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Symmetric Generalized Galois Logics

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Abstract. Symmetric generalized Galois logics (i.e., symmetric **gGl**s) are distributive **gGl**s that include weak distributivity laws between some operations such as fusion and fission. Motivations for considering distribution between such operations include the provability of cut for binary consequence relations, abstract algebraic considerations and modeling linguistic phenomena in categorial grammars. We *represent* symmetric **gGl**s by models on topological relational structures. On the other hand, topological relational structures are *realized* by structures of symmetric **gGl**s. We generalize the weak distributivity laws between fusion and fission to interactions of certain monotone operations within distributive *super* **gGl**s. We are able to prove appropriate generalizations of the previously obtained theorems—including *a functorial duality* result connecting classes of **gGl**s and classes of structures for them.

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0. Introduction

There are various ways to *combine logics*—for instance, various connectives, axioms and rules may be placed side by side into one logic. Symmetric **gGl**s may be viewed as a combination of two algebras each containing a monotone binary operation and its residuals on a semi-lattice. The semi-lattices are joined into a lattice with distributivity added, and similarly, certain distributivity is added to link the non-lattice operations. (We do not intend to provide some overall fabric for combining logics in general, however, cf. Carnielli and Coniglio [8] concerning pasting systems together.)

Alternatively, symmetric **gGl**s may be viewed as an example of adding to a distributive **gGl** a new group of operations comprising a monotone operation and its two residuals—none of which is definable in the **gGl** itself. Sometimes, the motivation to enrich the language of a logic with *non-interdefinable operations* is an interest in their interaction, whereas some other times, the intention is to increase the expressive power of the logic. Well-known examples of connectives that are not interdefinable include $\Diamond$ and $\Box$ from positive modal logics and relevant implication (or entailment) and negation from relevance logics.

Logical connectives may interact in different ways, and the interactions are usually more transparent in the models of the logic that in its proof system. *Gaggle theory*—another name for the framework of generalized Galois logics—makes precise the idea that *abstract residuation* allows connectives to be grouped together into families, although residuation itself is not sufficient to ensure the interdefinability of connectives.[1]

Another sort of interplay between connectives is captured by *distribution like* axioms. The most common axiom of distributivity is that between conjunction and disjunction (e.g., in intuitionistic logic). The effect of this axiom is that the Lindenbaum algebra of the logic is not only a lattice (perhaps, with some further operations), but it is a lattice in which meet and join distribute over each other. Of course, distribution like axioms may involve other connectives. It is standard in relevance logic to consider "fusion" as a connective that generalizes conjunction, and "fission" as a connective that generalizes disjunction. In the linear logic tradition this corresponds to "multiplicative" conjunction and disjunction. Formulas that express "some distribution" between connectives from the fusion and the fission families have been considered previously by Dunn and Hardegree [15, Sect. 6.9], Grishin [18] and Moortgat [22]—for very different reasons though. (The groups of operations that are called in the terminology of gaggle theory "fusion family" and "fission family" are introduced formally in Sect. 1. Note, in particular, the residuation clauses (re1) and (re2).)

Dunn and Hardegree showed that so-called "hemi-distributivity" principles between fusion and fission are necessary for a binary consequence relation to have the algebraic cut property. (In the case of the binary consequence relation, the premises and the conclusions are joined into formulas by two binary operations that are like fusion and fission, respectively.) *Grishin* arrived at distribution like inequations based on abstract algebraic features of the operations of a bi-Lambek algebra. Lastly, natural language phenomena prompted *Moortgat* and his colleagues to amend basic categorial grammar by adding rules that provide for distribution like interactions between fusion and the residuals of fission. As it turns out, intertwining fusion and fission (as well as their residuals) in a distributivity kind of pattern is neither trivial nor uninteresting from the point of view of the *semantics* of the resulting logics.

In this paper, we investigate distribution like interactions between pairs of operations in an increasingly abstract setting. We start with symmetric gaggles, and by the last section, we will have arrived at *super gaggles*, where those properties of the operations that are not relevant to our main results have been

[1] Gaggle theory was invented by Dunn [12]; see also Dunn [13] and Dunn [14]. A fairly comprehensive development of gaggle theory (save symmetric and super gaggles) is Bimbó and Dunn [7]. We follow the notation and the terminology used in our book.

discarded. Thereby, super gaggles provide a *universal logic* type framework for investigations of distribution like interactions.

First, we briefly overview the inequations that were introduced previously, together with various relationships between them. Then we define *relational semantics* for all the symmetric fusion–fission gaggles that are obtained by adding some of the distribution like inequations. Further, we prove *topological duality theorems* for them. In the last section, we give an inequation comprising two terms that has all the previous inequations as its special instances. This inequation captures a distribution like interaction between two operations that are monotone in some of their argument places (and possess some other properties too). We define relational semantics for these super gaggles too and prove duality theorems between the *category of super gaggles* and the *category of structures* for them. The *canonicity* of the distribution like axioms follows from the proofs.

1. Interactions Between Operations

A core observation behind *gaggle theory* is that operations in a partially ordered algebra may interact in ways that are advantageous from the point of view of their semantic modeling—even though the interaction falls short of ensuring interdefinability. Some well-known relationships between operations are residuation and Galois connections, and they are both special instances of *abstract residuation* and *colligation*. (Multiplication and division of rationals, conjunction and implication in intuitionistic logic, and fusion and implication in relevance logics are all examples of pairs of abstractly residuated operations.) Operations that are so connected (and additionally possess certain tonicity or distributivity properties) are grouped into families.[2]

To start with, we consider the ordered algebra introduced by Grishin [18].[3] $\mathfrak{A}_\mathfrak{g} = \langle A; \leq, \leftarrow, \circ, \rightarrow, \succ, +, \prec \rangle$ is a poset with six binary operations that are stipulated to satisfy the following eight quasi-inequations.

$$a \leq b \rightarrow c \ \ \Leftrightarrow \ \ a \circ b \leq c \ \ \Leftrightarrow \ \ b \leq c \leftarrow a \tag{re1}$$

$$a \geq b \prec c \ \ \Leftrightarrow \ \ a + b \geq c \ \ \Leftrightarrow \ \ b \geq c \succ a \tag{re2}$$

There are four other quasi-inequations that are immediate consequences of those listed above and express colligation between $\rightarrow$ and $\leftarrow$, and $\prec$ and $\succ$, respectively. They are easily seen to be implied here, hence we do not list them separately. However, it would be important to make them explicit, if $\circ$ (fusion) and $+$ (fission) were dropped from the families of operations. The set of operations $\{\leftarrow, \circ, \rightarrow\}$—the fusion family—is the algebraic analogue of

[2]See [7], especially, definitions 1.3.12, 4.3.7 and 5.3.4. for the precise conditions that operations have to satisfy in order to be considered a family of operations.

[3]More precisely, we replace the pre-order, that was stipulated by Grishin, by a partial order and we leave the rest of the algebra unchanged. We do not follow Grishin's notation, and we do not concern ourselves with the goal for which he introduced the algebra. But—out of respect to him—we add a subscript $\mathfrak{g}$ to the label of this algebra.

the connectives in Lambek's nonassociative sequent calculus.[4] Indeed, in the process of generalizing partially ordered groups and Heyting–Brouwer algebras into a common algebra, Grishin overtly relied on the algebraic counterpart of the nonassociative Lambek calculus that was foreshadowed by Ajdukiewicz.

Residuation between operations determines the tonicity of the operations in the argument places with respect to which they are residuated. For example, both $\circ$ and $+$ are *monotone* in both argument places. $\uparrow$ and $\downarrow$ indicate that an operation is monotone and antitone, respectively, in an argument place. The tonicity types of the operations in the two complete families of operations (i.e., in $\{\leftarrow, \circ, \rightarrow\}$ and in the fission family $\{\succ, +, \prec\}$) are as follows.

$$\leftarrow\colon\ \uparrow,\downarrow \qquad \circ\colon\ \uparrow,\uparrow \qquad \rightarrow\colon\ \downarrow,\uparrow \qquad\qquad \succ\colon\ \uparrow,\downarrow \qquad +\colon\ \uparrow,\uparrow \qquad \prec\colon\ \downarrow,\uparrow$$

There are eight argument places in the six operations in which an $\uparrow$ occurs, that is, where an operation is monotone. The six operations can be divided into two groups based on the side of the inequation on which they appear in (re1) and (re2) above. For instance, $+$ and $\rightarrow$ are placed to the right from $\leq$.

For conjunction and disjunction (or for meet and join), distributivity can be stated as

$$a \wedge (b \vee c) \leq (a \wedge b) \vee c.$$

Both $\wedge$ and $\vee$ are monotone in their arguments. However, $\wedge$ and $\vee$ appear on different sides of $\leq$ (when their residuals are included), and those sides coincide with the sides on which they are the main operation of some terms in the above inequation. This intertwining of $\wedge$ and $\vee$ may be viewed as a pattern after which distribution like inequations may be fashioned. In the above inequation, the second argument place of $\wedge$ and the first argument place of $\vee$ is selected—from the four possible ways to express distributivity of $\wedge$ and $\vee$ in short form (without relying on the commutativity of $\wedge$ and $\vee$).

There are 16 combinations when two operations are chosen similarly from the fusion and fission families, and the following list contains *all* the 16 inequations that result. At the same time, their labels encode which operations are put together and in which argument place. (For example, the hemidistributivity law 21 is obtained by combining $\circ$ in its second argument place with $+$ in its first argument place.) These labels will also allow us to refer easily to the inequations later on.

11.	$(a + b) \circ c \leq (a \circ c) + b$	21.	$a \circ (b + c) \leq (a \circ b) + c$
12.	$(a + b) \circ c \leq a + (b \circ c)$	22.	$a \circ (b + c) \leq b + (a \circ c)$
13.	$(a \rightarrow b) \circ c \leq a \rightarrow (b \circ c)$	23.	$a \circ (b \rightarrow c) \leq b \rightarrow (a \circ c)$
14.	$(b \leftarrow a) \circ c \leq (b \circ c) \leftarrow a$	24.	$a \circ (c \leftarrow b) \leq (a \circ c) \leftarrow b$
31.	$(a + b) \succ c \leq (a \succ c) + b$	41.	$a \prec (b + c) \leq (a \prec b) + c$
32.	$(a + b) \succ c \leq a + (b \succ c)$	42.	$a \prec (b + c) \leq b + (a \prec c)$
33.	$(a \rightarrow b) \succ c \leq a \rightarrow (b \succ c)$	43.	$a \prec (b \rightarrow c) \leq b \rightarrow (a \prec c)$
34.	$(b \leftarrow a) \succ c \leq (b \succ c) \leftarrow a$	44.	$a \prec (c \leftarrow b) \leq (a \prec c) \leftarrow b$

[4]See Lambek [19] for the associative Lambek calculus. Lambek [20] contains the nonassociative Lambek calculus with conjunction.

