RI)$ is not the universal relation over $\mathbb{B}(S)$; to see that $(CD \cup C) \cap (BI \cup RI) = \emptyset$, note that $(\varphi, \psi) \in (CD \cup C) \cap (BI \cup RI)$ entails that $S \models \neg(\varphi \wedge \psi)$ and $S \models \psi \rightarrow \varphi$, and thus $S \models \neg\psi$, which contradicts our assumption that ψ is (contingent and thus) satisfiable.

³⁷ Proof: it is trivial that $(BI \cup LI) \cup (CD \cup SC)$ is not the universal relation over $\mathbb{B}(S)$; to see that $(BI \cup LI) \cap (CD \cup SC) = \emptyset$, note that $(\varphi, \psi) \in (BI \cup LI) \cap (CD \cup SC)$ entails that $S \models \varphi \rightarrow \psi$ and $S \models \varphi \vee \psi$, and thus $S \models \psi$, which contradicts our assumption that ψ is (contingent and thus) not a tautology.

³⁸ Although the Aristotelian octagon in Fig. 26b was obtained by combining the two classical Aristotelian squares in Figs. 24b and 26a, it can also be seen as the result of combining the degenerated squares in the centers of the two Buridan octagons in Figs. 16 and 20.

³⁹ The hexagon in Fig. 27b is also intimately related to the information perspective on \mathcal{OG} and \mathcal{IG} developed in [79].

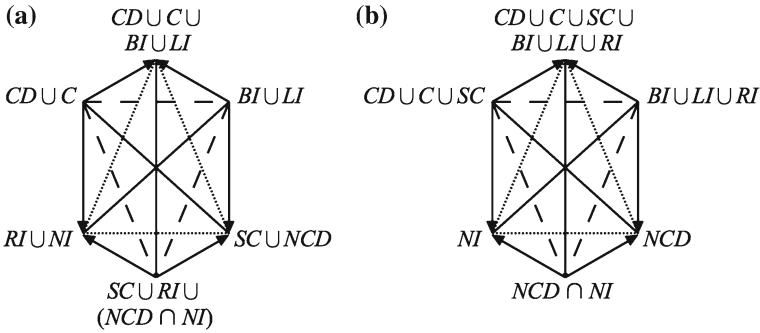


FIGURE 27. **a** Strong JSB hexagon (constraint: satisfiability of the first argument and contingency of the second argument), **b** another strong JSB hexagon (constraint: contingency of both arguments)

(for example, p and q do not stand in any duality relation at all), nor mutually exclusive (as was explained in Sect. 3.2, the existence of operators that are their own duals or internal negations entails that $ID \cap DUAL \neq \emptyset \neq ENEG \cap INEG$ and $ID \cap INEG \neq \emptyset \neq ENEG \cap DUAL$, respectively). We thus find ourselves in a situation that is highly similar to the one described at the beginning of the previous subsection, and we will therefore use the same strategy for dealing with it: we will proceed in a *local* fashion, and construct various interesting Aristotelian diagrams for the duality geometry \mathcal{DG} , without viewing them as subdiagrams of some larger, Boolean closed Aristotelian diagram. Furthermore, just as in the previous subsection, many of the Aristotelian diagrams that will be studied here do not hold in full generality, i.e. for *all* pairs of operators (O_1, O_2) , but only if we impose certain additional conditions on the operators, such as not being their own dual or internal negation.⁴⁰ For each diagram that is studied in this subsection, we will therefore again explicitly state the additional conditions it depends on.

We begin by defining an Aristotelian square for the duality relations INEG and ENEG, and their complements, which will be denoted as ‘not INEG’ and ‘not ENEG’.⁴¹ By definition, INEG and ‘not INEG’ are contradictory to each

⁴⁰ Recall that in Sect. 6.1, we imposed additional conditions on the Aristotelian relations’ two arguments *independently*—for example, it made sense to require the *first* formula to be *satisfiable*, and the *second* formula to be *contingent*; see Fig. 27a. Since the duality relations are functional, however, it can easily be shown that a duality relation’s *first* argument satisfies a certain condition iff its *second* argument satisfies that same condition; for example, if two operators O_1 and O_2 stand in some duality relation $R \in \mathcal{DG}$, then we have the following chain of equivalences: O_2 is self-dual iff $DUAL(O_2) = O_2$ iff $R(DUAL(O_2)) = R(O_2)$ iff $DUAL(R(O_2)) = R(O_2)$ iff $DUAL(O_1) = O_1$ iff O_1 is self-dual. Therefore, in this subsection we will always impose the same condition(s) on *both* of the duality relations’ arguments.

⁴¹ Note that because of our ‘local’ approach, we cannot write $INEG = \bigcup \mathcal{X}$ and ‘not INEG’ = $\bigcup \mathcal{Y}$ for some sets of relations \mathcal{X}, \mathcal{Y} . Of course, if we were working with a partition \mathcal{P} such that $\mathcal{DG} \subseteq \wp^\cup(\mathcal{P})$, then we would have $INEG = \bigcup \mathcal{X}$ for some $\mathcal{X} \subseteq \mathcal{P}$, and hence also ‘not INEG’ = $\bigcup(\mathcal{P} - \mathcal{X})$. Similar remarks apply to ENEG.

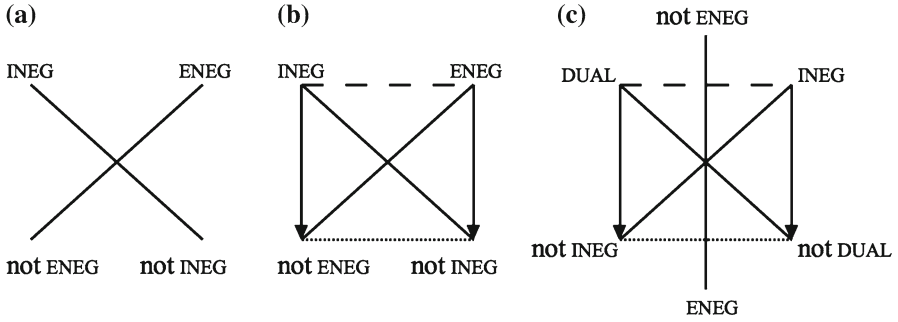


FIGURE 28. **a** Degenerated Aristotelian square for duality relations (constraints: none), **b** classical Aristotelian square for the same duality relations (constraint: no operators that are their own dual), **c** ‘unconnected-8’ hexagon for the duality relations (constraints: none)

other, and similarly for ENEG and ‘not ENEG’. Whether there are any other Aristotelian relations holding between these four relations, however, depends on the additional conditions we are willing to impose on the relations’ arguments. It can be shown that without additional conditions, there are *no* Aristotelian relations besides the two aforementioned contradictions, and we thus obtain the ‘degenerated’ square shown in Fig. 28a. In particular, note that INEG and ENEG are not contrary to each other, since if the operator O is self-dual, then $INEG(O) = ENEG(O)$, and thus $INEG \cap ENEG \neq \emptyset$. However, if we impose the additional condition that the relations’ arguments should be operators that are *not* self-dual, then it *does* hold that $INEG \cap ENEG = \emptyset$, which (together with the trivial fact that $INEG \cup ENEG$ is not the universal relation over $\mathbb{B}^{\mathbb{A}^n} := \{O \mid O: \mathbb{A}^n \rightarrow \mathbb{B}\}$) implies that INEG and ENEG are contrary to each other. Similarly, if we exclude self-dual operators, we also find two subalternations and a subcontrariety, and thus obtain the classical Aristotelian square shown in Fig. 28b.

