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Harmony in Multiple-Conclusion Natural-Deduction

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Abstract. The paper studies the extension of harmony and stability, major themes in proof-theoretic semantics, from single-conclusion natural-deduction systems to multiple-conclusions natural-deduction, independently of classical logic. An extension of the method of obtaining harmoniously-induced general elimination rules from given introduction rules is suggested, taking into account sub-structurality. Finally, the reductions and expansions of the multiple-conclusions natural-deduction representation of classical logic are formulated.

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1. Introduction

Proof-Theoretic Semantics (*PTS*) is a theory of meaning intended as an alternative to the orthodox *Model-Theoretic Semantics* (*MTS*) conception of the meaning of an affirmative (declarative) sentence as its truth-conditions (in arbitrary models) by its (*warranted*) *assertability conditions*, captured as *canonical* derivations (explained below), most often as formalized in a (single-conclusion) natural-deduction ($SCND$) proof-system, say $\mathcal{N}$, capturing the 'use' of the sentence the meaning of which is being defined. Typically, the introduction/elimination rules (I/E-rules) of the meaning-conferring $\mathcal{N}$-system are partitioned into two groups, one viewed as *self-justified*, by which meanings are defined, and the other justified by the self-justified rules via a *justification procedure*. In the current paper, only the approach called *inferentialism*, where, following Gentzen [16], the I-rules are meaning-constitutive, is considered. For an explicit presentation of the resulting *reified* meaning resulting by this approach, see [11]. I shall not discuss here the motivation for this view, for which a vast literature exists in the philosophy of logic and of language. Rather, after surveying the relevant notions and results in the $SCND$-framework (in some more detail then present in the literature for specific ND-systems), I will address several issues regarding the extension of those notions and results to *multiple-conclusion natural-deduction* ($MCND$) proof-systems. To facilitate this extension, the discussion of $SCND$ runs in parallel two threads, simple presentation and logistic presentation (defined below), as their mutual relationship in the $MCND$ case is of a lot of interest. The extra details, though not innovative in any way, sets the notation is such a way as to make the extension more transparent.

Typically, $MCND$-systems are considered in the PTS-context as a way to "rehabilitate" classical logic, making it proof-theoretically "kosher" also to PTS adherers, not yielding to the Prawitz–Dummett rejection of classical logic (CL) in favor of intuitionistic logic (IL). See [31], mainly which I will follow here. However, I am interested in the more general issue of $MCND$-systems as general means for conferring meaning (along the PTS-programme), not necessarily restricted to classical logic.

After Prior's attack on PTS in [30], it became clear that not any *arbitrary* collection of I/E-rules can qualify as meaning conferring. Two of the main criteria to be imposed on an $SCND$-system for such a qualification became known as *harmony* and *stability* (reviewed in Sect. 2.3), imposing a *balance* on the relative power of the I-rules w.r.t. the E-rules (see, among many other references, [8,29,40]). As a result of considerations of harmony/stability (and additional philosophical concerns of the realism/anti-realism debate), many of the leading figures of PTS, like Dummett, Prawitz, Tennant and others came to regard intuitionistic logic as preferable over classical logic, as the standard presentation of CL turned out not to satisfy harmony in its original formulation.

As a reaction, other adherers of PTS claimed that there are ways to "harmonize" CL and restore its status as proof-theoretically justified. I will concentrate here on one such approach, mainly following Read [31], where CL is "blamed" for its apparent lack of harmony, suggesting, following [2], an $MCND$-presentation regaining harmony not present in the $SCND$ presentation.

In a recent Ph.D thesis dedicated to PTS, Hjortland [18] expresses several concerns regarding the $MCND$ "solution" to the classical harmony problem.

- In spite of Read's claim in [31] that the $MCND$ presentation of classical logic is harmonious, no explicit specification of the needed detour-eliminating reductions is given. Those are not given in [2], as the normalization proof there is indirect, by mapping to a sequent-calculus with cut-elimination. However, such reductions are presented in [5], using them to show normalization. Also, in [4] a direct proof of weak normalisation is presented for the implicative fragment of classical logic, also using detour elimination reductions for implication.
- The $MCND$ presentation of classical logic is given without relating explicitly to the assumptions on which a node in a derivation depends. In an

attempt to "correct" the presentation and take open assumptions into consideration, Hjortland believes to have detected a difficulty, endowing the structural separator, the comma, with an ambiguous meaning.

- There is no proper procedure for the derivation of the E-rules from the I-rules that yields *general elimination* (GE-rules) that also accounts for the distinction between *additive* and *multiplicative* rules (not necessarily in a classical setting, where this distinction is elided).

The contributions of the current paper are:

- Consider $MCND$ in a general way, setached from the presentation of classical logic in it.
- Provide the missing detour-elimination reductions for $MCND$ both in simple and logistic presentations (see below). Thereby, the issue of the closure of ND-derivations (both $SCND$ and $MCND$) under composition[1] is brought to the forth as central to the existence of such reductions.
- Provide $MCND$ presentations in Gentzen's "logistic" $SCND$-style, using sequents as premises and conclusions of a rule, both for additive and multiplicative rules.
- Extend the definition of GE-rules to $MCND$ and adapt the procedure of obtaining GE-rules from I-rules in a $SCND$-system in [13] to an $MCND$ set-up, and analyze the impact of substructurality on this procedure.
- Show *stability* of the $MCND$ presentation (in addition to harmony), an issue not dealt with before in the literature on $MCND$.

2. Preliminaries

2.1. Single-Conclusion Propositional Natural-Deduction Proof-Systems

2.1.1. Object Languages, Contexts and Sequents. A propositional $SCND$-system, say $\mathcal{N}$, is defined over an *object language*, say L, that defines (usually, recursively) a freely-generated[2] collection $\mathcal{F}$ of *formulas*, ranged over by (possibly indexed) φ, ψ, etc. The basis of the recursion is usually[3] a (countably infinite) set $\mathcal{P} = \{p_i \mid i \geq 0\}$, referred to as *basic* (or *atomic*) sentences.[4] Sometimes, *propositional constants* are also included in the object language, most often '$\perp$' (falsum, absurdity) and its dual '$\top$' (verum). While in general an operator can have any arity, combining any number of formulas to form a new one, I shall employ as much as possible the use of binary (and unary) operators, facilitating the convenient infix notation. The *complexity* $|\varphi|$ of a formula φ is the number of operator occurrences it contains.

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1 Sometimes also called closure under (derivation) substitution.

2 Freely-generated here means that, for example a ternary operator '$*$' will generate a formula of the form '$*(\varphi, \psi, \chi)$', allowing arbitrary sub-formulas, and not, say, '$*(\varphi, \psi, \varphi)$', restricting to sub-formulas to be identical.

3 But not always! For example, in 1st-order logic, atomic sentences have a different form. Also, in the application of PTS to natural language in [12], a different kind of atomic sentences is used.

4 The base case in the recursive definition can also be viewed as *propositional variables*, amenable to substitution.

There are two common variants of presentation modes of $SCND$-systems.

Simple presentation. The objects of $\mathcal{N}$ in its simple presentation, serving as premises and conclusions of I/E-rules, are the formulas of L themselves. Dependence of formulas on assumptions in rules and derivations is left implicit.

Logistic presentation. The objects of $\mathcal{N}$ in its logistic presentation, serving as premises and conclusions of I/E-rules, are *sequents* of the form $\Gamma \vdash_{\mathcal{N}} \varphi$, where:

- Γ is a *context*, a (finite) list (i.e., ordered set[5]) of formulas of the object language, *assumptions* on which the derivation of φ may depend. Γ is also referred to as the *antecedent* of the sequent. A context Γ_1, Γ_2 is a combination of two contexts into one, formed by concatenation; the comma is sometimes omitted. I will not distinguish between a singleton context, containing only one formula, say φ, and the formula φ itself.
- φ, the *succedent* of the sequent, is a formula of the object language.

When $\mathcal{N}$ can be determined from context, or is immaterial, it is omitted from the sequent. I say more about contexts below. Below, I formulate the way a transition from one mode of presentation to other can take place, thus allowing the choice of mode as a matter of convenience. Simple presentation will mostly be used for presenting examples, to cut short notational clutter.

2.1.2. Rules. $SCND$-systems consist of a collection of *rules*, in contrast to the axiomatic nature of Hilbert-style proof systems that preceded them. The latter had two main roles:

1. To "capture" in a syntactic, effective way the consequence relation of the logic as defined model-theoretically.
2. To justify the axioms (as correctly performing the task above) by means of soundness and (preferably) completeness w.r.t. the model-theoretically specified meaning.

There is an important methodological difference between Hilbert's axiomatic proof-theory and Gentzen's structural proof-theory (in both variants—natural deduction and sequent calculus). While the former regards proofs (forming *categorical assertions*, depending on no open assumptions) as of primary interest for logic, the latter attribute priority to *derivations* from open assumptions (forming *hypothetical arguments*). Clearly, categorical arguments are a special case of the hypothetical ones, when the collection of open assumptions turns to be empty. In MTS terms, Hilbert systems capture *validities* (tautologies in the propositional case)—formulas yielding truth under every interpretation. This difference is reflected in several aspects of PTS, to be elaborated as the presentation advances.

A rule in a $SCND$-system, say (R), has *premises* (usually, finitely many), often presented as located over a horizontal line, and a *conclusion*, usually pre-

[5] For some logics, contexts may have a more complicated structure, for example trees; this will be specifically indicated if arising.

sented as located underneath the horizontal line. The premises and conclusions are objects depending on the presentation mode. To distinguish the two presentations of (the same) rule R, they are labelled R_s (simple presentation) and R_l (logistic presentation).

Simple presentation.

$$\frac{\begin{matrix}[\varphi_1^1,\ldots,\varphi_1^{m_1}]_1 & & [\varphi_p^1,\ldots,\varphi_p^{m_p}]_p\\ \vdots & & \vdots \\ \psi_1 & \cdots & \psi_p\end{matrix}}{\psi}\ (R_s^{\bar{i}}) \tag{1}$$

Here both the p premises and the conclusion are formulas of the object language L. A premise may (but need not) have discharged assumptions (also formulas in L), wrapped in square brackets when present. The index $\bar{i} = 1 \cdots p$ on the rule's name, called the *discharge label*, abbreviates the collection of indices of assumptions discharged by an application of the rule. I use $\Sigma_j = \{\varphi_j^1, \ldots, \varphi_j^{m_j}\}$ for the j'th *block of assumptions*, for $1 \leq j \leq p$. When $m_j = 0$, no assumptions are discharged by the jth premise. In addition to the dischargeable assumptions Σ_j on which ψ_j depends, it may depend on additional (lateral) assumptions which are implicit.

Rules are classified into two families, *additive* and *multiplicative*.

Additive. There is a restriction that all the premises depend on the *same* collection of (implicit) lateral assumptions.

Multiplicative. Each premise may depend on different collections of (implicit) lateral assumptions.

Logistic presentation. In this presentation, both the premises and the conclusion are all sequents as above, thereby making dependence on lateral assumptions explicit. The effect of a discharge of an assumption is exhibited here by the assumption present in the antecedent of a premise, but not in the antecedent of the conclusion. The presentation splits into two sub-cases, as follows.

Additive.

$$\frac{\Gamma, \varphi_1^1, \ldots, \varphi_1^{m_1} \vdash_{\mathcal{N}} \psi_1 \quad \cdots \quad \Gamma, \varphi_p^1, \ldots, \varphi_p^{m_p} \vdash_{\mathcal{N}} \psi_p}{\Gamma \vdash_{\mathcal{N}} \psi}\ (R_l) \tag{2}$$

The j'th logistically presented premise is abbreviated to $\Gamma, \Sigma_j \vdash_{\mathcal{N}} \psi_j$. The characteristic feature of additive rules is that all premises have the same context, Γ, hence they are also known as *context sharing* rules.

Multiplicative.

$$\frac{\Gamma_1, \varphi_1^1, \ldots, \varphi_1^{m_1} \vdash_{\mathcal{N}} \psi_1 \quad \cdots \quad \Gamma_p, \varphi_p^1, \ldots, \varphi_p^{m_p} \vdash_{\mathcal{N}} \psi_p}{\Gamma_1 \cdots \Gamma_p \vdash_{\mathcal{N}} \psi}\ (R_l) \tag{3}$$

Here the j'th logistically presented premise is abbreviated to $\Gamma_j, \Sigma_j \vdash_{\mathcal{N}} \psi_j$. The characteristic feature of multiplicative rules is that each premise may have its own context, Γ_j, hence they are also known as *context free* rules.

The difference between additive and multiplicative rules came to the forth most notably in Linear Logic [17].

Rules[6] are said to be *applied* to their premises, *yielding* their conclusion(s). Rules, in particularly as a tool for PTS, are to be read as *parametric* in the underlying object language. Thus, when a rule displayed as in (2) or (3) is contained in an $SCND$-system over object language L, Γ ranges over L-contexts. So, such a rule can be freely included in different meaning conferring $SCND$-systems. For that to serve its purpose as meaning conferring, the following *generality* properties should hold:

Formula generality. Formulas displayed in a premise or in the conclusion of a rule should appear in their most general form allowed by L. For example, in the conclusion, if φ has as its principal operator a binary operator, say '$*$', then φ should be presented as '$\psi * \chi$', allowing ψ and χ to differ, and not as, say, $\psi * \psi$, forcing the the two sub-formulas to be the same.

Context generality. In the logistic presentation (in both versions), contexts variables should be present in every premise and conclusion, ranging, as mentioned above, over arbitrary object language contexts.

In [20], those properties are also indicated as desirable for rules, under the names "generality in respect to constituent formulas" and "generality in respect to side formulas".

$SCND$-systems contain the following kinds of rules.

Operative rules. For every *logical constant* (e.g., connective or quantifier) of arity greater than 1, say[7] '$*$', of the object language, there are *two families of rules*, generally assumed disjoint.

Elimination rules (E-rules). $(*E)$, determining which formulas can be deduced from $(\varphi_1 * \varphi_2)$ in the simple presentation, or which sequents can be deduced from $\Gamma \vdash_{\mathcal{N}} (\varphi_1 * \varphi_2)$ in the logistic presentations. The premise containing '$*$' is the *major premise*, while all other premises of the rule (if there are any) are *minor premises*. E-rules should reflect *a direct conclusion* of $(\varphi_1 * \varphi_2)$.

Introduction rules (I-rules). $(*I)$, determining from which formulas, in the simple presentation, can the conclusion $(\varphi_1 * \varphi_2)$ be deduced, or from which sequents, in the logistic presentations, can the conclusion $\Gamma \vdash_{\mathcal{N}} (\varphi_1 * \varphi_2)$ be deduced. I-rules should reflect *a direct derivation* of $(\varphi_1 * \varphi_2)$.

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[6] Strictly speaking, *instances* of rules are applied, where the schematic meta-variables are instantiated to actual object language expressions. I will, by abuse of nomenclature, ignore this finer point, which carries over to $MCND$ and is not central to the discussion.

[7] As already mentioned, in general, logical constants can be of any arity. Recall that for readability, I most often present them as binary.

In both cases, $(\varphi_1 * \varphi_2)$ is the *principal formula* of the rule. Note that for 0-ary constants (to be encountered in the sequel), that have no principal operator, the I-rules introduce directly the constant. There is also a limit case of a constant that has *no* I-rules (cf. $\perp$ in intuitionistic logic). Let $\mathcal{E}_\mathcal{N}$ denote the collection of all E-rules in $\mathcal{N}$, $\mathcal{I}_\mathcal{N}$ the collection of all I-rules in $\mathcal{N}$ and $\mathcal{O}_\mathcal{N} = \mathcal{I}_N \cup \mathcal{E}_\mathcal{N}$. As usual, when $\mathcal{N}$ is clear from the context, it is omitted.

Structural rules. Those are rules not referring explicitly to any logical constant, and allow the manipulation of the context. They are always presented in the logistic presentation. Usually, such rules control the order and the multiplicity of formulas in a sequent. Typically, a structural rule, say (S), is depicted as follows.

$$\frac{\Gamma \vdash_\mathcal{N} \varphi}{\Gamma' \vdash_\mathcal{N} \varphi}\ (S)$$

implying that the context Γ' can replace, in a derivation of φ (in $\mathcal{N}$), that of Γ. Occasionally, it may be convenient to use the following notation,

$$\frac{\Gamma(\Gamma_1) \vdash_\mathcal{N} \varphi}{\Gamma(\Gamma_2) \vdash_\mathcal{N} \varphi}\ (S)$$

where $\Gamma(\Gamma_1)$ refers to a context Γ containing a sub-context (sublist of assumptions) Γ_1, and $\Gamma(\Gamma_2)$ is obtained from the above by replacing Γ_1 by Γ_2. Let $\mathcal{S}_\mathcal{N}$ denote the collection of the structural rules of $\mathcal{N}$.

There is a whole area of study, called *substructural logics*, dedicated to systems which contain only some (or none!) of the structural rules originally conceived by Gentzen. See, for example [31]. Structural rules are relative to a logical system, not just a logic, as the same logic may be defined using systems that differ w.r.t. the structural rules assumed primitive.

While typically $SCND$-systems have the structural rules implicitly built into the operative rules, when not so, I will present them as explicit primitive rules.

Below are defined two properties of an operative rule, that are important in the PTS programme.

Definition 2.1. (*Purity, simplicity*) An operative rule for an operator, say '$*$', is *pure* iff the rule does not mention any connectives other than '$*$'. The rule is *simple* iff it mentions '$*$' once only.