In the case of a Boolean algebra, which tightly constrains its operations, all (the Boolean variants of) the 16 inequations are equivalent and they are all true. (E.g., 23 is $a \wedge (b \supset c) \leq b \supset (a \wedge c)$ in Booleanese, and then via $a \wedge (-b \vee c) \leq -b \vee (a \wedge c)$ and $a \wedge (c \vee b) \leq b \vee (a \wedge c)$, we get the Boolean version of 21, i.e., $a \wedge (c \vee b) \leq b \vee (a \wedge c)$.) The situation is different in $\mathfrak{A}_\mathfrak{g}$—as the next lemma makes clear.

Lemma 1.1. *None of the inequations is interderivable from any other in* $\mathfrak{A}_\mathfrak{g}$.

Proof. There are various ways to prove the claim, one is by constructing concrete posets in which all but one of the inequations fail to hold. As sample cases we mention that 13 can be seen to express some permutation in fused terms, whereas 23 similarly expresses right associativity. A well-known fact from combinatory logic about the absence of interdefinability of the regular permutator C and of the associator B implies that algebras of combinatory terms (partially ordered by the weak reduction relation) can be utilized to show the independence of 13 and 23 over $\mathfrak{A}_\mathfrak{g}$.

The same algebras can prove the independence of 11 and 12 too, because these two inequations do not become equivalent even if both $+$ and $\circ$ are interpreted as the application operation.

A completely different sort of proof may be constructed after we will have defined a sound and complete relational semantics for the gaggles including those that are obtained by adding only one of the 16 inequations (see Sect. 2). □

We mentioned in the introduction that there were other motivations that vindicated some of the inequations from among 11–44. Investigations into desirable properties of binary consequence relations led to four *hemi-distributivity laws*, which are 11, 12, 21 and 22. The eight inequations (denoted as $G1, G1', \ldots, G4, G4'$ by Moortgat), which were proposed to handle *linguistic phenomena*, seem not to appear among those listed. However, they turn out to be provably equivalent—using residuation—to four of the inequations that are listed. Namely, $G1$ and $G1'$ are equivalent to 33, $G2$ and $G4'$ are equivalent to 34, $G3$ and $G3'$ are equivalent to 44, and lastly, $G4$ and $G2'$ are equivalent to 43.

Some of the inequations (or their provably equivalent versions) have been considered and motivated by reasons different from those we have listed so far. The inequations that are true in all *relation algebras*—when we think of $\circ$ as relational composition and $+$ as its Boolean dual—are 14, 23, 31 and 42, as well as two of the hemi-distributive laws 12 and 21. The lack of the two other hemi-distributive laws in relation algebras is explained by the absence of commutativity, in general, between binary relations. In other words, those inequations force a certain "permutation" or "switching" of some of the arguments of the operations (with respect to their order in the left-hand side term in the inequation).

We already mentioned the connection between 23 and the *combinator* B, as well as that between 13 and C. A (binary) groupoid operation can be straightforwardly viewed as *function application*. The following four inequations are equivalent (by residuation) to the four inequations 13, 14, 23 and 24, respectively.

13′.	$(a \circ b) \circ c \leq (a \circ c) \circ b$	14′.	$a \circ (b \circ c) \leq b \circ (a \circ c)$
23′.	$(a \circ b) \circ c \leq a \circ (b \circ c)$	24′.	$a \circ (b \circ c) \leq (a \circ b) \circ c$

The axioms for the combinators B, C, and the dual combinators b and c are as follows (with the application operation denoted by juxtaposition).

C.	$((\mathsf{C}x)y)z \rhd (xz)y$	c.	$x(y(z\mathsf{c})) \rhd y(zx)$
B.	$((\mathsf{B}x)y)z \rhd x(yz)$	b.	$x(y(z\mathsf{b})) \rhd (xy)z$

The similarity between the inequations above and the *combinatory axioms* (without the combinators) should be obvious now.

The inequations may be grouped together in various ways. For example, clusters may consist of all the inequations expressing forms of hemi-distributivity, all the inequations expressing associativity and certain commutativity of $+$ and all the inequations expressing associativity and certain commutativity of $\circ$. All these inequations hold of *addition and multiplication of natural numbers*, when the former is $+$ and the latter is $\circ$. The remaining four inequations are false under this interpretation. In other words, there are 12 "*arithmetical*" inequations and 4 "*linguistic*" inequations.

Another obvious division is to separate the inequations that involve operations that belong to *the same family* from those that *connect the two families*—which yields an even split of the 16 inequations. (This division is apparent in the conditions (f11–f44) in definition 2.4, where only two accessibility relations are used—one for the fusion family and another for the fission family.) Of course, the eight inequations that concern only one family can be further halved, since four of those inequations involve connectives from the fusion family, whereas the other inequations concern the fission family. The other eight inequations naturally fall into two groups—the hemi-distributive laws and the inequations that are equivalents of the linguistic principles $G1$–$G4$.

"Symmetry" may mean a lot of things. For example, a *function of two variables* f is called symmetric if its value is invariant under the permutation of its arguments, that is, $f(x, y) = f(y, x)$—see Church [9, pp. 17–18]. Distance functions are, perhaps, the best-known examples of functions that are symmetric in this sense. "Symmetry" in the title of this paper is derived from a transformation on inequations that was introduced by Grishin. Ultimately, his notion of symmetry is related to the notion of the converse of a relation, and indirectly, to the notion of symmetry of (binary) functions.

Grishin in [18] introduced two notions of "duality," and he called one of them *symmetry*. We denote these dualities by δ and c, respectively. The two transformations outlined by Grishin may be characterized by the following *algorithms*, where (d0) and (c0) are applicable only if no other clause from among (d1–d3) and (c1–c3), respectively, is applicable.

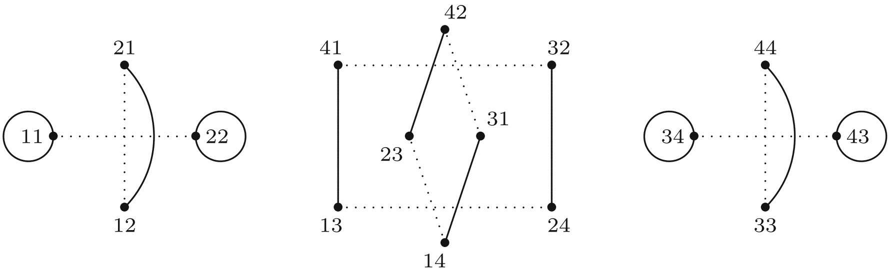

FIGURE 1. Dualities between the 16 inequations

Definition 1.2. The transformation δ is defined by (d0–d3), whereas c is defined by (c0–c3). (c2), (c3) and (d2), (d3) collect together three clauses each, those that concern operations from the same family.

(d0) $\delta a = a$,
(d1) $\delta(a_1 \leq a_2) = \delta a_1 \geq \delta a_2$,
(d2) $\delta(a_1 \circ a_2) = \delta a_1 + \delta a_2$, $\delta(a_1 \rightarrow a_2) = \delta a_1 \prec \delta a_2$,
$\delta(a_1 \leftarrow a_2) = \delta a_1 \succ \delta a_2$,
(d3) $\delta(a_1 + a_2) = \delta a_1 \circ \delta a_2$, $\delta(a_1 \prec a_2) = \delta a_1 \rightarrow \delta a_2$,
$\delta(a_1 \succ a_2) = \delta a_1 \leftarrow \delta a_2$.
(c0) $ca = a$,
(c1) $c(a_1 \leq a_2) = ca_1 \leq ca_2$,
(c2) $c(a_1 \circ a_2) = ca_2 \circ ca_1$, $c(a_1 \rightarrow a_2) = ca_2 \leftarrow ca_1$,
$c(a_1 \leftarrow a_2) = ca_2 \rightarrow ca_1$,
(c3) $c(a_1 + a_2) = ca_2 + ca_1$, $c(a_1 \prec a_2) = ca_2 \succ ca_1$,
$c(a_1 \succ a_2) = ca_2 \prec ca_1$.

Obviously, either transformation yields an inequation from any of the 16 inequations. Moreover, the set of the 16 inequations above is closed under δ and c. From a logical point of view, it is interesting to find out if there is a systematic relationship between δ, c and logical equivalence (denoted by $\equiv$) or its algebraic counterpart (i.e., equality in $\mathfrak{A}_\mathfrak{g}$). The picture that emerges is in Fig. 1. The dotted lines indicate c, and the continuous lines show δ. It is easy to verify that both functions are *invertible* and of *period two*, hence there are no arrows on the lines in the diagram.

As the diagram shows, there are four *fixed points* for δ $(11, 22, 34$ and $43)$, and on these points $\delta^n cc$ $(n \in \mathbb{N})$ is logical equivalence. On the pairs $\{ 21, 12 \}$ and $\{ 33, 44 \}$ δ and c *commute*, that is, δc and $c\delta$ are both $\equiv$. Lastly, on the sets $\{ 14, 23, 31, 42 \}$ and $\{ 13, 24, 32, 41 \}$ the *foursome compositions* $\delta c \delta c$ and $c \delta c \delta$ are logical equivalence.

The 16 points do not constitute a connected graph, rather they span six connected subgraphs. The four points on the left and on the right stand for inequations that involve operations from both families. The eight middle points are inequations with terms from just one family, however, δ switches from one family to another. The horizontal dotted lines connect inequations that include a certain amount of commutation (in the wide sense already mentioned above).

For our purposes, it is useful to enlarge the signature of $\mathfrak{A}_{\mathfrak{g}}$. We denote by $\mathfrak{A}_{\mathfrak{e}}$ a partially ordered algebra that is like $\mathfrak{A}_{\mathfrak{g}}$ with *four constants* $(0, 1, e_\circ$ and $e_+)$ added. In $\mathfrak{A}_{\mathfrak{e}}$ the (in)equations in (bt) and (ee) are stipulated to hold.

$$0 \leq a \qquad\qquad a \leq 1 \tag{bt}$$

$$e_\circ \circ a = a \qquad\qquad a = a + e_+ \tag{ee}$$

The elements 0 and 1 function as bounds of the poset—0 is bottom and 1 is top. The addition of these constants is straightforward and the addition of the two inequations does not produce new true inequations in the old signature. That is, if the terms in an inequation do not involve the constants, then the inequation can be derived in $\mathfrak{A}_{\mathfrak{e}}$ (or in the new quasi-variety containing the constants), just in case it can be derived in $\mathfrak{A}_{\mathfrak{g}}$ (or in the quasi-variety without the constants).