A more general picture can be obtained by considering Aristotelian diagrams that simultaneously contain the non-trivial (i.e. non-ID) duality relations ENEG, INEG and DUAL. By definition, these three relations are contradictory to their complements, i.e. ‘not ENEG’, ‘not INEG’ and ‘not DUAL’, respectively. Furthermore, it can be shown that DUAL and INEG are contrary to each other, since $DUAL \cap INEG = \emptyset$: if there were operators $O_1, O_2: \mathbb{A}^n \rightarrow \mathbb{B}$ such that $(O_1, O_2) \in DUAL \cap INEG$, then $DUAL(O_1) = O_2 = INEG(O_1)$, so $ENEG(O_2) = ENEG(INEG(O_1)) = DUAL(O_1) = O_2$, and thus \mathbb{B} would be the trivial Boolean algebra in which $\perp_{\mathbb{B}} = \top_{\mathbb{B}}$ (recall Footnote 5). For essentially the same reason, it also holds that ‘not INEG’ and ‘not DUAL’ are subcontrary to each other, and also that there are subalternations from DUAL to ‘not INEG’ and from INEG to ‘not DUAL’. However, if we do not make any additional assumptions about the operators, then there are no other Aristotelian relations among DUAL, INEG, ENEG and their complements besides the ones that have

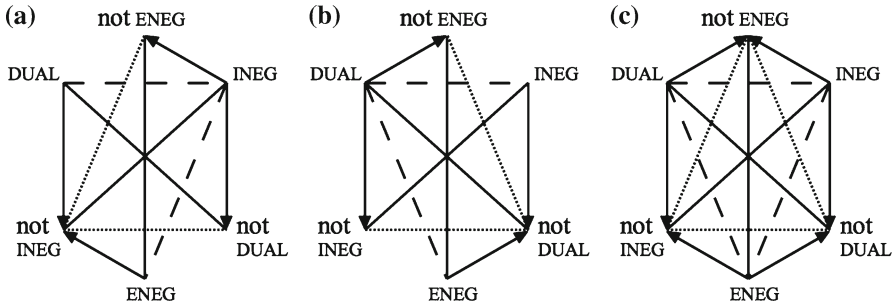


FIGURE 29. **a** ‘Unconnected-4’ hexagon for the duality relations (constraint: no operators that are their own dual), **b** ‘unconnected-4’ hexagon for the duality relations (constraint: no operators that are their own internal negation), **c** weak JSB hexagon for the duality relations (constraint: no operators that are their own dual or internal negation)

just been mentioned, and we thus end up with the Aristotelian hexagon shown in Fig. 28c. This type of hexagon has recently been called an ‘unconnected-8’ hexagon, since it contains exactly 8 pairs of vertices that do not stand in any Aristotelian relation at all [32].⁴² Such ‘unconnected-8’ hexagons have not been studied extensively so far, but it is known that representing them with bitstrings requires bitstring of length at least 5 (unlike all other types of Aristotelian hexagons, which can be represented with bitstrings of length 3 or 4).

If we now make the additional assumption that the duality relations’ arguments should be operators that are *not* their own duals, it holds that $INEG \cap ENEG = \emptyset$, and we thus find a contrariety between INEG and ENEG, and also the corresponding subcontrariety and two subalternations (recall the transition from the degenerated square in Fig. 28a to the classical square in Fig. 28b); the hexagon as a whole thus turns into an ‘unconnected-4’ hexagon, which is shown in Fig. 29a. Completely analogously, if we instead make the additional assumption that the duality relations’ arguments should be operators that are *not* their own internal negations, it holds that $DUAL \cap ENEG = \emptyset$, and we thus find a contrariety between DUAL and ENEG, and also the corresponding subcontrariety and two subalternations; the hexagon as a whole again turns into an ‘unconnected-4’ hexagon, which is shown in Fig. 29b. Note that although the Aristotelian hexagons in Fig. 29a, b both belong to the ‘unconnected-4’ family, there is still a crucial difference between them, which has to do with the ‘distribution’ of unconnectedness across the hexagon (i.e. which duality relations stand/do not stand in some Aristotelian relation to each other).

Finally, if we simultaneously impose the two additional assumptions that the duality relations’ arguments should be operators that are *neither* their

⁴² Note the terminological analogy with the ‘unconnected-4’ hexagon.

own duals *nor* their own internal negations, then the resulting hexagon turns out to be a JSB hexagon, which is shown in Fig. 29c. It should be noted that this is a *weak* JSB hexagon, since $\text{DUAL} \cup \text{INEG} \cup \text{ENEG}$ is not the universal relation over the class $\mathbb{B}^{\mathbb{A}^n}$ of all operators $O: \mathbb{A}^n \rightarrow \mathbb{B}$.

7. Duality Diagrams for the Opposition/Implication Relations

In Sects. 4, 5 and 6, we showed that the opposition, implication, Aristotelian, and duality geometries yield various interesting metalogical decorations of Aristotelian diagrams. In this section and the next one, we will discuss how these four geometries can also be used to decorate another type of logical diagrams, viz. duality diagrams. In this section, we will focus on duality diagrams for the opposition relations (Sect. 7.1), and also make some brief comments about duality diagrams for the implication relations (Sect. 7.2).

7.1. Duality Diagrams for the Opposition Relations

We start by studying duality diagrams for the opposition relations. At first sight, it might look like a ‘category mistake’ to talk about duality relations holding between opposition relations, since in Sect. 3.2 the duality relations were defined between *operators*, rather than *relations* (recall Definition 3.6). However, every relation R can naturally be associated with an operator, viz. its *characteristic function* χ_R ; when the characteristic functions χ_R and χ_S stand in some duality relation, then in a derived sense, the relations R and S themselves can also be said to stand in that duality relation. In particular, since each opposition relation $R \in \mathcal{OG}_S$ is a binary relation over the Lindenbaum–Tarski algebra $\mathbb{B}(S)$, its characteristic function χ_R looks as follows:

$$\chi_R: \mathbb{B}(S) \times \mathbb{B}(S) \rightarrow \{0, 1\}: ([\varphi], [\psi]) \mapsto \chi_R([\varphi], [\psi]) := \begin{cases} 1 & \text{if } ([\varphi], [\psi]) \in R, \\ 0 & \text{if } ([\varphi], [\psi]) \notin R. \end{cases}$$

Since $\{0, 1\}$ can be seen as a Boolean algebra (viz. the smallest non-trivial Boolean algebra), the operator χ_R is of the right ‘type’, and Definition 3.6 is applicable to it. To illustrate this, consider item 2a of Lemma 3.3, which states the following:

$$\text{for all formulas } \varphi, \psi: C(\varphi, \psi) \quad \text{iff} \quad SC(\neg\varphi, \neg\psi).$$

Reformulating this in terms of characteristic functions, we get:

$$\text{for all formulas } \varphi, \psi: \chi_C(\varphi, \psi) = 1 \quad \text{iff} \quad \chi_{SC}(\neg\varphi, \neg\psi) = 1.$$

Now, in the smallest non-trivial Boolean algebra $\{0, 1\}$, the biconditional $x = 1 \Leftrightarrow y = 1$ is equivalent to $x = y$, and thus we get:

$$\text{for all formulas } \varphi, \psi: \chi_C(\varphi, \psi) = \chi_{SC}(\neg\varphi, \neg\psi).$$

By Definition 3.6, this means exactly that $\text{INEG}(\chi_C, \chi_{SC})$, i.e. χ_C and χ_{SC} are each other’s internal negation. Moving from characteristic functions to the opposition relations themselves, we can also say, in a derived sense, that

$\text{INEG}(C, SC)$ ⁴³. Completely analogously, items 1a and 4a of Lemma 3.3 entail that $\text{INEG}(CD, CD)$ and $\text{INEG}(NCD, NCD)$, i.e. CD and NCD are their own internal negations.

Similar remarks also apply to *non-atomic* relations of $\varphi^{\cup}(\mathcal{OG})$. Consider, for example, the relations $CD \cup C \cup NCD$ and $CD \cup SC \cup NCD$: it follows immediately from items 1a, 2a and 4a of Lemma 3.3 that

for all formulas φ, ψ : $(\varphi, \psi) \in CD \cup C \cup NCD$ iff $(\neg\varphi, \neg\psi) \in CD \cup SC \cup NCD$.