Sometimes, rules are defined so as to have *side-conditions* to their application. For example, the following I-rule for the universal quantifier (not considered here further)

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \forall x.\varphi}\ (\forall I)$$

has as side-condition $x \notin free(\Gamma)$, that x does not occur free in any assumption of the context Γ. Similarly, the I-rule for necessitation in the modal logic S5

$$\frac{\Gamma\vdash \varphi}{\Gamma\vdash \Box\varphi}\ (\Box I)$$

has the side-condition that every assumption $\psi \in \Gamma$ is itself boxed, $\psi = \Box\psi'$ (for some ψ'). Note that this side-condition causes the rule to violate the context generality condition, making it in appropriate for serving as a meaning conferring rule for necessity. This violation was noted also in [20] in the context of uniqueness of an operator. One solution, suggested also in [1], is to change the form of the sequents over which rules for modalities are defined. As this issue is orthogonal to my current interests, I will not pursue it further.

Below, when I speak of an application of a rule, it is always presupposed that the side-condition (if any) is satisfied.

A typical characteristic of natural-deduction proof-systems is their use of *hypothetical reasoning*, which gives *ND* its force: A rule "temporarily" introduces *assumptions*, to be used in premises, and any number of their instances (including zero!) may be *discharged* by an application of the rule. In the logistic presentations, assumptions occur in the antecedents of one or more premises, and if all instances are discharged, the assumed formula does not occur anymore in the antecedent of the conclusion sequent. The natural view of a discharged assumption conforms to the above mentioned interpretation of a sequent, reflecting a dependency of the succedent on the antecedent, in terms of "holding under assumption". A discharged assumption is also called *closed*. In case of a discharge of zero occurrences, we refer to a *vacuous discharge* (a notion also relative to the logical system employed), *rejected by some logics*!

2.2. Rule Classification

The following classification of $SCND$-rules, taken from [13] (where only the categories directly relevant to the propositional case are kept), using some distinguishing criteria, will turn useful for the upcoming discussion.

1. An $SCND$-rule is *hypothetical* if it allows for at least one premise with *assumptions discharge*; otherwise, it is *categorical*. The latter seems to coincide with what Milne [21] calls a "*immediate inference*". I refer to the categorical part of a hypothetical rule as the '*grounds*', and refer to the discharged assumptions on which the grounds depend as the ground's *support*.
2. An $SCND$-rule is *combining* if has more than one premise; otherwise it is non-combining.

Note that the classification by the above criteria is orthogonal to the I-rule vs. E-rule classification. Thus, the the conjunction-introduction rule $\frac{\Gamma\vdash\varphi \quad \Gamma\vdash\psi}{\Gamma\vdash\varphi\wedge\psi}\ (\wedge I)$ and the conjunction-elimination rules $\frac{\Gamma\vdash\varphi\wedge\psi}{\Gamma\vdash\varphi}\ (\wedge E_1)$, $\frac{\Gamma\vdash\varphi\wedge\psi}{\Gamma\vdash\psi}\ (\wedge E_2)$ are categorical, while the implication-introduction rule $\frac{\Gamma,\varphi\vdash\psi}{\Gamma\vdash\varphi\to\psi}\ (\to I)$ and the disjunction-elimination rule $\frac{\Gamma\vdash\varphi\vee\psi \quad \Gamma,\varphi\vdash\xi \quad \Gamma,\psi\vdash\xi}{\Gamma\vdash\xi}\ (\vee E)$ are hypothetical. Also, the $(\wedge I)$ and $(\vee E)$ are combining, while $(\wedge E_i)$ and $(\to I)$ are non-combining.

2.2.1. Derivations. I assume the usual definition of (tree-shaped) displaying of $\mathcal{N}$-*derivations*, ranged over by $\mathcal{D}$, again defined separately for the two presentations of $\mathcal{N}$. Note, however, that my definition of a derivation deviates somewhat from the standard literature in that the names of rules the instances of which are applied at the nodes of the tree are retained in the tree. This evades certain difficulties orthogonal to my current concerns.

Definition 2.2. (*Derivations*)

Simple derivations. Here a derivation $\overset{\mathcal{D}}{\psi}$ has an explicit (single) conclusion ψ, and *implicit* assumptions on which the conclusion ψ depends, denoted by $\mathbf{d}_{\mathcal{D}}$, defined in parallel to $\mathcal{D}$.

- Every assumption φ is a derivation $\overset{\mathcal{D}}{\varphi}$, with $\mathbf{d}_{\mathcal{D}} = \{\varphi\}$.
- If $\overset{\mathcal{D}_j}{\psi_j}$, for $1 \leq j \leq p$, are derivations with dependency sets $\mathbf{d}_{\mathcal{D}_j}$ with $\Sigma_j \subseteq \mathbf{d}_{D_j}$, and if

$$\frac{\begin{matrix}[\Sigma_1]_1 \\ \vdots \\ \psi_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_p]_p \\ \vdots \\ \psi_p\end{matrix}}{\psi}\ (R_s^{\bar{i}})$$

is an instance of a (simply presented) rule in $\mathcal{R}_{\mathcal{N}}$ with a *fresh* discharge label $\bar{i}$, then

$$\overset{\mathcal{D}}{\psi} =^{\text{df.}} \frac{\begin{matrix}[\Sigma_1]_1 \\ \mathcal{D}_1 \\ \psi_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_p]_p \\ \mathcal{D}_p \\ \psi_p\end{matrix}}{\psi}\ (R_s^{\bar{i}})$$

is a derivation with $\mathbf{d}_{\mathcal{D}} = \cup_{1\leq j\leq p}\mathbf{d}_{\mathcal{D}_j} - \cup_{1\leq j\leq p}\hat{\Sigma}_j$, where $\hat{\Sigma}_j$ is the collection of formulas actually discharged by this instance of the applied rule. The derivations $\mathcal{D}_1$,..., $\mathcal{D}_p$ are the *direct sub-derivations* of $\mathcal{D}$.

As for the additive/multiplicative distinction, the following[8] holds.

Additive. Recursively, the sub-derivations $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are all additive, $\mathbf{d}_{\mathcal{D}_1} = \cdots = \mathbf{d}_{\mathcal{D}_1}$, and the rule R_s is additive. The resulting derivation $\mathcal{D}$ is additive too.

Multiplicative. Recursively, the sub-derivations $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are all multiplicative and the rule R_s is multiplicative. The resulting derivation $\mathcal{D}$ is multiplicative too.

It is convenient to use $\begin{matrix}\varphi \\ \mathcal{D} \\ \psi\end{matrix}$, when $\varphi \in \mathbf{d}_{\mathcal{D}}$, to focus on some assumption φ on which ψ depends.

Logistic derivations. Here the context Γ encodes explicitly, in a node of a derivation, the assumptions on which the succedent formula depends.

[8] For simplicity, I am ignoring here mixed derivations where additive and multiplicative rules are applied intermittently.

- Every instance of $\varphi \vdash_{\mathcal{N}} \varphi$ is a derivation.
- **Additive.** If $\overset{\mathcal{D}_1}{\Gamma, \Sigma_1 \vdash_{\mathcal{N}} \psi_1}, \ldots, \overset{\mathcal{D}_p}{\Gamma, \Sigma_p \vdash_{\mathcal{N}} \psi_p}$, for some $p \geq 1$, are (logistically presented) additive derivations, and if

$$\frac{\Gamma, \Sigma_1 \vdash_{\mathcal{N}} \psi_1 \quad \cdots \quad \Gamma, \Sigma_p \vdash_{\mathcal{N}} \psi_p}{\Gamma \vdash_{\mathcal{N}} \psi} \; (R_l)$$

is an instance of a (logistically presented) additive rule in $\mathcal{R}_{\mathcal{N}}$, then

$$\overset{\mathcal{D}}{\Gamma \vdash_{\mathcal{N}} \psi} =^{\text{df.}} \frac{\overset{\mathcal{D}_1}{\Gamma, \Sigma_1 \vdash_{\mathcal{N}} \psi_1} \quad \cdots \quad \overset{\mathcal{D}_p}{\Gamma, \Sigma_p \vdash_{\mathcal{N}} \psi_p}}{\Gamma \vdash_{\mathcal{N}} \psi} \; (R_l)$$

is a logistically presented additive derivation.

Multiplicative. If $\overset{\mathcal{D}_1}{\Gamma_1, \Sigma_1 \vdash_{\mathcal{N}} \psi_1}, \ldots, \overset{\mathcal{D}_p}{\Gamma_p, \Sigma_p \vdash_{\mathcal{N}} \psi_p}$, for some $p \geq 1$, are (logistically presented) multiplicative derivations, and if

$$\frac{\Gamma_1, \Sigma_1 \vdash_{\mathcal{N}} \psi_1 \quad \cdots \quad \Gamma_p, \Sigma_p \vdash_{\mathcal{N}} \psi_p}{\Gamma_1 \cdots \Gamma_n \vdash_{\mathcal{N}} \psi} \; (R_l)$$

is an instance of a (logistically presented) multiplicative rule in $\mathcal{R}_{\mathcal{N}}$, then

$$\overset{\mathcal{D}}{\Gamma_1 \cdots \Gamma_n \vdash_{\mathcal{N}} \psi} =^{\text{df.}} \frac{\overset{\mathcal{D}_1}{\Gamma_1, \Sigma_1 \vdash_{\mathcal{N}} \psi_1} \quad \cdots \quad \overset{\mathcal{D}_p}{\Gamma_p, \Sigma_p \vdash_{\mathcal{N}} \psi_p}}{\Gamma_1 \cdots \Gamma_n \vdash_{\mathcal{N}} \psi} \; (R_l)$$

is a logistically presented multiplicative derivation.

The derivations $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are the *direct subderivations* of $\mathcal{D}$.

Note that assumptions are introduced into a logistically presented derivation via identity derivations and remain as such as long as not discharged by some application of an instance of a rule.

In cases where structural rules are kept in $\mathcal{N}$ as explicit primitive rules, the definition of derivation has to incorporate their application too. As this does not introduce any significant difference to the generalisation of $MCND$ I skip the details of the modification needed.

There is an easy mutual conversion between the the two presentation modes:

Simple to logistic. Convert each node in $\overset{\mathcal{D}}{\psi}$ to $\overset{\mathcal{D}}{\Gamma \vdash_{\mathcal{N}} \psi}$, where $\Gamma = \mathbf{d}_{\mathcal{D}}$.

Logistic to simple. Convert each node in $\overset{\mathcal{D}}{\Gamma \vdash_{\mathcal{N}} \psi}$ to $\overset{\mathcal{D}}{\psi}$, setting $\mathbf{d}_{\mathcal{D}} = \Gamma$.

Definition 2.3. (*Derivability*) ψ is *derivable* from Γ in $\mathcal{N}$, denoted by $\Gamma \vdash_{\mathcal{N}} \psi$, iff there exist a simply presented $\mathcal{N}$-derivation $\overset{\mathcal{D}}{\psi}$ with $\mathbf{d}_{\mathcal{D}} = \Gamma$ (respectively, a logistically presented derivation $\overset{\mathcal{D}}{\Gamma \vdash_{\mathcal{N}} \psi}$).

Thus, the logistic presentation keeps track *explicitly* of the assumptions Γ on which ψ depends. Note that $\vdash_{\mathcal{N}} \psi$ indicates derivability of ψ from an empty context, in which case ψ is referred to as a (*formal*) *theorem* of $\mathcal{N}$.

Derivations in an $SCND$-system are depicted as trees (of formulas, or of sequents, according to the presentation mode), where a node and its descendants are an instance (of an application of) one of the rules. In derivation-trees for simply presented derivations, the discharged occurrences of the assumptions are enclosed in square brackets, and marked, $[...]_i$, for some index i, to match i on the applied rule-name, and are leaves in the tree depicting the derivation. Importantly, any *instance* of an assumption-discharging rule has a *unique* index! Rules of an $SCND$-system operate on the succedent of a sequent *only*. I also use $\mathcal{D} : \Gamma \vdash_{\mathcal{N}} \varphi$ to indicate a specific derivation of that sequent. In such a derivation, Γ is also referred to as the *open assumptions* of $\mathcal{D}$. If Γ is empty, the derivation is *closed*.

An important property of $SCND$-derivations is their *closure under composition* (known also as closure under substitution of a derivation for an assumption). This closure establishes the *transitivity* of $SCND$-derivability, namely '$\vdash_{\mathcal{N}}$'. It underlies the definition of derivation reduction used to define harmony (local-soundness), introduced below in Sect. 2.3.

Definition 2.4. (*Closure under derivation composition*)

Simple derivations. Let $\begin{matrix}\varphi\\ \mathcal{D}\\ \psi\end{matrix}$ be any simply presented $\mathcal{N}$-derivation of ψ from φ (and assumptions $\mathbf{d}_{\mathcal{D}}$, left implicit), and let $\begin{matrix}\mathcal{D}'\\ \varphi\end{matrix}$ be any simply presented $\mathcal{N}$-derivation of φ (from assumptions $\mathbf{d}_{\mathcal{D}'}$, left implicit). Then $\mathcal{N}$ is *closed under derivation composition* iff the result of replacing every occurrence of (the leaf) φ in $\mathcal{D}$ by the sub-tree $\mathcal{D}'$ is also a derivation, denoted $\begin{matrix}\mathcal{D}''\\ \psi\end{matrix} = \begin{matrix}\mathcal{D}[\varphi := \begin{matrix}\mathcal{D}'\\ \varphi\end{matrix}]\\ \psi\end{matrix}$, with $\mathbf{d}_{\mathcal{D}''} = (\mathbf{d}_{\mathcal{D}} - \{\varphi\}) \cup \mathbf{d}_{\mathcal{D}'}$.

Logistic derivations. Let $\overset{\mathcal{D}}{\Gamma, \varphi \vdash_{\mathcal{N}} \psi}$ be any logistically presented $\mathcal{N}$-derivation (of ψ from Γ, φ), and let $\overset{\mathcal{D}'}{\Gamma' \vdash_{\mathcal{N}} \varphi}$ be any logistically presented $\mathcal{N}$-derivation (of φ, from Γ'). Then $\mathcal{N}$ is *closed under derivation composition* iff the result of

1. Replacing every occurrence of (the leaf) $\varphi \vdash_{\mathcal{N}} \varphi$ in $\mathcal{D}$ by the sub-tree $\overset{\mathcal{D}'}{\Gamma \vdash_{\mathcal{N}} \varphi}$.
2. Replacing the antecedent of the sequent labelling any node in $\mathcal{D}$, say $\overline{\Gamma}$, by the new antecedent $\overline{\Gamma} - \varphi, \Gamma'$ (leaving the succedent of the sequent intact).

is also a derivation, denoted $\overset{\mathcal{D}''}{\Gamma - \varphi, \Gamma' \vdash_{\mathcal{N}} \psi} = \overset{\mathcal{D}[\varphi \vdash_{\mathcal{N}} \varphi := \overset{\mathcal{D}'}{\Gamma' \vdash_{\mathcal{N}} \varphi}]}{\Gamma - \varphi, \Gamma' \vdash_{\mathcal{N}} \psi}$ (of ψ, from $\Gamma - \varphi, \Gamma'$).

Thus, in $\mathcal{D}''$, any use in $\mathcal{D}$ of the assumption φ is replaced by re-deriving φ according to $\mathcal{D}'$ (from possibly additional assumptions, on which the conclusion of $\mathcal{D}''$ (which is the same as the conclusion of $\mathcal{D}$) now depends). This conclusion, however, depends no more on the assumption φ. Note that if the assumption φ is not actually used in $\mathcal{D}$, then $\underset{\psi}{\mathcal{D}[\varphi := \stackrel{\mathcal{D}'}{\varphi}] \equiv \mathcal{D}}$ (and similarly for the logistic counterpart).

Example 2.1. Suppose $\mathcal{D}$ is the following simply presented derivation for $\varphi, \varphi{\rightarrow}\psi, \varphi{\rightarrow}\chi \vdash \psi \wedge \chi$, and $\mathcal{D}'$ is the following simply presented derivation for $\xi, \xi{\rightarrow}\varphi \vdash \varphi$ (both in intuitionistic logic).

$$\mathcal{D}: \quad \dfrac{\dfrac{\varphi \quad \varphi{\rightarrow}\psi}{\psi}\,(\rightarrow E) \quad \dfrac{\varphi \quad \varphi{\rightarrow}\chi}{\chi}\,(\rightarrow E)}{\psi \wedge \chi}\,(\wedge I) \qquad \mathcal{D}': \quad \dfrac{\xi \quad \xi{\rightarrow}\varphi}{\varphi}\,(\rightarrow E) \tag{4}$$

Then,

$$\underset{\varphi\wedge\chi}{\mathcal{D}[\varphi := \stackrel{\mathcal{D}'}{\varphi}]} = \dfrac{\dfrac{\dfrac{\xi \quad \xi{\rightarrow}\varphi}{\varphi}\,(\rightarrow E) \quad \varphi{\rightarrow}\psi}{\psi}\,(\rightarrow E) \quad \dfrac{\dfrac{\xi \quad \xi{\rightarrow}\varphi}{\varphi}\,(\rightarrow E) \quad \varphi{\rightarrow}\chi}{\chi}\,(\rightarrow E)}{\psi \wedge \chi}\,(\wedge I) \tag{5}$$

Note that *both* occurrences of φ in $\mathcal{D}$ where replaced during the substitution.