The operations in the fusion and fission families preserve or reverse the extremal elements as follows.

$$\leftarrow : 1, 0 \longrightarrow 1 \qquad \circ : 0, 0 \longrightarrow 0 \qquad \rightarrow : 0, 1 \longrightarrow 1$$

$$\succ : 0, 1 \longrightarrow 0 \qquad + : 1, 1 \longrightarrow 1 \qquad \prec : 1, 0 \longrightarrow 0$$

For example, $\circ$ is a *normal* operation, that is, if either argument place of $\circ$ is filled with 0 (and the other argument is filled with any element of A), then the whole term reduces to 0. That is, $\circ$ preserves 0 in both argument places. On the other hand, $\succ$ reverses 1 to 0 in its second argument place, that is, $a \succ 1 = 0$. The inequations that are "summarized" by the above type-like notation may be derived without much difficulty utilizing residuation in an essential way.[5]

The addition of the two other constants, $e_\circ$ and e_+, is less important for our purposes. However, the inclusion of identity elements for $\circ$ and $+$ can be easily motivated. If $\mathfrak{A}_{\mathfrak{e}}$ is viewed as arising as a generalization of ordered groups then retaining identities is a must. If $\mathfrak{A}_{\mathfrak{e}}$ is thought to emerge in connection to nonclassical logics, e.g., relevance logics, then $e_\circ$ becomes especially important from the point of view of proof theory (especially, consecution calculi). Furthermore, $e_\circ$ is paramount in the algebraization of relevance logics. The equations in (ee) above mean that the constants $e_\circ$ and e_+ are left and right *identity* elements for the two operations $\circ$ and $+$, respectively.

2. Fusion–Fission Gaggles

Gaggles were originally defined by Dunn [12] to inhabit on a distributive lattice.[6] Sometimes, we use the term 'gaggles' to refer to a whole range of algebraic structures that lie within the limits of gaggle theory. However, in this and the next section, we more often use the same term to refer to algebras that

[5] We introduced this notation in [7, Ch. 6]—with more detailed explanations.

[6] 'Generalized Galois logics' is abbreviated as 'ggl's'. In turn, the acronym 'ggl' is pronounced as "gaggle." (See also Dunn [12].)

contain a family of operations on an underlying bounded distributive lattice.[7] Distributive gaggles turned out to be the best-behaved kind of gaggles. This suggests that we start our investigations with considering **gGl**s that contain the fusion and the fission families, as well as some of the inequations listed as 11–44.

The hemi-distributivity laws that hold for "multiplicative" conjunction and disjunction in the Lindenbaum algebra of linear logic were modeled in Allwein and Dunn [1]. ("Additive" conjunction and disjunction do not distribute over each other in linear logic, that is, the Lindenbaum algebra of linear logic does not contain a distributive lattice reduct.) This leads to considerable complications (and to new definitions for the operations on sets of situations) in the relational semantics of linear logic.[8] The inequations $G1$–$G4$ were considered by Moortgat and his colleagues within an algebra that we would call a partial gaggle. The two inequations 13 and 23, that express a certain commutativity and the right associativity of fusion, were often dealt with in the literature on relevance logics. (See e.g., Meyer and Routley [21], Anderson, Belnap and Dunn [2], Dunn and Restall [17].) The closely related combinatory axioms were modeled in the setting of partial gaggles in Dunn and Meyer [16], and in the setting of **gGl**s and nondistributive **gGl**s in Bimbó and Dunn [6] and Bimbó [3], respectively.

Definition 2.1. A *fusion–fission gaggle* $\mathfrak{A}_\mathfrak{d}$ is an algebra $\langle A; \wedge, \vee, 0, 1, \leftarrow, \circ, \rightarrow, e_\circ, \succ, +, \prec, e_+\rangle$ (of similarity type $\langle 2, 2, 0, 0, 2, 2, 2, 0, 2, 2, 2, 0\rangle$), where (dl) as well as (re1), (re2), (bt) and (ee) (from Sect. 1) hold.

(dl) $\langle A; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice.

The gaggles $\mathfrak{A}_{11}$–$\mathfrak{A}_{44}$ are defined by adding to $\mathfrak{A}_\mathfrak{d}$ the identically labeled inequation.

Of course, further **gGl**s may be defined by adding to $\mathfrak{A}_\mathfrak{d}$ more than one of the distribution like inequations. We do not define all those **gGl**s separately, because they can be represented by adding the corresponding frame conditions to the structure defined in 2.3 (below). However, we keep in mind that those gaggles fall into the class of symmetric gaggles too. The partial order $\leq$ that was present in $\mathfrak{A}_\mathfrak{g}$ and $\mathfrak{A}_\mathfrak{e}$ is retained in $\mathfrak{A}_\mathfrak{d}$, however, it no longer appears explicitly in the previous definition, because $\leq$ is definable from either meet or join, in the standard lattice-theoretical way. This means that $\mathfrak{A}_\mathfrak{d}$ is rightly seen as an extension of both $\mathfrak{A}_\mathfrak{g}$ and $\mathfrak{A}_\mathfrak{e}$.

The next lemma shows that the lattice operations interact with the fusion and fission families in a pleasant way. For example, $\rightarrow$ distributes over $\vee$ in its first argument place and over $\wedge$ in its second argument place into $\wedge$. That is, $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$ and $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$. (See also [7, pp. 28–30].)

[7] These algebras are nearly the same as the gaggles defined by Dunn in [12]. They are called "distributive gaggles" in Bimbó and Dunn [7]—to distinguish them from other classes of gaggles.

[8] Situations (sometimes called information states) are the analogues of the possible worlds, that are the objects in the Kripke-style semantics of normal modal logics.

Lemma 2.2. *In $\mathfrak{A}_{\mathfrak{d}}$ the binary operations from the fusion and fission families have the same* tonicity type *as in $\mathfrak{A}_{\mathfrak{g}}$, and they preserve or reverse the bounds as in $\mathfrak{A}_{\mathfrak{e}}$. Furthermore, in $\mathfrak{A}_{\mathfrak{d}}$ the same operations possess* distribution types*, which are given in* (d∘) *and* (d+)*.*

$$\leftarrow\colon \wedge,\vee \longrightarrow \wedge \qquad \circ\colon \vee,\vee \longrightarrow \vee \qquad \rightarrow\colon \vee,\wedge \longrightarrow \wedge \tag{d$\circ$}$$

$$\succ\colon \vee,\wedge \longrightarrow \vee \qquad +\colon \wedge,\wedge \longrightarrow \wedge \qquad \prec\colon \wedge,\vee \longrightarrow \vee \tag{d+}$$

We do not give a proof for the lemma, however, we remark that the proof is quite straightforward.

The distribution types of the operations are interesting not only in themselves, but also for the choice of the modeling of the operations. To describe the modeling of all the gaggles introduced in definition 2.1, we define a semantics in two steps. First, we describe a structure for $\mathfrak{A}_{\mathfrak{d}}$, and then we list conditions that correspond to the inequations 11–44 (on the background of $\mathfrak{A}_{\mathfrak{d}}$).

We extend to relations our use of ↑ and ↓ to show the tonicity of operations (from Sect. 1), putting ↑ and ↓ into the argument places of a relation to show the tonicity (monotone or antitone, respectively) of the relation in that argument place. $R_\circ$ and R_+ are "accessibility relations," and we follow the customary convention of omitting the commas and parentheses around their arguments. Priestley in [23] introduced *ordered Stone spaces* in order to obtain a representation of bounded distributive lattices. (Her representation is different from the representation given by Stone in [25].) We call a *compact* and *totally order disconnected* topological space a *Priestley space*, which is Priestley's ordered Stone space. (Cf. also Davey and Priestley [10, Ch. 11].) As a notation for a topological space, we give the base set of the topology (e.g., U), and we denote the set of open sets by $\mathfrak{O}$. If the space includes further components, such as relations, then we list those too. The set of *clopen* sets (i.e., the set of those sets that are both open and closed in the topology) is denoted by $\mathcal{CO}(\mathfrak{O})$. We add the superscript $^\uparrow$ to restrict the set to contain upward closed subsets of U only (where the closure is understood with respect to the partial order $\leq$). To simplify the statement of (f8–f10), we use the operations on sets of situations that are defined by (v1–v6) in definition 2.5. [For instance, (f9) could be fully written out as $\bar{R}_\circ\alpha\beta\gamma \Rightarrow \exists O_1, O_2 \in \mathcal{CO}(\mathfrak{O})^\uparrow.\, \alpha \in O_1\ \&\ \beta \in O_2\ \&\ \gamma \notin \{\,\delta\colon \exists\alpha,\beta.\, R_\circ\alpha\beta\delta\ \&\ \alpha \in O_1\ \&\ \beta \in O_2\,\}$.]

Definition 2.3. A *structure for* $\mathfrak{A}_{\mathfrak{d}}$ is $\mathfrak{F} = \langle U; \leq, R_\circ, R_+, I_\circ, I_+, \mathfrak{O}\rangle$, where (f1)–(f10) hold.

(f1) $I_\circ, I_+ \subseteq U, \quad \leq\ \subseteq U^2, \quad R_\circ, R_+ \subseteq U^3, \quad \mathfrak{O} \subseteq \wp(U),$

(f2) $\alpha \leq \alpha, \quad \alpha \leq \beta\ \&\ \beta \leq \gamma. \Rightarrow \alpha \leq \gamma, \quad \alpha \leq \beta\ \&\ \beta \leq \alpha. \Rightarrow \alpha = \beta,$

(f3) $R_\circ \downarrow\downarrow\uparrow, \quad R_+ \downarrow\downarrow\uparrow,$

(f4) $\langle U, \leq, \mathfrak{O}\rangle$ is a Priestley space,

(f5) $\exists\iota \in I_\circ.\, R_\circ\iota\alpha\beta \ \Leftrightarrow\ \alpha \leq \beta,$

(f6) $\bar{R}_+\alpha\beta\gamma\ \&\ \beta \notin I_+. \Rightarrow \gamma \leq \alpha, \quad \exists\beta.\, \bar{R}_+\alpha\beta\alpha\ \&\ \beta \notin I_+,$

(f7) $I_\circ = I_\circ^\uparrow \in \mathfrak{O}, \quad \bar{I}_\circ \in \mathfrak{O}, \quad I_+ = I_+^\uparrow \in \mathfrak{O}, \quad \bar{I}_+ \in \mathfrak{O},$

(f8) $O_1, O_2 \in \mathcal{CO}(\mathfrak{O})^\uparrow \ \Rightarrow\ O_1 * O_2 \in \mathcal{CO}(\mathfrak{O}),$ *where* $* \in \{\leftarrow, \circ, \rightarrow, \succ, +, \prec\}$, (f9) $\bar{R}_\circ\alpha\beta\gamma \;\Rightarrow\; \exists O_1, O_2 \in \mathbb{C}(\mathfrak{O})^\uparrow.\, \alpha \in O_1 \mathbin{\&} \beta \in O_2 \mathbin{\&} \gamma \notin O_1 \circ O_2,$

(f10) $R_+\alpha\beta\gamma \;\Rightarrow\; \exists O_1, O_2 \in \mathbb{C}(\mathfrak{O})^\uparrow.\, \alpha \notin O_1 \mathbin{\&} \beta \notin O_2 \mathbin{\&} \gamma \in O_1 + O_2.$

The two accessibility relations are antitone in their first two argument places and isotone in their third arguments. The stipulations in (f5) and (f6) guarantee that $I_\circ$ and I_+ will turn out to be identity elements for $\circ$ and $+$ [that are defined in (v2) and (v5)]. (f7) provides that these two subsets of U are indeed elements of the algebra of clopen cones on the structure $\mathfrak{F}$.

