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A Strong and Rich 4-Valued Modal Logic Without Łukasiewicz-Type Paradoxes

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Abstract. The aim of this paper is to introduce an alternative to Łukasiewicz's 4-valued modal logic Ł. As it is known, Ł is afflicted by "Łukasiewicz (modal) type paradoxes". The logic we define, PŁ4, is a strong paraconsistent and paracomplete 4-valued modal logic free from this type of paradoxes. PŁ4 is determined by the degree of truth-preserving consequence relation defined on the ordered set of values of a modification of the matrix MŁ characteristic for the logic Ł. On the other hand, PŁ4 is a rich logic in which a number of connectives can be defined. It also has a simple bivalent semantics of the Belnap–Dunn type.

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1. Introduction

The aim of this paper is to provide an alternative to Łukasiewicz's 4-valued modal logic Ł that will lack the "Łukasiewicz type (modal) paradoxes" derivable in Ł. This alternative essentially consists in maintaining the strong conditional of Ł but introducing a paraconsistent De Morgan negation (instead of the Boolean one characteristic of Ł) along with a new definition of the necessity operator.

As it is known, the motivation underlying Łukasiewicz's many-valued systems lies in (1) the rejection of deterministic philosophy; (2) the aim to provide an adequate logical foundation to the notions of *possibility* and *necessity* (cf. [24]). Thus, for example, Łukasiewicz points out the following about the 3-valued logic Ł3, the first one of the many-valued logics he defined: "The indeterministic philosophy [...] is the metaphysical substratum of the new logic" [17, p. 88]. "The third logical value may be interpreted as "possibility" " [17, p. 87].

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Łukasiewicz presented two different analyses of modal notions by means of many-valued logics: (a) the linearly ordered systems Ł3,..., Łn,..., Łω he defined since 1920 (cf. [20]); (b) the 4-valued modal logic Ł he defined in the last years of his career (cf. [18,19]). In the family Ł3,..., Łn,..., Łω the modal operators L (necessity) and M (possibility) can be defined as follows: $LA =_{\text{df}} \neg(A \rightarrow \neg A)$, $MA =_{\text{df}} \neg A \rightarrow A$ (these definitions were suggested by Tarski when he was Łukasiewicz's student; the symbols L and M are Łukasiewicz's—cf. [13], notes 2 and 3; cf. Definition 2.1 on the languages used in this paper). On the other hand, L and M are defined in Ł independently of the rest of the connectives of the system (cf. [13,18,19]).

Unfortunately, both the systems of the sequence Ł3,...,Łn,...,Łω and the logic Ł validate such theses as the following (cf. Proposition 7.13 below): (F7) $L(A \vee B) \rightarrow (LA \vee LB)$ and (F8) $(MA \wedge MB) \rightarrow M(A \wedge B)$, which are in principle difficult to accept from an intuitive point of view. Moreover, in addition to F7 and F8, the following are provable in Ł: (F5) $(A \rightarrow B) \rightarrow (MA \rightarrow MB)$; (F6) $(A \rightarrow B) \rightarrow (LA \rightarrow LB)$; (F9) $LA \rightarrow (B \rightarrow LB)$; (F10) $LA \rightarrow (MB \rightarrow B)$. Theses F9 and F10 are especially counterintuitive, a fact that leads the authors of [13] to conclude that Ł is a "dead end" as a modal logic of necessity and possibility (the reader can find an analysis of Ł explaining why these counterintuitive consequences arise in the system in [23]). Thus, it must be concluded that neither the family Ł3,...,Łn,...,Łω nor Ł can be taken as a many-valued analysis of the notions of necessity and possibility when understood in their customary sense.

The aim of this paper is to introduce the logic PŁ4 ('a paraconsistent version of Łukasiewicz's 4-valued modal logic Ł'). The logic PŁ4 is determined by the degree of truth-preserving consequence relation defined on the ordered set of values of the matrix MPŁ4 (cf. Definition 2.6), which is a modification of Łukasiewicz's matrix MŁ (cf. Definition 2.5) by keeping the conditional table while modifying both the tables for negation and necessity (cf. Definitions 2.5, 2.6 and Proposition 7.5). It will be proved that the Tarskian definitions of L and M work in MPŁ4 in the following sense: (1) Łukasiewicz type modal paradoxes such as F5-F10 remarked above are falsified; (2) PŁ4 is a strong and genuine 4-valued modal logic (cf. Propositions 7.11–7.13 below).

In the rest of the Introduction, we shall limit ourselves to explain some properties of PŁ4 as briefly as possible and how the paper is organized.

PŁ4 has the following properties:

1. It lacks Łukasiewicz-type paradoxes as F5–F10 mentioned above (cf. Proposition 7.13).
2. PŁ4 is a strong logic as shown by the following facts:
   (a) In addition to all theorems of classical positive logic, it contains, for example, the double negation axioms, all forms of the De Morgan laws and Contraposition as admissible rule, as well as the most prominent characteristic theses of Lewis' S5 together with the rule Necessitation (it is admissible; cf. Proposition 7.12). A consequence

of the admissibility of Contraposition is that 'replacement of equivalents' holds for PŁ4; a consequence of the admissibility of Necessitation is that PŁ4 is a quasi-normal modal logic. On the other hand, classical propositional logic is definable in PŁ4 (cf. Proposition 7.9).

(b) There are four possible significant ways of extending PŁ4. These four ways are the result of adding any of the following theses: (i) $A \vee \neg A$; (ii) $\neg A \rightarrow (A \rightarrow B)$; (iii) $(\neg A \vee B) \rightarrow (A \rightarrow B)$ (or, equivalently, $\neg(A \rightarrow B) \rightarrow (A \wedge \neg B)$); (iv) $(A \rightarrow B) \rightarrow (\neg A \vee B)$ (or, equivalently, $(A \wedge \neg B) \rightarrow \neg(A \rightarrow B)$). But the extension of PŁ4 by addition of any of (i)–(iv) is classical propositional logic (cf. Proposition 7.16 and Corollary 7.17).

3. PŁ4 has a great expressive power: a number of interesting connectives are definable, some examples of which are given in Sect. 7.
4. PŁ4 can be endowed with a bivalent Belnap–Dunn semantics of the type defined for FDE by Dunn. (Cf. [11,12]. These semantics go back to Dunn's doctoral dissertation [10], but as remarked by Dunn himself [11, p. 150], essentially equivalent semantics are defined in [31,37]). The essential idea in this semantics is the following. Let T and F represent the (truth) values truth and falsity. Then, propositions can be assigned T , F, both values or none of them. These four possibilities are represented in the present paper as follows: $0 = F$ but not T; $1 =$ neither T nor F ; $2 =$ both T and F; and $3 = T$ but not F. (Cf. Remark 2.8). The fact that PŁ4 can be endowed with this type of semantics makes it possible to provide an easy completeness proof for this logic. (This part of our work in the present paper has been inspired by Brady's method for axiomatizing 3-valued and 4-valued matrices developed in his excellent [7]—cf. also [8], Chapter 9).
5. PŁ4 fills a place in the family of paraconsistent logics extending positive classical propositional logic C_+ among which the most famous exemplars in the family of paraconsistent logics are to be found, such as Da Costa's systems, Pac or J3, for example (cf. [9,16,27] about these logics). PŁ4 distinguishes itself from all these logics in the fact that the rule contraposition, Con, is admissible and so it holds as a *rule of proof* (cf. Remark 6.1). However, Con is not assumed as a rule of proof in the aforementioned systems. Furthermore, it is not even admissible in such systems as Pac (notice that a convenient consequence of Con as a rule of proof is that 'replacement of equivalents' holds). Therefore, PŁ4 is a strong paraconsistent (and paracomplete) logic useful in situations of inconsistent and/or incomplete information.
6. As it has been remarked, PŁ4 can be endowed with a simple bivalent semantics of the type defined by Dunn for FDE. This fact connects ŁB4 with relevant logics, actually, with the very basic foundations of relevant logics, the logic FDE and Dunn's semantics for it. Other facts pointing in the same direction are the following: (i) PŁ4 enjoys an easy intuitive Routley–Meyer type ternary relational semantics (cf. [30]); (ii) PŁ4 can be interpreted with a binary Routley semantics of the kind defined in [29].