Example 2.2. The composition of the above derivations when logistically presented looks as follows.

$$\mathcal{D}: \quad \dfrac{\dfrac{\varphi\vdash\varphi \quad \varphi{\rightarrow}\psi\vdash\varphi{\rightarrow}\psi}{\varphi, \varphi{\rightarrow}\psi\vdash\psi}\,(\rightarrow E) \quad \dfrac{\varphi\vdash\varphi \quad \varphi{\rightarrow}\chi\vdash\varphi{\rightarrow}\chi}{\varphi, \varphi{\rightarrow}\chi\vdash\chi}\,(\rightarrow E)}{\varphi, \varphi{\rightarrow}\psi, \varphi{\rightarrow}\chi\vdash\psi\wedge\chi}\,(\wedge I)$$

$$\mathcal{D}': \quad \dfrac{\xi\vdash\xi \quad \xi{\rightarrow}\varphi\vdash\xi{\rightarrow}\varphi}{\xi, \xi{\rightarrow}\varphi\vdash\varphi}\,(\rightarrow E) \tag{6}$$

Then,

$$\underset{\varphi, \varphi\rightarrow\psi, \varphi\rightarrow\chi\vdash\varphi\wedge\chi}{\mathcal{D}[\varphi\vdash\varphi := \stackrel{\mathcal{D}'}{\xi, \xi\rightarrow\varphi\vdash\varphi}]} = \dfrac{\dfrac{\dfrac{\xi\vdash\xi \quad \xi\rightarrow\varphi\vdash\xi\rightarrow\varphi}{\xi, \xi\rightarrow\varphi\vdash\varphi}\,(\rightarrow E) \quad \varphi\rightarrow\psi\vdash\varphi\rightarrow\psi}{\xi, \xi\rightarrow\varphi, \varphi\rightarrow\psi\vdash\psi}\,(\rightarrow E) \quad \dfrac{\dfrac{\xi\vdash\xi \quad \xi\rightarrow\varphi\vdash\xi\rightarrow\varphi}{\xi, \xi\rightarrow\varphi\vdash\varphi}\,(\rightarrow E) \quad \varphi\rightarrow\chi\vdash\varphi\rightarrow\chi}{\xi, \xi\rightarrow\varphi, \varphi\rightarrow\chi\vdash\chi}\,(\rightarrow E)}{\xi, \xi\rightarrow\varphi, \varphi\rightarrow\psi, \varphi\rightarrow\chi\vdash\psi\wedge\chi}\,(\wedge I) \tag{7}$$

The definitions are naturally extended to multiple simultaneous substitution $\mathcal{D}[\varphi_1 := \stackrel{\mathcal{D}'_1}{\varphi_1}, \ldots, \varphi_m := \stackrel{\mathcal{D}'_m}{\varphi_m}]$, for any $m \geq 1$. The satisfaction of the closure under derivation composition needs to be shown for the specific $SCND$-systems used for meaning conferring, as it underlies establishing harmony (see below). Most often, it is proved by induction on the structure of derivations.

Note that there is difference between this property of an $SCND$-system $\mathcal{N}$ and the property of *admissibility* in $\mathcal{N}$ of the *cut*-rule

$$\frac{\Gamma_1 \vdash \varphi \quad \Gamma_2, \varphi \vdash \psi}{\Gamma_1 \Gamma_2 \vdash \psi}\ (cut)$$

also reflecting the transitivity of $\mathcal{N}$-derivability. The admissibility of (cut) establishes a property at the level of sequents: any sequent $\mathcal{N}$-derivable using (cut) has a *direct* $\mathcal{N}$-derivation without the use of (cut), which usually differs in form. The closure property above is a property of derivations, not of sequents (see more about this issue in [24]).

As a simple example of the failure of the closure under derivation composition, suppose that the definition of a derivation is modified, by adding a requirement that Γ, the collection of leaves in $\mathcal{D}$, is consistent. In such a case, $\begin{smallmatrix}\Gamma, \varphi \\ \mathcal{D} \\ \psi\end{smallmatrix}$ may have a consistent Γ, φ, as will $\begin{smallmatrix}\Gamma' \\ \mathcal{D}' \\ \varphi\end{smallmatrix}$; however, this does not ensure the consistency of Γ, Γ', the leaves of $\begin{smallmatrix}\mathcal{D}[\varphi := \begin{smallmatrix}\mathcal{D}' \\ \varphi\end{smallmatrix}] \\ \psi\end{smallmatrix}$, whereby the latter fails to be a legal derivation, causing failure of closure under derivation.

For an example of an ND-system (for a relevant logic) for which closure under composition does not hold with a "blind replacement", see [9]. For another example of an $SCND$-system not closed under derivation composition see [43].

There is a family of derivations that play a central role in the PTS programme, as being the vehicle through which meaning is conferred by the operational rules, called *canonical derivations*, defined[9] below. See [10] for a detailed motivation and discussion of this definition.

Definition 2.5. (*Canonical derivation from open assumptions*) A $\mathcal{N}$-derivation $\mathcal{D}$ for $\Gamma \vdash \psi$ is *canonical* iff it satisfies one of the following two conditions.

- The last rule applied in $\mathcal{D}$ is an I-rule (for the main operator of ψ).
- The last rule applied in $\mathcal{D}$ is an assumption-discharging E-rule, the major premise of which is some φ in Γ, and its encompassed sub-derivations $\mathcal{D}_1, \ldots, \mathcal{D}_n$ are all canonical derivations of ψ.

In a canonical derivation of φ from Γ, the conclusion φ results by an application of an I-rule of the main operator of φ. This is viewed as the *most direct* way to derive φ, a derivation *according to the meaning of its principal operator*. It is important to note, that the sub-derivations of a canonical derivation need *not* be canonical themselves. I use $\Gamma \vdash^c_{\mathcal{N}} \varphi$ for canonical derivability in $\mathcal{N}$.

2.3. Harmony and Stability

The notion of harmony can be traced back to the following famous comment by Gentzen (in [16, p. 80]), one of the fathers of modern (structural) proof-theory:

9 In the literature, canonicity is seen as a property of *proofs* (derivation from an empty context). I extend it in [10] to arbitrary derivations from open assumptions.

> ... The introductions represent, as it were, the 'definitions' of the symbol concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions....

Thus, by this comment, the E-rules can be "read off" the I-rules.

In [28], going back to his original formulation of [27], Prawitz formulates the following principle,[10] that came to be known as the *inversion principle*, which supposedly captures Gentzen's remark that the E-rules should be "read off" the I-rules.

> Let ρ be an application of an elimination rule that has ψ as consequence. Then, the derivation that justify the sufficient condition [...] for deriving the major premiss of ρ, when combined with the derivations of the minor premises of ρ (if any), already "contain" a derivation of ψ; the derivation of ψ is thus obtainable directly from the given derivations without the addition of ρ.

and [28]

> The corresponding introductions and eliminations are inverses of each other, in the sense that the conclusion obtained by an elimination does not state anything more than what must have already been obtained if the major premise of the elimination was inferred by an introduction. In other words, a proof of the conclusion of an elimination is already contained in the proofs of the premisses when the major premiss is inferred by introduction.

In a sense, this principle embodies the old conception of a valid argument, one the conclusions of which are "contained" in its assumptions. For a historic survey of this principle, see [22] or [41].

Dummett [8] basically adheres to the same, coining its obtaining as "*harmony*".

One can discern two main notions of harmony that might be intended for the purpose for which harmony is needed: a criterion for I/E-rules to qualify as meaning conferring.

Intrinsic harmony. This is a property of the I/E-rules for a logical constant, say '$*$', that depends on the $SCND$-system as a whole, as rules for other logical constants may have to be appealed to establish that the property holds for '$*$'.

Harmony in form. This is a property that supposedly depends *only on the form* of the I/E-rules, independent of any underlying ND-system augmented with the rules of '$*$'.

2.3.1. Intrinsic Harmony. Following [6,26], intrinsic harmony is formalized by *local-soundness*, existence of reductions removing maximal formulas (known also as 'detour removal').

[10] The quotation is notationally modified, to fit the notation used here.

Definition 2.6. (*Maximal formula*) A maximal formula in an $SCND$-derivation $\mathcal{D}$ is a node in $\mathcal{D}$ that is the conclusion of an application of an I-rule, and, simultaneously, a major premise of an application of an E-rule.

Definition 2.7. (*Local-soundness*) An $SCND$-system $\mathcal{N}$ is *locally-sound* iff every derivation $\mathcal{D}$ having an occurrence of a maximal formula can be transformed into an equivalent derivation $\mathcal{D}'$ (with the same conclusion and open assumptions) in which that occurrence[11] of the maximal formula is removed. Such a transformation is called a (*proof*) *reduction*.

If $\mathcal{N}$ is locally-sound, it means that the E-rules in $\mathcal{N}$ are *not too strong* w.r.t. the I-rules, as nothing can be "gained" by first introducing and then immediately eliminating. Clearly, the rules for *tonk fail* local-soundness, as the maximal formula φ *tonk* ψ in (8) cannot be removed.

$$\dfrac{\dfrac{\varphi}{\varphi\ tonk\ \psi}\ (tonkI)}{\psi}\ (tonkE) \tag{8}$$

As is well-known, intuitionistic logic (in its standard $SCND$-presentation) is intrinsically harmonious [27]. The removal of an implicative maximal formula is by means of the reduction in (9) (with all lateral assumptions omitted).

$$\dfrac{\dfrac{\begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \psi\end{array}}{\varphi\to\psi}\ (\to I^i) \quad \begin{array}{c}\mathcal{D}'\\ \varphi\end{array}}{\psi}\ (\to E) \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}[\varphi := \begin{array}{c}\mathcal{D}'\\ \varphi\end{array}]\\ \psi\end{array} \tag{9}$$

The reduction is well-defined due to the closure of intuitionistic logic under derivation composition.

2.3.2. Harmony in Form. This kind of harmony reflects the "reading off" the E-rules from the I-rules in a different, more direct way, by requiring the E-rules to have *a specific form*, given the I-rules. This specific form is induced by another principle, called "a stronger inversion principle" [23].

A stronger Inversion Principle: Whatever follows from the *direct grounds* for deriving a proposition must follow from that proposition.

To understand this specific form, that allows for the derivation of an *arbitrary* conclusion, consider the (logistically presented) E-rule for intuitionistic disjunction.

$$\dfrac{\Gamma\vdash_{NJ} (\varphi\vee\psi) \quad \Gamma,\varphi\vdash_{NJ} \chi \quad \Gamma,\psi\vdash_{NJ} \chi}{\Gamma\vdash_{NJ} \chi}\ (\vee E)$$

We see that in order for an arbitrary consequence χ to be drawn from a disjunction $\varphi\vee\psi$, this conclusion has to be derivable (with the aid of the auxiliary assumptions Γ), *from each* of the grounds for the introduction of $\varphi\vee\psi$ (serving as a discharged assumption), namely from φ (first minor premise) and from ψ

[11] Thus, this is a weaker property than normalisation, requiring iterated reduction until *no* maximal formula remains.

(second minor premise). Mimicking this form of rule for other operators leads to what became to be known[12] as *general elimination rules* (*GE-rules*). A rule of this form was also proposed by [42] for independent reasons, related to the relationship between normal ND-derivations, and cut-free sequent-calculus derivations, and also by several other authors. As already mentioned, in [11] a procedure was proposed, for $SCND$, to *generate* harmoniously-induced GE-rules from given I-rules. In Sect. 4.1 this procedure is extended to $MCND$.

A digression: There is an ongoing debate in the community about this procedure, that aims, in the terminology of Read [33], to produce "flattened" GE-rules, that only discharge formulas as assumptions (in their simple presentation). Recently, a new approach to harmony emerged (see [25,37,38]) with a different view of PTS, called a *reductive* approach by Schroeder-Heister. Under this reductive approach, a certain logic is taken to have a *given* meaning of its connectives, 2nd order intuitionistic logic $IP2$ (with universal quantification over propositional variables) in this case; then, the rules of any other connective are assumed to have a certain form allowing expressing its two meanings (an I-meaning and an E-meaning) as $IP2$ *formulas*. Harmony is then *defined* as inter-derivability of those two formulas in $IP2$. Under *this* definition, [25] proves the non-existence of "flat" GE-rules for a certain ternary connective.

It is important to note, and this was emphasized in [38] (including an explicit footnote) and in the Conclusions section of [37], that the reductive approach is different from the "foundational approach" to which I adhere, aiming at different notions of harmony. The existence of a GE-rule whose expression in $IP2$ is inter-derivable with the expression of a certain I-rule in $IP2$ is a different question than the existence of a GE-rule obtained from an arbitrary I-rule so that local-soundness and local-completeness hold. I am aiming at much more general settings, where the rules need not at all have meaning(s) representable in $IP2$. An attentive reader might have noticed that when the object language was discussed, I only said that *usually* (but definitely not *always*!) formulas are built freely over propositional variables. I am interested also in harmony as applicable to *natural language* (see [12,14]), the sentences of which are certainly not built that way, and to which the reductive approach in [38] does not apply. Even in logic, I am interested in rules with side-conditions, like those in the Relevant Logic **R** (see [9]) which cannot be translated to $IP2$ formulas because of the side-conditions.

Thus, I regard the generalisation of the procedure to induce harmonious GE-rules to $MCND$ as a worthy task, in spite of the ongoing debate on flattening GE-rules.

2.3.3. Stability. The notion of stability did not receive a precise definition in [8], but the intention is clear: the E-rules should not be *too weak* w.r.t. the I-rules. Again, following [6,26], *an approximation* to this property can be formalised by means of the existence of *expansions*.

[12] In the literature, such rules were only proposed in the framework of $SCND$. In Sect. 4 I generalise it to the $MCND$ setting.

Definition 2.8. (*Local-completeness*) An *SCND*-system $\mathcal{N}$ is *locally-complete* iff every derivation $\mathcal{D}$ in $\mathcal{N}$ for $\varphi\vdash_{\mathcal{N}}\varphi$ (for a compound φ) there exists a derivation $\mathcal{D}'$ for $\varphi\vdash_{\mathcal{N}}\varphi$ that contains applications of E-rules to φ as a major premise, as well as application of all the I-rules (for the operator dominating φ). $\mathcal{D}'$ is called *an expansion* of $\mathcal{D}$.

The local-completeness implies that the I-rules are strong enough (w.r.t. the E-rules) to allow a "reconstruction" of φ after it was "taken apart". Local-completeness is only an approximation to stability. For example (from [8]), *quantum logic disjunction* has an E-rule more restricted than the intuitionistic $(\vee E)$-rule in allowing no lateral assumptions in the minor premises, but still having an expansion.

As is also well-known, intuitionistic logic is locally-complete.

3. Multiple-Conclusion Natural-Deduction

In this section, I adapt the definitions of all the *SCND* notions from single conclusion to *multiple-conclusion natural-deduction* (*MCND*). The extra detailed presentation of *SCND* above allows for a transparent parallelism of the *MCND* presentation. Note again that I am aiming to a general presentation, not restricted to classical logic, the latter discussed after the general discussion.

3.1. Object Language, Contexts and Sequents

The object language L is defined exactly as for *SCND*-systems. Sequents for *MCND* have the form $\Gamma\vdash_{\mathcal{N}}\Delta$, where Γ is the *left context*, a finite (possibly empty) sequence of L-formulas, and Δ is the *right context*, equally structured.

3.2. Rules

In contrast to the *SCND* case, where the distinction between additivity and multiplicativity applied only to left contexts, here they apply both to left and to right contexts. I assume the following assumption, that will affect the construction of a harmoniously-induced *MCND GE*-rule.

Assumption (structural rule-uniformity): In a *MCND*-system $\mathcal{N}$, a rule is multiplicative on both Γ and Δ, or is additive on both Γ and Δ.

This assumption excludes rules that are additive on the left context and multiplicative on the right context, or vice versa. Note that this assumption *does not* exclude that some rules in $\mathcal{N}$ are additive, while other rules are multiplicative—the assumption restricts the form of a single rule only.

Here too there are two modes of presentation, also referred to as 'simple' and 'logistic'.

A rule in a *MCND*-system, say (R), again has (finitely many) *premises* and (finitely many) *conclusions*, again all objects depending on the presentation mode.

Simple presentation. The objects of $\mathcal{N}$ in its simple presentation, serving as premises and conclusions of rules, are finite (possibly empty) sequences Δ of formulas of L.

Additive rule.

$$\frac{\begin{array}{c}[\Sigma_1]_1 \\ \vdots \\ \varphi_1, \Delta\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_p]_p \\ \vdots \\ \varphi_p, \Delta\end{array}}{\varphi, \Delta}\ (R_s^{\bar{i}}) \tag{10}$$

Here both the premises and the conclusion are finite sequences of formulas of the object language L. Here too premises may depend on additional, non-discharged, lateral assumptions, left implicit also, where there is a constraint that all premises depend on the same lateral assumptions.

How should[13] the notation $[\Sigma]_k$ be read? If $\Sigma = \alpha_1, \ldots, \alpha_n$, then $[\Sigma]_k$ is to be read as $[\alpha_1]_k \cdots [\alpha_n]_k$. That is, a collection of assumptions each being a single formula, collectively discharged by any application of an instance of the rule. It *is not* to be read s a single assumption constituting a sequence of formulas, an assumption discharged by applications of instances of the rule. This convention allows for a certain compaction of the notation, already fairly complicated; once explained, it should not cause any confusion.

Multiplicative rule.

$$\frac{\begin{array}{c}[\Sigma_1]_1 \\ \vdots \\ \varphi_1, \Delta_1\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_p]_p \\ \vdots \\ \varphi_p, \Delta_p\end{array}}{\varphi, \Delta_1, \ldots, \Delta_p}\ (R_s^{\bar{i}}) \tag{11}$$

Here the restriction on the equality of lateral assumptions is not imposed, each premise possibly depending on a different collection of lateral assumptions.

Logistic presentation. The objects of $\mathcal{N}$ in its logistic presentation, serving as premises and conclusions of rules, are *sequents* of the above form $\Gamma \vdash_{\mathcal{N}} \Delta$.