The group of gaggles that includes $\mathfrak{A}_\mathfrak{d}$ and one or more of $\mathfrak{A}_{11}$–$\mathfrak{A}_{44}$ may be thought to be analogous to a certain extent to a group of normal modal logics, for example, to K, KT, KB, $S4$ and $S5$. These logics differ from each other in respect to which axioms from among T, 4 and B they include. In order to be able to refer to all the gaggles defined in 2.1 or to any one of them, we will use the notation $\mathfrak{A}_\mathfrak{d}^*$. In the case of normal modal logics, T, 4 and B each has its own *corresponding* frame condition. For the inequations 11–44, the corresponding conditions (in the context of $\mathfrak{A}_\mathfrak{d}$) are (f11–f44).

Definition 2.4. A *structure for* a gaggle $\mathfrak{A}_{\mathrm{mn}}$ (from among $\mathfrak{A}_{11}$–$\mathfrak{A}_{44}$) is a structure for $\mathfrak{A}_\mathfrak{d}$ together with the postulate (f mn) added.

(f11) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\varepsilon\gamma. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\vartheta\varepsilon\alpha \mathbin{\&} R_\circ\vartheta\beta\delta,$
(f12) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\varepsilon\gamma. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\delta\vartheta\alpha \mathbin{\&} R_\circ\vartheta\beta\varepsilon,$
(f13) $R_\circ\alpha\beta\gamma \mathbin{\&} R_\circ\gamma\delta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, R_\circ\vartheta\beta\varepsilon \mathbin{\&} R_\circ\alpha\delta\vartheta,$
(f14) $R_\circ\alpha\beta\gamma \mathbin{\&} R_\circ\delta\gamma\varepsilon. \;\Rightarrow\; \exists\vartheta.\, R_\circ\vartheta\beta\varepsilon \mathbin{\&} R_\circ\delta\alpha\vartheta,$
(f21) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\varepsilon\gamma. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\vartheta\varepsilon\beta \mathbin{\&} R_\circ\alpha\vartheta\delta,$
(f22) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\varepsilon\gamma. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\delta\vartheta\beta \mathbin{\&} R_\circ\alpha\vartheta\varepsilon,$
(f23) $R_\circ\alpha\beta\gamma \mathbin{\&} R_\circ\gamma\delta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, R_\circ\alpha\vartheta\varepsilon \mathbin{\&} R_\circ\beta\delta\vartheta,$
(f24) $R_\circ\alpha\beta\gamma \mathbin{\&} R_\circ\delta\gamma\varepsilon. \;\Rightarrow\; \exists\vartheta.\, R_\circ\alpha\vartheta\varepsilon \mathbin{\&} R_\circ\delta\beta\vartheta,$
(f31) $\bar{R}_+\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\gamma\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\vartheta\beta\varepsilon \mathbin{\&} \bar{R}_+\delta\alpha\vartheta,$
(f32) $\bar{R}_+\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\gamma\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\alpha\vartheta\varepsilon \mathbin{\&} \bar{R}_+\delta\beta\vartheta,$
(f33) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\alpha\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\delta\gamma\vartheta \mathbin{\&} R_\circ\varepsilon\beta\vartheta,$
(f34) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\delta\beta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\delta\gamma\vartheta \mathbin{\&} R_\circ\alpha\varepsilon\vartheta,$
(f41) $\bar{R}_+\alpha\beta\gamma \mathbin{\&} \bar{R}_+\gamma\delta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\vartheta\beta\varepsilon \mathbin{\&} \bar{R}_+\alpha\delta\vartheta,$
(f42) $\bar{R}_+\alpha\beta\gamma \mathbin{\&} \bar{R}_+\gamma\delta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\alpha\vartheta\varepsilon \mathbin{\&} \bar{R}_+\beta\delta\vartheta,$
(f43) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\alpha\delta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\gamma\delta\vartheta \mathbin{\&} R_\circ\varepsilon\beta\vartheta,$
(f44) $R_\circ\alpha\beta\gamma \mathbin{\&} \bar{R}_+\beta\delta\varepsilon. \;\Rightarrow\; \exists\vartheta.\, \bar{R}_+\gamma\delta\vartheta \mathbin{\&} R_\circ\alpha\varepsilon\vartheta.$

The *operations* and the *constants* of the algebra of clopen cones on a structure are defined in the same way in all these gaggles, that is, the next definition can be applied to any of the above frames.

Definition 2.5. A *model for* an $\mathfrak{A}_\mathfrak{d}^*$ gaggle is $\mathfrak{M} = \langle \mathfrak{F}, v \rangle$, where $\mathfrak{F}$ is a structure for the $\mathfrak{A}_\mathfrak{d}^*$ gaggle, and v is a valuation function. v is of type $v\colon A \longrightarrow \mathbb{C}(\mathfrak{O})^\uparrow$ and v satisfies (v1–v12).

(v1) $v(c \leftarrow a) = \{\, \beta\colon \forall\alpha,\gamma.\, R_\circ\alpha\beta\gamma \mathbin{\&} \alpha \in va. \Rightarrow \gamma \in vc \,\},$
(v2) $v(a \circ b) = \{\, \gamma\colon \exists\alpha,\beta.\, R_\circ\alpha\beta\gamma \mathbin{\&} \alpha \in va \mathbin{\&} \beta \in vb \,\},$
(v3) $v(b \rightarrow c) = \{\, \alpha\colon \forall\beta,\gamma.\, R_\circ\alpha\beta\gamma \mathbin{\&} \beta \in vb. \Rightarrow \gamma \in vc \,\},$
(v4) $v(c \succ a) = \{\, \beta\colon \exists\alpha,\gamma.\, \bar{R}_+\alpha\beta\gamma \mathbin{\&} \alpha \notin va \mathbin{\&} \gamma \in vc \,\},$

(v5) $v(a+b) = \{\,\gamma\colon \forall\alpha,\beta.\,\bar{R}_+\alpha\beta\gamma \Rightarrow .\,\alpha \in va \vee \beta \in vb\,\}$,
(v6) $v(b \prec c) = \{\,\alpha\colon \exists\beta,\gamma.\,\bar{R}_+\alpha\beta\gamma \;\&\; \beta \notin vb \;\&\; \gamma \in vc\,\}$,
(v7) $v(a \wedge b) = \{\,\alpha\colon \alpha \in va \;\&\; \alpha \in vb\,\}$,
(v8) $v(a \vee b) = \{\,\alpha\colon \alpha \in va \vee \alpha \in vb\,\}$,
(v9) $ve_\circ = \{\,\alpha\colon \alpha \in I_\circ\,\}$,
(v10) $ve_+ = \{\,\alpha\colon \alpha \in I_+\,\}$,
(v11) $v1 = \{\,\alpha\colon \alpha \in U\,\}$,
(v12) $v0 = \emptyset$.

We note two potential *simplifications* in the definitions of structures and models that are applicable depending on whether the $\mathfrak{A}^*_\eth$ gaggle is thought to arise from a proof system for a logic or not. If there is set of generators B for the gaggle such that $B \subsetneq A$ and B includes the constants $0, 1, e_\circ$ and e_+, then, first, v may be restricted to B with (v9–v12) held fixed. Then (v1–v8) can be viewed as defining an extension of v from B to A. If the $\mathfrak{A}^*_\eth$ gaggles emerge from a logic, then typically, the set of propositional variables (together with the constants) serves as a set of generators, that is a proper subset of the set of formulas. On the other hand, if there is no generator set readily available (other than A itself), then (f7) and (f8) follow from the type assumption on v together with clauses (v1–v6) and (v9–v10). Similarly, the tonicity conditions for $R_\circ$ and R_+, that is, (f3) may be dropped when the co-domain of v is specified as above.

Having defined a relational semantics for the gaggles, we can develop another comparison between some groups of inequations. We noted earlier that some of the inequations involve "permutation" (in a wide sense). The motivation behind the use of the term 'permutation' comes from the fact that some of the inequations express permutation in the usual sense of the term. Concretely, 13 is equivalent to the inequation $(a \circ b) \circ c \leq (a \circ c) \circ b$, and 24 can be proven to be equivalent to $a \circ (b \circ c) \leq b \circ (a \circ c)$. Dual combinators are less well-known than combinators, however, they are well-motivated by algebraic and proof theoretic considerations, as well as by generalizations of the Curry–Howard isomorphism. Combinators and dual combinators can be straightforwardly associated to formulas and algebraic terms that contain $\circ$, and we have shown above that the four inequations that include operations only from the fusion family become analogues of combinatory axioms. There seems to be no similar connection between $+$ and some sort of constants that have been investigated for independent reasons. However, 41 parallels 13 and so does 32 24 within the fission family; 42 and 31 express the associativity of $+$.

An interesting property of the inequations without permutation (in the wide sense) is that the corresponding relational conditions can be pictured by simple *square shaped diagrams*, where the existentially quantified ϑ is almost contained in the diagram of the antecedent. As illustrations, we give diagrams for 14, 23 and 44. (We leave to the reader the discovery of the five other inequations that can be so visualized.)

The arrows are composed as in serial commuting diagrams of binary relations. The $\circ$ or $+$ at the corner of a triangle indicates the ternary relation that induces the triangle. For example, $R_\circ\alpha\beta\gamma$ is pictured as the "south-east" (SE)

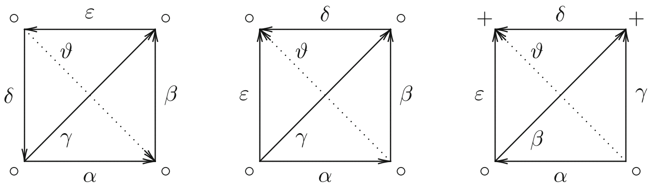

FIGURE 2. Diagrams of frame conditions for 14, 23 and 44

triangle of the first two square diagrams in Fig. 2, whereas $\bar{R}_{+}\beta\delta\varepsilon$ gives rise to the NW triangle of the third square diagram. In terms of binary relations, α and β may be thought to compose into γ in the former case, and β and δ compose into ε in the latter one. The dotted line—always a diagonal of the square—is the line of the emergent arrow.

We note that the coincidence of easy picturing and the lack of permutation is somewhat arbitrary when the inequation involves operations from two families. The order in which the situations are linked by the accessibility relation is—in a sense—arbitrary. More precisely, the order is (systematically) derived from the residuation patterns within a family of operations, and so it depends on the way the family has been standardized.[9] However, families of operations may be standardized independently from each other, which produces a certain arbitrariness.