Reformulating this in terms of characteristic functions, we get

for all formulas φ, ψ : $\chi_{CD \cup C \cup NCD}(\varphi, \psi) = 1$ iff $\chi_{CD \cup SC \cup NCD}(\neg\varphi, \neg\psi) = 1$

and hence

for all formulas φ, ψ : $\chi_{CD \cup C \cup NCD}(\varphi, \psi) = \chi_{CD \cup SC \cup NCD}(\neg\varphi, \neg\psi)$,

which means exactly that $\text{INEG}(CD \cup C \cup NCD, CD \cup SC \cup NCD)$.

We can also study other duality relations holding between the opposition relations. For example, since \mathcal{OG} is a partition of $\mathbb{B}(S) \times \mathbb{B}(S)$ (recall Lemma 3.5), it follows that

for all formulas φ, ψ : $(\varphi, \psi) \in CD$ iff $(\varphi, \psi) \notin C \cup SC \cup NCD$.

Reformulating this in terms of characteristic functions, we get

for all formulas φ, ψ : $\chi_{CD}(\varphi, \psi) = 1$ iff $\chi_{C \cup SC \cup NCD}(\varphi, \psi) = 0$.

Now, in the smallest non-trivial Boolean algebra $\{0, 1\}$, the biconditional $x = 1 \Leftrightarrow y = 0$ is equivalent to $x = \neg y$ (where \neg is the complement operator of the $\{0, 1\}$ Boolean algebra, i.e. $\neg y := 1 - y$), and thus we get:

for all formulas φ, ψ : $\chi_{CD}(\varphi, \psi) = \neg\chi_{C \cup SC \cup NCD}(\varphi, \psi)$.

By Definition 3.6, this means exactly that $\text{ENEG}(\chi_{CD}, \chi_{C \cup SC \cup NCD})$, i.e. CD and $C \cup SC \cup NCD$ are each other's external negation. In general, it can be shown that $\text{ENEG}(\bigcup \mathcal{X}, \bigcup(\mathcal{OG} - \mathcal{X}))$ for all $\mathcal{X} \subseteq \mathcal{OG}$.

We have already established that $\text{INEG}(C) = SC$ and $\text{ENEG}(SC) = CD \cup C \cup NCD$. Since $\text{ENEG} \circ \text{INEG} = \text{DUAL}$, it should follow that also $\text{DUAL}(C) = \text{ENEG}(\text{INEG}(C)) = \text{ENEG}(SC) = CD \cup C \cup NCD$. To verify this, note that Lemmas 3.5 and 3.3 imply that

for all formulas φ, ψ : $(\varphi, \psi) \in C$ iff $(\neg\varphi, \neg\psi) \notin CD \cup C \cup NCD$,

⁴³The idea that contrariety and subcontrariety are each other's internal negation is not entirely new. A closely related idea can already be found in the *Summulae Logicales* of the 13th century philosopher Petrus Hispanus: after he has given the definitions (which he calls 'laws') of contrariety and subcontrariety, Hispanus writes that "the law of subcontraries is the reverse of the law of contraries" [15, p. 113, our emphasis]. Interestingly, the original Latin text says that the law of subcontraries is *contrary* to the law of subcontraries ("lex subcontrariarum *contrario* modo se habet legi contrariarum" [15, p. 112, our emphasis]), which is a particularly ironic example of the confusion between Aristotelian relations and duality relations that was mentioned in Subsect. 3.2. The English translation as 'reverse' is thus not literally correct, but it probably better captures what Hispanus had in mind (see [15, p. 113, Footnote 16] for the translators' remarks about this issue).

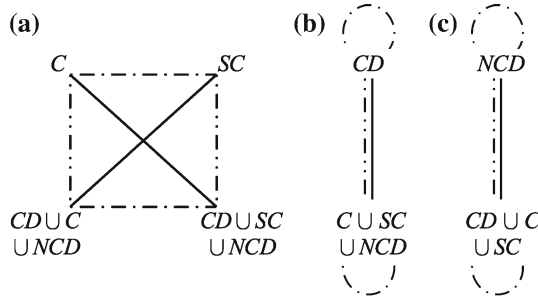


FIGURE 30. **a** Duality square for C and SC , **b**, **c** degenerated duality diagrams for CD and NCD

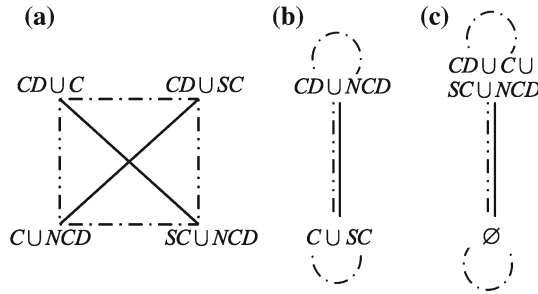


FIGURE 31. **a** Duality square for $CD \cup C$ and other opposition relations, **b** degenerated duality diagrams for $CD \cup NCD$ and $C \cup SC$, **c** degenerated duality diagram for the empty and universal relation

and hence

$$\text{for all formulas } \varphi, \psi : \chi_C(\varphi, \psi) = \neg \chi_{CD \cup C \cup NCD}(\neg \varphi, \neg \psi),$$

which by Definition 3.6 means indeed that $\text{DUAL}(C, CD \cup C \cup NCD)$.

We are now in a position to construct actual duality diagrams for the relations in $\wp^{\cup}(\mathcal{O}\mathcal{G})$. First of all, based on the fact that $\text{INEG}(C, SC)$, we can construct the duality square shown in Fig. 30a. Note that this square also visualizes facts such as $\text{ENEG}(C, CD \cup SC \cup NCD)$ and $\text{DUAL}(SC, CD \cup SC \cup NCD)$, all of which follow from Lemmas 3.3 and 3.5. Next, since $\text{INEG}(CD, CD)$, or, functionally speaking, $\text{INEG}(CD) = CD$, it follows that $\text{DUAL}(CD) = \text{ENEG}(\text{INEG}(CD)) = \text{ENEG}(CD) = C \cup SC \cup NCD$, and hence we find the degenerated duality diagram in Fig. 30b, which is thus a metalogical instance of the generic degenerated duality diagram in Fig. 4c. The fact that $\text{INEG}(NCD, NCD)$ leads to a similar degenerated duality diagram, which is shown in Fig. 30c.

Next, the facts that $\text{INEG}(CD \cup C, CD \cup SC)$ and $\text{ENEG}(CD \cup SC, C \cup NCD)$ lead to the duality square in Fig. 31a, and the fact that $CD \cup NCD$ is its own internal negation—and hence $\text{DUAL}(CD \cup NCD) = \text{ENEG}(CD \cup$

$NCD) = C \cup SC$ —leads to the degenerated duality diagram in Fig. 31b. The five duality diagrams shown in Figs. 30a–c and 31a, b jointly exhibit all the duality relations that hold between the 14 opposition relations in the Aristotelian RDH for $\wp^{\cup}(\mathcal{O}\mathcal{G})$ (recall Fig. 8b).⁴⁴ The only elements of $\wp^{\cup}(\mathcal{O}\mathcal{G})$ that are not present in this RDH are the empty relation and the universal relation over $\mathbb{B}(\mathbf{S})$ (recall Footnote 9); these two relations turn out to be their own internal negations and each other’s external negations, and thus lead to a final degenerated duality diagram, which is shown in Fig. 31c.

It is interesting to note that several (sets of) opposition relations constitute Aristotelian as well as duality diagrams. For example, we have just shown that C , SC , $CD \cup C \cup NCD$ and $CD \cup SC \cup NCD$ yield the duality square shown in Fig. 30a, but in Sect. 4.4 it was shown that these same four opposition relations also yield a classical Aristotelian square, which was shown in Fig. 14a, and is embedded as the ‘vertically stretched’ square inside the Buridan octagon in Fig. 16a. By contrast, $CD \cup C$, $CD \cup SC$, $C \cup NCD$ and $SC \cup NCD$ yield the duality square shown in Fig. 31a, but the corresponding Aristotelian diagram is not a classical square, but rather a ‘degenerated’ square (or ‘cross’), which is embedded as the ‘horizontally stretched’ square inside the Buridan octagon in Fig. 16a.