In this, those objects resemble more the ones used for the Sequent Calculi presentation of logics. However, the central ingredients of $SCND$-systems, namely the use of I/E-rules and the discharge of assumptions, are preserved.

Again, the presentation splits into an additive and a multiplicative sub cases.

Additive.

$$\frac{\Gamma, \Sigma_1 \vdash_{\mathcal{N}} \varphi_1, \Delta \quad \cdots \quad \Gamma, \Sigma_p \vdash_{\mathcal{N}} \varphi_p, \Delta}{\Gamma \vdash_{\mathcal{N}} \varphi, \Delta}\ (R_l) \tag{12}$$

[13] I thank an anonymous referee who noted that this notation might have an unintended reading when derivations are defined as below.

Here both left context and right context are shared among the premises.

Multiplicative.

$$\frac{\Gamma_1, \Sigma_1 \vdash_{\mathcal{N}} \varphi_1, \Delta_1 \quad \cdots \quad \Gamma_p, \Sigma_p \vdash_{\mathcal{N}} \varphi_p, \Delta_p}{\Gamma_1, \ldots, \Gamma_p \vdash_{\mathcal{N}} \varphi, \Delta_1, \ldots, \Delta_p}\ (R_l) \tag{13}$$

Here both left context and right context may vary with the premises.

The notions of rule-generality are inherited from the $SCND$-setting, and apply both to Γ and to Δ. The classification of the operational rules in Sect. 2.2 remains intact for $MCND$-rules too.

Anticipating the presentation of classical logic in Sect. 5, the following $(\wedge I)$ rule is an example of a logistically-presented additive categorical rule in $MCND$-form.

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ (\wedge I) \tag{14}$$

For an example of a logistically-presented additive hypothetical rule, the I-rule for implication, can be used.

$$\frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ (\to I) \tag{15}$$

A note about (abuse of) notation: Because sequences of formulas may have repetitions of the same formulas, a strict formulation of a rule discharging assumptions has to indicate which instances (if any) of an assumption is discharged. This might be done, for example, by using $\Gamma(\varphi)$ instead of Γ, φ as above. Under this strict notation, the $(\to I)$ rule would appear as follows:

$$\frac{\Gamma(\varphi) \vdash \Delta(\psi)}{\Gamma \vdash \ \Delta(\varphi \to \psi)}\ (\to I)$$

where the parenthetical occurrence is a distinguished one. Since the notation below becomes complicated anyway, I will relax this strictness. Furthermore, the notation will pretend as if the exchange structural rule is in force, and display the principal formulas as peripheral. Since those extra complication are orthogonal to the issue of harmony, no harm should be caused by this abuse of notation.

As for structural rules, they can be applied on *both* sides of '$\vdash$', i.e., both on Γ and on Δ. Note that while both Γ and Δ use commas as formula separators, the meaning of the comma differs according to which context it is a part of.

In [18, p. 141], Hjortland finds an apparent difficulty when explicit assumption recording (as an explicit left context) is desired. Because of a wrong representation of this context, where Γ is just added to nodes in a derivation, that according to him look like Γ, Δ, the comma turns ambiguous. However, by strictly adhering to Gentzen's logistic-$MCSC$ form used in the sequent-calculi [15, p. 150], where the nodes are *sequents* $\Gamma \vdash \Delta$, there is no ambiguity, and, as mentioned above, commas in Γ are interpreted differently than commas in Δ: for systems intermediate between intuitionistic logic and classical logic (though not necessarily for arbitrary $MCND$-systems), the comma in Γ is read conjunctively, while the comma in Δ is read disjunctively. See [3] (and further references therein) for $MCND$-systems for intermediate logics.

3.3. Derivations

I assume also for $MCND$-derivations the usual definition of (tree-shaped) $\mathcal{N}$-*derivations*, ranged over by $\mathcal{D}$, again defined separately for the two presentations of $\mathcal{N}$.

Definition 3.1. (*Derivations*) The presentation splits into the simple and logistic presentations.

Simple derivations. Here a derivation $\overset{\mathcal{D}}{\Delta}$ has an explicit (multiple) conclusion Δ, and *implicit* assumptions on which the conclusion Δ depends, again denoted by $\mathbf{d}_{\mathcal{D}}$, defined in parallel to $\mathcal{D}$.

- Every assumption φ is a derivation $\overset{\mathcal{D}}{\varphi}$, with $\mathbf{d}_{\mathcal{D}} = \{\varphi\}$.

Additive. If $\overset{\mathcal{D}_j}{\varphi_j, \Delta}$, for $1 \leq j \leq p$, are simply presented derivations with dependency sets $\mathbf{d}_{\mathcal{D}_1} = \cdots = \mathbf{d}_{\mathcal{D}_p} = \mathbf{d}$ with $\Sigma_j \subseteq \mathbf{d}$, and if

$$\frac{\begin{matrix}[\Sigma_1]_1 \\ \vdots \\ \varphi_1, \Delta\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_p]_p \\ \vdots \\ \varphi_p, \Delta\end{matrix}}{\varphi, \Delta} \ (R_s^{\bar{i}}) \tag{16}$$

is an instance of (simply presented) additive rule in $\mathcal{R}_{\mathcal{N}}$ with a fresh discharge label, then

$$\overset{\mathcal{D}}{\varphi, \Delta} =^{\text{df.}} \frac{\begin{matrix}[\Sigma_1]_1 \\ \mathcal{D}_1 \\ \varphi_1, \Delta\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_p]_p \\ \mathcal{D}_p \\ \varphi_p, \Delta\end{matrix}}{\varphi, \Delta} \ (R_s^{\bar{i}}) \tag{17}$$

is a derivation with $\mathbf{d}_{\mathcal{D}} = \mathbf{d} - \cup_{1 \leq j \leq p} \hat{\Sigma}_j$
The derivations $\mathcal{D}_1, \ldots, \mathcal{D}_p$ are the *direct sub-derivations* of $\mathcal{D}$.

Multiplicative. If $\overset{\mathcal{D}_j}{\varphi_j, \Delta_j}$, for $1 \leq j \leq p$, are simply presented derivations with dependency sets $\mathbf{d}_{\mathcal{D}_j}$ with $\Sigma_j \subseteq \mathbf{d}_{\mathcal{D}_j}$, and if

$$\frac{\begin{matrix}[\Sigma_1]_1 \\ \vdots \\ \varphi_1, \Delta_1\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_p]_p \\ \vdots \\ \varphi_p, \Delta_p\end{matrix}}{\varphi, \Delta_1, \ldots, \Delta_p} \ (R_s^{\bar{i}}) \tag{18}$$

is an instance of (simply presented) multiplicative rule in $\mathcal{R}_{\mathcal{N}}$ with a fresh discharge label, then

$$\begin{array}{c}\mathcal{D}\\ \varphi,\Delta\end{array} =^{\text{df.}} \dfrac{\begin{array}{c}[\Sigma_1]_1\\ \mathcal{D}_1\\ \varphi_1,\Delta_1\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_p]_p\\ \mathcal{D}_p\\ \varphi_p,\Delta_p\end{array}}{\varphi,\Delta_1,\ldots,\Delta_p}\ (R_s^{\bar{i}}) \tag{19}$$

is a derivation with $\mathbf{d}_{\mathcal{D}} = \cup_{1\leq i\leq p}\mathbf{d}_{\mathcal{D}_i} - \cup_{1\leq j\leq p}\hat{\Sigma}_j$
The derivations $\mathcal{D}_1,\ldots,\mathcal{D}_p$ are the *direct sub-derivations* of $\mathcal{D}$.

Reminder: Recall the convention as to how $[\Sigma]_k$ is to be read. Thus, the derivation $\mathcal{D}$ has $|\Sigma_1|+\cdots+|\Sigma_p|$ leaves (each being just a formula).

Logistic derivations. Here a derivation $\Gamma \overset{\mathcal{D}}{\vdash}_{\mathcal{N}} \Delta$ has explicit assumptions Γ, on which a (multiple) conclusion Δ depends.

- Every instance of an identity sequent $\varphi \vdash_{\mathcal{N}} \varphi$ is a derivation.
- **Additive.** If $\Gamma,\Sigma_1 \overset{\mathcal{D}_1}{\vdash}_{\mathcal{N}} \varphi_1,\Delta, \cdots, \Gamma,\Sigma_p \overset{\mathcal{D}_p}{\vdash}_{\mathcal{N}} \varphi_p,\Delta$, for some $p \geq 1$, are logistically presented derivations, and if

$$\dfrac{\Gamma,\Sigma_1 \vdash_{\mathcal{N}} \varphi_1,\Delta \quad \cdots \quad \Gamma,\Sigma_p \vdash_{\mathcal{N}} \varphi_p,\Delta}{\Gamma \vdash_{\mathcal{N}} \varphi,\Delta}\ (R_l)$$

is an instance of a (logistically presented) additive rule in $\mathcal{R}_{\mathcal{N}}$, then

$$\Gamma \overset{\mathcal{D}}{\vdash}_{\mathcal{N}} \varphi,\Delta =^{\text{df.}} \dfrac{\Gamma,\Sigma_1 \overset{\mathcal{D}_1}{\vdash}_{\mathcal{N}} \varphi_1,\Delta \quad \cdots \quad \Gamma,\Sigma_p \overset{\mathcal{D}_p}{\vdash}_{\mathcal{N}} \varphi_p,\Delta}{\Gamma \vdash_{\mathcal{N}} \varphi,\Delta}\ (R_l)$$

is a logistically presented derivation.

Multiplicative. If $\Gamma_1,\Sigma_1 \overset{\mathcal{D}_1}{\vdash}_{\mathcal{N}} \varphi_1,\Delta_1, \ldots, \Gamma_p,\Sigma_p \overset{\mathcal{D}_p}{\vdash}_{\mathcal{N}} \varphi_p,\Delta_p$, for some $p \geq 1$, are logistically presented derivations, and if

$$\dfrac{\Gamma_1,\Sigma_1 \vdash_{\mathcal{N}} \varphi_1,\Delta_1 \quad \cdots \quad \Gamma_p,\Sigma_p \vdash_{\mathcal{N}} \varphi_p,\Delta_p}{\Gamma_1,\ldots,\Gamma_p \vdash_{\mathcal{N}} \varphi,\Delta_1,\ldots,\Delta_p}\ (R_l)$$

is an instance of a (logistically presented) multiplicative rule in $\mathcal{R}_{\mathcal{N}}$, then

$$\Gamma_1,\ldots,\Gamma_p \overset{\mathcal{D}}{\vdash}_{\mathcal{N}} \varphi,\Delta_1,\ldots,\Delta_p$$
$$=^{\text{df.}} \dfrac{\Gamma_1,\Sigma_1 \overset{\mathcal{D}_1}{\vdash}_{\mathcal{N}} \varphi_1,\Delta_1 \quad \cdots \quad \Gamma_p,\Sigma_p \overset{\mathcal{D}_p}{\vdash}_{\mathcal{N}} \varphi_p,\Delta_p}{\Gamma_1,\ldots,\Gamma_p \vdash_{\mathcal{N}} \varphi,\Delta_1,\ldots,\Delta_p}\ (R_l)$$

is a logistically presented derivation.

The derivations $\mathcal{D}_1,\ldots,\mathcal{D}_p$ are the *direct subderivations* of $\mathcal{D}$.

Note that here too, assumptions are introduced into a derivation via identity axioms and remain as such as long as not discharged by some application of a rule. Note also that the definition adheres to the structural rule-uniformity assumption.

For $MCND$-derivations, there is also an easy mutual conversion between the the two presentation modes:

Simple to logistic. Convert each node in $\overset{\mathcal{D}}{\psi}, \Delta$ to $\Gamma \vdash_{\mathcal{N}} \overset{\mathcal{D}}{\psi}, \Delta$, where $\Gamma = \mathbf{d}_{\mathcal{D}}$.

Logistic to simple. Convert each node in $\Gamma \vdash_{\mathcal{N}} \overset{\mathcal{D}}{\psi}, \Delta$ to $\overset{\mathcal{D}}{\psi}, \Delta$, setting $\mathbf{d}_{\mathcal{D}} = \Gamma$.

Definition 3.2. (*Derivability*) Δ is *derivable* from Γ in $\mathcal{N}$, denoted by $\Gamma \vdash_{\mathcal{N}} \Delta$, iff there exist a simple $\mathcal{N}$-derivation $\overset{\mathcal{D}}{\Delta}$ (respectively, a logistically presented derivation $\Gamma \overset{\mathcal{D}}{\vdash_{\mathcal{N}}} \Delta$) of Δ from Γ in $\mathcal{N}$.

Thus, the logistic presentation keeps track *explicitly* of the assumptions Γ on which Δ depends. Note that $\vdash_{\mathcal{N}} \Delta$ indicates derivability of Δ from an empty context, in which case Δ is referred to[14] as a (*formal*) *theorem* of $\mathcal{N}$.

Derivations in an $MCND$-system are also depicted as trees (of formulas, or of sequents, according to the presentation mode), where a node and its descendants are an instance (of an application of) one of the rules. In simply presented derivations, the discharged occurrences of the assumptions are again enclosed in square brackets, and marked, $[...]_i$, for some (unique) index i, to match i on the applied rule-name, and are leaves in the tree depicting the derivation. Operative rules of an $MCND$-system also operate on the right context of a sequent *only*. I also use $\mathcal{D}: \Gamma \vdash_{\mathcal{N}} \Delta$ to indicate a specific derivation of that sequent. In such a derivation, Γ is also referred to as the *open assumptions* of $\mathcal{D}$. If Γ is empty, the derivation is *closed*.

The property of *closure under composition* is very important also for $MCND$-derivations, again manifesting the *composability* of $MCND$-derivations. While derivation composition in $SCND$-systems take place at the leaves only, for $MCND$-derivations a whole path (defined below) is modified during derivation composition.

Definition 3.3. (Paths) A *path* in a $MCND$-derivation $\mathcal{D}$ is a sequence d_i, $1 \leq i \leq m$ (for some natural number $m \geq 1$) of nodes (which depend on the presentation mode) in $\mathcal{D}$, s.t.:

- d_1 is a leaf, (an assumption formula φ for a simple $\mathcal{D}$, and a sequent $\varphi \vdash_{\mathcal{N}} \varphi$ for a logistically presented $\mathcal{D}$).
- for $1 \leq i < m$, d_i is a premise of some rule application in $\mathcal{D}$, the conclusion of which is d_{i+1}.
- d_m is the conclusion of $\mathcal{D}$.

For a leaf d, let $\Pi_{\mathcal{D}}(d)$ be the path in $\mathcal{D}$ starting at d.

Definition 3.4. (*Closure under derivation composition*) Let $\mathcal{N}$ be an $MCND$-system.

[14] Usually, this notion is only used for $\Delta = \{\varphi\}$, a single conclusion.

Simple derivations. Let $\overset{\mathcal{D}'}{\varphi, \Delta'_1}$ and $\overset{\overset{\varphi}{\mathcal{D}}}{\Delta_1}$ be two simply presented $\mathcal{N}$ derivations. $\mathcal{N}$ is *closed under derivation composition* iff the result of prefixing $\mathcal{D}'$ to the result of adding Δ'_1 to every node in $\Pi_{\mathcal{D}}(\varphi)$ (in $\mathcal{D}$) is a legal $\mathcal{N}$ derivation, denoted by $\mathcal{D}'' = \overset{\mathcal{D}[\Pi_{\mathcal{D}}(\varphi) := \overset{\mathcal{D}'}{\varphi, \Delta'_1}]}{\Delta_1, \Delta'_1}$.

Logistic derivations. Let $\overset{\mathcal{D}'}{\Gamma' \vdash_{\mathcal{N}} \varphi, \Delta'}$ and $\overset{\mathcal{D}}{\Gamma, \varphi \vdash_{\mathcal{N}} \Delta}$ be two logistically presented $\mathcal{N}$-derivation. $\mathcal{N}$ is *closed under derivation composition* if the result of prefixing $\mathcal{D}'$ to the result of adding Γ' to every antecedent and Δ' to every succedent of every node in $\Pi_{\mathcal{D}}(\varphi)$ (in $\mathcal{D}$) is also a derivation, denoted $\mathcal{D}'' = \overset{\mathcal{D}[\Pi_{\mathcal{D}}(\varphi) := \overset{\mathcal{D}'}{\Gamma' \vdash_{\mathcal{N}} \varphi, \Delta'}]}{\Gamma - \varphi, \Gamma' \vdash_{\mathcal{N}} \Delta, \Delta'}$.

Thus, in $\mathcal{D}''$, any use in $\mathcal{D}$ of the assumption φ is replaced by re-deriving φ according to $\mathcal{D}'$ (from possibly additional assumptions, on which the conclusion of $\mathcal{D}''$ (which is the same as the conclusion of $\mathcal{D}$) now depends). This conclusion, however, depends no more on the assumption φ. Note that if the assumption φ is not actually used in $\mathcal{D}$, then $\mathcal{D}[\Pi_{\mathcal{D}}(\varphi) := \overset{\mathcal{D}'}{\varphi, \Delta'}] \equiv \mathcal{D}$.

The definition of derivation substitution is naturally extended to multiple simultaneous derivation substitutions $\mathcal{D}[\Pi_{\mathcal{D}}(\varphi_1) := \overset{\mathcal{D}'_1}{\varphi_1, \Delta'_1}, \ldots, \Pi_{\mathcal{D}}(\varphi_m) := \overset{\mathcal{D}'_m}{\varphi_m, \Delta'_m}]$, for any $m \geq 1$. The satisfaction of the closure property needs to be shown for the specific $MCND$-systems used for meaning conferring.

Closure under derivation composition is a necessary condition for reductions in $MCND$ derivations, and has to be established whenever needed.