The paper is organized as follows: Sect. 2: after some preliminary definitions, the matrices MŁ and MPŁ4 are defined. Section 3: the logic PŁ4 is defined and some facts about PŁ4-theories are established. The section ends with the proof of the primeness lemma. Section 4: Belnap–Dunn semantics for PŁ4 are defined and the soundness theorems w.r.t. this semantics and the semantics based upon the matrix MPŁ4 are proved. Section 5: completeness theorems w.r.t. both the semantics mentioned in the preceding section are proved. Section 6: some facts about PŁ4 are proved. For example, that it is a paraconsistent logic. Section 7: we remark some connectives definable in PŁ4 among which three types of negation and the necessity and possibility operators are to be noted. Section 8: we briefly point out some remarks on the results obtained and about some possible further work to be done in the same line.

2. The Matrix MPŁ4

The aim of this section is to define the 4-valued matrix MPŁ4. We begin by defining the logical languages and the notion of logic used in the paper.

Definition 2.1. (*Languages*) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, \ldots, p_n, \ldots$, and some or all of the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), L (necessity), M (possibility). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B, etc. are metalinguistic variables. By $\mathcal{P}$ and $\mathcal{F}$, we shall refer to the set of all propositional variables and the set of all wffs, respectively.

Definition 2.2. (*Logics*) A logic S is a structure (L, $\vdash_{\mathrm{S}}$) where L is a propositional language and $\vdash_{\mathrm{S}}$ is a (proof-theoretical) consequence relation defined on L by a set of axioms and a set of rules of derivation. The notions of 'proof' and 'theorem' are understood as it is customary in Hilbert-style axiomatic systems ($\Gamma \vdash_{\mathrm{S}} A$ means that A is derivable from the set of wffs Γ in S; and $\vdash_{\mathrm{S}} A$ means that A is a theorem of S).

Next, the notion of a logical matrix and related notions are defined.

Definition 2.3. (*Logical matrix*) A (logical) matrix is a structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) D is a non-empty proper subset of $\mathcal{V}$ (the set of designated values); and (3) $\mathtt{F}$ is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in \mathtt{F}$ such that $f_c : \mathcal{V}^n \to \mathcal{V}$.

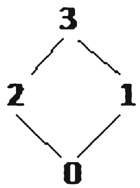
Definition 2.4. (*M-interpretations, M-consequence, M-validity*) Let M be a matrix for (a propositional language) L. An M-interpretation I is a function from $\mathcal{F}$ to $\mathcal{V}$ according to the functions in $\mathtt{F}$. Then, there are essentially two different ways of defining a consequence relation in M: truth-preserving relation (denoted by $\vDash^1_{\mathrm{M}}$) and degree of truth-preserving relation (denoted by $\vDash^{\leq}_{\mathrm{M}}$). These relations are defined as follows for any set of wffs Γ and $A \in \mathcal{F}$: (1)

$\Gamma \vDash^1_{\mathrm{M}} A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I; (2) $\Gamma \vDash^{\leq}_{\mathrm{M}} A$ iff $a \leq I(A)$ whenever $a \leq I(\Gamma)$ for all $a \in \mathcal{V}$ and M-interpretations I ($I(\Gamma) = \inf\{I(B) \mid B \in \Gamma\}$). In particular, $\vDash^1_{\mathrm{M}} A$ iff $I(A) \in D$ for all M-interpretations I, and $\vDash^{\leq}_{\mathrm{M}} A$ iff $a \leq I(A)$ for all $a \in \mathcal{V}$ and M-interpretations I. ($\Gamma \vDash^1_{\mathrm{M}} A$ ($\Gamma \vDash^{\leq}_{\mathrm{M}} A$) can be read "$A$ is a consequence of Γ according to M in the truth-preserving (degree of truth-preserving) sense". And $\vDash^1_{\mathrm{M}} A$ ($\vDash^{\leq}_{\mathrm{M}} A$) can be read as A is M-valid or A is valid in the matrix M in the truth-preserving (degree of truth-preserving) sense).

Notice that the set $\{A \mid \vDash^{\leq}_{\mathrm{M}} A\}$ is not empty iff the order in $\mathcal{V}$ has a *maximum*.

Next, we define (our version of) Łukasiewicz's matrix MŁ (cf. [13,34]) and then the matrix MPŁ4:

Definition 2.5. (*The matrix MŁ*) The proposition language consists of the connectives $\rightarrow, \neg, L$. The matrix MŁ is the structure $(\mathcal{V}, D, f_{\rightarrow}, f_{\neg}, f_L)$ where $\mathcal{V} = \{0, 1, 2, 3\}$ and they are partially ordered as it is shown in the following diagram



$D = \{3\}$ and $f_{\rightarrow}$, $f_{\neg}$ and f_L are defined according to the following tables:

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	2	3	2	3
2	1	1	3	3
3	0	1	2	3

	$\neg$
0	3
1	2
2	1
3	0

	L
0	0
1	0
2	2
3	2

The related notions of MŁ-interpretation, etc. are defined according to the general Definition 2.4.

Definition 2.6. (*The matrix MPŁ4*) The propositional language consists of the connectives $\rightarrow$ and $\neg$. The matrix MPŁ4 for the logic PŁ4 is the structure $(\mathcal{V}, D, f_{\rightarrow}, f_{\neg})$ where $\mathcal{V}$, D and $f_{\rightarrow}$ are as in MŁ and $f_{\neg}$ is defined according to the following table:

	$\neg$
0	3
1	1
2	2
3	0

The notions of a MPŁ4-interpretation, etc. are defined according to the general Definition 2.4.

Remark 2.7. ($\vDash^{\leq}_{\mathrm{MPŁ4}} A$ *iff* $\vDash^{1}_{\mathrm{MPŁ4}} A$) Notice that $\vDash^{\leq}_{\mathrm{MPŁ4}} A$ iff $I(A) = 3$ for all MPŁ4-interpretations I. Thus, for every wff A, $\vDash^{\leq}_{\mathrm{MPŁ4}} A$ iff $\vDash^{1}_{\mathrm{MPŁ4}} A$

Remark 2.8. (*On the intuitive meaning of the truth values in MPŁ4*) The truth values $0, 1, 2$ and 3 can intuitively be interpreted in MPŁ4 as follows. Let T and F represent truth and falsity. Then, $0 = F$, $1 = N$ (either), $2 = B$ (oth) and $3 = T$ (cf. [4,5]) Or, in terms of subsets of $\{T, F\}$, we have: $0 = \{F\}$, $1 = \emptyset$, $2 = \{T, F\}$ and $3 = \{T\}$ (cf. [12] and references therein). It is in this sense that we speak of "bivalent semantics" when referring to the Belnap–Dunn semantics: there are only two truth values and the possibility of assigning both or neither to propositions. (We use the symbols $0, 1, 2$ and 3 because they are convenient for using the tester in [14] in case the reader needs one).

3. The Logic PŁ4

Definition 3.1. (*The logic PŁ4*) The propositional language of PŁ4 has $\to$ and $\neg$ as the sole primitive connectives. The logic PŁ4 is axiomatized as follows:

Axioms: (A1) $A \to (B \to A)$; (A2) $[A \to (B \to C)] \to [(A \to B) \to (A \to C)]$; (A3) $[(A \to B) \to A] \to A$; (A4) $A \to \neg\neg A$; (A5) $\neg\neg A \to A$; (A6) $\neg(A \to B) \to (\neg A \to C)$; (A7) $\neg(A \to B) \to \neg B$; (A8) $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)]$.

Rules of derivation: Modus Ponens (MP): $A \ \& \ A \to B \Rightarrow B$ (That is, A and $A \to B$ imply B).

Notice that A1–A3 (together with MP) axiomatize the implicative fragment, $\mathrm{C}_{\to}$, of classical propositional logic (this was firstly proved in [21]). Also, remark that A8 is equivalent to A8′ $\neg B \to [\neg A \vee \neg(A \to B)]$ (when disjunction is understood according to the definition $A \vee B =_{\mathrm{df}} (A \to B) \to B$; cf. Proposition 7.6). In any standard axiomatic system for propositional classical logic (e.g. [22]), A8′ is an immediate consequence of the *Modus Ponens axiom* $[(A \to B) \wedge A] \to B$ by contraposition and the De Morgan laws.