These observations can be used to argue for the conceptual independence between the Aristotelian and duality geometries. From a duality perspective, there is no difference whatsoever between the duality squares in Figs. 30a and 31a. The corresponding Aristotelian squares, however, are radically different from each other: the first one is a classical Aristotelian square, whereas the second one is a ‘degenerated’ square, i.e. an Aristotelian ‘cross’. As to individual relations, note, for example, that $\text{DUAL}(C, CD \cup C \cup NCD)$ as well as $\text{DUAL}(CD \cup C, C \cup NCD)$. Switching from the duality to the Aristotelian perspective, however, we see that there is a subalternation from C to $CD \cup C \cup NCD$, whereas $CD \cup C$ and $C \cup NCD$ are unconnected (i.e. they do not stand in any Aristotelian relation at all). This clearly shows that the duality relation DUAL cannot straightforwardly be identified with the Aristotelian relation of subalternation. This argument for the conceptual independence between the Aristotelian and duality geometries has already been made earlier in the literature (recall the relevant discussion in Sect. 3.2), but what we have shown here, is that the argument can be made not only based on object-logical notions (as has been done so far in the literature), but also based on metalogical notions.

We will finish this subsection with a more ‘lightweight’, terminological observation. Recall that in Sect. 4.4, we studied two ‘weak’ notions of contrariety, viz. $C_w = CD \cup C$ and $C_w^* = C \cup NCD$, and informally said that these two notions are ‘dual’ to each other, since they can both be obtained from the strong notion of contrariety ($C_s = C$) by dropping the latter’s $\not\models$ - or \models -condition, respectively. However, in this subsection we have shown that

⁴⁴ Analogously, at the object-logical level, [76] studies the duality relations that hold in the Aristotelian RDH decorated with the 14 binary propositional connectives.

$\text{DUAL}(CD \cup C, C \cup NCD)$, which means exactly that $\text{DUAL}(C_w, C_w^*)$, and hence, C_w and C_w^* turn out to be also ‘dual’ to each other in the technically precise sense of the word!⁴⁵

7.2. Duality Diagrams for the Implication Relations

By now, it should no longer come as a surprise that all results from the previous subsection about duality diagrams for the *opposition* relations can straightforwardly be transposed to duality diagrams for the *implication* relations (also recall the relation between Sects. 4 and 5 on Aristotelian diagrams for \mathcal{OG} and \mathcal{IG} , respectively). In particular, it follows from Lemma 3.3 that BI and NI are their own internal negation, and that LI and RI are each other’s internal negation. Constructing duality diagrams for the implication relations is thus entirely straightforward; however, to avoid being overly repetitive, we will not do this in full detail at this point.

8. Duality Diagrams for Aristotelian and Duality Relations

In this section, we finish our exploration of metalogical decorations of duality diagrams. Section 8.1 studies duality diagrams for the Aristotelian relations, while Sect. 8.2 studies duality diagrams for the duality relations themselves.

8.1. Duality Diagrams for Aristotelian Relations

Since the Aristotelian geometry is hybrid between the opposition and implication geometries ($\mathcal{AG} \subseteq \mathcal{OG} \cup \mathcal{IG}$), many of the diagrams that were studied in Sect. 7 can be viewed not only as duality diagrams for \mathcal{OG} or \mathcal{IG} , but also as duality diagrams for \mathcal{AG} itself (a similar observation was made at the beginning of Sect. 6.1 about $\mathcal{OG}/\mathcal{IG}$ - and \mathcal{AG} -based decorations of *Aristotelian* diagrams). However, these diagrams do not entirely exemplify the hybrid nature of \mathcal{AG} , since none of them has a decoration that contains opposition *and* implication relations simultaneously. This is not a coincidence, since the INEG-relations in these diagrams are all based on Lemma 3.3, which describes the effects of negating both arguments of an opposition or implication relation. This lemma does not establish any connection between \mathcal{OG} and \mathcal{IG} : if both arguments of an opposition (resp. implication) relation are negated, the result is again an opposition (resp. implication) relation. This shows that there do not exist any duality diagrams (in the classical sense of the word ‘duality’, as captured by Definition 3.6) whose decoration contains opposition *and* implication relations simultaneously.

From this perspective, Lemma 3.3 stands in sharp contrast to Lemma 3.4. The latter describes the effects of negating only a single (either the first or the second) argument of an opposition or implication relation, and thereby establishes various links between \mathcal{OG} and \mathcal{IG} : if a single argument of an opposition relation is negated, the result is an implication relation, and vice versa. In terms of duality, this means that by moving from *classical* duality (Definition 3.6) to

⁴⁵ Similar remarks can be made, of course, about $SC_w = CD \cup SC$ and $SC_w^* = SC \cup NCD$.

generalized Post duality (Definition 3.7), we will be able to construct duality diagrams whose decoration contains opposition as well as implication relations.

To illustrate this, consider item 2b of Lemma 3.4, which states the following:

$$\text{for all formulas } \varphi, \psi : C(\varphi, \psi) \text{ iff } LI(\varphi, \neg\psi).$$

Reformulating this in terms of characteristic functions, we get:

$$\text{for all formulas } \varphi, \psi : \chi_C(\varphi, \psi) = 1 \text{ iff } \chi_{LI}(\varphi, \neg\psi) = 1$$

and hence:

$$\text{for all formulas } \varphi, \psi : \chi_C(\varphi, \psi) = \chi_{LI}(\varphi, \neg\psi).$$

By Definition 3.7, this means exactly that $INEG_2(\chi_C, \chi_{LI})$, and in a derived sense, $INEG_2(C, LI)$. In exactly the same way, it can be shown that $INEG_2(CD, BI)$, $INEG_2(SC, RI)$, and so on. Moving to *non-atomic* relations of $\wp^\cup(\mathcal{OG})$ and $\wp^\cup(\mathcal{IG})$, we find, for example:

$$\text{for all formulas } \varphi, \psi : (\varphi, \psi) \in CD \cup C \text{ iff } (\varphi, \neg\psi) \in BI \cup LI.$$

and hence

$$\text{for all formulas } \varphi, \psi : \chi_{CD \cup C}(\varphi, \psi) = \chi_{BI \cup LI}(\varphi, \neg\psi),$$

which means exactly that $INEG_2(CD \cup C, BI \cup LI)$. Analogously, it also holds that $INEG_2(SC \cup NCD, RI \cup NI)$.

We have already established that $INEG_2(CD \cup C) = BI \cup LI$ and (in Sect. 7.2) that $ENEG(BI \cup LI) = RI \cup NI$. Since $ENEG \circ INEG_2 = DUAL_2$, it should follow that also $DUAL_2(CD \cup C) = ENEG(INEG_2(CD \cup C)) = ENEG(BI \cup LI) = RI \cup NI$. To verify this, note that Lemmas 3.5 and 3.4 imply that

$$\text{for all formulas } \varphi, \psi : (\varphi, \psi) \in CD \cup C \text{ iff } (\varphi, \neg\psi) \notin RI \cup NI,$$

and hence

$$\text{for all formulas } \varphi, \psi : \chi_{CD \cup C}(\varphi, \psi) = \neg\chi_{RI \cup NI}(\varphi, \neg\psi),$$

which by Definition 3.7 indeed means that $DUAL_2(CD \cup C, RI \cup NI)$.