Example 3.1. Suppose we have the following two simple $MCND$-derivations $\mathcal{D}$, $\mathcal{D}'$, using some rules that are not further specified.

$$\mathcal{D}: \quad \frac{\dfrac{\varphi, \Delta_1 \quad \Delta_2}{\Delta_3}\,(R_1) \qquad \Delta_4}{\Delta_5}\,(R_2) \qquad\qquad \mathcal{D}': \quad \frac{\dfrac{\Sigma_1 \quad \Sigma_2 \quad \Sigma_3}{\Sigma_4}\,(R_3) \qquad \Sigma_5}{\varphi, \Sigma_6}\,(R_4)$$

In $\mathcal{D}$, $\Pi_{\mathcal{D}}(\varphi)$ consists of the nodes (φ, Δ_1), Δ_3 and Δ_5. The resulting $\mathcal{D}[\Pi_{\mathcal{D}}(\varphi) := \overset{\mathcal{D}'}{\varphi, \Sigma_6}]$ is as follows.

$$\frac{\dfrac{\dfrac{\dfrac{\Sigma_1 \quad \Sigma_2 \quad \Sigma_3}{\Sigma_4}\,(R_3) \qquad \Sigma_5}{\Delta_1, \Sigma_6}\,(R_4) \qquad \Delta_2}{\Delta_3, \Sigma_6}\,(R_1) \qquad \Delta_4}{\Delta_5, \Sigma_6}\,(R_2)$$

The richness of structure in $MCND$ emphasizes even more strongly how structural rules participate in meaning conferring. As observed in [7], the very same operational rule in (15) can give rise to three different meanings of implication, by varying the structural assumptions.

Classical. No structural restriction imposed.

Intuitionistic. Abolishing weakening on the right, on Δ (rendering the system single-conclusion).

Relevant. Abolishing weakening on the left, on Γ (with two variants, classical and intuitionistic) depending whether weakening on the right is retained or abolished.

4. Harmony and Stability in $MCND$-Systems

In view of the move from premises and conclusions as formulas to premises and conclusions as contexts, a reconsideration of local-soundness and local-completeness is due. The impact of the fact that in $MCND$-systems both premises and conclusions consist of (finite) sequences of formulas is, as observed by Hjortland [18, p. 139], that a maximal formula is *disjunctively situated* (D-situated) w.r.t. a right context Δ. This affects the form of the reductions needed to establish local-soundness. The situation is similar for the arbitrary conclusion to be drawn by a harmoniously-induced GE-rule, needing a finer analysis of the dependency on right and left contexts, affecting its construction.

This can be best understood in terms of the following generalization of Prawitz's inversion principle to the $MCND$-environment, to be called the D-inversion principle.

Definition 4.1. A collection of direct grounds $\varphi'_1, \Delta_1, \ldots, \varphi'_m, \Delta_m$ for the introduction of φ, Δ (premises of a suitable I-rule) is D-*exhaustive* iff $\cup_{1 \leq j \leq m} \Delta_j = \Delta$.

The two relevant cases of D-exhaustiveness to emerge below are:

- Every single premise of an additive I-rule.
- The collection of *all* premises of a multiplicative I-rule.

The D-inversion Principle: Every conclusion D-situated w.r.t. a right context Δ drawn from φ, itself D-situated w.r.t. Δ, can already be drawn from any D-exhaustive grounds of introducing φ.

This principle will underly the GE-rules constructed below, allowing the reductions that establish harmony, and the expansions that establish stability.

4.1. $MCND$ Harmoniously-Induced GE-Rules

In this sub-section, I adapt and extend the procedure described in [13] for generating the harmoniously-induced GE-rules (harmonious in form) from given I-rules, to an $MCND$-environment. Thereby, Gentzen's remark quoted above, about "reading off" the E-rules from the I-rules, is extended to cover also $MCND$-systems. Only propositional rules are handled here. Recall that while in $SCND$-systems a conclusion is a formula, for $MCND$-systems, a conclusion is a (finite) sequence of formulas. So, deriving an *arbitrary* conclusion means deriving an *arbitrary such sequence*, to be denoted Δ'. The examples anticipate the *MCND* presentations of classical logic in Sect. 5.

As already mentioned, the key observation, already anticipated by Hjortland [18, p. 139], is that the arbitrary conclusion Δ' inferred by applying a *GE*-rule has to be *disjunctively situated* ($D-situated$) w.r.t. the right context Δ of the major premise. In other words, right contexts are propagated from a D-situated formula to its arbitrary conclusions. The construction is done for both simple and logistic presentations, where the latter reflects possible structural impact by distinguishing multiplicative and additive I-rules. I consider separately the two modes of presentation (simple, logistic), and within each first the special case of categorical I-rules (more easily comprehended), to be followed by hypothetical I-rules. The resulting harmoniously-induced *GE*-rules reflect those distinctions.

4.2. Simple Presentation

Additive I-rule. For additive rules, by definition, since all premises and the conclusion share the same right context Δ, each premise is, on its own, a D-exhaustive ground for introduction of the conclusion.

Categorical I-rule. Suppose the simple additive categorical I-rules of an operator δ, the main operator of φ, can be schematically presented

$$\frac{\varphi_i^1, \Delta \ \cdots \ \varphi_i^{m_i}, \Delta}{\varphi, \Delta}\ (\delta I)_i, \quad 0 \leq i \leq n \tag{20}$$

The premises of (δI_i), each a finite collection of formulas, are denoted $\Delta_i^j = \varphi_i^j, \Delta,\ 1 \leq j \leq m_i$. Note again that φ, as well as all φ_i^j, are D-situated w.r.t. the *same* right context Δ, whence the additivity of the I-rules.

Then, to be harmonious in form, a harmoniously-induced *GE*-rule should draw an arbitrary conclusion Δ', D-situated w.r.t. Δ, from every D-exhaustive ground of introducing φ (a single premise of the I-rule) using that ground as a discharged assumption. Thus, the harmoniously-induced *GE*-rules have the following form.

$$\frac{\varphi, \Delta \quad \begin{matrix}[\varphi_1^{j_1}]_{l_1} \\ \mathcal{D}'_{1,j_1} \\ \Delta'\end{matrix} \quad \cdots \quad \begin{matrix}[\varphi_n^{j_n}]_{l_n} \\ \mathcal{D}'_{n,j_n}, \\ \Delta'\end{matrix}}{\Delta', \Delta}\ (\delta GE^{l_1,\ldots,l_n}_{j_1,\ldots,j_n}), \quad 0 \leq i \leq n,\ 1 \leq j_i \leq m_i \tag{21}$$

This construction gives rise to the following reductions.

$$\frac{\dfrac{\begin{matrix}\hat{\mathcal{D}}_1 \\ \varphi_i^1, \Delta\end{matrix} \ \cdots \ \begin{matrix}\hat{\mathcal{D}}_{m_i} \\ \varphi_i^{m_i}, \Delta\end{matrix}}{\varphi, \Delta}\ (\delta I)_i \quad \begin{matrix}[\varphi_1^{j_1}]_{l_1} \\ \mathcal{D}'_{1,j_1} \\ \Delta'\end{matrix} \quad \cdots \quad \begin{matrix}[\varphi_n^{j_n}]_{l_n} \\ \mathcal{D}'_{n,j_n} \\ \Delta'\end{matrix}}{\Delta', \Delta}\ (\delta GE^{l_1,\ldots,l_n}_{j_1 \cdots j_n}) \quad \rightsquigarrow_r \tag{22}$$

$$\begin{matrix} \hat{\mathcal{D}}_{j_i} \\ \mathcal{D}'_{i,j_i}[\Pi_{\mathcal{D}'_{i,j_i}}(\varphi_i^{j_i}) := \varphi_i^{j_i}, \Delta] \\ \Delta', \Delta \end{matrix}$$

Here, $\varphi_i^{j_i}$ is one of the premises of $(\delta I)_i$, so that the js span all those premises. The total number of harmoniously-induced GE-rules is $\Pi_{1\leq i\leq n} m_i$.

For a simple categorical additive I-rule, this form of a harmoniously-induced GE-rule indeed reflects the D-inversion idea: any arbitrary conclusion Δ' that can be drawn from (the major premise) φ, Δ, can already be drawn (D-situated w.r.t. Δ) from each collection of its grounds of introduction, being singletons in this additive case. Every one of the generated harmoniously-induced GE-rules "prepares itself", so to speak, to "confront" every one of the (δI)-rules (via one of its premises), as reflected by the reductions.

Recall that the availability of such a reduction constitutes part of the definition of intrinsic harmony. Note also that the availability of this reduction rests on the closure under derivation composition. An instance of this rule (for additive conjunction) appears in [32]. In the examples to follow, I allow myself some relaxation of the strict indexing used everywhere whenever no confusion should arise.

Example 4.1. Consider the simply presented additive categorical I-rules for disjunction.

$$\frac{\varphi, \Delta}{\varphi\vee\psi, \Delta}\ (\vee I)_1 \qquad \frac{\psi, \Delta}{\varphi\vee\psi, \Delta}\ (\vee I)_2 \tag{23}$$

Here $n = 2$ (two $(\vee I)$ rules), $m_1 = m_2 = 1$ (one premise for each $(\vee I)$ rule), $\Delta_1^1 = \varphi, \Delta$ and $\Delta_2^1 = \psi, \Delta$. By applying the construction above, the resulting one $(\vee GE)$-rule is the following.

$$\frac{\varphi\vee\psi, \Delta \quad \begin{matrix}[\varphi]_{l_1}\\ \mathcal{D}_1'\\ \Delta'\end{matrix} \quad \begin{matrix}[\psi]_{l_2}\\ \mathcal{D}_2'\\ \Delta'\end{matrix}}{\Delta', \Delta}\ (\vee E^{l_1,l_2}) \tag{24}$$

Example 4.2. Consider the simply presented additive categorical I-rule for conjunction.

$$\frac{\varphi, \Delta \quad \psi, \Delta}{\varphi\wedge\psi, \Delta}\ (\wedge I) \tag{25}$$

Here $n = 1$ (one $(\wedge I)$ rule), $m_1 = 2$ (two premises for this single rule), where $\Delta_1^1 = \varphi, \Delta$ and $\Delta_1^2 = \psi, \Delta$. By applying the construction above, the resulting two $(\wedge GE)$-rules are the following.

$$\frac{\varphi\wedge\psi, \Delta \quad \begin{matrix}[\varphi]_{l_1}\\ \mathcal{D}_1'\\ \Delta'\end{matrix}}{\Delta', \Delta}\ (\wedge GE^{l_1})_1 \qquad \frac{\varphi\wedge\psi, \Delta \quad \begin{matrix}[\psi]_{l_2}\\ \mathcal{D}_2'\\ \Delta'\end{matrix}}{\Delta', \Delta}\ (\wedge GE^{l_2})_2 \tag{26}$$

Continuing examples (4.1) and (4.2), the resulting reductions are as follows.

$$\dfrac{\dfrac{\begin{array}{c}\hat{\mathcal{D}}\\ \varphi,\Delta\end{array}}{\varphi\vee\psi,\Delta}\,(\vee I)_1 \quad \begin{array}{c}[\varphi]_{l_1}\\ \mathcal{D}'_1\\ \Delta'\end{array} \quad \begin{array}{c}[\psi]_{l_2}\\ \mathcal{D}'_2\\ \Delta'\end{array}}{\Delta',\Delta}\,(\vee GE^{l_1,l_2}) \qquad \rightsquigarrow_r \qquad \begin{array}{c}\hat{\mathcal{D}}\\ \mathcal{D}'_1[\Pi_{\mathcal{D}'_1}(\varphi):=\varphi,\Delta]\\ \Delta',\Delta\end{array} \tag{27}$$

(and similarly for $(\vee I)_2$).

$$\dfrac{\dfrac{\begin{array}{c}\hat{\mathcal{D}}_1\\ \varphi,\Delta\end{array} \quad \begin{array}{c}\hat{\mathcal{D}}_2\\ \psi,\Delta\end{array}}{\varphi\wedge\psi,\Delta}\,(\wedge I) \quad \begin{array}{c}[\varphi]_{l_1}\\ \mathcal{D}'_1\\ \Delta'\end{array}}{\Delta',\Delta}\,(\wedge GE^{l_1})_1 \qquad \rightsquigarrow_r \qquad \begin{array}{c}\hat{\mathcal{D}}_1\\ \mathcal{D}'_1[\Pi_{\mathcal{D}'_1}(\varphi):=\varphi,\Delta]\\ \Delta',\Delta\end{array} \tag{28}$$

(and similarly for $(GE\wedge)_2$).

Hypothetical *I*-rule. Suppose the simply presented additive hypothetical I-rules for an operator δ, the main operator in φ, are of the following form.

$$\dfrac{\begin{array}{c}[\Sigma_i^1]_1\\ \mathcal{D}_1\\ \psi_i^1,\Delta\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_i^{m_i}]_{m_i}\\ \mathcal{D}_{m_i}\\ \psi_i^{m_i},\Delta\end{array}}{\varphi,\Delta}\,(\delta I^{1,\ldots,m_i})_i, \qquad 1\leq i\leq n \tag{29}$$

The i'th rule has m_i *possibly* discharging premises, each discharging a collection Σ_i^j of assumptions, the support of the ground. When $m_j=0$ (for some j), the j'th premise discharges no assumptions (a categorical premise). Once again, recall the convention as to how $[\Sigma]_k$ is read. If all premises are categorical, the hypothetical rule reduces to the special case of a categorical rule. These I-rules generate harmoniously-induced GE-rules based on the same D-exhaustive collections of grounds, each GE-rule corresponding to one premise discharging assumptions in the i'th I-rule. Note that the arbitrary conclusion can be drawn (D-situated w.r.t. Δ) from the grounds *provided* the corresponding support has been derived (as D-situated w.r.t. Δ). The total number of harmoniously-induced GE-rules is the same as in the categorical case. Thus, the contribution of hypotheticality in an I-rule is two-folded. Each of the supports becomes a premise (in the corresponding GE-rule), and all the grounds become dischargeable assumptions (in all GE-rules).

The general form of the GE-rule is as follows.

$$\dfrac{\begin{array}{c}\hat{\mathcal{D}}\\ \varphi,\Delta\end{array} \quad \begin{array}{c}\mathcal{D}^*_{1,j_1}\\ \Sigma_1^{j_1}\end{array} \quad \cdots \quad \begin{array}{c}\mathcal{D}^*_{n,j_n}\\ \Sigma_n^{j_n}\end{array} \quad \begin{array}{c}[\psi_1^{j_1}]_{l_1}\\ \mathcal{D}'_{1,j_1}\\ \Delta'\end{array} \quad \cdots \quad \begin{array}{c}[\psi_n^{j_n}]_{l_n}\\ \mathcal{D}'_{n,j_n}\\ \Delta'\end{array}}{\Delta',\Delta}\,(\delta GE^{l_1,\ldots,l_n}_{j_1,\ldots,j_n}) \tag{30}$$

$1\leq i\leq n,\ 1\leq j_i\leq m_i$. If $m_i=0$, there is no derivation of the ith support.

This construction leads to the following reductions, where each (δGE) is again "confronted" against each (δI). Note that the notation $\Pi_{\mathcal{D}}(\varphi)$ is naturally extended to $\Pi_{\mathcal{D}}(\Sigma)$ pointwise.

$$\dfrac{\dfrac{\begin{matrix}[\Sigma_i^1]_1 \\ \mathcal{D}_i^1 \\ \psi_i^1,\Delta\end{matrix} \quad \cdots \quad \begin{matrix}[\Sigma_i^{m_i}]_{m_i} \\ \mathcal{D}_i^{m_i} \\ \psi_i^{m_i},\Delta\end{matrix}}{\varphi,\Delta}\,(\delta I^{1,\ldots,m_i})_i \quad \begin{matrix}\mathcal{D}^*_{1,j_1} \\ \Sigma_1^{j_1}\end{matrix} \quad \cdots \quad \begin{matrix}\mathcal{D}^*_{n,j_n} \\ \Sigma_n^{j_n}\end{matrix} \quad \begin{matrix}[\psi_1^{j_1}]_{l_1} \\ \mathcal{D}'_{1,j_1} \\ \Delta'\end{matrix} \quad \cdots \quad \begin{matrix}[\psi_n^{j_n}]_{l_n} \\ \mathcal{D}'_{n,j_n} \\ \Delta'\end{matrix}}{\Delta',\Delta}\,(\delta GE^{l_1,\ldots,l_n}_{j_1,\ldots,j_n})$$

$$\rightsquigarrow_r \quad D'_{i,j_i}[\Pi_{D'_{i,j_i}}(\psi_i^{j_i}) := \begin{matrix}\mathcal{D}_i^{j_i}[\Pi_{\mathcal{D}_i^{j_i}}(\Sigma_i^{j_i}) := \begin{matrix}\mathcal{D}^*_{i,j_i} \\ \Sigma_i^{j_i}\end{matrix}] \\ \psi_i^{j_i},\Delta \\ \Delta',\Delta\end{matrix}] \tag{31}$$

Note the nestedness of the reduction. The availability of this reduction depends on closure of derivations under derivation composition.