Proposition 3.2. (Some theorems of PŁ4) *The following theorems of* $C_{\to}$ *will be useful:* (t1) $A \to A$; (t2) $[A \to (A \to B)] \to (A \to B)$; (t3) $(A \to C) \to [(B \to C) \to [[(A \to B) \to B] \to C]$.

The Deduction Theorem (DT) is provable in PŁ4.

Proposition 3.3. (The deduction theorem—DT) *For any set of wffs* Γ *and wff* A, B, *if* $\Gamma, A \vdash_{PŁ4} B$, *then* $\Gamma \vdash_{PŁ4} A \to B$.

Proof. As it is known, DT is provable in any extension of the implicative fragment of propositional intuitionistic logic (axiomatized by A1, A2 and MP) with MP as the sole rule of inference (cf. e.g., [22]). □

Definition 3.4. (*Logics determined by matrices*) Let L be a propositional language, M a matrix for L and $\vdash_{\mathrm{S}}$ a (proof theoretical) consequence relation defined on L. Then, the logic S (cf. Definition 2.2) is determined by M iff for every set of wffs Γ and wff A, $\Gamma \vdash_{\mathrm{S}} A$ iff $\Gamma \vDash_{\mathrm{M}} A$ ($\vDash_{\mathrm{M}}$ is here understood either as a truth-preserving or as a degree of truth-preserving consequence relation). In particular, the logic S (considered as the set of its theorems) is determined by M iff for every wff A, $\vdash_{\mathrm{S}} A$ iff $\vDash_{\mathrm{M}} A$ (cf. Definition 2.4).

We shall prove that the logic PŁ4 is determined by the matrix MPŁ4 when $\vDash_{\mathrm{M}}$ is understood as the degree of truth-preserving consequence relation.

In the rest of this section, we prove some facts about the theories built upon PŁ4. These facts are used in the completeness proofs of the following sections. Firstly, the notion of a theory is defined.

Definition 3.5. (*PŁ4-theories*) A PŁ4-theory (theory, for short) is a set of formulas containing all theorems of PŁ4 and closed under Modus Ponens (MP). That is, a is a theory iff (1) if $\vdash_{\mathrm{PŁ4}} A$ then $A \in a$; and (2) if $A \to B \in a$ and $A \in a$, then $B \in a$.

The following classes of theories are of interest for the aims of the paper.

Definition 3.6. (*Classes of theories*) Let a be a theory. We set (1) a is prime iff whenever $(A \to B) \to B \in a$, then $A \in a$ or $B \in a$; (2) a is trivial iff a contains every wff; (3) a is a-consistent ("consistent in an absolute sense") iff a is not trivial.

Next, we prove some properties of theories and prime theories w.r.t. the conditional and negation and, finally, the primeness lemma.

Lemma 3.7. (Theories and double negation) *Let a be a theory. For $A \in \mathcal{F}$, $A \in a$ iff $\neg\neg A \in a$.*

Proof. Immediate by A4 and A5. □

Lemma 3.8. (The conditional in prime, a-consistent theories) *Let a be a prime, a-consistent theory. For $A, B \in \mathcal{F}$, we have* (1) *$A \to B \in a$ iff $A \notin a$ or $B \in a$;* (2) *$\neg(A \to B) \in a$ iff $\neg A \notin a$ and $\neg B \in a$.*

Proof. (1a) Suppose $A \to B \in a$ and $A \in a$. Then, $B \in a$ by MP. (1b) Suppose $A \notin a$. By t2, $[A \to (A \to B)] \to (A \to B) \in a$. By primeness, $A \in a$ or $A \to B \in a$. So, $A \to B \in a$. On the other hand, suppose $B \in a$. By A1, $A \to B \in a$. (2a) Suppose $\neg(A \to B) \in a$ but $\neg A \in a$. By A6, $\neg(A \to B) \to (\neg A \to C) \in a$ for arbitrary C. Then, a is trivial, contradicting the hypothesis. So, $\neg A \notin a$. On the other hand, by A7, $\neg(A \to B) \to \neg B \in a$. So, $\neg B \in a$ and consequently, $\neg A \notin a$ and $\neg B \in a$, as was to be proved. (2b) Suppose $\neg A \notin a$ and $\neg B \in a$. By A8, $[\neg A \to \neg(A \to B)] \to \neg(A \to B) \in a$. By primeness, $\neg A \in a$ or $\neg(A \to B) \in a$. So, $\neg(A \to B) \in a$, as was to be proved. □

Lemma 3.9. (Primeness) *Let a be a theory and A a wff such that $A \notin a$. Then, there is a prime* (*and a-consistent*) *theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. Extend a to a maximal theory x such that $A \notin x$. Now, suppose that x is not prime. Then, there are wffs B and C such that $(B \to C) \to C \in x$ but $B \notin x$ and $C \notin x$. Define the sets $[x, B] = \{D \mid B \to D \in x\}$, $[x, C] =$

$\{D \mid C \to D \in x\}$. By using the Deduction Theorem, it is clear that $[x, B]$ and $[x, C]$ are theories such that $x \subseteq [x, B]$, $x \subseteq [x, C]$ and (by t1) $B \in [x, B]$ and $C \in [x, C]$. Thus, $x \not\supseteq [x, B]$, $x \not\supseteq [x, C]$ (since by hypothesis, $B \notin x$ and $C \notin x$). Consequently, (by the maximality of x) we have $A \in [x, B]$, $A \in [x, C]$ whence $A \in x$ (by t3 and the hypothesis $(B \to C) \to C \in x$), which is impossible. Therefore, x is prime. On the other hand, x is a-consistent ($A \notin x$). □

Notice that the proof of Lemma 3.9 just given holds for any extension of $C_{\to}$ with MP as the sole rule of inference.

In what follows, we shall provide a Belnap–Dunn type bivalent semantics for PŁ4.

4. Belnap–Dunn Type Semantics for PŁ4

In this section, a Belnap–Dunn type semantics for PŁ4 is provided and the soundness theorem is proved. This semantics is "bivalent" in the sense of Remark 2.8. Firstly, PŁ4-models and notions of PŁ4-consequence and PŁ4-validity are defined.

Definition 4.1. (*PŁ4-models*) An PŁ4-model is a structure $(K4, I)$ where (i) $K4 = \{\{T\}, \{F\}, \{T, F\}, \emptyset\}$; (ii) I is an PŁ4-interpretation from $\mathcal{F}$ to K4, this notion being defined according to the following conditions for all $p \in \mathcal{P}$ and $A, B \in \mathcal{F}$: (1) $I(p) \in K4$; (2a) $T \in I(\neg A)$ iff $F \in I(A)$; (2b) $F \in I(\neg A)$ iff $T \in I(A)$; (3a) $T \in I(A \to B)$ iff $T \notin I(A)$ or $T \in I(B)$; (3b) $F \in I(A \to B)$ iff $F \notin I(A)$ and $F \in I(B)$.

Definition 4.2. (*PŁ4-consequence; PŁ4-validity*) For any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ (A is a consequence of Γ in the PŁ4-model M) iff (1) $T \in I(A)$ whenever $T \in I(\Gamma)$; and (2) $F \notin I(A)$ whenever $F \notin I(\Gamma)$ ($T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$). In particular, $\vDash_{\mathrm{M}} A$ (A is true in M) iff $T \in I(A)$ and $F \notin I(A)$. Then, $\Gamma \vDash_{\mathrm{PŁ4}} A$ (A is a consequence of Γ in PŁ4-semantics) iff $\Gamma \vDash_{\mathrm{M}} A$ for each PŁ4-model M. In particular, $\vDash_{\mathrm{PŁ4}} A$ (A is valid in PŁ4-semantics) iff $\vDash_{\mathrm{M}} A$ for each PŁ4-model M (by $\vDash_{\mathrm{PŁ4}}$, we shall refer to the relation just defined).