In sum, then, we find that the opposition relations $CD \cup C$ and $SC \cup NCD$ together with the implication relations $BI \cup LI$ and $RI \cup NI$ can be used to decorate a generalized Post duality square, which is shown in Fig. 32a. Note that these are exactly the four relations defined by Löbner [52, p. 55],

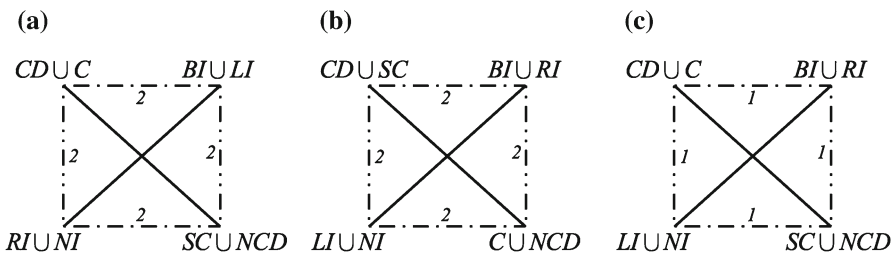


FIGURE 32. Three generalized Post duality squares

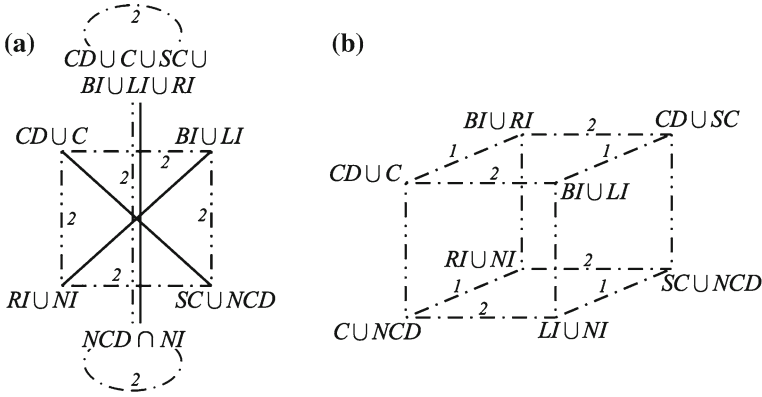


FIGURE 33. **a** Shield and spear generalized Post duality diagram corresponding to Seuren’s Aristotelian hexagon in Fig. 25, **b** generalized Post duality cube corresponding to the Aristotelian octagon in Fig. 26b

who explicitly recognized that they “themselves form a duality square *with respect to the predicate* [i.e. using $INEG_2$ and $DUAL_2$ instead of $INEG$ and $DUAL$]” (emphasis added). In Sect. 6.1 we showed that these four relations also yield an Aristotelian square (recall Fig. 24), and pointed out that this, too, was already recognized by Löbner. The exact nature of this Aristotelian square turned out to depend on whether or not the relations’ first argument is assumed to be satisfiable—compare Fig. 24a, b. By contrast, the (generalized Post) duality square in Fig. 32a does not depend on any additional assumptions.

In an analogous fashion, it can be shown that $CD \cup SC$, $C \cup NCD$, $BI \cup RI$ and $LI \cup NI$ yield another generalized Post duality square, which is shown in Fig. 32b. Note that these are exactly the four relations that were used in Sect. 6.1 to define a ‘variant’ to Löbner’s square (recall Fig. 26a). Furthermore, the exact nature of this Aristotelian square turned out to depend on whether or not the relations’ first argument is not a tautology; by contrast, the corresponding generalized Post duality square in Fig. 32b does not depend on any additional assumptions.

Similar results can be obtained if we work with $INEG_1$ instead of $INEG_2$. For example, it follows from items 1a and 2a of Lemma 3.4 that $INEG_1(CD \cup C, BI \cup RI)$, and from items 3a and 4a of the same lemma that $INEG_1(SC \cup NCD, LI \cup NI)$. Furthermore, also making use of Lemma 3.5, it is easy to check that $DUAL_1(CD \cup C, LI \cup NI)$ and $DUAL_1(SC \cup NCD, BI \cup RI)$. The resulting generalized Post duality square is shown in Fig. 32c.

Recall that in Sect. 6.1, we also studied Seuren’s Aristotelian meta-hexagon, which can be obtained by adding $CD \cup C \cup SC \cup BI \cup LI \cup RI$ and $NCD \cap NI$ to Löbner’s four relations (recall Fig. 25). Now, it follows from Lemma 3.4 that

$$\text{for all formulas } \varphi, \psi : (\varphi, \psi) \in NCD \cap NI \quad \text{iff} \quad (\varphi, \neg\psi) \in NCD \cap NI,$$

and hence $\text{INEG}_2(NCD \cap NI, NCD \cap NI)$. Analogously, it holds that $\text{INEG}_2(CD \cup C \cup SC \cup BI \cup LI \cup RI, CD \cup C \cup SC \cup BI \cup LI \cup RI)$ and $\text{DUAL}_2(NCD \cap NI, CD \cup C \cup SC \cup BI \cup LI \cup RI)$. Seuren’s Aristotelian hexagon in Fig. 25 thus turns out to correspond to the generalized Post duality diagram shown in Fig. 33a. This diagram consists of two ‘independent’ parts, viz. the duality square which was already shown in Fig. 32a, and a degenerated duality diagram containing $NCD \cap NI$ and $CD \cup C \cup SC \cup BI \cup LI \cup RI$.⁴⁶ Finally, recall that in Sect. 6.1 it was shown that the exact nature of Seuren’s Aristotelian hexagon depends on whether or not the relations’ first argument is assumed to be satisfiable—compare Fig. 25a, b. By contrast, the (generalized Post) duality diagram in Fig. 33a does not depend on any additional assumptions.

We showed in Sect. 3.2 that generalized Post duality generally gives rise to *cube* diagrams. Starting with $CD \cup C$ and applying the INEG_1 -, INEG_2 -, DUAL_1 - and DUAL_2 -operators (and their combinations) to it, we find exactly the 8 relations that were shown in Sect. 6.1 to yield the Aristotelian octagon in Fig. 26b. In other words, from a duality perspective, the Aristotelian octagon in Fig. 26b turns out to correspond to a generalized Post duality cube, which is shown in Fig. 33b. This cube has been studied before in [51, Figure 3], and it contains the (classical) duality square in Fig. 31a and the three (generalized Post) duality squares in Fig. 32 as subdiagrams on its diagonal planes. Finally, recall that in Sect. 6.1 it was shown that the exact nature of the Aristotelian octagon in Fig. 26b depends on whether or not the relations’ first and second argument are assumed to be contingent. By contrast, the corresponding generalized Post duality cube in Fig. 33b does not depend on any additional assumptions.

8.2. Duality Diagrams for Duality Relations

In this subsection we will show that the duality relations can themselves be used to decorate duality diagrams. The key idea is that the duality relations are all essentially their own internal negations. Consider, for example, the ENEG -relation: for any two operators $O_1, O_2: \mathbb{A}^n \rightarrow \mathbb{B}$, the Boolean nature of $\neg_{\mathbb{B}}$ and Definition 3.6 imply that⁴⁷

$$\begin{aligned} \text{ENEG}(O_1, O_2) &\text{ iff for all } \bar{a} \in \mathbb{A}^n: O_1(\bar{a}) = \neg_{\mathbb{B}} O_2(\bar{a}) \\ &\text{ iff for all } \bar{a} \in \mathbb{A}^n: \neg_{\mathbb{B}} O_1(\bar{a}) = \neg_{\mathbb{B}} \neg_{\mathbb{B}} O_2(\bar{a}) \\ &\text{ iff } \text{ENEG}(\neg_{\mathbb{B}} O_1, \neg_{\mathbb{B}} O_2). \end{aligned}$$

To make this fully precise, we will write \mathbb{F} for the class of all operators $O: \mathbb{A}^n \rightarrow \mathbb{B}$. It is well-known that \mathbb{F} is itself a Boolean algebra, whose Boolean operators are defined *pointwise*. For example, given operators $O_1, O_2 \in \mathbb{F}$, their meet $O_1 \wedge_{\mathbb{F}} O_2$ is defined to be the operator $O_1 \wedge_{\mathbb{F}} O_2: \mathbb{A}^n \rightarrow \mathbb{B}: \bar{a} \mapsto (O_1 \wedge_{\mathbb{F}} O_2)(\bar{a}) := O_1(\bar{a}) \wedge_{\mathbb{B}} O_2(\bar{a})$, and similarly, the complement $\neg_{\mathbb{F}} O_1$ is defined to be the operator $\neg_{\mathbb{F}} O_1: \mathbb{A}^n \rightarrow \mathbb{B}: \bar{a} \mapsto (\neg_{\mathbb{F}} O_1)(\bar{a}) := \neg_{\mathbb{B}}(O_1(\bar{a}))$. Relations such as ENEG are binary relations on \mathbb{F} and hence, their characteristic

⁴⁶ The duality diagram corresponding to a JSB hexagon was called a “shield and spear” diagram by Smessaert [77, p. 180].