Example 4.3. Consider implication, with the following $(\rightarrow I)$ simply presented hypothetical rule.

$$\dfrac{\begin{matrix}[\varphi]_l \\ \mathcal{D} \\ \psi,\Delta\end{matrix}}{\varphi\rightarrow\psi,\Delta}\,(\rightarrow I^l) \tag{32}$$

Here $n = 1$ (one rule, with no categorical premise), $m_1 = 1$ (one discharging premise), $\Sigma_1^1 = \varphi$. This simple additive hypothetical I-rule gives rise to one GE-rule, as follows.

$$\dfrac{\begin{matrix}\hat{\mathcal{D}} \\ \varphi\rightarrow\psi,\Delta\end{matrix} \quad \begin{matrix}\mathcal{D}^* \\ \varphi\end{matrix} \quad \begin{matrix}[\psi]_l \\ \mathcal{D}'_1 \\ \Delta'\end{matrix}}{\Delta',\Delta}\,(\rightarrow GE^l) \tag{33}$$

Continuing example (4.3), the reduction for implication is the following.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]_1 \\ \mathcal{D} \\ \psi,\Delta\end{matrix}}{\varphi\rightarrow\psi,\Delta}\,(\rightarrow I^1) \quad \begin{matrix}\mathcal{D}^* \\ \varphi\end{matrix} \quad \begin{matrix}[\psi]_l \\ \mathcal{D}'_1 \\ \Delta'\end{matrix}}{\Delta',\Delta}\,(\rightarrow GE^l) \quad \rightsquigarrow_r \quad \mathcal{D}'_1[\Pi_{\mathcal{D}'_1}(\psi) := \begin{matrix}\mathcal{D}[\Pi_{\mathcal{D}}(\varphi) := \begin{matrix}\mathcal{D}^* \\ \varphi\end{matrix}] \\ \psi,\Delta \\ \Delta',\Delta\end{matrix}] \tag{34}$$

Example 4.4. Consider another simply presented hypothetical additive I-rule for '*if... then... else*' (*ite*), where $n = 1$, and the single I-rule has no categorical premise, and has two premises discharging assumptions;

$$\dfrac{\begin{matrix}[\varphi]_{l_1} \\ \mathcal{D}_1 \\ \psi,\Delta\end{matrix} \quad \begin{matrix}[\neg\varphi]_{l_2} \\ \mathcal{D}_2 \\ \chi,\Delta\end{matrix}}{ite(\varphi,\psi,\chi),\Delta}\,(iteI^{l_1,l_2}) \tag{35}$$

thus, the two following GE-rules are harmoniously-induced.

$$\frac{ite(\varphi,\psi,\chi),\Delta \quad \begin{array}{c}\mathcal{D}^*\\ \varphi\end{array} \quad \begin{array}{c}[\psi]_1\\ \mathcal{D}'_1\\ \Delta'\end{array}}{\Delta',\Delta}\ (iteGE^1)_1 \qquad \frac{ite(\varphi,\psi,\chi),\Delta \quad \begin{array}{c}\mathcal{D}^*\\ \neg\varphi\end{array} \quad \begin{array}{c}[\chi]_1\\ \mathcal{D}'_1\\ \Delta'\end{array}}{\Delta',\Delta}\ (iteGE^1)_2 \tag{36}$$

The reductions are as follows

$$\frac{\dfrac{\begin{array}{c}[\varphi]_1\\ \mathcal{D}_1\\ \psi,\Delta\end{array} \quad \begin{array}{c}[\neg\varphi]_2\\ \mathcal{D}_2\\ \chi,\Delta\end{array}}{ite(\phi,\psi,\chi),\Delta}\ (iteI^{1,2}) \quad \begin{array}{c}\mathcal{D}^*\\ \varphi\end{array} \quad \begin{array}{c}[\psi]_3\\ \mathcal{D}'_1\\ \Delta'\end{array}}{\Delta',\Delta}\ (iteGE^3)_1 \ \rightsquigarrow_r \ \begin{array}{c}\mathcal{D}'_1[\Pi_{\mathcal{D}'_1}(\psi) := \begin{array}{c}\mathcal{D}_1[\Pi_{\mathcal{D}_1}(\varphi) := \begin{array}{c}\mathcal{D}^*\\ \varphi\end{array}]\\ \psi,\Delta\end{array}]\\ \Delta',\Delta\end{array} \tag{37}$$

and

$$\frac{\dfrac{\begin{array}{c}[\varphi]_1\\ \mathcal{D}_1\\ \psi,\Delta\end{array} \quad \begin{array}{c}[\neg\varphi]_2\\ \mathcal{D}_2\\ \chi,\Delta\end{array}}{ite(\phi,\psi,\chi),\Delta}\ (iteI^{1,2}) \quad \begin{array}{c}\mathcal{D}^*\\ \neg\varphi\end{array} \quad \begin{array}{c}[\chi]_3\\ \mathcal{D}'_1\\ \Delta'\end{array}}{\Delta',\Delta}\ (iteGE^3)_2 \ \rightsquigarrow_r \ \begin{array}{c}\mathcal{D}'_1[\Pi_{\mathcal{D}'_1}(\psi) := \begin{array}{c}\mathcal{D}_1[\Pi_{\mathcal{D}_1}(\neg\varphi) := \begin{array}{c}\mathcal{D}^*\\ \neg\varphi\end{array}]\\ \psi,\Delta\end{array}]\\ \Delta',\Delta\end{array} \tag{38}$$

Clearly, the harmoniously-induced GE-rule for a categorical additive I-rule is the assumption-less special case, yielding the original formulation.

Multiplicative I-rule. Recall that here only the collection of all premises of an I-rule *together* are D-exhaustive.

Categorical I-rule. The simple multiplicative categorical I-rules of an operator δ, the main operator in φ, can be schematically presented as

$$\frac{\varphi_i^1,\Delta_i^1 \ \cdots \ \varphi_i^{m_i},\Delta_i^{m_i}}{\varphi,\Delta_i^1,\ldots,\Delta_i^{m_i}}\ (\delta I)_i, \quad 1\leq i\leq n \tag{39}$$

Note that φ and all φ_i^j are D-situated w.r.t. possibly *different* right contexts Δ_i^j, whence the multiplicativity of the rule.

As mentioned above, no single premise can form a D-exhaustive ground, only all of the premises taken together. Intuitively, it is undeterminable which subset of $\Delta = \Delta_i^1,\ldots,\Delta_i^{m_i}$ is contributed by each separate premise. Therefore, the harmoniously-induced GE-rule should combine all the grounds for introducing δ (by all of δ's I-rules) and use each of those grounds as (discharged) assumptions for deriving an *arbitrary* conclusion Δ', thus having the following form:

$$\frac{\varphi,\Delta \quad \begin{array}{c}[\varphi_1^1,\ldots,\varphi_1^{m_1}]_{l_1}\\ \mathcal{D}'_1\\ \Delta'\end{array} \quad \cdots \quad \begin{array}{c}[\varphi_n^1,\ldots,\varphi_n^{m_n}]_{l_n}\\ \mathcal{D}'_n\\ \Delta'\end{array}}{\Delta',\Delta}\ (\delta GE^{l_1,\ \ldots,\ l_n}) \tag{40}$$

Note that the decomposition of Δ into $\Delta_i^1, \ldots, \Delta_i^{m_i}$ is unavailable to the GE-rule, but becomes available during a reduction (see below), when the I-rule becomes available too.

For a simple multiplicative categorical rule, this form of the harmoniously-induced GE-rule indeed reflects the D-inversion idea: any arbitrary consequence Δ' that can be drawn (D-situated w.r.t. Δ) from (the major premise, once δI) is determined) $\varphi, \Delta_i^1, \ldots, \Delta_i^{m_i}$, can already be drawn from the D-exhaustive grounds of introduction (all of them!) $[\varphi_i^j]$, $i = 1, \ldots, n$, $j = 1, \ldots, m_i$. Note again that all those assumed grounds are discharged by the rule.

The construction above leads directly to the following reduction:

$$\frac{\dfrac{\begin{matrix}\hat{\mathcal{D}}_1 & & \hat{\mathcal{D}}_{m_i}\\ \varphi_i^1, \Delta_i^1 & \cdots & \varphi_i^{m_i}, \Delta_i^{m_i}\end{matrix}}{\varphi, \Delta_i^1, \ \ldots, \Delta_i^{m_i}}\ (\delta I)_i \quad \begin{matrix}[\varphi_1^1, \ \ldots, \ \varphi_1^{m_1}]_{l_1}\\ \mathcal{D}_1'\\ \Delta'\end{matrix} \quad \cdots \quad \begin{matrix}[\varphi_n^1, \ \ldots, \ \varphi_n^{m_n}]_{l_n}\\ \mathcal{D}_n'\\ \Delta'\end{matrix}}{\Delta', \Delta_i^1, \ \ldots, \ \Delta_i^{m_i}}\ (\delta GE^{l_1, \ \ldots, \ l_n}) \quad \rightsquigarrow_r$$

$$\begin{matrix} & \hat{\mathcal{D}}_1 & & \hat{\mathcal{D}}_{m_i}\\ \mathcal{D}_i'[\Pi_{\mathcal{D}_i'}(\varphi_i^1) := & \varphi_i^1, \Delta_i^1, & \ldots, \ \Pi_{\mathcal{D}_{m_i}'}(\varphi_i^{m_i}) := & \varphi_i^{m_i}, \Delta_i^{m_i}]\\ & \Delta', \Delta_i^1, \ldots, \Delta_i^{m_i} & & \end{matrix} \tag{41}$$

Each such individual substitution adds *a possibly different* $\Delta_i^{m_i}$ to every node in every path $\Pi_{\mathcal{D}_{m_i}'}(\varphi_i^j)$.

Example 4.5. Consider multiplicative conjunction $\otimes$ (known in Linear Logic as the tensor product, and in Relevant Logic as the intensional conjunction), with the following I-rule.

$$\frac{\varphi, \Delta_1 \quad \psi, \Delta_2}{\varphi \otimes \psi, \Delta_1, \Delta_2}\ (\otimes I) \tag{42}$$

Here $n = 1$ (one $(\otimes I)$ rule), $m_1 = 2$ (two premises for this single rule), where $\Delta_1^1 = \varphi, \Delta_1$ and $\Delta_1^2 = \psi, \Delta_2$. By applying the construction above, the resulting (single!) $(\wedge GE)$-rule is the following.

$$\frac{\varphi \otimes \psi, \Delta \qquad \begin{matrix}[\varphi, \psi]_l\\ \mathcal{D}_1'\\ \Delta'\end{matrix}}{\Delta', \Delta}\ (\otimes GE^l)_1 \tag{43}$$

Recall again that $[\varphi, \psi]_l$ represents two formula assumptions simultaneously discharged, not one assumption of a sequence (here of length 2). This is clearly seen by the reduction for multiplicative conjunction, presented below.

$$\frac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \varphi, \Delta_1 & \psi, \Delta_2\end{matrix}}{\varphi \otimes \psi, \Delta_1, \Delta_2}\ (\otimes I) \quad \begin{matrix}[\varphi, \psi]_l\\ \mathcal{D}_1'\\ \Delta'\end{matrix}}{\Delta', \Delta_1, \Delta_2}\ (\otimes GE^l)_1 \quad \rightsquigarrow_r \quad \begin{matrix} & \mathcal{D}_1 & \mathcal{D}_2\\ \mathcal{D}_1'[\Pi_{\mathcal{D}_1'}(\varphi) := & \varphi, \Delta_1, \Pi_{\mathcal{D}_1'}(\psi) := & \psi, \Delta_2]\\ & \Delta', \Delta_1, \Delta_2 & \end{matrix} \tag{44}$$

In comparing the multiplicative conjunction with the additive one in (4.2) in a sub-structural logic lacking contraction and weakening, like Linear Logic, we clearly get

$$\varphi \wedge \psi \vdash \varphi, \qquad \text{but} \quad \varphi \otimes \psi \nvdash \varphi \tag{45}$$

The observation that the additive and multiplicative I-rules for conjunction yield different harmoniously-induced GE-rules (also for conjunction) appears already in [33] in a $SCND$-context, without resorting to additivity/multiplicativity of the right context Δ in general, and without any reference to D-exhaustiveness. I believe the latter provides a clearer insight to the source of the different results of the construction of harmoniously-induced GE-rules.

Hypothetical I-rule. Suppose a simply presented hypothetical multiplicative I-rule is the following:

$$\dfrac{\begin{array}{ccc} [\Sigma_i^1]_1 & & [\Sigma_i^{m_i}]_{m_i} \\ \mathcal{D}_1 & & \mathcal{D}_{m_i} \\ \psi_i^1, \Delta_i^1 & \cdots & \psi_i^{m_i}, \Delta_i^{m_i} \end{array}}{\varphi, \Delta_i^1, \ \ldots, \ \Delta_i^{m_i}} \ (\delta I^{1,\ \ldots,\ m_i})_i \tag{46}$$

The resulting harmoniously-induced simple multiplicative GE-rule is the following.

$$\dfrac{\begin{array}{cccccccc} & & & & [\psi_1^1, \ \ldots, \ \psi_1^{m_1}]_{l_1} & & [\psi_n^1, \ \ldots, \ \psi_n^{m_n}]_{l_n} \\ & \mathcal{D}_1^* & & \mathcal{D}_n^* & \mathcal{D}_1' & & \mathcal{D}_n' \\ \varphi, \Delta & \Sigma_i^1 & \cdots & \Sigma_n^1 & \Delta' & \cdots & \Delta' \end{array}}{\Delta', \Delta} \ (\delta GE^{l_1,\ \ldots,\ l_n}) \tag{47}$$

The resulting reduction is as follows.

$$\begin{array}{l} \dfrac{\dfrac{\begin{array}{ccc} [\Sigma_i^1]_1 & & [\Sigma_i^{m_i}]_{m_i} \\ \mathcal{D}_1 & & \mathcal{D}_{m_i} \\ \psi_i^1, \Delta_i^1 & \cdots & \psi_i^{m_i}, \Delta_i^{m_i} \end{array}}{\varphi, \Delta_i^1, \ldots, \Delta_i^{m_i}} \ (\delta I^{1,\ldots,m_i})_i \quad \begin{array}{cccccc} & & & [\psi_1^1, \ldots, \psi_1^{m_1}]_{l_1} & & [\psi_n^1, \ \ldots, \ \psi_n^{m_n}]_{l_n} \\ \mathcal{D}_1^* & & \mathcal{D}_n^* & \mathcal{D}_1' & & \mathcal{D}_n' \\ \Sigma_i^1 & \cdots & \Sigma_n^1 & \Delta' & \cdots & \Delta' \end{array}}{\Delta', \Delta_i^1, \ldots, \Delta_i^{m_i}} \ (\delta GE^{l_1,\ldots,l_n}) \\ \\ \rightsquigarrow_r \quad D_i'[\Pi_{D_i'}(\psi_i^1) := \begin{array}{c} \mathcal{D}_1^* \\ \mathcal{D}_i[\Pi_{\mathcal{D}_i}(\Sigma_i^1) := \Sigma_i^1] \\ \psi_i^1, \Delta_i^1 \end{array}, \ \ldots, \ \Pi_{D_i'}(\psi_i^{m_i}) := \begin{array}{c} \mathcal{D}_{m_i}^* \\ \mathcal{D}_{m_i}[\Pi_{\mathcal{D}_{m_i}}(\Sigma_i^{m_i}) := \Sigma_i^{m_i}] \\ \psi_i^{m_i}, \Delta_i^{m_i} \end{array}] \\ \qquad\qquad\qquad \Delta', \Delta_i^1, \ \ldots, \ \Delta_i^{m_i} \end{array} \tag{48}$$

Note that while the generation of harmoniously-induced GE-rules are functional, this function is not injective, and different sets of I-rules may lead to the same set of harmoniously-induced GE-rules; however, there is no reason to expect a one-one relationship.

4.3. Logistic Presentation

In this section, I turn to the construction of harmoniously-induced logistically presented GE-rules, incorporating also the left context Γ. In the multiplicative case, the context-uniformity assumption comes into place. I omit the specification of the resulting reductions, that become notationally cumbersome.

Additive I-rule. Once again, the discussion splits into the categorical case and hypothetical case. The subscript $\mathcal{N}$ (on '$\vdash$') is omitted to avoid notational clutter.

Categorical I-rule. Suppose the ith logistically presented additive categorical I-rule for δ has the following form, where δ is the main operator of φ:

$$\frac{\Gamma\vdash\varphi_i^1,\Delta \quad \cdots \quad \Gamma\vdash\varphi_i^{m_i},\Delta}{\Gamma\vdash\varphi,\Delta}\ (\delta I)_i \tag{49}$$

Then, the harmoniously-induced logistically presented GE-rules (again based on each premise forming a D-exhaustive ground) have the following form:

$$\frac{\Gamma\vdash\varphi,\Delta \quad \Gamma,\varphi_1^{j_1}\vdash\Delta' \quad \cdots \quad \Gamma,\varphi_n^{j_n}\vdash\Delta'}{\Gamma\vdash\Delta',\Delta}\ (\delta GE_{j_1,\ldots,j_n}) \tag{50}$$

Hypothetical I-rule. Suppose the ith logistically presented additive hypothetical I-rule for δ has the following form, where δ is the main operator of φ:

$$\frac{\Gamma,\Sigma_i^1\vdash\varphi_i^1,\Delta \quad \cdots \quad \Gamma,\Sigma_i^{m_i}\vdash\varphi_i^{m_i},\Delta}{\Gamma\vdash\varphi,\Delta}\ (\delta I)_i \tag{51}$$

The general form of the harmoniously-induced logistically presented GE-rule is as follows:

$$\frac{\Gamma\vdash\varphi,\Delta \quad \Gamma\vdash\Sigma_1^{j_1} \quad \cdots \quad \Gamma\vdash\Sigma_n^{j_n} \quad \Gamma,\psi_{j_1}^1\vdash\Delta' \quad \cdots \quad \Gamma,\psi_{j_n}^n\vdash\Delta'}{\Gamma\vdash\Delta',\Delta}\ (\delta GE_{j_1,\ldots,j_n}) \tag{52}$$

Multiplicative I-rule. Again, categorical and hypothetical I-rules are separately considered.