The situation is not wholly arbitrary when the operations belong to the same family. To illustrate how the notion of permutation may be conceptualized based on residuation together with replacing terms, we give two examples to show how certain manipulations of inequations that hold in $\mathfrak{A}_{\mathfrak{g}}$ can yield 13 and 23. $(b \to c) \circ b \leq c$ and $c \leq a \to (c \circ a)$ are true in $\mathfrak{A}_{\mathfrak{g}}$. Then postulating $(b \to c) \circ b \leq a \to (c \circ a)$ does not add anything new to $\mathfrak{A}_{\mathfrak{g}}$. Furthermore, the terms in the inequation involve only two variables (rather that three, as in 11–44). This observation suggests that we move some subterms to the other side of the inequation using residuation (which yielded the inequation itself). We may obtain either $b \to c \leq b \to (a \to (c \circ a))$ or $((b \to c) \circ b) \circ a \leq c \circ a$ in this way. However, either residuation step blocks the other one. But replacing the left-hand side term in the latter inequation by $((b \to c) \circ a) \circ b$ allows us to move b to the right-hand side to get the inequation $(b \to c) \circ a \leq b \to (c \circ a)$. The former inequation does not lead to a similarly "obvious" step, although right associativity amounts to replacing the term $b \to (a \to (c \circ a))$ by $(b \to (a \circ c)) \leftarrow a$. A more transparent manipulation leading to 23 starts with another inequation that is true in $\mathfrak{A}_{\mathfrak{g}}$, namely, $(b \to c) \circ b \leq (a \circ c) \leftarrow a$. Then $a \circ ((b \to c) \circ b) \leq a \circ c$, and by rearranging the parentheses, we arrive at $(a \circ (b \to c)) \circ b \leq a \circ c$. One more step gives 23, that is, the inequation $a \circ (b \to c) \leq b \to (a \circ c)$. The examples clearly show the difference between swapping two subterms such as a and b (while the whole structure

[9]The notion of standardization was introduced by Bimbó and Dunn in [7, Ch. 4]. The order of the arguments of accessibility relations is also considered in Chapter 2.

of the terms remains the same) and regrouping a term by moving a pair of parentheses around.

We do not pursue further the problem of picturing relational conditions here; rather we turn to our models.

Theorem 2.6. *A model on a structure* $\mathfrak{F}$ *for an* $\mathfrak{A}^*_\mathfrak{d}$ *gaggle is* a concrete $\mathfrak{A}^*_\mathfrak{d}$ gaggle*, that is, an* $\mathfrak{A}^*_\mathfrak{d}$ *gaggle consisting of clopen cones.*

Proof. We sketch the proof of the theorem, including details of two steps. The clopen cones of a Priestley space form a distributive lattice with $\emptyset$ and $\mathcal{C}(\mathfrak{O})^\uparrow$ being the extremal elements. The residuation patterns between the operations are immediate from their definitions in 2.5—once we note that $\mathcal{C}(\mathfrak{O})^\uparrow$ is closed under all the operations in the families. $I_\circ$ and I_+ are easily seen to be identity elements for $\circ$ and $+$ due to the frame conditions stipulated in (f5–f6) and (f7).

In **1.** and **2.** we show that (f21) suffices for 21 to hold in the algebra of clopen cones, and so does (f44) for 44.

1. A straightforward verification of $v(a \circ (b + c)) \subseteq v((a \circ b) + c)$ (that is, of 21) using (f21) can be carried out as follows. The assumption $\gamma \in v(a \circ (b+c))$ expands into $\exists \alpha, \beta.\, R_\circ \alpha\beta\gamma \,\&\, \alpha \in va \,\&\, \beta \in v(b+c)$. Having assumed $\bar{R}_+\zeta\vartheta\gamma$, we can detach $\bar{R}_+\zeta\vartheta\gamma \& R_\circ\alpha\beta\gamma$ from (f21), and then we obtain $\exists\delta.\, \bar{R}_+\delta\vartheta\beta \& R_\circ\alpha\delta\zeta$. From $\beta \in v(b + c)$ then—together with $\bar{R}\delta\vartheta\beta$—$\delta \in vb$ or $\varepsilon \in vc$ follows. If $\delta \in vb$, then $\zeta \in v(a \circ b)$, and so $\zeta \in v(a \circ b)$ or $\vartheta \in vc$. If $\vartheta \in vc$ then it is immediate that $\zeta \in v(a \circ b)$ or $\vartheta \in vc$. In sum, $\zeta \in v(a \circ b)$ or $\vartheta \in vc$. By eliminating the second assumption, $\gamma \in v((a \circ b) + c)$ by the definition of $+$.

2. The other inequation that we selected is 44, and it may be proven to be true in the gaggle of clopen cones on a frame for $\mathfrak{A}_{44}$ as follows. Let us assume that $\gamma \in v(a \circ (b \prec c))$. Then we get $\exists \alpha, \beta.\, R_\circ\alpha\beta\gamma \,\&\, \alpha \in va \,\&\, \beta \in v(b \prec c)$, and further, from the last conjunct we obtain $\exists \delta, \varepsilon.\, \bar{R}_+\beta\delta\varepsilon \,\&\, \delta \notin vb \,\&\, \varepsilon \in vc$. After existential instantiation and conjunction elimination, we can introduce $\&$ to get $R_\circ\alpha\beta\gamma \,\&\, \bar{R}_+\beta\delta\varepsilon$. By yet another instantiation and modus ponens, we obtain $\exists\vartheta.\, R_\circ\alpha\varepsilon\vartheta \,\&\, \bar{R}_+\gamma\delta\vartheta$ from (f44). After rearranging some conjuncts and reintroducing the existential quantifiers, we arrive at the formula $\exists \alpha, \varepsilon.\, R_\circ\alpha\varepsilon\vartheta \,\&\, \alpha \in va \,\&\, \varepsilon \in vc$, which means that $\vartheta \in v(a \circ c)$. Having added two more conjuncts and quantifiers, we have $\exists \delta, \vartheta.\, \bar{R}_+\gamma\delta\vartheta \,\&\, \delta \notin vb \,\&\, \vartheta \in v(a \circ c)$, which is the same as $\gamma \in v(b \prec (a \circ c))$, according to (v4). □

Definitions 2.3, 2.4 and 2.5 describe how to build a concrete gaggle (that is, a **gGl** comprising sets of situations) from a frame and Theorem 2.6 shows that the definitions accomplish what they were intended to do. Now we describe how to define a structure—the canonical frame—from an $\mathfrak{A}^*_\mathfrak{d}$ gaggle. (We denote by $\tau(\mathfrak{B})$ the topology generated by the basis $\mathfrak{B}$.)

Definition 2.7. The *canonical frame* of an $\mathfrak{A}^*_\mathfrak{d}$ gaggle is $\mathfrak{F}_\mathfrak{c} = \langle \mathcal{P}_o, \subseteq, R_\circ, R_+, I_\circ, I_+, \mathfrak{O} \rangle$, where the elements of the tuple are as follows. $(\alpha, \beta, \gamma \in \mathcal{P}_o$ everywhere.)

(b1) $B \in \mathcal{P}_o \;\Leftrightarrow\; .\, B \subseteq A \,\&\, A \neq B \,\&\, B \neq \emptyset \,\&$
$(\forall a, b.\, a \wedge b \in B \Leftrightarrow a \in B \,\&\, b \in B) \,\&\, (\forall a, b. a \vee b \in B \Leftrightarrow .\, a \in B \vee b \in B)$,

(b2) $\forall B, C \in \mathcal{P}_o.\, B \subseteq C \;\Leftrightarrow\; \forall a \in B.\, a \in C$,

(b3) $R_\circ = \{\, \langle \alpha, \beta, \gamma \rangle \colon \forall a, b.\, a \in \alpha \;\&\; b \in \beta. \Rightarrow a \circ b \in \gamma \,\}$,
(b4) $R_+ = \{\, \langle \alpha, \beta, \gamma \rangle \colon \exists a, b.\, a \notin \alpha \;\&\; b \notin \beta \;\&\; a + b \in \gamma \,\}$,
(b5) $I_\circ = \{\, \alpha \colon e_\circ \in \alpha \,\}$,
(b6) $I_+ = \{\, \alpha \colon e_+ \in \alpha \,\}$,
(b7) $\mathfrak{O} = \tau(\mathfrak{B})$, where $\mathfrak{B} = \{\, C \cap D \colon C, D \in h[A] \,\}$ and $ha = \{\, \alpha \colon a \in \alpha \,\}$.

The canonical frame of an $\mathfrak{A}^*_\mathfrak{d}$ gaggle may be explained in words as follows. The underlying set of elements of the frame is the set of proper nonempty prime filters, which is denoted by $\mathcal{P}_o$. The elements of $\mathcal{P}_o$ are denoted by $\alpha, \beta, \gamma, \ldots$, that is, the situations are prime filters. The partial order of the structure is set inclusion. The two ternary relations $R_\circ$ and R_+, that are associated with the fusion and the fission families, are defined in (b3) and (b4) from the two monotone operations of the families. However, the *same* relations may be defined using either of the two other operations of the families. The two distinguished subsets of $\mathcal{P}_o$ are related to the identity elements, and the form of the defining conditions (i.e., "$y \in Y$") automatically ensures that the resulting subsets are upward closed. Lastly, the set of open sets is defined from a basis for the set of open sets. (A basis is closed under finite intersections, hence $\mathfrak{B}$ only has to be closed under arbitrary unions.) The definition of $\mathfrak{B}$ utilizes the function h [which is also given in (b7)]. The function h will play a further rôle in the proof of theorem 2.10.

The next theorem parallels the previous one to the extent that it shows that starting with an $\mathfrak{A}^*_\mathfrak{d}$ gaggle, we can build a frame for that gaggle from the gaggle itself.

Theorem 2.8. *The canonical frame of* $\mathfrak{A}^*_\mathfrak{d}$ *is a* structure for $\mathfrak{A}^*_\mathfrak{d}$*.*

Proof. The proof may be divided into several parts. First, $\langle \mathcal{P}_o, \subseteq, \mathfrak{O} \rangle$ can be shown to be a Priestley space without any concern about the other operations and constants. It is obvious that the type of the other elements is as required due to (b3–b6).