⁴⁷ We write \bar{a} for the n -tuple $(a_1, \dots, a_n) \in \mathbb{A}^n$, and also $\overline{\neg_{\mathbb{A}} \bar{a}}$ for $(\neg_{\mathbb{A}} a_1, \dots, \neg_{\mathbb{A}} a_n) \in \mathbb{A}^n$.

function is itself an operator $\chi_{\text{ENEG}}: \mathbb{F}^2 \rightarrow \{0, 1\}$. Since \mathbb{F} as well as $\{0, 1\}$ are Boolean algebras, the operator χ_{ENEG} is of the right ‘type’, and Definition 3.6 is applicable to it. In particular, recall that we have just showed above that

$$\text{for all } O_1, O_2 \in \mathbb{F} : \text{ENEG}(O_1, O_2) \quad \text{iff} \quad \text{ENEG}(\neg_{\mathbb{B}}O_1, \neg_{\mathbb{B}}O_2).$$

Putting it in terms of characteristic functions, this becomes:

$$\text{for all } O_1, O_2 \in \mathbb{F} : \chi_{\text{ENEG}}(O_1, O_2) = 1 \quad \text{iff} \quad \chi_{\text{ENEG}}(\neg_{\mathbb{B}}O_1, \neg_{\mathbb{B}}O_2) = 1.$$

Now, in the Boolean algebra $\{0, 1\}$, the biconditional $x = 1 \Leftrightarrow y = 1$ is equivalent to $x = y$, and thus we get:

$$\text{for all } O_1, O_2 \in \mathbb{F} : \chi_{\text{ENEG}}(O_1, O_2) = \chi_{\text{ENEG}}(\neg_{\mathbb{B}}O_1, \neg_{\mathbb{B}}O_2).$$

Because of the pointwise definition of $\neg_{\mathbb{F}}$ in terms of $\neg_{\mathbb{B}}$, this can be reformulated one final time, as follows:

$$\text{for all } O_1, O_2 \in \mathbb{F} : \chi_{\text{ENEG}}(O_1, O_2) = \chi_{\text{ENEG}}(\neg_{\mathbb{F}}O_1, \neg_{\mathbb{F}}O_2).$$

By Definition 3.6, this means exactly that $\text{INEG}(\chi_{\text{ENEG}}, \chi_{\text{ENEG}})$. Moving from the characteristic function χ_{ENEG} to the duality relation ENEG itself, we can also say, in a derived sense, that $\text{INEG}(\text{ENEG}, \text{ENEG})$, i.e. ENEG is its own internal negation.

In exactly the same way, it can also be shown that the other duality relations, ID , INEG and DUAL , are their own internal negations, based on the following chains of equivalences:

$$\begin{aligned} \text{ID}(O_1, O_2) & \quad \text{iff} \quad \text{for all } \bar{a} \in \mathbb{A}^n : O_1(\bar{a}) = O_2(\bar{a}) \\ & \quad \text{iff} \quad \text{for all } \bar{a} \in \mathbb{A}^n : \neg_{\mathbb{B}}O_1(\bar{a}) = \neg_{\mathbb{B}}O_2(\bar{a}) \\ & \quad \text{iff} \quad \text{ID}(\neg_{\mathbb{B}}O_1, \neg_{\mathbb{B}}O_2), \\ \text{INEG}(O_1, O_2) & \quad \text{iff} \quad \text{for all } \bar{a} \in \mathbb{A}^n : O_1(\bar{a}) = O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\ & \quad \text{iff} \quad \text{for all } \bar{a} \in \mathbb{A}^n : \neg_{\mathbb{B}}O_1(\bar{a}) = \neg_{\mathbb{B}}O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\ & \quad \text{iff} \quad \text{INEG}(\neg_{\mathbb{B}}O_1, \neg_{\mathbb{B}}O_2), \\ \text{DUAL}(O_1, O_2) & \quad \text{iff} \quad \text{for all } \bar{a} \in \mathbb{A}^n : O_1(\bar{a}) = \neg_{\mathbb{B}}O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\ & \quad \text{iff} \quad \text{for all } \bar{a} \in \mathbb{A}^n : \neg_{\mathbb{B}}O_1(\bar{a}) = \neg_{\mathbb{B}}\neg_{\mathbb{B}}O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\ & \quad \text{iff} \quad \text{DUAL}(\neg_{\mathbb{B}}O_1, \neg_{\mathbb{B}}O_2). \end{aligned}$$

We will also consider the complement of the ENEG -relation, i.e. ‘not ENEG ’ $:= \mathbb{F}^2 - \text{ENEG}$. By definition of ‘not ENEG ’, it holds that

$$\text{for all } O_1, O_2 \in \mathbb{F} : (O_1, O_2) \in \text{‘not ENEG’} \quad \text{iff} \quad (O_1, O_2) \notin \text{ENEG}.$$

Putting this in terms of characteristic functions, we get:

$$\text{for all } O_1, O_2 \in \mathbb{F} : \chi_{\text{‘not ENEG’}}(O_1, O_2) = 1 \quad \text{iff} \quad \chi_{\text{ENEG}}(O_1, O_2) = 0.$$

Now, in the Boolean algebra $\{0, 1\}$, the biconditional $x = 1 \Leftrightarrow y = 0$ is equivalent to $x = \neg y$ (where \neg is the complement operator of the $\{0, 1\}$ Boolean algebra, i.e. $\neg y := 1 - y$), and thus we get:

$$\text{for all } O_1, O_2 \in \mathbb{F} : \chi_{\text{‘not ENEG’}}(O_1, O_2) = \neg\chi_{\text{ENEG}}(O_1, O_2).$$

By Definition 3.6, this means exactly that $\text{ENEG}(\chi_{\text{‘not ENEG’}}, \chi_{\text{ENEG}})$, or in a derived sense: $\text{ENEG}(\text{‘not ENEG’}, \text{ENEG})$.

We have already established above that $\text{INEG}(\text{ENEG}) = \text{ENEG}$ and just now also that $\text{ENEG}(\text{ENEG}) = \text{‘not ENEG’}$; it now follows that $\text{DUAL}(\text{ENEG}) =$

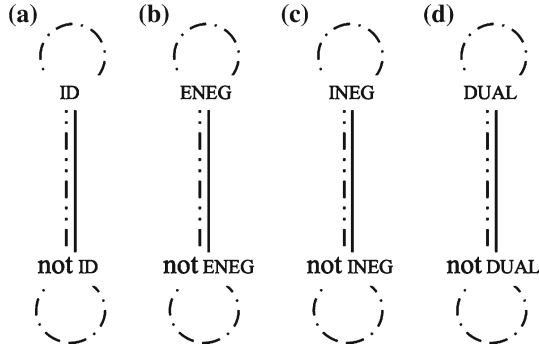


FIGURE 34. Degenerated duality diagrams for the duality relations

$(\text{ENEG} \circ \text{INEG})(\text{ENEG}) = \text{ENEG}(\text{INEG}(\text{ENEG})) = \text{ENEG}(\text{ENEG}) = \text{'not ENEG'}$, and hence $\text{DUAL}(\text{ENEG}, \text{'not ENEG'})$.