Categorical I-rule. Suppose the ith logistically presented multiplicative categorical I-rule for δ has the following form, where δ is the main operator of φ:

$$\frac{\Gamma_i^1\vdash\varphi_i^1,\Delta_i^1 \quad \cdots \quad \Gamma_i^{m_i}\vdash\varphi_i^{m_i},\Delta_i^{m_i}}{\Gamma_i^1,\ldots,\Gamma_i^{m_i}\vdash\varphi,\Delta_i^1,\ldots,\Delta_i^{m_i}}\ (\delta I)_i \tag{53}$$

Then, the harmoniously-induced logistically presented GE-rule has the following form:

$$\frac{\Gamma\vdash\varphi,\Delta \quad \Gamma_1,\varphi_1^1,\ \ldots,\ \varphi_1^{m_1}\vdash\Delta' \quad \cdots \quad \Gamma_n,\varphi_n^1,\ldots,\varphi_n^{m_n}\vdash\Delta'}{\Gamma_1,\ \ldots,\ \Gamma_n\vdash\Delta',\Delta}\ (\delta GE) \tag{54}$$

Hypothetical I-rule. Suppose the ith logistically presented multiplicative hypothetical I-rule for δ has the following form, where δ is the main operator of φ:

$$\frac{\Gamma_i^1,\Sigma_i^1\vdash\varphi_i^1,\Delta_i^1 \quad \cdots \quad \Gamma_i^{m_i},\Sigma_i^{m_i}\vdash\varphi_i^{m_i},\Delta_i^{m_i}}{\Gamma_i^1,\ldots,\Gamma_i^{m_i}\vdash\varphi,\Delta_i^1,\ \ldots,\ \Delta_i^{m_i}}\ (\delta I^{1,\ldots,m_i})_i \tag{55}$$

Then, the general form of the logistically presented harmoniously-induced GE-rule is as follows:

$$\frac{\Gamma\vdash\varphi,\Delta \quad \Gamma_1\vdash\Sigma_i^1 \quad \cdots \quad \Gamma_n\vdash\Sigma_i^n \quad \Gamma_1,\ \psi_1^1,\ \ldots,\ \psi_1^{m_1}\vdash\Delta' \quad \cdots \quad \Gamma_n,\ \psi_n^1,\ \ldots,\ \psi_n^{m_n}\vdash\Delta'}{\Gamma_1,\ \ldots,\ \Gamma_n\vdash\Delta',\Delta}\,(\delta GE) \tag{56}$$

4.4. GE-Harmony Implies Local Intrinsic Harmony

I now reestablish for $MCND$ the relationship between the two notions of harmony considered, and show that GE-harmony (under the extended harmoniously-induced GE-rule construction) is stronger than intrinsic harmony: the form of the GE-rules *guarantees* both local-soundness and local-completeness w.r.t. to the I-rules. I present only the case of simple presentation, logistic presentation being similar but notationally cumbersome.

Theorem. (Harmony implication) *If $\mathcal{N}$ is closed under derivation composition, then for any operator δ, its GE-rules harmoniously-induced by its I-rules are intrinsically-harmonious.*

Proof. Assume the closure of $\mathcal{N}$ under composition and that the GE-rules for δ are harmoniously-induced by its I-rules. We proceed as follows.

Local-soundness. The reductions for a δ-maximal formula were already presented in (22), (41), (31) and (48).

Local-completeness. First, note that the definition needs an adaption to $MCND$: it now requires a derivation of φ,Δ from itself, not just derive φ from itself. Otherwise, the property will not relate $MCND$ I/E-rules as required for stability.

I have to show *some* way to expand a derivation of φ,Δ (with main operator δ). The way to do it differs for the additive and multiplicative cases, as the harmoniously-induced GE-rules generated differ. Only the simple presentation is shown.

Additive rules. Again, the categorical case is separated, for convenience.

Categorical rules. The exact specification of the expansion involves a lot of multiple indices, so instead of presenting it, I describe the idea of its construction, in stages, and then present an example. First, construct the following 1st-layer subderivation $\mathcal{D}_{j_1,\ldots,j_n}^{l_1,\ldots,l_n}$. The arbitrary conclusion Δ' is chosen as φ itself (with an empty Δ).

$$\frac{\varphi,\Delta \quad \dfrac{[\varphi_1^1]_{l_{j_1}} \quad \cdots \quad [\varphi_1^{m_1}]_{l_{j_1}}}{\varphi}\,(\delta I)_1 \quad \ldots \quad \dfrac{[\varphi_n^1]_{l_{j_n}} \quad \cdots \quad [\varphi_n^{m_n}]_{l_{j_n}}}{\varphi}\,(\delta I)_n}{\varphi,\Delta}\,(GE_{j_1,\ldots,j_n}^{l_{j_1},\ldots,l_{j_n}})$$

This 1st-layer subderivation uses one (δGE)-rule, and all the (δI)-rules. Note that it discharges *one* assumption[15] from

[15] This is not reflected in the notation; however, because of its complexity, I preferred not to complicate it more in order to enforce this, and just state it outside the rule itself.

each block of grounds of introduction via each $(\delta I)_i$. In the next layer, combine n 1st-layer subderivations into a 2nd-layer derivation, choosing the (GE)-rules applied (with a major premise as another copy of φ, Δ) so as again to discharge one (yet undischarged) assumption from each block of grounds. The second layer looks like

$$\dfrac{\varphi, \Delta \quad \begin{matrix} \varphi, \Delta \\ \mathcal{D}\vdots \\ \varphi, \Delta \end{matrix} \quad \cdots \quad \begin{matrix} \varphi, \Delta \\ \mathcal{D}\vdots \\ \varphi, \Delta \end{matrix}}{\varphi, \Delta}\ (\delta GE^{\cdots}_{\cdots})$$

At the nth (last layer), one appropriately chosen (GE)-rule combines the results of the previous layer, discharging the remaining assumptions in each block.

Hypothetical rules. Here the idea is similar, but the construction of the 1st-layer requires also establishing the supports $\Sigma_i^{j_i}$ in order to apply a (GE)-rule. The supports are assumed, and discharged by the applications of the (δI)-rules. The form of the 1st layer subderivation becomes the following.

$$\dfrac{\varphi, \Delta \quad \dfrac{[\Sigma^1_{j_1}]_1, [\varphi^1_1]_{l_{j_1}} \quad \cdots \quad [\varphi_1^{m_1}]_{l_{j_1}}}{\varphi}\ (\delta I^1_1) \quad \cdots \quad \dfrac{[\Sigma^1_{j_n}]_n, [\varphi^1_n]_{l_{j_n}} \quad \cdots \quad [\varphi_n^{m_n}]_{l_{j_n}}}{\varphi}\ (\delta I^n_n)}{\varphi, \Delta}\ (GE^{l_{j_1}, \ldots, l_{j_n}}_{j_1, \ldots, j_n})$$

The rest of the layered construction is like for the categorical case.

Multiplicative rules. Here the construction is simpler.

Categorical I-rule. Recall that only one harmoniously-induced GE-rule is generated, having as premises *all* the grounds of introduction, for all the (δI)-rules. The expansion is as follows.

$$\dfrac{\varphi, \Delta \quad \dfrac{[\varphi^1_1, \cdots \varphi^{m_1}_1]_{l_1}}{\varphi}\ (\delta I_1) \quad \cdots \quad \dfrac{[\varphi^1_n, \ldots, \varphi^{m_n}_n]_{l_n}}{\varphi}\ (\delta I_n)}{\varphi, \Delta}\ (\delta GE^{l_1, \ldots, l_n})$$

Hypothetical I-rule. Similar, with the Σ-supports assumed by the GE-rule and discharged by the I-rules. I omit the details.

This ends the proof.

Example 4.6. Returning to conjunction, its expansions are as follows.

Additive. According to the construction, there are two layers only.

$$\begin{matrix} \mathcal{D} \\ \varphi \wedge \psi, \Delta \end{matrix} \quad \rightsquigarrow_e$$

$$\dfrac{\begin{matrix} \mathcal{D} \\ \varphi \wedge \psi, \Delta \end{matrix} \quad \dfrac{\begin{matrix} \mathcal{D} \\ \varphi \wedge \psi, \Delta \end{matrix} \quad \dfrac{[\varphi]_{l_1} \quad [\psi]_{l_2}}{\varphi \wedge \psi}\ (\wedge I)}{\varphi \wedge \psi, \Delta}\ (\wedge GE^{l_1}_1)}{\varphi \wedge \psi, \Delta}\ (\wedge GE^{l_2}_2) \tag{57}$$

Note, however, that there is in this case a simpler expansion.

$$\frac{\dfrac{\overset{\mathcal{D}}{\varphi\wedge\psi,\Delta}\quad[\varphi]_i}{\varphi,\Delta}\,(\wedge GE^i)\qquad \dfrac{\overset{\mathcal{D}}{\varphi\wedge\psi,\Delta}\quad[\psi]_j}{\psi,\Delta}\,(\wedge GE^j)}{\varphi\wedge\psi,\Delta}\,(\wedge I) \tag{58}$$

Multiplicative.

$$\begin{array}{c}\mathcal{D}\\ \varphi\otimes\psi,\Delta\end{array} \quad\rightsquigarrow_e\quad \frac{\overset{\mathcal{D}}{\varphi\otimes\psi,\Delta}\qquad \dfrac{[\varphi,\psi]_i}{\varphi\otimes\psi}\,(\otimes I)}{\varphi\otimes\psi,\Delta}\,(\otimes GE^i) \tag{59}$$

Example 4.7. To see an additive expansion[16] with more layers, consider 'exclusive or' (x). The additive and multiplicative cases are separately considered.

Additive. Let the two simple additive (xI)-rules be the following.

$$\frac{\varphi,\Delta\quad \neg\psi,\Delta}{\varphi x\psi,\Delta}\,(xI)_1 \qquad \frac{\neg\varphi,\Delta\quad \psi,\Delta}{\varphi x\psi,\Delta}\,(xI)_2 \tag{60}$$

Four harmoniously-induced GE-rules are generated.

$$\frac{\varphi x\psi,\Delta\quad \begin{array}{c}[\varphi]_i\\ \mathcal{D}_1\\ \Delta'\end{array}\quad \begin{array}{c}[\psi]_j\\ \mathcal{D}_2\\ \Delta'\end{array}}{\Delta',\Delta}\,(xGE^{i,j})_1 \qquad \frac{\varphi x\psi,\Delta\quad \begin{array}{c}[\varphi]_i\\ \mathcal{D}_1\\ \Delta'\end{array}\quad \begin{array}{c}[\neg\varphi]_j\\ \mathcal{D}_2\\ \Delta'\end{array}}{\Delta',\Delta}\,(xGE^{i,j})_2$$

$$\frac{\varphi x\psi,\Delta\quad \begin{array}{c}[\neg\psi]_i\\ \mathcal{D}_1\\ \Delta'\end{array}\quad \begin{array}{c}[\psi]_j\\ \mathcal{D}_2\\ \Delta'\end{array}}{\Delta',\Delta}\,(xGE^{i,j})_3 \qquad \frac{\varphi x\psi,\Delta\quad \begin{array}{c}[\neg\psi]_i\\ \mathcal{D}_1\\ \Delta'\end{array}\quad \begin{array}{c}[\neg\varphi]_j\\ \mathcal{D}_2\\ \Delta'\end{array}}{\Delta',\Delta}\,(xGE^{i,j})_4 \tag{61}$$

An expansion is as follows. For typographical reasons, it is displayed in parts, separating the layers. First, there are four 1st-level derivations. All weakenings are omitted.

$$\mathcal{D}_{1,1}:\ \frac{\varphi x\psi,\Delta\qquad \dfrac{[\varphi]_1\quad[\neg\psi]_2}{\varphi x\psi}\,(xI)_1\qquad \dfrac{\neg[\varphi]_3\quad[\psi]_4}{\varphi x\psi}\,(xI)_2}{\varphi x\psi,\Delta}\,(xGE^{1,4})_1$$

$$\mathcal{D}_{1,2}:\ \frac{\varphi x\psi,\Delta\qquad \dfrac{[\varphi]_5\quad[\neg\psi]_6}{\varphi x\psi}\,(xI)_1\qquad \dfrac{[\neg\varphi]_7\quad[\psi]_2}{\varphi x\psi}\,(xI)_2}{\varphi x\psi,\Delta}\,(xGE^{5,7})_2$$

$$\mathcal{D}_{1,3}:\ \frac{\varphi x\psi,\Delta\qquad \dfrac{[\varphi]_8\quad[\neg\psi]_9}{\varphi x\psi}\,(xI)_1\qquad \dfrac{\neg[\varphi]_{10}\quad[\psi]_{11}}{\varphi x\psi}\,(xI)_2}{\varphi x\psi,\Delta}\,(xGE^{8,11})_1$$

[16] The expansion for the additive case, in the single-conclusion case, was suggested to me by Stephen Read.

$$\mathcal{D}_{1,4}: \quad \dfrac{\varphi x\psi, \Delta \quad \dfrac{[\varphi]_{12}, \Delta \quad [\neg\psi]_{13}, \Delta}{\varphi x\psi, \Delta}(xI)_1 \quad \dfrac{\neg[\varphi]_{10}, \Delta \quad [\psi]_{14}, \Delta}{\varphi x\psi, \Delta}(xI)_2}{\varphi x\psi, \Delta}(xGE^{13,14})_3$$

At the 2nd layer, there are two derivations, each one combining two 1st layer ones.

$$\mathcal{D}_{2,1}: \quad \dfrac{\varphi x\psi, \Delta \quad \begin{matrix}\mathcal{D}_{1,1}\\ \varphi x\psi, \Delta\end{matrix} \quad \begin{matrix}\mathcal{D}_{1,2}\\ \varphi x\psi, \Delta\end{matrix}}{\varphi x\psi, \Delta}(xGE^{2,6})_3$$

$$\mathcal{D}_{2,2}: \quad \dfrac{\varphi x\psi, \Delta \quad \begin{matrix}\mathcal{D}_{1,3}\\ \varphi x\psi, \Delta\end{matrix} \quad \begin{matrix}\mathcal{D}_{1,4}\\ \varphi x\psi, \Delta\end{matrix}}{\varphi x\psi, \Delta}(xGE^{10,13})_1$$

Finally, at the 3rd layer, there is one derivation, combining the two 2nd layer derivation, completing the discharge of the yet undischarged assumptions.

$$\mathcal{D}_{3,1}: \quad \dfrac{\varphi x\psi, \Delta \quad \begin{matrix}\mathcal{D}_{2,1}\\ \varphi x\psi, \Delta\end{matrix} \quad \begin{matrix}\mathcal{D}_{2,2}\\ \varphi x\psi, \Delta\end{matrix}}{\varphi x\psi, \Delta}(xGE^{3,9})_4$$

Assumptions discharge: There are seven applications of GE-rules in the full derivations, and sixteen assumptions to discharge, where two pairs of assumptions are equi-labelled, discharged together.

- The applications of GE-rules in 1st level subderivations discharge two (different) assumptions each.
- The applications of GE-rules in 2nd level subderivations discharge three assumptions each: xGE_3 one ψ and two $\neg\psi$s, and xGE_2 one φ and two $\neg\varphi$s.
- The application of the GE-rule on (the one) third level subderivation (ending the whole derivation) discharges two (different) assumptions.

Multiplicative. Here, only one harmoniously-induced GE-rule is generated.

$$\dfrac{\varphi x\psi, \Delta \quad \begin{matrix}[\varphi, \neg\psi]_i\\ \mathcal{D}_1\\ \Delta'\end{matrix} \quad \begin{matrix}[\neg\varphi, \psi]_j\\ \mathcal{D}_2\\ \Delta'\end{matrix}}{\Delta', \Delta}(xGE^{i,j}) \tag{62}$$

The expansion establishing local-completeness is as follows.

$$\begin{matrix}\mathcal{D}\\ \varphi x\psi, \Delta\end{matrix} \quad \rightsquigarrow_e \quad \dfrac{\begin{matrix}\mathcal{D}\\ \varphi x\psi, \Delta\end{matrix} \quad \dfrac{[\varphi, \neg\psi]_i}{\varphi x\psi}(xI)_1 \quad \dfrac{[\neg\varphi, \psi]_j}{\varphi x\psi}(xI)_2}{\varphi x\psi, \Delta}(xGE^{i,j})$$

$$\frac{\varphi,\Delta \quad \psi,\Delta}{\varphi\wedge\psi,\Delta}\ (\wedge I) \qquad \frac{\varphi\wedge\psi,\Delta}{\varphi,\Delta}\ (\wedge_1 E) \qquad \frac{\varphi\wedge\psi,\Delta}{\psi,\Delta}\ (\wedge_2 E)$$

$$\frac{\varphi,\Delta}{\varphi\vee\psi,\Delta}\ (\vee_1 I) \qquad \frac{\psi,\Delta}{\varphi\vee\psi,\Delta}\ (\vee_2 I) \qquad \frac{\varphi\vee\psi,\Delta}{\varphi,\psi,\Delta}\ (\vee E)$$

$$\frac{\begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \psi,\Delta\end{array}}{\varphi\to\psi,\Delta}\ (\to I^i) \qquad \frac{\varphi\to\psi,\Delta \quad \varphi,\Delta}{\psi,\Delta}\ (\to E)$$

$$\frac{\begin{array}{c}[\varphi]_i\\ \mathcal{D}\\ \Delta\end{array}}{\neg\varphi,\Delta}\ (\neg I^i) \qquad \frac{\neg\varphi,\Delta \quad \varphi,\Delta}{\Delta}\ (\neg E)$$

FIGURE 1. The simple MCND-presentation NC of propositional classical logic

Some more expansions

For implication, the resulting expansion is

$$\begin{array}{c}\mathcal{D}\\ \varphi\to\psi,\Delta\end{array} \quad \rightsquigarrow_e \quad \cfrac{\cfrac{\begin{array}{c}\mathcal{D}\\ \varphi\to\psi,\Delta\end{array} \quad [\varphi]_1 \quad [\psi]_2}{\psi,\Delta}\ (\to GE^2)}{\varphi\to\psi,\Delta}\ (\to I^1) \tag{63}$$

Finally, consider the expansion for (ite), where $n = 1$, the I-rules of which are given in (35), and GE-rules in (36).

$$\cfrac{\cfrac{ite(\varphi,\psi,\chi),\Delta \quad [\varphi]_1 \quad [\psi]_2}{\psi,\Delta}\ (iteGE_1^2) \quad \cfrac{ite(\varphi,\psi,\chi),\Delta \quad [\neg\varphi]_3 \quad [\chi]_4}{\chi,\Delta}\ (iteGE_2^4)}{ite(\varphi,\psi,\chi),\Delta}\ (iteI^{1,3}) \tag{64}$$

5. A Multiple-Conclusion Presentation of Classical Logic

Following [2,31], the operational rules for the connectives for a simple $MCND$-presentation NC of classical logic is in Fig. 1. In addition, the previous structural rules are assumed too. Since in their presence the multiplicative and additive rules are equivalent, only the latter[17] are presented. I restrict the discussion to the propositional fragment, which suffices to make the point of proof-theoretical justification.