Next, we prove that $\vDash^{\leq}_{\mathrm{MPŁ4}}$ (the relation defined in the matrix MPŁ4—cf. Definition 2.6) and $\vDash_{\mathrm{PŁ4}}$ (the consequence relation just defined in PŁ4-semantics) are coextensive.

Proposition 4.3. (Coextensiveness of $\vDash^{\leq}_{\mathrm{MPŁ4}}$ and $\vDash_{\mathrm{PŁ4}}$) *For any set of wffs* Γ *and wff* A, $\Gamma \vDash_{PŁ4} A$ *iff* $\Gamma \vDash^{\leq}_{MPŁ4} A$.

Proof. It is trivial given the correspondence between the points of K4 and those of the set values of MPŁ4 established in Remark 2.8 (cf. [7], Lemmas 1 and 7). □

Theorem 4.4. (Soundness of PŁ4 w.r.t. $\vDash^{\leq}_{\mathrm{MŁP4}}$) *For any set of wffs* Γ *and wff* A, *if* $\Gamma \vdash_{PŁ4} A$, *then* $\Gamma \vDash^{\leq}_{MPŁ4} A$.

Proof. Induction on the length of the derivation. The proof is left to the reader. (In case a tester is needed, the reader can use that in [14]). □

An immediate corollary of Theorem 4.4 is the following:

Corollary 4.5. (Soundness of PŁ4 w.r.t. $\vDash_{\text{PŁ4}}$) *For any set of wffs* Γ *and wff* A*, if* $\Gamma \vdash_{PŁ4} A$*, then* $\Gamma \vDash_{PŁ4} A$*.*

Proof. Immediate by Theorem 4.4 and Proposition 4.3. □

5. Completeness of PŁ4

We shall define the notion of a preferred model upon a-consistent and prime theories. By using the primeness lemma, it is then shown that each non-consequence A of a set of formulas Γ fails to belong to some a-consistent and prime theory that includes Γ; that is, it is shown that each non-consequence A of a set of formulas Γ is not true in some preferred model of Γ. We begin by defining the basic notion of a $\mathcal{T}$-interpretation.

Definition 5.1. ($\mathcal{T}$*-interpretation*) Let $K4$ be the set $\{\{T\},\{F\},\{T,F\},\emptyset\}$ as in Definition 4.1. And let $\mathcal{T}$ be an a-consistent and prime theory. Then, the function I from $\mathcal{F}$ to $K4$ is defined as follows: for each $p \in \mathcal{P}$, we set (a) $T \in I(p)$ iff $p \in \mathcal{T}$; (b) $F \in I(p)$ iff $\neg p \in \mathcal{T}$. Next, I assigns a member of $K4$ to each $A \in \mathcal{F}$ according to conditions 2 and 3 in Definition 4.1. Then, it is said that I is a $\mathcal{T}$-interpretation. (As in Definition 4.1, $T \in I(\Gamma)$ iff $\forall A \in \Gamma(T \in I(A))$; $F \in I(\Gamma)$ iff $\exists A \in \Gamma(F \in I(A))$).

Definition 5.2. (*Preferred PŁ4-models*) A preferred PŁ4-model is a structure $(K4, I_{\mathcal{T}})$ where $K4$ is defined as in Definition 4.1 (or as in Definition 5.1) and $I_{\mathcal{T}}$ is a $\mathcal{T}$-interpretation built upon an a-consistent and prime theory $\mathcal{T}$.

Proposition 5.3. (Any preferred PŁ4-model is a PŁ4-model) *Let* $M = (K4, I_{\mathcal{T}})$ *be a preferred PŁ4-model. Then,* M *is indeed a PŁ4-model.*

Proof. It follows immediately by Definitions 4.1 and 5.2 (by the way, notice that each propositional variable—and so, each wff A—can be assigned $\{T\}$, $\{F\}, \{T,F\}$ or $\emptyset$, since $\mathcal{T}$ is required to be a-consistent but nor complete or consistent in the classical sense). □

The following lemma generalizes conditions a and b in Definition 5.1 to the set $\mathcal{F}$ of all wffs.

Lemma 5.4. ($\mathcal{T}$-interpreting the set of wffs $\mathcal{F}$) *Let* I *be a* $\mathcal{T}$*-interpretation defined on the theory* $\mathcal{T}$*. For each* $A \in \mathcal{F}$*, we have:* (1) $T \in I(A)$ *iff* $A \in \mathcal{T}$*;* (2) $F \in I(A)$ *iff* $\neg A \in \mathcal{T}$*.*

Proof. Induction on the length of A (the clauses cited in points a, b and c below refer to the clauses in Definition 5.1—Definition 4.1—H.I abbreviates "hypothesis of induction"). (a) A is a propositional variable: by conditions a and b in Definition 5.1. (b) A is of the form $\neg B$: (i) $T \in I(\neg B)$ iff (clause 2a) $F \in I(B)$ iff (H.I) $\neg B \in \mathcal{T}$. (ii) $F \in I(\neg B)$ iff (clause 2b) $T \in I(B)$

iff (H.I) $B \in \mathcal{T}$ iff (Lemma 3.7) $\neg\neg B \in \mathcal{T}$. (c) A is of the form $B \to C$: (i) $T \in I(B \to C)$ iff (clause 3a) $T \notin I(A)$ or $T \in I(B)$ iff (H.I) ($A \notin \mathcal{T}$ or $B \in \mathcal{T}$) iff (Lemma 3.8) $B \to C \in \mathcal{T}$. (ii) $F \in I(B \to C)$ iff (clause 3b) $F \notin I(B)$ and $F \in I(C)$ iff (H.I) $\neg(B \to C) \in \mathcal{T}$ (Lemma 3.8). □

In what follows, we turn to the completeness proof. The standard concept of "set of consequences of a set of wffs" is useful and it is defined as follows for the logic treated in this paper.

Definition 5.5. (*The set Cn*Γ[*PŁ*4]) The set of consequences in PŁ4 of a set Γ, CnΓ[PŁ4] is defined as follows: $\text{Cn}\Gamma[\text{PŁ4}] = \{A \mid \Gamma \vdash_{\text{PŁ4}} A\}$ (cf. Definitions 2.2, 3.1).

It is clear that CnΓ[PŁ4] is a theory, for any Γ.

Theorem 5.6. (Completeness of PŁ4 w.r.t. $\vDash_{\text{PŁ4}}$) *For any set of wffs* Γ *and wff* A, *if* $\Gamma \vDash_{PŁ4} A$, *then* $\Gamma \vdash_{PŁ4} A$.

Proof. We prove the contrapositive of the claim. For some set of wffs Γ and wff A, suppose $\Gamma \nvdash_{\text{PŁ4}} A$. Then, $A \notin \text{Cn}\Gamma[\text{PŁ4}]$. So, by Definition 5.5 and Lemma 3.9, there is a prime (and a-consistent) theory $\mathcal{T}$ such that $\text{Cn}\Gamma[\text{PŁ4}] \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. By Definition 5.1 and Lemma 5.4, $\mathcal{T}$ induces a $\mathcal{T}$-interpretation I such that (1) $T \notin I(A)$ and (2) $T \in I(\Gamma)$ ($\Gamma \subseteq \text{Cn}\Gamma[\text{PŁ4}] \subseteq \mathcal{T}$). Thus, by 1 and 2, we have $\Gamma \nvDash_{\mathcal{T}} A$ (Definition 5.2), hence, by Definition 4.2 and Proposition 5.3, $\Gamma \nvDash_{\text{PŁ4}} A$, as it was required. □

Corollary 5.7. (Strong sound. and comp. w.r.t. $\vDash_{\text{PŁ4}}$ and $\vDash_{\text{MPŁ4}}$) *For any set of wffs* Γ *and wff* A, *we have* (1) $\Gamma \vdash_{PŁ4} A$ *iff* $\Gamma \vDash_{PŁ4} A$; (2) $\Gamma \vdash_{PŁ4} A$ *iff* $\Gamma \vDash^{\leq}_{MPŁ4} A$.