The tonicity of $R_\circ$ and R_+ is nearly obvious, and it is easily verifiable. This means that (f3) holds on the canonical frame. To show that $R_\circ$ and $\subseteq$ interact as desired, let us assume that $\alpha \subseteq \beta$ (again, $\alpha, \beta \in \mathcal{P}_o$). The filter generated by $e_\circ$, that is, $[e_\circ)$ satisfies the defining condition of $R_\circ$ (with α in place of β and β in place of γ). Furthermore, $[e_\circ) \in \{\, F \colon \forall a, b.\, a \in F \& b \in \alpha. \Rightarrow a \circ b \in \beta \,\}$ (where F ranges over the set of proper nonempty filters). A union of a chain of nonempty filters is a filter, and belongs to the set we have just defined whenever all the elements of the chain are elements of the same set. By Zorn's lemma, there is a maximal element—let us say ι—in this partially ordered set of filters. Based on the maximality of ι and the distribution type of fusion, ι may be shown to be a prime filter. (We omit the details.)

The conditions guaranteeing that the set of clopen cones is closed under the operations can be proven once we observe that h is an isomorphism, and it maps every element of the gaggle into a clopen cone. It is also true that every clopen cone is of this form, that is, h is onto. (The proof of the latter claim is part of the proof that $\mathfrak{O}$ with $\subseteq$ is a Priestley space.) Incidentally, this implies that $I_\circ$ and I_+ are clopen cones too, thereby, showing that all the conditions in (f7) hold on $\mathfrak{F}_\mathfrak{c}$. These steps suffice for a proof of completeness theorem without a topology.

The inspection of all the details of the proof of theorem 2.6 reveals that (f9) and (f10) have not been used in that proof at all. (f9) and (f10) are postulated to ensure that the topological structures have other desirable properties beyond giving rise to a concrete $\mathfrak{A}_\mathfrak{d}^*$ gaggle. At the same time, these stipulations are unproblematic in the present theorem, because the two conditions may be proven to be true on $\mathfrak{F}_\mathfrak{c}$. One has to take into account that the co-domain of h is $\mathcal{CC}(\mathfrak{O})^\uparrow$ and h is an isomorphism together with the definitions of the accessibility relations. (We leave filling out the details to the reader.)

The proof that we outlined so far suffices to prove the claim of the theorem for $\mathfrak{A}_\mathfrak{d}$. For the other $\mathfrak{A}_\mathfrak{d}^*$ gaggles, their characteristic frame condition has to be shown to hold on the canonical structure. We separate this step into the next lemma. This proof is concluded by remarking that (f11–f44) hold on $\mathfrak{F}_\mathfrak{c}$ (when appropriate) by Lemma 2.9. □

Lemma 2.9. *If an $\mathfrak{A}_\mathfrak{d}^*$ gaggle satisfies an inequation from among* 11–44*, then $\mathfrak{F}_\mathfrak{c}$, the canonical frame of the gaggle, satisfies the condition with the same number from among* (f11–f44)*.*

Proof. The inequations fall into four groups from the point of view of the proof of this lemma, and we illustrate proofs in three of those groups. Proofs of the frame conditions from the fourth group may be found in the literature on nonclassical logics.[10]

1. First we show that condition (f21) holds on the canonical frame of $\mathfrak{A}_{21}$. The consequent of (f21) is existentially quantified, therefore, we define a subset of A that we then extend to a prime ideal. We also show that the complement of the latter bears $R_\circ$ and $\overline{R}_+$ to certain prime filters, as required.

Let x be defined as $\{\, a\colon \forall b.\, b \notin \vartheta \Rightarrow a+b \notin \beta \,\}$. Because of the tonicity of $+$, this definition yields a co-cone (a downward closed subset). Furthermore, $+$ distributes over $\wedge$ into $\wedge$, and ϑ and β are prime filters, which together mean that x is prime too. The complement of a prime co-cone is a filter. Let R'_+ be defined as R_+, except that some of the filters occupying the argument places of R'_+ may not to be prime. The definition of x guarantees that $\overline{R}'_+\overline{x}\vartheta\beta$ holds. We note that R_+'s tonicity type is $R_+{\downarrow}{\downarrow}{\uparrow}$, whereas $\overline{R}_+$'s tonicity type is $\overline{R}_+{\uparrow}{\uparrow}{\downarrow}$.

If x were an ideal, then the proof could be concluded by showing that $\overline{x}$ is in the $R_\circ$ relation as needed. Instead, we first show the latter, and then we create a superset of $\overline{x}$ that is a prime filter. Let us assume that $a \in \alpha$, $b \in \overline{x}$, but $a \circ b \notin \varepsilon$. Since $b \notin x$, there is some $c \notin \vartheta$ such that $b + c \in \beta$. However, $(a \circ b) + c \notin \gamma$ due to the second conjunct in the antecedent of (f21). By 21, then $a \circ (b + c) \notin \gamma$, since γ is a cone. We are given that $R_\circ\alpha\beta\gamma$ and $a \in \alpha$, thus $b+c$ cannot be an element of β. From the contradiction, we conclude that

[10] See e.g., Dunn [11, Sect. 4.7] for a proof of 13 and 23. A conceptually different proof of the "squeeze lemma" is given in Bimbó [4, Sect. 3.8], on the basis of which associativity may be proven.

$R'_\circ\alpha\overline{x}\varepsilon$ (where $R'_\circ$ is like $R_\circ$ without a primeness requirement imposed upon the arguments).

Next we note that $R_\circ\downarrow\downarrow\uparrow$. The expansion of $\overline{x}$ to a prime filter may be carried out via maximizing on a set of filters that include $\overline{x}$ and preserve the relation $R'_\circ$. A filter that is maximal with respect to the defining properties is guaranteed to exist by Zorn's lemma. We may denote such a maximal element by δ. $R'_\circ\alpha\delta\varepsilon$ is immediate, and $\overline{R}'_+\delta\vartheta\beta$ follows by the tonicity of $\overline{R}'_+$ that we noted above. δ may be proven to be prime in the usual way using the distribution type of fusion, the maximality of δ as well as the distributivity of meet and join in the underlying lattice.[11]

2. Next we prove that (f44) is true on the canonical frame of $\mathfrak{A}_{44}$. Again, there is an interest in this proof, because of the way the appropriate prime filters are constructed. We include some of the details here. Suppose the antecedent of (f44). We set $x = [\alpha \circ \varepsilon)$. We may observe that $R'_\circ\alpha\varepsilon x$ holds (where $R'_\circ$ is like $R_\circ$ except that the requirement that the arguments are prime filters is omitted). We claim that $Q'_+\overline{\gamma}\overline{\delta}x$ holds as well, where Q'_+ is the uniform accessibility relation associated to R_+.[12] To show that $Q'_+\overline{\gamma}\overline{\delta}x$ holds, let us assume that $c \notin \gamma$, $d \notin \delta$ and $c + d \in x$. By the definition of x, and using that $+$ is monotone, plus that α and ε are prime filters, it follows that $a \circ e \leq c + d$ for some $a \in \alpha$, $e \in \varepsilon$. By residuation, $d \prec (a \circ e) \leq c$, hence—a fortiori—$a \circ (d \prec e) \leq c$. Then $a \circ (d \prec e) \notin \gamma$, since $c \notin \gamma$. But then $d \prec e \notin \beta$, because $a \in \alpha$ and $R_\circ\alpha\beta\gamma$. The residuation between $\prec$ and $+$ implies that $e \leq (d \prec e) + d$ (for any e and d). However, $(d \prec e) + d \notin \varepsilon$ because $\overline{R}_+\beta\delta\varepsilon$ and $d \notin \delta$, by the second assumption. Then $e \notin \varepsilon$, which is a contradiction.

We note that Q'_+ is antitone in its last argument place, therefore, x can be maximized to obtain a ϑ by defining a set of filters x' such that $x \subseteq x' \,\&\, Q'_+\overline{\gamma}\overline{\delta}x'$. ϑ can be proven to be prime, hence $\overline{R}_+\gamma\delta\vartheta$—due to the relationship between R_+ and Q'_+. The construction guarantees the truth of the other conjunct $R_\circ\alpha\varepsilon\vartheta$ in the consequent of (f44), since $R_\circ$ is monotone in its last argument place, that is, from $R_\circ\alpha\varepsilon x$ and $x \subseteq \vartheta$, $R_\circ\alpha\varepsilon\vartheta$ follows.

3. Lastly, as an illustration of proofs from the third group, we prove that (f42) holds on the canonical frame of $\mathfrak{A}_{42}$. Let us assume that $\overline{R}_+\alpha\beta\gamma$ as well as $\overline{R}_+\gamma\delta\varepsilon$. Obviously, $\overline{\beta}$ and $\overline{\delta}$ are (nonempty proper) prime ideals. We take $(\overline{\beta} + \overline{\delta}\,]$ for ζ, that is, $\zeta = \{\, c\colon \exists b_1, b_2 \in \overline{\beta}\; \exists d_1, d_2 \in \overline{\delta}.\, (b_1 + d_1) \vee (b_2 + d_2) \geq c \,\}$. We claim that $\forall a \notin \alpha\, \forall c \in \zeta.\, a + c \notin \varepsilon$. To see that this is indeed the case, let us assume $a + c \in \varepsilon$ for some a and c that instantiate the quantifiers. Then $\exists b \in \overline{\beta}\, \exists d \in \overline{\delta}.\, b + d \geq c$. $+$ is an isotone operation, hence $a + (b + d) \geq a + c$. 42 is equivalent to half of associativity for $+$, namely, to $a + (b + d) \leq (a + b) + d$. Therefore, from the assumption $a + c \in \varepsilon$, we may conclude that $a + (b + d) \in \varepsilon$ and $(a + b) + d \in \varepsilon$ too. By the definition of R_+, $a + b \in \gamma$ or $d \in \delta$. However, $d \in \overline{\delta}$ which means that $a + b \in \gamma$. This is a contradiction, because $\overline{R}_+\alpha\beta\gamma$, $a \in \overline{\alpha}$ as well as $b \in \overline{\beta}$ are assumed to be true.

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[11] A primeness lemma together with some of its variants may be found in [7]—see lemma 2.3.29 and the following remarks.

[12] The uniform accessibility relation here is used according to its definition 2.3.18 in [7].

The set ζ, that we defined above, is an ideal, however, there is no reason for ζ to be prime. To remedy this situation, first, we define $E = \{ I \colon \zeta \subseteq I \,\&\, \forall a \notin \alpha \; \forall c \in I .\, a + c \notin \varepsilon \}$. ζ is an element of E—due to the definitions of ζ and E. It is easy to see that the other hypotheses for the applicability of Zorn's lemma (for posets) are also satisfied. The universally quantified formula in the definition of E is in fact the formula that defines the uniform accessibility relation Q'_+ (that is associated to R_+), since $a \notin \alpha$ is the same as $a \in \overline{\alpha}$. The tonicity of Q'_+ harmonizes with the maximization of ζ, and the resulting ideal ζ' may be shown to be prime relying on the distribution type of $+$. Then $\bar{R}_+\alpha\vartheta\varepsilon$, where $\vartheta = -\zeta'$. The construction of ζ and the tonicity of $\bar{R}_+$ in its third argument place implies that $\bar{R}_+\beta\delta\vartheta$ is also true. □

The next theorem is the completeness theorem or—in other terms—the embedding theorem for $\mathfrak{A}^*_\mathfrak{d}$ gaggles.