[17] Boričić (in [2]) and Cellucci (in [5]) present the former.

Remarks.

- All the rules are both pure and simple.
- In [5], Cellucci presents the following single $(\vee I)$ rule, that can be shown equivalent to the two rules of Boričić [2].

$$\frac{\varphi, \psi, \Delta}{\varphi\vee\psi, \Delta}\ (\vee I) \tag{65}$$

- Note that by substituting in all the rules of $MCND$ an empty Δ (except for $(\neg I)$), familiar single-conclusion $SCND$-rules are obtained.
- (LEM) is derivable as follows (contexts omitted).

$$\dfrac{\dfrac{\dfrac{\dfrac{[\varphi]_1}{\varphi\vee\neg\varphi}\ (\vee_1 I)}{\varphi\vee\neg\varphi, \neg\varphi}\ (\neg I^1)}{\varphi\vee\neg\varphi, \varphi\vee\neg\varphi}\ (\vee_2 I)}{\varphi\vee\neg\varphi}\ (C) \tag{66}$$

- As observed in [31] (sub-section 3.3), the definition of $\neg\varphi$ as $\varphi\to\bot$ is not peculiar to intuitionistic logic, as has been generally believed. Whether this implication "is intuitionistic" or "is classical" depends on the rules for '$\to$'. The implication is classical here, yielding classical negation with the following E-rule for $\bot$.

$$\frac{\bot, \Delta}{\varphi, \Delta}\ (\bot E) \tag{67}$$

- In continuation to the previous comment, it was also observed in [31, section 3.3] that the source of the non-conservativity of NK (Gentzen's classical ND system) is that the theory does not specify completely the meaning of the classical implication; the current NC does! For example, below is an NC-derivation of Peirce's rule within the *positive* '$\to$'-fragment, without any appeal to negation.

$$\dfrac{\dfrac{\dfrac{[(\varphi\to\psi)\to\varphi]_2 \qquad \dfrac{\dfrac{[\varphi]_1}{\varphi, \psi}\ (W)}{\varphi, \varphi\to\psi}\ (\to I^1)}{\varphi, \varphi}\ (\to E)}{\varphi}\ (C)}{((\varphi\to\psi)\to\varphi)\to\varphi}\ (\to I^2) \tag{68}$$

- The Double-negation elimination rule (DNE) is derivable too, as shown below.

$$\dfrac{\dfrac{\dfrac{\neg\neg\varphi \qquad \dfrac{\dfrac{[\varphi]_1}{\bot, \varphi}\ (W)}{\neg\varphi, \varphi}\ (\to I^1)}{\bot, \varphi}\ (\to E)}{\varphi, \varphi}\ (\bot E)}{\varphi}\ (C) \tag{69}$$

Theorem 5.1. (Closure of NC under derivation composition) *NC is closed under derivation composition.*

The proof is essentially the same as in the single conclusion case for intuitionistic logic, by induction on the derivation.

By applying the construction above, the following additive harmoniously-induced GE-rules are obtained (Fig. 2).

The shared contexts (additive) formulation of NC with explicit left contexts is presented in Fig. 3. Eliminations are via GE-rules. Formulating a multiplicative presentation is straightforward and omitted, as the two are equivalent in the presence of the standard structural rules.

Remarks.

- Since the object language of CL contains both '∧', '∨' and '→', it is possible to regard sequents $\Gamma\vdash\Delta$ here as expressing $\wedge_\Gamma\to\vee_\Delta$ (not necessarily possible for arbitrary object languages), with $\wedge_\emptyset = \bot$ and $\vee_\emptyset = \top$.
- As an example of a derivation from assumptions, below is a shared-context derivation for $\neg(\varphi\wedge\psi)\vdash_{NC}\neg\varphi\vee\neg\psi$, which is not acceptable in intuitionistic logic.

$$\frac{\varphi\wedge\psi,\Delta \qquad \begin{matrix}[\varphi]_i,[\psi]_j\\ \mathcal{D}_2\\ \chi\end{matrix}}{\chi,\Delta}\ (\wedge GE^{i,j})$$

$$\frac{\varphi\vee\psi,\Delta \qquad \begin{matrix}[\varphi]_i\\ \mathcal{D}_1\\ \chi\end{matrix} \qquad \begin{matrix}[\psi]_j\\ \mathcal{D}_2\\ \chi\end{matrix}}{\chi,\Delta}\ (\vee GE^{i,j})$$

$$\frac{\varphi\to\psi,\Delta \qquad \varphi \qquad \begin{matrix}[\psi]_i\\ \mathcal{D}\\ \chi\end{matrix}}{\chi,\Delta}\ (\to GE^{i})$$

Figure 2. The harmoniously-induced GE-rules for NC

$$\frac{}{\varphi\vdash\varphi}\ (Ax)$$

$$\frac{\Gamma\vdash\varphi,\Delta \quad \Gamma\vdash\psi,\Delta}{\Gamma\vdash\varphi\wedge\psi,\Delta}\ (\wedge I) \qquad \frac{\Gamma\vdash\varphi\wedge\psi,\Delta \quad \Gamma,\varphi,\psi\vdash\chi,\Delta}{\Gamma\vdash\chi,\Delta}\ (\wedge E)$$

$$\frac{\Gamma\vdash\varphi,\Delta}{\Gamma\vdash\varphi\vee\psi,\Delta}\ (\vee_1 I) \qquad \frac{\Gamma\vdash\psi,\Delta}{\Gamma\vdash\varphi\vee\psi,\Delta}\ (\vee_2 I) \qquad \frac{\Gamma\vdash\varphi\vee\psi,\Delta \quad \Gamma,\varphi\vdash\chi,\Delta \quad \Gamma,\psi\vdash\chi,\Delta}{\Gamma\vdash\chi,\Delta}\ (\vee E)$$

$$\frac{\Gamma,[\varphi]_i\vdash\psi,\Delta}{\Gamma\vdash\varphi\to\psi,\Delta}\ (\to I^i) \qquad \frac{\Gamma\vdash\varphi\to\psi,\Delta \quad \Gamma\vdash\varphi,\Delta \quad \Gamma,\psi\vdash\chi,\Delta}{\Gamma\vdash\chi,\Delta}\ (\to E)$$

$$\frac{\Gamma,\varphi\vdash\Delta}{\Gamma\vdash\neg\varphi,\Delta}\ (\neg I) \qquad \frac{\Gamma\vdash\neg\varphi,\Delta \quad \Gamma\vdash\varphi,\Delta}{\Gamma\vdash\Delta}\ (\neg E)$$

Figure 3. The additive logistically presentation of NC

$$\dfrac{\neg(\varphi\wedge\psi)\vdash_{NC}\neg(\varphi\wedge\psi) \quad \dfrac{\dfrac{}{\varphi\vdash_{NC}\varphi}\ (Ax) \quad \dfrac{}{\psi\vdash_{NC}\psi}\ (Ax)}{\varphi,\psi\vdash_{NC}\varphi\wedge\psi}\ (\wedge I)}{\dfrac{\neg(\varphi\wedge\psi),\varphi,\psi\vdash_{NC}}{\dfrac{\neg(\varphi\wedge\psi)\vdash_{NC}\neg\varphi,\neg\psi}{\dfrac{\neg(\varphi\wedge\psi)\vdash_{NC}\neg\varphi\vee\neg\psi,\neg\psi}{\dfrac{\neg(\varphi\wedge\psi)\vdash_{NC}\neg\varphi\vee\neg\psi,\neg\varphi\vee\neg\psi}{\neg(\varphi\wedge\psi)\vdash_{NC}\neg\varphi\vee\neg\psi}\ (C)}\ (\vee_2 I)}\ (\vee_1 I)}\ (\neg I\ \times\ 2)}\ (\neg E) \quad (70)$$

- NC restores the *symmetry* characteristic of classical logic. For example, parallel to $\varphi,\psi\vdash_{NC}\varphi\wedge\psi$, we also have $\varphi\vee\psi\vdash_{NC}\varphi,\psi$, in contrast to NK, in which no direct immediate conclusion can be drawn from a disjunction.

5.1. Reevaluating Classical Logic for Meeting the Meaning Conferring Criteria

First, as noted above, the addition of negation to the positive fragment *is* conservative.

GE-harmony obtains, by the form of the E-rules. The adaptation of the construction in [13] shown above can "read off" the GE-rules from the I-rules. Note that, pace Hjortland [18, p. 139], $(\neg E)$ *is* the GE-rule harmonically-induced by $(\neg I)$. This can be realized the easiest by comparison to the '$\rightarrow$'-rules. In '$(\rightarrow I)$', there are both a support (namely, φ), and a ground (categorical component, ψ). The former produces the premise $\Gamma\vdash\varphi$ (of '$(\rightarrow E)$'), and the latter the premise $\Gamma,\psi\vdash\chi,\Delta$. However, '$(\neg I)$' has only a support, again φ, producing the premise $\Gamma\vdash\varphi$ for '$(\neg E)$' too, but *no explicit ground*, hence no premise with an explicit arbitrary conclusion.

As for intrinsic harmony and stability, the reductions (which are not presented neither in [2] nor in [31], but are presented in [5]) and expansions (not considered at all in [2,5] and [31]) are shown below.

Theorem 5.2. (Local-soundness of NC) *NC is locally-sound.*

Proof. Bellow are the detour-eliminating reductions, with left contexts omitted. They are shown using the regular E-rules, that are derivable in NC from the GE-rules, and render the reductions easier to understand.

Reducing conjunction.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \varphi,\Delta\end{matrix} \quad \begin{matrix}\mathcal{D}_2\\ \psi,\Delta\end{matrix}}{\varphi\wedge\psi,\Delta}\ (\wedge I)}{\varphi,\Delta}\ (\wedge E_1) \quad \rightsquigarrow_r \quad \begin{matrix}\mathcal{D}_1\\ \varphi,\Delta\end{matrix} \quad (71)$$

The case of $(\wedge\hat{E}_2)$ is similar.

Reducing implication.

$$\dfrac{\dfrac{\begin{matrix}[\varphi]_i\\ \mathcal{D}_1\\ \psi,\Delta\end{matrix}}{(\varphi\rightarrow\psi),\Delta}\ (\rightarrow I^i) \quad \begin{matrix}\mathcal{D}_2\\ \varphi\end{matrix}}{\psi,\Delta}\ (\rightarrow E) \quad \rightsquigarrow_r \quad \begin{matrix}\mathcal{D}_1[\Pi_{\mathcal{D}_1}(\varphi) := \begin{matrix}\mathcal{D}_2\\ \varphi\end{matrix}]\\ \psi,\Delta\end{matrix} \quad (72)$$

Note the dependence on the closure of NC under derivation substitution, similarly to the single-conclusion case for intuitionistic logic.

Reducing negation.

$$\cfrac{\cfrac{\begin{matrix}[\varphi]_i\\ \mathcal{D}_1\\ \Delta\end{matrix}}{\neg\varphi, \Delta}\,(\neg I^i) \quad \begin{matrix}\mathcal{D}_2\\ \varphi, \Delta\end{matrix}}{\Delta}\,(\neg E) \quad \rightsquigarrow_r \quad \begin{matrix}\mathcal{D}_2\\ \mathcal{D}_1[\Pi_{\mathcal{D}_1}(\varphi) := \varphi, \Delta]\\ \Delta\end{matrix} \tag{73}$$

There is an implicit use of contraction in this derivation, as the substitution produces Δ, Δ in the r.h.s. of the conclusion.

Theorem 5.3. (Local-completeness of NC) *NC is locally-complete.*

Proof. Below are the required expansions.

Expanding conjunction.

$$\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi, \Delta\end{matrix} \quad \rightsquigarrow^e \quad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi, \Delta\end{matrix}}{\varphi, \Delta}\,(\wedge_1 E) \quad \cfrac{\begin{matrix}\mathcal{D}\\ \varphi\wedge\psi, \Delta\end{matrix}}{\psi, \Delta}\,(\wedge_2 E)}{\varphi\wedge\psi, \Delta}\,(\wedge I) \tag{74}$$

Expanding implication.

$$\begin{matrix}\mathcal{D}\\ (\varphi\rightarrow\psi), \Delta\end{matrix} \quad \rightsquigarrow^e \quad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ (\varphi\rightarrow\psi), \Delta\end{matrix} \quad [\varphi]_i, \Delta}{\psi, \Delta}\,(\rightarrow E)}{(\varphi\rightarrow\psi), \Delta}\,(\rightarrow I^i) \tag{75}$$

Expanding negation.

$$\begin{matrix}\mathcal{D}\\ \neg\varphi, \Delta\end{matrix} \quad \rightsquigarrow^e \quad \cfrac{\cfrac{\begin{matrix}\mathcal{D}\\ \neg\varphi, \Delta\end{matrix} \quad [\varphi]_i, \Delta}{\Delta}\,(\neg E)}{\neg\varphi, \Delta}\,(\neg I^i) \tag{76}$$

5.2. Criticizing Multiple-Conclusion Natural-Deduction as a Way to Confer Meaning

While the appeal to multiple-conclusion natural-deduction restores harmony for classical logic, it raises the issue of adhering to the principle of answerability [39]—does such a system of rules conform to our deductive inferential practice? In what sense does it reflect use? After all, harmony per se is a goal of PTS only as much as it serves as a qualification criterion for being meaning conferring.

A detailed negative answer to this question is presented in [39], where the final conclusion is that conclusions should remain single in order for the ND-system to be used to confer meaning. While in [34] Restall presents an argument in favor of multiple conclusions, it is based on one example—a certain way of proving by cases—claimed by Steinberger to be better handled by single-conclusion ND, appealing to disjunction elimination. A similar criticism is expressed by Dummett [8, p. 187], as he inherently connects the conclusions set Δ with the disjunction of its elements, claiming that $MCND$ *enforces* the learning of the meaning of '$\vee$'. Yet another similar opposition is expressed by Tennant [40, p. 320].

However, such a criticism is based on a rather narrow view of what constitutes our inferential practices. Under a broader view, a positive answer to the above question might still be available. One attempt (admitted by the author to consist in a first approximation) is presented in [36]. It views a multiple conclusion sequent (in a slightly modified notation), for an arbitrary object language,

$$\Gamma = \{\gamma_1, \ldots, \gamma_m\} \vdash \{\delta_1, \ldots, \delta_n\} = \Delta \tag{77}$$

as a *meta-rule*

$$\frac{\Theta \vdash \gamma_1 \quad \cdots \quad \Theta \vdash \gamma_m \quad \Theta, \delta_1 \vdash \varphi \quad \cdots \quad \Theta, \delta_n \vdash \varphi}{\Theta \vdash \varphi} \tag{78}$$

For arbitrary Θ, φ (and for technical reasons, $n \geq 1$).

If one views a single-conclusion sequent $\Gamma \vdash \varphi$ as a rationality requirement of a doxastic agent to accept φ whenever (s)he accepts every member of Γ, then the above rule expresses a more general rationality requirement of the agent. In case disjunction is present in the object language, as it is for classical logic, then the meta-rule is in line with the view of Δ as the disjunction of its elements, and with the GE-rule for disjunction. For a detailed presentation of this view, with some technical results, the reader is referred to [36]. Another way of rebutting the offence of $MCND$ is presented by Hjortland (in [18, p. 89], relating it to *Bilateralism* [35].

While I find both of the arguments above as a convincing advance, I consider this issue, of relating multiple-conclusions natural-deduction to actual inferential practices, as still open, awaiting a fully satisfactory solution, which, I conjecture, will be found in due time.

6. Conclusions

The purpose of this paper was to study the notions of harmony and stability, central to the proof-theoretic semantics programme, in the context of multiple-consequence natural-deduction proof systems, not necessarily as a justification of classical logic.

The main result is an adaptation and extension to multiple-conclusions natural-deduction systems of the procedure of constructing harmoniously-induced general-elimination rules from given introduction rules, originally presented for single-conclusion natural-deduction system in [13]. In doing so, special attention was paid to the effect on this construction of the distinction between additive and multiplicative rule (in spite of their equivalence in classical logic). For a general discussion of the role of sub-structurality and emphasis on its role in the study of harmony see [19]. It is hoped that this study enhances our understanding of harmony and stability in general, and adds another voice to the claim that choosing among "rival" logics, if to be done at all, should not be based on their proof-theoretic justification (or an alleged lack thereof).

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