Proof. (1) By Corollary 4.5 and Theorem 5.6. (2) By Theorem 4.4 and Theorem 5.6 with Proposition 4.3. □

6. Some Facts About PŁ4

We begin with the proof of an important fact for the significance of PŁ4: the rule *Contraposition* is *admissible* in PŁ4 (cf. Point 5 in the Introduction to this paper). Firstly, the notions of 'admissible rule' and 'derivable rule' and the related ones 'rule of proof' and 'rule of inference' are defined.

Remark 6.1. (*On 'rules of inference' and 'rules of proof'*) The distinction between 'rules of proof' and 'rules of inference' is essential in the context of some substructural logics formulated in the Hilbert-style way. Let S be a logic formulated in this way (cf. Definition 2.2) and r: A_1 & A_2 & , . . . , & $A_n \Rightarrow B$ be a rule of S. Then, r is a 'rule of inference' if it can be applied to no matter which premises formulated in the language of S; but r is a 'rule of proof' if r is applied only in case $A_1, \ldots, A_n$ are theorems of S.

The case of relevant logics is paradigmatic. Actually, to the best of our knowledge, Ackermann (the father of relevant logics) was the first logician who defined a logic (his systems Π and Π′, cf. [1]) whose formulation leans essentially on the aforementioned distinction. Although he did not define the notions involved, Ackermann strongly stressed (cf. [1], pp. 119–120) that the rule δ, i.e., $B \ \& \ A \to (B \to C) \Rightarrow A \to C$ can only be applied if B is a "Logische Identität" (a logical theorem). A number of relevant logics are formulated with both rules of inference and rules of proof. For example, Anderson and Belnap's logic of Entailment E can be axiomatized with MP and Adj as rules of inference and Assertion ($A \Rightarrow (A \to B) \to B$) as a rule of proof (cf. [32]). On the other hand, Routley and Meyer's basic logic B (cf. [32]) or Brady's weak relevant logics (cf. [8]) have the rules Prefixing ans Suffixing as rules of proof.

Rules of inference and rules of proof can be assumed as primitives in a system S or else they can be proved as derived rules in S, which takes us to the next remark.

Remark 6.2. (*On 'admissible rules' and 'derivable rules'*) Anderson and Belnap remark: "We will say that a rule: from $A_1, \ldots, A_n$ to infer B, is derivable when it is possible to proceed from the premisses to the conclusion with the help of the axioms and primitive rules alone" [2, p. 54]. On the other hand, a rule is admissible if "whenever there is a proof of the premisses, there is also a proof of the conclusion" [2, p. 54].

Thus, for example, the rule Prefixing (Pref) ($B \to C \Rightarrow (A \to B) \to (A \to C)$) is derivable in standard relevant logics such as T, E or R (cf. [2]) but Disjunctive Syllogism (DS) ($A \vee B \quad \& \quad \neg A \Rightarrow B$) is only admissible (not derivable) in the said logics. From another perspective, Pref is a rule of inference and DS is a rule of proof in T, E or R. Concerning Con in PŁ4, it is an admissible rule, as it is shown below. So, it could have been added as a rule of proof to the formulation of PŁ4 (cf. Definition 3.1), although such an addition would be unnecessary. Nevertheless, Con is not derivable in PŁ4 as proved in Proposition 6.4.

Proposition 6.3. (Admissibility of Con and Efq in PŁ4) *The rules Contraposition* (*Con*) *and 'E falso quodlibet'* (*'Any proposition follows from a false proposition'—Efq*)*, that is,* (*Con*) $A \to B \Rightarrow \neg B \to \neg A$ ($A \to B$ *implies* $\neg B \to \neg A$)*;* (*Efq*) $A \Rightarrow \neg A \to B$ (A *implies* $\neg A \to B$) *are admissible in PŁ4.*

Proof. (1) Suppose that $A \to B$ is a theorem of PŁ4. We prove that $\neg B \to \neg A$ is a theorem as well. Let I be any MPŁ4-interpretation (cf. Definition 2.6). By Corollary 5.7 (soundness), $I(A \to B) = 3$. Then, it is easy to check that $I(\neg B \to \neg A) = 3$. So, $\neg B \to \neg A$ is a theorem by applying Corollary 5.7 again (completeness). (2) If A is a theorem of PŁ4, then $\neg A \to B$ is a theorem as well. The proof of 2 is similar to that of 1 and is left to the reader (actually, it can easily be proved (on purely proof-theoretic grounds) that, given PŁ4, if Con is admissible, then Efq is admissible as well—use A1). □

Proposition 6.4. (Con is not derivable in PŁ4) *The rule Con is not derivable in PŁ4.*

Proof. Let $\Gamma = \{B_1, \ldots, B_n\}$. It is obvious that we have $\Gamma \vDash_{\text{PŁ4}} A$ iff $\vDash_{\text{PŁ4}}$ $(B_1 \wedge \cdots \wedge B_n) \to A$. Consequently, $\{A \to B\} \nvDash_{\text{PŁ4}} \neg B \to \neg A$ since $\nvDash_{\text{PŁ4}}$

$(A \to B) \to (\neg B \to \neg A)$. (Anyway, we have, of course, $\vDash_{\text{PŁ4}} A \to B \Rightarrow \vDash_{\text{PŁ4}} \neg B \to \neg A$). □

Next, we briefly investigate the properties of PŁ4 as a paraconsistent logic.

As it is well-known, the notion of paraconsistency can be rendered as follows (cf. [9] or [27]).

Definition 6.5. (*Paraconsistent logics*) Let $\Vdash$ represent a consequence relation (may it be defined either semantically or proof-theoretically). Then, a logic S is paraconsistent if, for any wffs A, B, the rule Ecq $A, \neg A \Vdash B$ does not hold in S.

In other words, a logic is paraconsistent if theories built upon S are not necessarily trivial when a contradiction arises. Then, concerning the logic studied in this paper, we prove:

Proposition 6.6. (PŁ4 is paraconsistent) *The logic PŁ4 is a paraconsistent logic.*

Proof. Consider an MPŁ4-interpretation I such that $I(p_i) = 2$ and $I(p_m) = 1$ for the *ith* and *mth* propositional variables p_i and p_m. Then, $I(\{p_i, \neg p_i\}) \nleq I(p_m)$. So, $p_i, \neg p_i \nvDash^{\leq}_{\text{MPŁ4}} p_m$ and consequently, Ecq does not hold in PŁ4. □

Sometimes a logic is defined to be paraconsistent if at least one of its inconsistent theories (i.e. theories containing a wff and its negation) is not trivial. In this sense, we prove:

Proposition 6.7. (Inconsistent theories that are not trivial) *There are prime, inconsistent theories* (*i.e. theories containing a wff and its negation*) *that are not trivial.*

Proof. Let p_i and p_m $(i \neq m)$ be propositional variables and consider the set $y = \{A \mid \{p_i, \neg p_i\} \vdash_{\text{PŁ4}} A\}$. It is clear that y is a theory and that it is inconsistent since p_i and $\neg p_i$ belong to y. Anyway, y is not trivial: $\{p_i, \neg p_i\}$ is assigned the value 2 and p_m the value 1 for any MPŁ4-interpretation I such that $I(p_i) = 2$ and $I(p_m) = 1$. So, $\{p_i, \neg p_i\} \nvdash_{\text{PŁ4}} p_m$ by soundness (Corollary 5.7). Consequently, $p_m \notin y$. Now, by Lemma 3.9, there is a prime (and a-consistent) theory x such that $y \subseteq x$ and $p_m \notin x$. Therefore, x is inconsistent, but not trivial. □

Consider now the following definition:

Definition 6.8. (*Weak consistency*) Let a be a theory. Then, a is w-inconsistent ("inconsistent in a weak sense") iff for some theorem A of PŁ4, $\neg A \in a$; a is w-consistent ("consistent in a weak sense") iff a is not w-inconsistent (cf. [28] on the label "w-consistent").

We prove:

Proposition 6.9. (a-consistency and w-consistency) *Let a be a theory. Then, a is a-consistent iff a is w-consistent.*

Proof. It is immediate by the admissibility of Efq in PŁ4 (cf. Proposition 6.3): if a contains the negation of theorem, then a contains every wff. □

Remark 6.10. (*On the collapse of theories into triviality*) As we have seen, a theory containing the negation of a theorem is trivial. Also, a theory is trivial if it contains a negated conditional together with the negation of the antecedent (cf. A6 in Definition 3.1).