Theorem 2.10. *An* $\mathfrak{A}^*_\mathfrak{d}$ *gaggle is* isomorphic *to a concrete gaggle defined on its canonical frame.*

Proof. The proof goes along lines similar to other completeness proofs—though the details are, of course, different. The first portion of the proof is theorem 2.8. Afterward, it suffices to define a function that has the properties of v (from definition 2.5). $h(a) = \{ \alpha \colon a \in \alpha \,\&\, \alpha \in \mathcal{P}_o \}$ is a suitable choice, moreover, h is 1–1 between A and $\mathcal{CC}(\mathfrak{O})^\uparrow$. (We do not include the rest of the details here.) □

Now we turn to a theorem that is rarely stated in connection to a relational semantics of a logic. Indeed, it often could not be stated, because it would be simply false. Above, in Definition 2.3, we gave a tighter (than the usual) characterization of the frames for the $\mathfrak{A}^*_\mathfrak{d}$ gaggles. In a sense, we precisely described the frames for the $\mathfrak{A}^*_\mathfrak{d}$ gaggles—as may be seen from the following theorem.

Theorem 2.11. *A frame for an* $\mathfrak{A}^*_\mathfrak{d}$ *gaggle is* homeomorphic *and* relationally isomorphic *to the canonical frame of the concrete gaggle arising on the frame.*

Proof. To prove the claim we have to find a function from a structure into the canonical structure of its gaggle so that the function can be proven to be a homeomorphism, which possesses the required isomorphism properties as well. We define the function f for this purpose as follows.

$$f\alpha = \{ O \colon O \in \mathcal{CC}(\mathfrak{O})^\uparrow \,\&\, \alpha \in O \}$$

The function is obviously well-defined and it has the right type. We omit the details of showing that f is a homeomorphism, however, we demonstrate in some detail that f is a relational isomorphism for the two ternary relations.

1. Let us assume that $R_\circ\alpha\beta\gamma$ holds. To prove that $R_\circ f\alpha f\beta f\gamma$, we have to show that $\forall O_1 \in f\alpha \; \forall O_2 \in f\beta.\; O_1 \circ O_2 \in f\gamma$. The two antecedents of the implication yield—by the definition of f—that $\alpha \in O_1$ and $\beta \in O_2$. Having applied (v2), we get that $\gamma \in O_1 \circ O_2$, and then by the definition of f, $O_1 \circ O_2 \in f\gamma$.

To prove the converse, we start with the assumption that $\bar{R}_\circ\alpha\beta\gamma$ obtains. By (f9), there are clopen cones O_1 and O_2 such that $\alpha \in O_1$ and $\beta \in O_2$ though

$\gamma \notin O_1 \circ O_2$. From the definition of f, we get that $O_1 \in f\alpha$, $O_2 \in f\beta$ and $O_1 \circ O_2 \notin f\gamma$. Using (b3) and the definition of $R_\circ$, we arrive at $\bar{R}f\alpha f\beta f\gamma$, as needed.

2. To prove relational isomorphism with respect to R_+, first, we assume that $R_+\alpha\beta\gamma$. Then by (f10), $\alpha \notin O_1$, $\beta \notin O_2$ whereas $\gamma \in O_1 + O_2$ for some clopen cones O_1, O_2. The definition of f ensures that $O_1 \notin f\alpha$, $O_2 \notin f\gamma$ but $O_1 + O_2 \in f\gamma$. The definition of R_+ is (b4), and it gives—together with the previously established facts—that $R_+f\alpha f\beta f\gamma$.

The other direction of the equivalence is $R_+f\alpha f\beta f\gamma$ implies $R_+\alpha\beta\gamma$. This time we do not contrapose, but assume $R_+f\alpha f\beta f\gamma$. By (b4), this means that $\exists O_1, O_2.\, O_1 \notin f\alpha \,\&\, O_2 \notin f\beta \,\&\, O_1 + O_2 \in f\gamma$. The definition of f means that $O_1, O_2 \in \mathcal{CC}(\mathfrak{O})^\uparrow$, as well as that $\alpha \notin O_1$, $\beta \notin O_2$ and $\gamma \in O_1 + O_2$. Then by (v5), $R_+\alpha\beta\gamma$ or $\alpha \in O_1$ or $\beta \in O_2$ follows. The latter two disjuncts are false, therefore, $R_+\alpha\beta\gamma$ holds, as desired. □

Algebras are often considered together with homomorphisms, just as logics are usually investigated together with their interpretations. This means that it is straightforward to define categories for the $\mathfrak{A}^*_\mathfrak{d}$ gaggles. Obviously, the definition of each $\mathfrak{A}^*_\mathfrak{d}$ gaggle induces a variety, and so for each of them we can have a category of algebras and a category of structures. Accordingly, Definitions 2.12 and 2.13 should be thought to be parametrized by a gaggle from the class of the $\mathfrak{A}^*_\mathfrak{d}$ gaggles.

Definition 2.12. The *category of $\mathfrak{A}^*_\mathfrak{d}$ gaggles* contains the algebras that are $\mathfrak{A}^*_\mathfrak{d}$ gaggles (as objects) together with homomorphisms, which are functions that preserve all the operations and the four constants (as maps).

Definition 2.13. The *category of $\mathfrak{F}$ frames for an $\mathfrak{A}^*_\mathfrak{d}$ gaggle* contains structures for the $\mathfrak{A}^*_\mathfrak{d}$ gaggle (as objects) together with frame morphisms (as maps).

Frame morphisms are defined for all the structures for the $\mathfrak{A}^*_\mathfrak{d}$ gaggles in the same way. In particular, frame morphisms are *continuous order preserving* functions ψ, where $\psi\colon \mathfrak{F} \longrightarrow \mathfrak{F}'$ and the conditions (∘1–∘4) as well as (+1–+4) are satisfied.

(∘1) $R_\circ\alpha\beta\gamma \;\Rightarrow\; R'_\circ\psi\alpha\psi\beta\psi\gamma$,
(∘2) $R'_\circ\alpha\beta\psi\gamma \;\Rightarrow\; \exists\delta\varepsilon.\, R_\circ\delta\varepsilon\gamma \,\&\, \alpha \le \psi\delta \,\&\, \beta \le \psi\varepsilon$,
(∘3) $R'_\circ\psi\alpha\beta\gamma \;\Rightarrow\; \exists\delta\varepsilon.\, R_\circ\alpha\delta\varepsilon \,\&\, \beta \le \psi\delta \,\&\, \psi\varepsilon \le \gamma$,
(∘4) $R'_\circ\alpha\psi\beta\gamma \;\Rightarrow\; \exists\delta\varepsilon.\, R_\circ\delta\beta\varepsilon \,\&\, \alpha \le \psi\delta \,\&\, \psi\varepsilon \le \gamma$,
(+1) $\bar{R}_+\alpha\beta\gamma \;\Rightarrow\; \bar{R}'_+\psi\alpha\psi\beta\psi\gamma$,
(+2) $\bar{R}'_+\alpha\beta\psi\gamma \;\Rightarrow\; \exists\delta\varepsilon.\, \bar{R}_+\delta\varepsilon\gamma \,\&\, \psi\delta \le \alpha \,\&\, \psi\varepsilon \le \beta$,
(+3) $\bar{R}'_+\psi\alpha\beta\gamma \;\Rightarrow\; \exists\delta\varepsilon.\, \bar{R}_+\alpha\delta\varepsilon \,\&\, \psi\delta \le \beta \,\&\, \gamma \le \psi\varepsilon$,
(+4) $\bar{R}'_+\alpha\psi\beta\gamma \;\Rightarrow\; \exists\delta\varepsilon.\, \bar{R}_+\delta\beta\varepsilon \,\&\, \psi\delta \le \alpha \,\&\, \gamma \le \psi\varepsilon$.

Now we assume the same definition of h and of f as above, and we use the notation g^{-1} for the *inverse image* of g. The following lemma is paramount to the duality between the categories of gaggles and structures.

Lemma 2.14. *If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are both $\mathfrak{A}^*_\mathfrak{d}$ gaggles (of the same kind), and $\varphi\colon A_1 \longrightarrow A_2$ is a homomorphism, then $h\varphi a = {\varphi^{-1}}^{-1}ha$.*

If $\mathfrak{F}_1$ *and* $\mathfrak{F}_2$ *are structures for the same* $\mathfrak{A}^*_{\mathfrak{d}}$ *gaggle, and* $\psi\colon U_1 \longrightarrow U_2$ *is a frame morphism, then* $f\psi\alpha = \psi^{-1\,-1} f\alpha$.

Proof. After Theorems 2.10 and 2.11 have been proven, the proof of this lemma is straightforward. The proof steps rely on the definitions of h and f in an essential way, as well as on the fact that φ^{-1} and ψ^{-1} have the right co-domain. (We do not include the rest of the details here.) □

Theorem 2.15. *The pairs of categories defined above are duals of each other.*

Proof. This theorem is the culmination of the results in this section. A proof can be composed from the proofs of theorems 2.10 and 2.11, and Lemma 2.14 together with the observation that the categories are duals of each other. The latter is a result of the inverse image construction that is apparent from Lemma 2.14. □

The canonical construction leading from $\mathfrak{A}^*_{\mathfrak{d}}$ gaggles to structures for gaggles, and the canonical construction leading from structures for an $\mathfrak{A}^*_{\mathfrak{d}}$ gaggle to a concrete $\mathfrak{A}^*_{\mathfrak{d}}$ gaggle may be proven to be *functors*. The core of the proof is given by Theorem 2.15. A few additional standard requirements concerning identities and function composition are easily seen to be satisfied.

3. Multiplicative-Additive Interaction

The uniform approach provided by gaggle theory allows us to *generalize* the 16 inequations connecting operations from the fusion and fission families to certain other operations of arbitrary finite arity. Moreover, with the concept of the distribution type of an operation, we can clarify some of the less well-motivated choices that were made by Grishin.[13]

Let $\circledast_1$ and $\circledast_2$ be two operations of arity z and z' (where $z, z' \in \mathbb{Z}^+$). Additionally, we suppose that they respect the bounds and are isotone in one of their arguments, whereas they distribute into join and into meet, respectively. (We use $⩙$ to denote $\wedge$ or $\vee$, and $\mathbb{0}$ to denote 0 or 1. A bracket with a subscript singles out an argument place, whereas $\rightarrow$ indicates that the other argument places are filled in as well.)