In exactly the same way, we can also define the relations $\text{'not ID'} := \mathbb{F}^2 - \text{ID}$, $\text{'not INEG'} := \mathbb{F}^2 - \text{INEG}$ and $\text{'not DUAL'} := \mathbb{F}^2 - \text{DUAL}$, and show the following:

$$\begin{aligned} &\text{ENEG}(\text{ID}, \text{'not ID'}), && \text{DUAL}(\text{ID}, \text{'not ID'}), \\ &\text{ENEG}(\text{INEG}, \text{'not INEG'}), && \text{DUAL}(\text{INEG}, \text{'not INEG'}), \\ &\text{ENEG}(\text{DUAL}, \text{'not DUAL'}), && \text{DUAL}(\text{DUAL}, \text{'not DUAL'}). \end{aligned}$$

Each of the four duality relations $R \in \mathcal{DG}$ thus gives rise to a degenerated duality diagram, which is shown in Fig. 34. Unfortunately, these degenerated duality diagrams are not particularly interesting. A much richer perspective can be obtained by considering the generalized Post duality relations holding between the duality relations.⁴⁸ For example, note that for all operators $O_1, O_2: \mathbb{A}^n \rightarrow \mathbb{B}$, we have:

$$\begin{aligned} \text{ID}(O_1, O_2) &\text{ iff for all } \bar{a} \in \mathbb{A}^n: O_1(\bar{a}) = O_2(\bar{a}) \\ &\text{ iff for all } \bar{a} \in \mathbb{A}^n: \neg_{\mathbb{B}} O_1(\bar{a}) = \neg_{\mathbb{B}} O_2(\bar{a}) \\ &\text{ iff } \text{ENEG}(\neg_{\mathbb{B}} O_1, O_2) \\ &\text{ iff for all } \bar{a} \in \mathbb{A}^n: O_1(\bar{a}) = \neg_{\mathbb{B}} \neg_{\mathbb{B}} O_2(\bar{a}) \\ &\text{ iff } \text{ENEG}(O_1, \neg_{\mathbb{B}} O_2). \end{aligned}$$

These chains of equivalences can be used to show that $\text{INEG}_1(\text{ID}, \text{ENEG})$ and $\text{INEG}_2(\text{ID}, \text{ENEG})$. Furthermore, we can also show that $\text{INEG}_1(\text{INEG}, \text{DUAL})$ and $\text{INEG}_2(\text{INEG}, \text{DUAL})$, based on the following chains of equivalences:

⁴⁸ In this paper we will focus on generalized Post duality relations holding between the ‘classical’ duality relations; in other words, we will define ‘classical’ duality decorations for generalized Post duality diagrams. Of course, one could also study (classical and/or generalized Post) duality relations holding between the generalized Post duality relations themselves, or, in other words, define generalized Post duality decorations for (classical and/or generalized Post) duality diagrams. This, however, will be left for another paper.

$$\begin{aligned}
 \text{INEG}(O_1, O_2) & \text{ iff for all } \bar{a} \in \mathbb{A}^n: O_1(\bar{a}) = O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\
 & \text{ iff for all } \bar{a} \in \mathbb{A}^n: \neg_{\mathbb{B}}O_1(\bar{a}) = \neg_{\mathbb{B}}O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\
 & \text{ iff } \text{DUAL}(\neg_{\mathbb{B}}O_1, O_2) \\
 & \text{ iff for all } \bar{a} \in \mathbb{A}^n: O_1(\bar{a}) = \neg_{\mathbb{B}}\neg_{\mathbb{B}}O_2(\overline{\neg_{\mathbb{A}}\bar{a}}) \\
 & \text{ iff } \text{DUAL}(O_1, \neg_{\mathbb{B}}O_2).
 \end{aligned}$$

We have just shown that $\text{INEG}_i(\text{ID}) = \text{ENEG}$ and $\text{INEG}_i(\text{INEG}) = \text{DUAL}$ (for $i = 1, 2$). Using this, we find

$$\begin{aligned}
 \text{DUAL}_i(\text{ID}) & = \text{ENEG}(\text{INEG}_i(\text{ID})) = \text{ENEG}(\text{ENEG}) = \text{‘not ENEG’}, \\
 \text{DUAL}_i(\text{INEG}) & = \text{ENEG}(\text{INEG}_i(\text{INEG})) = \text{ENEG}(\text{DUAL}) = \text{‘not DUAL’},
 \end{aligned}$$

and hence $\text{DUAL}_i(\text{ID}, \text{‘not ENEG’})$ and $\text{DUAL}_i(\text{INEG}, \text{‘not DUAL’})$ (for $i = 1, 2$).

By going from ‘classical’ to generalized Post duality, the degenerated duality diagrams for ID and ENEG in Fig. 34a, b thus ‘click together’ to yield a generalized Post duality square, which is shown in Fig. 35a. Since $\text{INEG}_1(\text{ID}) = \text{ENEG} = \text{INEG}_2(\text{ID})$, this square can actually be seen as a degenerated generalized Post duality cube (see Fig. 6 in Sect. 3.2). Furthermore, if we ignore the classical INEG- and DUAL-relations, then this square can be decomposed into a square for $\text{INEG}_1/\text{DUAL}_1$ and one for $\text{INEG}_2/\text{DUAL}_2$, which are shown in Fig. 35b, c, respectively.

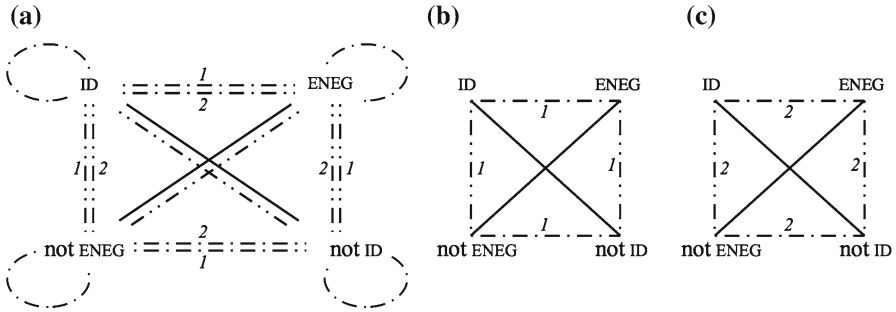


FIGURE 35. Generalized Post duality square for ID and ENEG

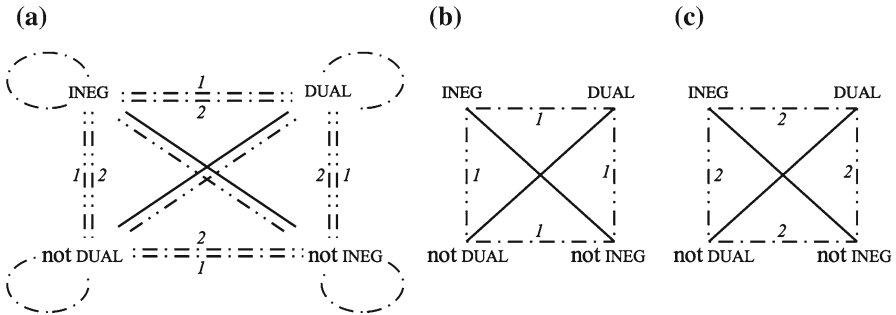


FIGURE 36. Generalized Post duality square for INEG and DUAL

Finally, the degenerated duality diagrams for INEG and DUAL in Fig. 34c, d also ‘click together’ to yield another generalized Post duality square, which is shown in Fig. 36a. We can make the same remarks about this square as about the one in Fig. 35a; in particular, if we ignore the classical INEG- and DUAL-relations, it again decomposes into a square for INEG₁/DUAL₁ and one for INEG₂/DUAL₂, which are shown in Fig. 36b, c, respectively.

9. Conclusion

When dealing with logical diagrams, we can distinguish between a diagram’s *type* and its *decoration*. The diagram’s decoration consists of the formulas/notions that it represents, while its type is determined by the kind of logical relations holding between those formulas/notions.⁴⁹ Consider, for example, the square diagram in Fig. 2b: this is an Aristotelian square for (formulas from) the modal logic S5, so we can say that its type is Aristotelian, while its decoration is based on S5.