On the other hand, let us recall that, as it is well-known, Ecq does not hold in minimal intuitionistic logic $\mathrm{I_m}$. So, according to Definition 6.5, $\mathrm{I_m}$ is a paraconsistent logic, no matter the fact that the following restriction of Ecq (Ecq′) $A \wedge \neg A \Vdash \neg B$ does hold in $\mathrm{I_m}$. This is not the case with PŁ4, where Ecq does not hold in general when B presents one of the forms $C \wedge D$, $C \vee D$, $C \rightarrow D$ or $\neg C$, as it is shown below (for any $A, B \in \mathcal{F}$, $A \wedge B$ abbreviates $\neg[(\neg A \rightarrow \neg B) \rightarrow \neg B]$ and $A \vee B$, $(A \rightarrow B) \rightarrow B$; cf. Proposition 7.6).

Proposition 6.11. (Restricted forms of Ecq not holding in PŁ4) *Let* A, B, C *be any wffs and let* $\overset{n}{\neg} A$ *represent that* n $(n \geq 0)$ *symbols of negation* $(\neg)$ *are preceding* A. *Consider now the following restricted forms of Ecq:* (1) $\overset{n}{\neg} A \wedge \overset{n+1}{\neg} A \Vdash B \rightarrow C$; (2) $\overset{n}{\neg} A \wedge \overset{n+1}{\neg} A \Vdash B \wedge C$; (3) $\overset{n}{\neg} A \wedge \overset{n+1}{\neg} A \Vdash B \vee C$; (4) $\overset{n}{\neg} A \wedge \overset{n+1}{\neg} A \Vdash \neg B$. (*Cf. Definition* 6.5). *Then, the rules* (1)–(4) *do not hold in PŁ4.*

Proof. Consider a MPŁ4-interpretation I such that $I(p_j) = 1$, $I(p_i) = 3$, $I(p_n) = 0$, $I(p_m) = 0$, for distinct propositional variables p_j, p_i, p_n and p_m. This interpretation is such that $I(\overset{n}{\neg} p_j \wedge \overset{n+1}{\neg} p_j) = 1$. Then, (1) $I[(\overset{n}{\neg} p_j \wedge \overset{n+1}{\neg} p_j) \rightarrow (p_i \rightarrow p_m)] = 2$; (2) $I[(\overset{n}{\neg} p_j \wedge \overset{n+1}{\neg} p_j) \rightarrow (p_i \wedge p_m)] = 2$; (3) $I[(\overset{n}{\neg} p_j \wedge \overset{n+1}{\neg} p_j) \rightarrow (p_n \vee p_m)] = 2$; (4) $I[(\overset{n}{\neg} p_j \wedge \overset{n+1}{\neg} p_j) \rightarrow \neg p_i] = 2$. □

7. Some Connectives Definable in PŁ4

In this section we remark some of the connectives definable in PŁ4. In particular, we define the following: three different negation operators ($\overset{\bullet}{\neg}$, $\overset{\circ}{\neg}$, $\overset{\square}{\neg}$), and possibility (M), necessity (L), conjunction ($\wedge$) and disjunction ($\vee$) operators. Some of these are generally definable in negation expansions (by means of the negation operator $\neg$ of MPŁ4) of "natural implicative 4-valued matrices" (cf. Definition 7.3). Since the proof of the general case does not add any special difficulty to the particular case of the matrix MPŁ4, we will derive the proof of the latter from that of the former.

Following Tomova in [33], we define 'natural conditionals' as follows:

Definition 7.1. (*Natural conditionals*) Let L be a propositional language with $\rightarrow$ among its connectives and M be a matrix for L where the values x and y represent the maximum and the infimum in $\mathcal{V}$. Then, an $f_{\rightarrow}$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:

1. $f_{\rightarrow}$ coincides with (the $f_{\rightarrow}$-function for) the classical conditional when restricted to the subset $\{x, y\}$ of $\mathcal{V}$.
2. $f_{\rightarrow}$ satisfies Modus Ponens, that is, for any $a, b \in \mathcal{V}$, if $a \rightarrow b \in D$ and $a \in D$, then $b \in D$.
3. For any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a \leq b$.

Proposition 7.2. (Natural conditionals in 4-valued matrices) *Let L be a propositional language and M a 4-valued matrix for L where $\mathcal{V}$ and D are defined exactly as in MŁ (or as in MPŁ4). Now, consider the 2304 $f_{\rightarrow}$-functions defined in the following general table:*

	$\rightarrow$	0	1	2	3
	0	3	3	3	3
TI	1	a_1	3	a_2	3
	2	a_3	a_4	3	3
	3	0	b_1	b_2	3

where $a_i(1 \leq i \leq 4) \in \{0, 1, 2, 3\}$ and $b_j(j = 1$ or $j = 2) \in \{0, 1, 2\}$. The set of functions (contained) in TI is the set of all natural conditionals definable in M.

Proof. (1) $f_{\rightarrow}(0, 0) = f_{\rightarrow}(0, 1) = f_{\rightarrow}(0, 2) = f_{\rightarrow}(0, 3) = f_{\rightarrow}(1, 1) = f_{\rightarrow}(1, 3) = f_{\rightarrow}(2, 2) = f_{\rightarrow}(2, 3) = f_{\rightarrow}(3, 3) = 3$ are needed in order to fulfill clause 3 in Definition 7.1. (2) $f_{\rightarrow}(3, 0) = 0$ is required by clause 1 in the same definition. (3) Finally, $f_{\rightarrow}(3, 1) \in \{0, 1, 2\}$ and $f_{\rightarrow}(3, 2) \in \{0, 1, 2\}$ are necessary by clause 2 in Definition 7.1. □

Leaning on the notion just defined, we set:

Definition 7.3. (*Natural implicative* 4-*valued matrices*) Let L be a propositional language with the connective $\rightarrow$. And let M be a 4-valued matrix where $\mathcal{V}$ and D are defined as in Definition 2.6. Moreover, let $f_{\rightarrow}$ be one of the functions (defining one of the conditionals) in TI (in Proposition 7.2). Then, it is said that M is a natural implicative 4-valued matrix. (Notice that we are supposing that $\mathcal{V}$ is partially ordered as stated in Definition 2.5).

In what follows, we investigate some of the negation and modal connectives definable in negation expansions (by means of the negation operator $\neg$ of MPŁ4) of natural implicative 4-valued matrices.

Proposition 7.4. (Additional negations) *Let M be a natural implicative matrix where $f_{\neg}$ is defined as in MPŁ4. Then, the additional negation connectives $\overset{\bullet}{\neg}$ and $\overset{\circ}{\neg}$ given by the following tables:*

	0	1	2	3
$\overset{\bullet}{\neg}$	3	3	3	0

	0	1	2	3
$\overset{\circ}{\neg}$	3	0	0	0

are definable in M.

Proof. For any wff A, let $\overset{\bullet}{\neg}A =_{\text{df}} A \rightarrow \neg A$ and $\overset{\circ}{\neg}A =_{\text{df}} \neg(\neg A \rightarrow A)$. □

Proposition 7.5. (Possibility and necessity) *Let M be a natural implicative matrix where $f_\neg$ is defined as in MPŁ4. Then, the possibility (M) and necessity (L) operators given by the following tables:*

	0	1	2	3
M	0	3	3	3

	0	1	2	3
L	0	0	0	3

are definable in M.

Proof. For any wff A, let $MA =_{\text{df}} \overset{\times}{\neg}\overset{\circ}{\neg}A$ and $LA =_{\text{df}} \overset{\times}{\neg}\overset{\bullet}{\neg}A$ where $\overset{\times}{\neg}$ is any of the three negation operators at our disposal (i.e., $\neg$, $\overset{\bullet}{\neg}$ and $\overset{\circ}{\neg}$; the operator $\overset{\Box}{\neg}$ (defined in Proposition 7.8) would also work). □

Thus, the operators $\overset{\bullet}{\neg}$, $\overset{\circ}{\neg}$, M and L are definable in any natural implicative 4-valued matrix where $f_\neg$ is defined as in MPŁ4. Consequently, they are definable in PŁ4. Furthermore, Proposition 7.8 shows that the Boolean negation characteristic of MŁ is definable, whence classical propositional logic is also definable in PŁ4 as it is the case with Ł. But firstly, conjunction ($\wedge$) and disjunction ($\vee$) are defined.

Proposition 7.6. (Conjunction and disjunction) *The conjunction ($\wedge$) and disjunction ($\vee$) connectives given by the following tables are definable in MPŁ4.*

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

Proof. For any $A, B \in \mathcal{F}$, let $A \vee B =_{\text{df}} (A \to B) \to B$ and $A \wedge B =_{\text{df}} \neg(\neg A \vee \neg B)$. (We note that these definitions of $\wedge$ and $\vee$ give the same tables for these connectives in the matrix MŁ as a result). □

We note the following proposition:

Proposition 7.7. (All theorems of C_+ are theorems of PŁ4) *All theorems of classical positive logic C_+ are theorems of PŁ4.*

Proof. The following theses are MPŁ4-valid, and so, provable in PŁ4 by Corollary 5.7: $A \to (A \vee B)$; $B \to (A \vee B)$; $(A \to C) \to [(B \to C) \to [(A \vee B) \to C]]$; $(A \wedge B) \to A$; $(A \wedge B) \to B$; $A \to [B \to (A \wedge B)]$. But these theses axiomatize C_+ together with MP and A1, A2 and A3 of PŁ4. □

In addition to the negation operators $\overset{\bullet}{\neg}$ and $\overset{\circ}{\neg}$, the following proposition shows that the Boolean negation characteristic of MŁ is definable.

Proposition 7.8. (Another additional negation definable in PŁ4) *The additional negation connective given by the following table:*

	0	1	2	3
$\overset{\Box}{\neg}$	3	2	1	0

is definable in PŁ4.

Proof. For any A, set $\overset{\Box}{\neg}A =_{\text{df}} A \to \neg(A \to A)$. □

Then, we have:

Proposition 7.9. (All theorems of CL are theorems of PŁ4) *All theorems of classical propositional logic CL are theorems of PŁ4.*

Proof. $(\overset{\Box}{\neg}A \to \overset{\Box}{\neg}B) \to (B \to A)$ is provable in PŁ4 since it is MPŁ4-valid (Corollary 5.7). But this thesis axiomatizes CL together with MP and A1, A2 and A3 of PŁ4. □

In what follows, we note some more theorems of PŁ4 expressed in the new connectives. It will suffice to check that they are MPŁ4-valid since they are then theorems by Corollary 5.7 (in case a tester is needed, the reader can use that in [14]).

Proposition 7.10. (The De Morgan laws are provable in PŁ4) *The following are provable in PŁ4:* $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$; $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$; $(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$; $(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$.

Next, we record some modal theses provable in PŁ4.

Proposition 7.11. (Some modal theses provable in PŁ4) *The following are provable in PŁ4:* (T1) $LA \leftrightarrow \neg M\neg A$; (T2) $MA \leftrightarrow \neg L\neg A$; (T3) $LA \to A$; (T4) $A \to MA$; (T5) $LA \to LLA$; (T6) $MA \to LMA$; (T7) $MLA \to LA$; (T8) $L(A \to B) \to (LA \to LB)$; (T9) $L(A \wedge B) \leftrightarrow (LA \wedge LB)$; (T10) $M(A \vee B) \leftrightarrow (MA \vee MB)$; (T11) $M(A \to B) \leftrightarrow (LA \to MB)$; (T12) $(MA \to LB) \to L(A \to B)$; (T13) $(MA \to MB) \to M(A \to B)$; (T14) $(LA \vee LB) \to L(A \vee B)$; (T15) $M(A \wedge B) \to (MA \wedge MB)$; (T16) $L(A \vee B) \to (LA \vee MB)$; (T17) $(MA \wedge LB) \to M(A \wedge B)$; (T18) $A \to (\neg A \vee LA)$; (T19) $(\neg LA \wedge A) \to \neg A$.

The reader has undoubtedly recognized T1–T17 as some significant theorems of Lewis' S5. Theses T18 and T19 (that will briefly be commented on in Remark 7.14) are not, however, provable in S5. Actually, addition of any of them to S5 would cause the collapse of S5 into classical propositional logic.

In addition to the provable theses just recorded, the following important modal rules are admissible in PŁ4.

Proposition 7.12. (Admissibility of Nec, d$M_{\to}$ and d$L_{\to}$) *Consider the following rules. Necessitation (Nec)* $A \Rightarrow LA$, *Distribution of* M *in* $\to$ (d$M_{\to}$) $A \to B \Rightarrow MA \to MB$ *and Distribution of* L *in* $\to$ (d$L_{\to}$) $A \to B \Rightarrow LA \to LB$. *The rules Nec, d*$M_{\to}$ *and d*$L_{\to}$ *are admissible in PŁ4. That is, if* A $(A \to B)$ *is a theorem of PŁ4, then* LA $(MA \to MB,\ LA \to LB)$ *is a theorem as well.*

Proof. It is immediate by using the soundness and completeness theorems (cf. Corollary 5.7). Let us prove the admissibility of Nec as a way of an example. Suppose $\vdash_{\text{PŁ4}} A$. By soundness, $\vDash_{\text{MPŁ4}} A$. Then, it is easy to check that $\vDash_{\text{MPŁ4}} LA$, whence $\vdash_{\text{PŁ4}} LA$ follows by completeness (cf. Corollary 5.7; cf. the proof of the admissibility of Con in Proposition 6.3). □

On the other hand, we record some schemes not provable in PŁ4.

Proposition 7.13. (Modal wffs not provable in PŁ4) *The following wffs are not provable in PŁ4:* (F1) $A \to LA$; (F2) $MA \to A$; (F3) $LMA \to A$; (F4) $A \to MLA$; (F5) $(A \to B) \to (MA \to MB)$; (F6) $(A \to B) \to (LA \to LB)$; (F7) $(MA \wedge MB) \to M(A \wedge B)$; (F8) $L(A \vee B) \to (LA \vee LB)$; (F9) $LA \to (B \to LB)$; (F10) $LA \to (MB \to B)$.

Proof. It is easy to check that each one of these wffs is invalidated in the matrix MPŁ4. Consequently, they are not provable in PŁ4 by the soundness theorems (cf. Corollary 5.7). Provability of F1-F4 would result in collapse, that is, in the provability of $A \leftrightarrow LA$ ($A \leftrightarrow MA$). Concerning F5–F10, provable in Łukasiewicz's 4-valued modal logic Ł (cf. Definition 2.5; cf. [23]), they are instances of the Łukasiewicz-type modal paradoxes referred to in the title of the paper. □

The section is ended with a brief comment on theorems T18 and T19 of PŁ4 on the one hand, and with a proof that PŁ4 cannot be extended with the "Principle of excluded middle" and other strong theses, on the other.

Remark 7.14. (*On two modal theses of PŁ4*) In his nice paper [24], Minari notes that Łukasiewicz "skillfully takes advantage of the failure of the contraction law in Ł3" [24, p. 164] "and proposes the Ł3-thesis $A \to (A \to LA)$ as a (partially) adequate formal version of the classical principle Unumquodque, quando est, oportet esse" (cf. [24], p. 164). The PŁ4-thesis $A \to (\neg A \vee LA)$ (and the equivalent $(\neg LA \wedge A) \to \neg A$) can be similarly viewed: PŁ4 does not collapse in classical logic because of the failure of $(\neg A \vee B) \to (A \to B)$ in PŁ4 ($(A \to B) \to (\neg A \vee B)$ also fails, cf. Corollary 7.17 below). Only there is a difference: in Ł3 the paradoxes F7 and F8 are provable; in PŁ4, they are not (cf. Proposition 7.13).

In sum, Propositions 7.11–7.13 support the conclusion that PŁ4 can be understood as a (strong) genuine (4-valued) modal logic.