In this paper, we have studied metalogical diagrams, whose types and decorations consist of logical relations. In particular, we have studied two diagram types, viz. Aristotelian diagrams and duality diagrams, and four kinds of diagram decorations, viz. based on relations from the opposition geometry (\mathcal{OG}), the implication geometry (\mathcal{IG}), the Aristotelian geometry (\mathcal{AG}) and the duality geometry (\mathcal{DG}). A high-level overview of the core sections of the paper can therefore be given by means of the following table:

	\mathcal{OG}	\mathcal{IG}	\mathcal{AG}	\mathcal{DG}
Aristotelian diagrams	Sect. 4	Sect. 5	Sect. 6.1	Sect. 6.2
Duality diagrams	Sect. 7.1	Sect. 7.2	Sect. 8.1	Sect. 8.2

This overview table suggests a natural question: why are the opposition and implication relations only used in the diagrams’ decorations, and not in the diagram types? In other words, shouldn’t it also be possible to study opposition diagrams and implication diagrams for the opposition, implication, Aristotelian and duality relations? The answer is that this is indeed perfectly possible. In Sect. 3.1 it was explained that for every Aristotelian diagram, we can define a corresponding opposition diagram and a corresponding implication diagram (with the original Aristotelian diagram being informationally optimal among the three types of diagrams). For example, given the Aristotelian square for S5 shown in Fig. 2b, we can define the corresponding opposition and implication squares for S5, which are shown in Fig. 2a, c, respectively. This account perfectly transfers from object- to metalogical decorations, and hence, for every Aristotelian diagram that has been studied in Sects. 4, 5 and 6 of this paper, it is possible to define the corresponding opposition and implication diagrams. However, for reasons of space (and because the

⁴⁹ If a logical diagram is viewed as a graph [34], its type and decoration correspond to the graph’s edges and vertices, respectively.

Aristotelian diagrams are more informative than their corresponding opposition and implication diagrams anyway), we have focused on the Aristotelian diagrams.

To conclude, we will highlight some general themes that have shown up at various places throughout the paper. The first such consideration is that despite their more abstract nature, metalogical decorations of logical diagrams give rise to the same kinds of linguistic lexicalization patterns as object-logical decorations. For example, in Sect. 4.2 we compared the non-lexicalization of the metalogical notions of ‘non-tautology’ and ‘non-contingency’ (Fig. 11) to the non-lexicalization of the quantifiers *nall* and *allno*. Furthermore, in Sect. 4.3 we pointed out that the strong and weak notions of contrariety are analogous to the bilateral and unilateral readings of quantifiers such as *some* and *many* (see Fig. 13). Observations such as these further reinforce Seuren’s claim that “universal lexicalization phenomena are also present in the terminology of the metalogic of logical relations. We surely have not heard the last of this” [72, p. 11].

A second remark concerns the relative importance of non-contradiction (*NCD*) and non-implication (*NI*). In [79] these are shown to be the least informative relations of their respective geometries, i.e. *NCD* is the least informative opposition relation, and *NI* is the least informative implication relation. From this perspective, these relations seem to be mathematical artefacts—needed solely to turn \mathcal{OG} and \mathcal{IG} into partitions—without much independent significance. In this paper, by contrast, *NCD* and *NI* have played an important role at various places.⁵⁰ For example, in Sect. 4.2 it was shown that $NCD \cap \Delta = \{(\varphi, \varphi) \mid NCD(\varphi, \varphi)\}$ corresponds to the largest class of formulas, viz. the contingencies. This allowed us to define the strong JSB hexagon in Fig. 11b, instead of merely the square in Fig. 12. Furthermore, in Sect. 4.5 we argued that the errors in Béziau’s hexagon in Fig. 17a, b are the result of illicitly ignoring *NCD*, and showed how these errors can be corrected by taking this relation into account (see Fig. 18). Similarly, in Sect. 5.2 we explained the difference between the strong JSB hexagon for total orders in Fig. 21a and the RDH for partial orders in Fig. 22 in terms of *NI* (or rather, its abstract analogue, the incomparability relation $\#$). Finally, as was discussed in Sect. 6.1, *NCD* and *NI* can be used to define the notion of unconnectedness or independence (viz. as $NCD \cap NI$). In ongoing work [26], we are exploring the interplay between unconnectedness and other notions of logical (in)dependence, which gives rise to the same kind of metalogical diagrams as discussed in this paper.

⁵⁰ For yet another metalogical illustration of the importance of *NCD*, recall that in 2003, Béziau [1] famously argued that the Aristotelian relations *CD*, *C* and *SC* are analogous to classical, paracomplete and paraconsistent negation, respectively. However, in more recent work, Béziau has also considered so-called *paranormal* negations [2, 3, 7]. These are characterized by the fact that a proposition and its paranormal negation can be true together, and can also be false together. But this means exactly that paranormal negation is analogous to *NCD*. Béziau’s original analogy can thus be completed as follows: the opposition relations *CD*, *C*, *SC* and *NCD* are analogous to classical, paracomplete, paraconsistent and paranormal negation, respectively.

A third topic of interest is the distinction between ‘classical’ and ‘degenerated’ Aristotelian squares. It is well-known that the four categorical statements of syllogistics constitute a ‘classical’ Aristotelian square. However, it is equally well-known that if we move from syllogistics to contemporary first-order logic, by dropping the assumption of *existential import*, only the two contradictions are left, and hence the classical Aristotelian square degenerates into a mere Aristotelian cross [30, Section 4]. As we have seen throughout this paper, similar phenomena also occur for metalogical instead of object-logical decorations. For example, in Sect. 4.2 we defined an Aristotelian square for metalogical notions such as tautology and satisfiability, which is shown in Fig. 12a. However, if we drop the assumption that the underlying logical system is *consistent*, then this square degenerates into a cross, as shown in Fig. 12b. Furthermore, in Sect. 6.1 we showed that Löbner’s [52, p. 55] four relations yield an Aristotelian square, which is shown in Fig. 24b. However, if we drop the assumption that the first argument of Löbner’s relations is *satisfiable*, then this square also degenerates into a cross, as shown in Fig. 24a. Finally, in Sect. 6.2 we studied an Aristotelian square for the duality relations INEG and ENEG, which is shown in Fig. 28b. However, if we drop the assumption that operators cannot be *self-dual*, then this square, once more, degenerates into a cross, as shown in Fig. 28a.

A fourth and final recurrent theme is the subtle relation between the different types of diagrams. We have seen that several sets of relations can be used to decorate Aristotelian as well as duality diagrams. In some cases, however, the Aristotelian diagram turns out to depend on certain additional assumptions, whereas the corresponding duality diagram does not. For example, in Sect. 8.1 we studied the duality square for Löbner’s relations, which is shown in Fig. 32a. In Sect. 6.1, however, we showed that these relations can also be used to define an Aristotelian square, but whether this square is classical or degenerated, depends on whether the relations’ first argument is assumed to be satisfiable; compare Fig. 24a, b. Similarly, the six relations in the duality diagram shown in Fig. 33a can also be used to define an Aristotelian hexagon, but whether this is a JSB or a U4 hexagon again depends on whether the relations’ first argument is assumed to be satisfiable; compare Fig. 25a, b.

This sharp contrast between Aristotelian and duality diagrams can be seen as the metalogical manifestation of a more general phenomenon that is well-understood at the object-logical level: the Aristotelian relations are *sensitive* to the deductive power of the underlying logical system, but the duality relations are entirely *insensitive* to this [23].⁵¹ Consider, for example, the formulas $\Box p$ and $\Box \neg p$ from modal logic. The Aristotelian relation holding between these formulas heavily depends on the particular modal system that we happen to be working in. For example, (i) in the system D, these formulas are contrary, (ii) in the system $D + \{\Diamond \rightarrow \Box p\}$, they are contradictory, and (iii) in the minimal normal system K, they are unconnected, i.e. they do not stand in any Aristotelian relation at all [18]. However, switching from Aristotelian

⁵¹ Also see Sect. 2, in particular, the motivation for going from Definition 2.1 to 2.2.

to duality relations, note that in each of these three modal systems, it holds that $\Box p$ and $\Box \neg p$ are each other's internal negation, which shows that the duality relation holding between these formulas is independent of any additional factors. This contrast between 'context-sensitive' Aristotelian relations and 'context-insensitive' duality relations manifests itself at the metalogical as well as at the object-logical level.

All these remarks and observations point in the direction of a final, overarching conclusion: there exists a fundamental continuity between object- and metalogical decorations of logical diagrams. The mathematical background of this continuity was described in Sect. 2 (for Aristotelian diagrams) and in Sect. 3.2 (for duality diagrams). Furthermore, since metalogical decorations give rise to the same kinds of rich logical and linguistic behavior as object-logical decorations, it is to be expected that over time, the former will come to be as widely studied as the latter.

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