Proposition 7.15. (Given PŁ4, PEM and AEfq are equivalent) *The "Principle of excluded middle"* (*PME*) *is the thesis* $A \vee \neg A$; *the Efq-axiom* (*AEfq*) *is the thesis* $\neg A \to (A \to B)$. *Given the logic PŁ4, PME and AEfq are deducible from each other.*

Proof. (a) Let $A \vee \neg A$ be added to PŁ4. Then, we have (1) $\neg B \to (A \vee \neg A)$ by A1; by 1, Con, double negation and the De Morgan laws, (2) $(A \wedge \neg A) \to B$. Finally, $\neg A \to (A \to B)$ is immediate by using the thesis $[(A \wedge B) \to C] \leftrightarrow [A \to (B \to C)]$. (b) Let $\neg A \to (A \to B)$ be added to PŁ4. Then, $A \vee \neg A$ is proved by proceeding backwards in the proof of case (a). □

Proposition 7.16. (PŁ4 plus PEM or AEfq is CL) *The result of adding PEM or AEfq to PŁ4 is a logic equivalent to classical propositional logic CL.*

Proof. Suppose that PEM (so AEfq) is added to PŁ4. It will be proved that the contraposition axiom $(\neg B \to \neg A) \to (A \to B)$ is derivable and thus, that classical propositional logic is derivable as well. By $C_{\to}$ (cf. Definition 3.1), we have (1) $[B \to (A \to B)] \to [[\neg B \to (A \to B)] \to [(B \vee \neg B) \to (A \to B)]]$. By MP (with A1), we have (2) $[\neg B \to (A \to B)] \to [(B \vee \neg B) \to (A \to B)]$. By the Permutation axiom and MP (with PEM), we obtain (3) $[\neg B \to (A \to B)] \to (A \to B)$. By AEfq and the Prefixing rule, we get (4) $(\neg B \to \neg A) \to [\neg B \to (A \to B)]$. Finally, the desired result follows by 3 and 4. □

Notice that an immediate corollary of Proposition 7.16 is the following:

Corollary 7.17. (PŁ4 plus $(A \to B) \to (\neg A \vee B)$) *The result of adding* $(A \to B) \to (\neg A \vee B)$ *or* $(\neg A \vee B) \to (A \to B)$ *to PŁ4 is equivalent to CL.*

Proof. Immediate by Proposition 7.16. □

8. Concluding Remarks

We briefly record a few remarks around the results obtained and a conclusion to the paper.

1. The basic paralogics extending C_{+f} (C_+ with a falsity constant f added) are the following, according to [3]: (CLuN) C_{+f} plus $(A \to \neg A) \to \neg A$ or $A \vee \neg A$; (CLaN) C_{+f} plus $\neg A \to (A \to B)$; (CLoN) C_{+f} without further axioms added. (Recall that PŁ4 contains all axioms of C_+. Cf. Proposition 7.7).

   Maximal paralogics are obtained by adding to each one of these systems the following axioms: (a) $A \leftrightarrow \neg\neg A$; (b) $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$; (c) $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$; and (d) $\neg(A \to B) \leftrightarrow (A \wedge \neg B)$. According to this classification, PŁ4 is a paraconsistent and paracomplete extension of CLoN with (in addition to Con as a rule of proof) axioms a, b and c. Recall that addition of $A \vee \neg A$, $\neg A \to (A \to B)$, $\neg(A \to B) \to (A \wedge \neg B)$ or $(A \wedge \neg B) \to \neg(A \to B)$ results in a collapse into CL—cf. Proposition 7.16 and Corollary 7.17). It would be interesting to investigate if PŁ4 is a maximal paraconsistent and/or paracomplete logic w.r.t CL.
2. There is a number of paraconsistent logics extending C_+ (cf., e.g., [9] or [27]; cf. also [15]); of these, [25,26,35,36] have some relation with PŁ4, especially the systems investigated in [25,26] in which Con holds as a rule of proof. It seems worthwhile to investigate the structure of the family of paraconsistent logics extending C_+ and the place PŁ4 occupies in it.
3. In [6], some suggestions are advanced for defining 4-valued modal logics. We think that PŁ4 can be considered as a version of Beziau's Partial Modal 4-valued logic, PM4, with a paraconsistent negation and when $D = \{3\}$. (Recall that the set of all $\vDash^1_{\text{PŁ4}}$-valid formulas is exactly the set of theorems of PŁ4—cf. Definition 2.6 and Remark 2.7).
4. PŁ4 can be more conspicuously axiomatized by using the connectives $\wedge, \vee$ and L in addition to $\to$ and $\neg$. Then, we can add the following axioms to A1-A8 in Definition 3.1: A9 $(A \wedge B) \to A / (A \wedge B) \to B$; A10 $A \to [B \to (A \wedge B)]$; A11 $A \to (A \vee B) / B \to (A \vee B)$; A12 $(A \to C) \to [(B \to C) \to [(A \vee B) \to C]]$; A13 $LA \to A$; A14 $(LA \wedge \neg A) \to B$; A15 $A \to (\neg A \vee LA)$. A Belnap–Dunn semantics can be defined for this axiomatization of PŁ4 by adding the

following clauses to 1–3 in Definition 4.1: (4a) $T \in I(A \wedge B)$ iff $T \in I(A)$ and $T \in I(B)$; (4a) $F \in I(A \wedge B)$ iff $F \in I(A)$ or $F \in I(B)$; (5a) $T \in I(A \vee B)$ iff $T \in I(A)$ or $T \in I(B)$; (5b) $F \in I(A \vee B)$ iff $F \in I(A)$ and $F \in I(B)$; (6a) $T \in I(LA)$ iff $T \in I(A)$ and $F \notin I(A)$; (6b) $F \in I(LA)$ iff $T \notin I(A)$ or $F \in I(A)$. (The truth tables for $\wedge$ and $\vee$ are displayed in Proposition 7.6; the table for L, in Proposition 7.5). It would not be difficult to show that the system just described and PŁ4 are (definitionally) equivalent, but we leave the matter for another paper.

5. Smiley's 4-valued matrix MSm4 is the structure $(\mathcal{V}, D, \mathtt{F})$ where $\mathcal{V}$ and D are defined exactly as in MŁ (or in MPŁ4) and $\mathtt{F} = \{f_{\rightarrow}, f_{\wedge}, f_{\vee}, f_{\neg}\}$ where $f_{\neg}$ is the function in Definition 2.5, $f_{\wedge}$ and $f_{\vee}$ are defined according to the tables in Proposition 7.6, and, finally, $f_{\rightarrow}$, according to the following table:

$\rightarrow$	0	1	2	3
0	3	3	3	3
1	0	3	0	3
2	0	0	3	3
3	0	0	0	3

MSm4 is characteristic for Anderson and Belnap's First Degree Entailment Logic, FDE (cf. [2], pp. 161–162). Now, PŁ4 can be viewed as the logic determined by the matrix MSm4′ where MSm4′ is defined exactly as MSm4 except for the $f_{\rightarrow}$ function, defined according to the $\rightarrow$-table in Definition 2.5 (and in Definition 2.6). In other words, MSm4 and MPŁ4 are two different implicative expansions of the matrix $(\mathcal{V}, D, \mathtt{F})$ where $\mathtt{F}= \{f_{\wedge}, f_{\vee}, f_{\neg}\}$ and $\mathcal{V}, D, f_{\wedge}, f_{\vee}$ and $f_{\neg}$ are defined as indicated above. From another point of view, MSm4 and MPŁ4 are natural 4-valued implicative matrices.

6. It is conjectured that the logic characterized by the relation $\vDash^{1}_{\text{MPŁ4}}$ (cf. Definitions 2.4 and 2.6) can be axiomatized by adding to PŁ4 Con as a rule of inference, whence Ecq is immediately derivable. The investigation about this logic is left for another paper.

We conclude by stating our belief that, as the title of the paper reads, PŁ4 is a strong, rich (in expressive power) genuine modal logic that is quasi-normal in the sense that the rule Necessitation (Nec) is admissible. Furthermore, it is a paraconsistent and paracomplete logic that (we hope) may be useful in inconsistent and/or incomplete situations.

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