Concisely, the distribution type of $\circledast_1$ and $\circledast_2$ is

$$\circledast_1\colon \overrightarrow{⩙}, [\vee]_i \longrightarrow \vee \qquad \text{and} \qquad \circledast_2\colon \overrightarrow{⩙}, [\wedge]_j \longrightarrow \wedge. \tag{dt}$$

The operations preserve or reverse the extremal elements as follows.

$$\circledast_1\colon \overrightarrow{\mathbb{0}}, [0]_i \longrightarrow 0 \qquad \text{and} \qquad \circledast_2\colon \overrightarrow{\mathbb{0}}, [1]_j \longrightarrow 1. \tag{et}$$

To create an inequation, we compose the two operations in their ith and jth argument places. Obviously, there are two terms that can result. We insert $\leq$ between the two terms so that the main operation of the term is on the same side in this inequation as in the (quasi-)inequation that expresses the residuation of the operation with respect to some operation (in the *free* family

[13] In [7] we introduced *partial* and *extrapolated distribution types*, that may replace distribution types in algebras that do not contain a lattice.

of operations). In other words, the term in which $\circledast_1$ is the main operation is placed to the left from $\leq$, and the other term is put on the right-hand side of $\leq$.

The inequation (M), the "Master Distribution," is what we obtain given $\circledast_1$ and $\circledast_2$ with the previously assumed types.

$$\circledast_1(\vec{a}, [\circledast_2(\vec{c}, [b]_j)]_i) \leq \circledast_2(\vec{c}, [\circledast_1(\vec{a}, [b]_i)]_j) \tag{M}$$

Lemma 3.1. *All the 16 inequations considered by Grishin are* instances *of* (M) *with the operations chosen from the* $\{\leftarrow, \circ, \rightarrow\}$ *and* $\{\succ, +, \prec\}$ *families. Given these two families, the 16 inequations* exhaust *the range of possible instantiations of* (M).

The proof of this lemma is easy and we leave the proof (as an exercise) to the reader.

Now we may consider a gaggle that contains two operations like $\circledast_1$ and $\circledast_2$. We note that the operations may belong to one family (and then have the same arity) or they may belong to different families (and then they may have the same or different arities). In either case, the families are not assumed to be complete. Now we define a *super* **gGl** and a relational topological structure for that super gaggle. We use some of the same notation as in Definition 2.3. Additionally, we use $\updownarrow$ and $\pitchfork$ to refer to $\uparrow$ or $\downarrow$, and to $\in$ or $\notin$, respectively. The actual shape of $\updownarrow$ and $\pitchfork$ depends on the distribution type of the operation that has the same subscript as R has. The two operations that appear in (f6) may be viewed as abbreviations at this point—they may be eliminated via (v3) and (v4).

Definition 3.2. Let $\mathfrak{A}_{\mathfrak{M}}$ be a *distributive bounded gaggle with* 0–1 *operations*, that is, $\mathfrak{A}_{\mathfrak{M}} = \langle A; \wedge, \vee, 0, 1, \circledast_1, \circledast_2\rangle$, where the operations have types as in (dt) and (et) above, and let (M) be true.

A *structure for* $\mathfrak{A}_{\mathfrak{M}}$ is $\mathfrak{F} = \langle U, \leq, R_1, R_2, \mathfrak{O}\rangle$, where (f1–f6) and (fm) are true.

(f1) $U \neq \emptyset, \quad R_1 \subseteq U^{z+1}, \quad R_2 \subseteq U^{z'+1}, \quad \mathfrak{O} \subseteq \wp(U),$
(f2) $\langle U, \leq, \mathfrak{O}\rangle$ is a Priestley space,
(f3) $\alpha \leq \alpha, \quad \alpha \leq \beta \,\&\, \beta \leq \gamma. \Rightarrow \alpha \leq \gamma, \quad \alpha \leq \beta \,\&\, \beta \leq \alpha. \Rightarrow \alpha = \beta,$
(f4) $R_1 \overset{\rightarrow}{\updownarrow} [\downarrow]_i [\uparrow]_{z+1}, \quad R_2 \overset{\rightarrow}{\updownarrow} [\downarrow]_j [\uparrow]_{z'+1},$
(f5) $\vec{O} \in \mathcal{C}\!\mathcal{O}(\mathfrak{O})^{\uparrow} \Rightarrow \circledast_1(\vec{O}) \in \mathcal{C}\!\mathcal{O}(\mathfrak{O}), \quad \vec{O} \in \mathcal{C}\!\mathcal{O}(\mathfrak{O})^{\uparrow} \Rightarrow \circledast_2(\vec{O}) \in \mathcal{C}\!\mathcal{O}(\mathfrak{O}),$
(f6) $\bar{R}_1(\vec{\alpha}, [\beta]_i, [\gamma]_{z+1}) \Rightarrow \exists \vec{O}, P. \bigwedge \overrightarrow{\alpha \pitchfork O} \,\&\, \beta \in P \,\&\, \gamma \notin \circledast_1(\vec{O}, [P]_i),$
$R_2(\vec{\alpha}, [\beta]_j, [\gamma]_{z'+1}) \Rightarrow \exists \vec{O}, P. \bigwedge \overrightarrow{\alpha \pitchfork O} \,\&\, \beta \notin P \,\&\, \gamma \in \circledast_2(\vec{O}, [P]_j).$
(fm) $R_1(\vec{\alpha}, [\beta]_i, [\gamma]_{z+1}) \,\&\, \bar{R}_2(\vec{\eta}, [\varepsilon]_j, [\gamma]_{z'+1}). \Rightarrow$
$\exists \vartheta. \bar{R}_2(\vec{\eta}, [\vartheta]_j, [\beta]_{z'+1}) \,\&\, R_1(\vec{\alpha}, [\vartheta]_i, [\varepsilon]_{z+1}).$

Based on any structure for $\mathfrak{A}_{\mathfrak{M}}$ we may construct a model by adding a valuation function.

Definition 3.3. A *model for* $\mathfrak{A}_{\mathfrak{M}}$ is defined by adding a valuation function v ($v \colon A \longrightarrow \mathcal{C}\!\mathcal{O}(\mathfrak{O})^{\uparrow}$) such that it satisfies the following four clauses.

(v1) $v(a \wedge b) = v(a) \cap v(b)$,
(v2) $v(a \vee b) = v(a) \cup v(b)$,
(v3) $v \circledast_1(\vec{a}, [b]_i) = \{ \gamma \colon \exists \vec{\alpha}, \beta.\, R_1(\vec{\alpha}, [\beta]_i, [\gamma]_{z+1})\ \&\ \bigwedge \overrightarrow{\alpha \pitchfork va}\ \&\ \beta \in vb \}$,
(v4) $v \circledast_2(\vec{a}, [b]_i) = \{ \gamma \colon \forall \vec{\alpha}, \beta.\, \bar{R}_2(\vec{\alpha}, [\beta]_j, [\gamma]_{z'+1})\ \&\ \bigwedge \overrightarrow{\alpha \not\pitchfork va}. \Rightarrow \beta \in vb \}$.

Lemma 3.4. *Given a frame for* $\mathfrak{A}_{\mathfrak{M}}$, *there is* a concrete gaggle *of sets of elements of the structure that form an* $\mathfrak{A}_{\mathfrak{M}}$ *gaggle, that is,* (M) *holds in an algebra of clopen cones.*

Proof. We outline only the step that shows that (M) is true in a model on $\mathfrak{F}$. (We leave the reconstruction of the rest of the proof to the reader.) Let $\gamma \in v(\circledast_1(\vec{a}, [\circledast_2(\vec{c}, [b]_j)]_i))$ but $\gamma \notin v(\circledast_2(\vec{c}, [\circledast_1(\vec{a}, [b]_i)]_j))$. By (v3) and (v4) we get that $\exists \vec{\alpha}, \beta.\, R_1(\vec{\alpha}, [\beta]_i, [\gamma]_{z+1})$ such that $\alpha \pitchfork va$ and $\beta \in v(\circledast_2(\vec{c}, [b]_j))$, and $\exists \vec{\alpha}, \beta.\, \bar{R}_2(\vec{\eta}, [\varepsilon]_j, [\gamma]_{z'+1})$ such that $\eta \not\pitchfork vc$ for each η and c, and $\varepsilon \notin v(\circledast_1(\vec{a}, [b]_i))$, respectively. Two of the conjuncts, which include β and ε, expand into universally quantified formulas. By two detachments from (fm)—after the universal quantifiers have been instantiated—we get both $\vartheta \in b$ and $\vartheta \notin b$, which is a contradiction. □

Going in the other direction, we build a structure from a **gGl**. (We extend the previous notation $\tau(\mathfrak{T})$ to the case where $\mathfrak{T}$ is a subbasis.)

Definition 3.5. The *canonical frame* of an $\mathfrak{A}_{\mathfrak{M}}$ gaggle is $\mathfrak{F}_{\mathfrak{c}} = \langle \mathcal{P}_o, \subseteq, R_1, R_2, \mathfrak{O} \rangle$, where the elements of the quintuple are characterized as follows.
(a1) $\mathcal{P}_o$ is the set of proper nonempty prime filters on A,
(a2) $\subseteq$ is set inclusion,
(a3) $R_1(\vec{\alpha}, [\vartheta]_i, [\varepsilon]_{z+1}) \ \Leftrightarrow\ \forall \vec{a}, b.\ \bigwedge \overrightarrow{a \pitchfork \alpha}\ \&\ b \in \vartheta. \Rightarrow \circledast_1(\vec{a}, [b]_i) \in \varepsilon$,
(a4) $R_2(\vec{\eta}, [\vartheta]_j, [\beta]_{z'+1}) \ \Leftrightarrow\ \exists \vec{a}, b.\ \bigwedge \overrightarrow{a \not\pitchfork \eta}\ \&\ b \notin \vartheta\ \&\ \circledast_2(\vec{a}, [b]_j) \in \beta$,
(a5) $\mathfrak{O} = \tau(\mathfrak{S})$ where $\mathfrak{S} = \{ O \colon O = \{ \alpha \colon a \in \alpha\ \&\ \alpha \in \mathcal{P}_o \} \}$.

This definition is successful in the sense that it is suitable for our purposes as the next lemma shows.

Lemma 3.6. *The canonical frame of an* $\mathfrak{A}_{\mathfrak{M}}$ *gaggle is* a structure for the $\mathfrak{A}_{\mathfrak{M}}$ gaggle, *in particular,* $\mathfrak{F}_{\mathfrak{c}}$ *satisfies* (fm).

Proof. We omit most of the details here. However, we outline the step that shows that (fm) is true on the canonical frame.
1.1 Let us assume that $R_1(\vec{\alpha}, [\beta]_i, [\gamma]_{z+1})$ and $\bar{R}_2(\vec{\eta}, [\varepsilon]_j, [\gamma]_{z'+1})$. The uniform accessibility relation Q'_2 (that is a counterpart of R_2) is defined as follows—with the μ's ranging over filters and ideals according to the (actual) distribution type of $\circledast_2$, and with ϑ and β being filters.

$$Q'_2(\vec{\mu}, [\overline{\vartheta}]_j, [\beta]_{z'+1}) \ \Leftrightarrow\ \forall \vec{a}, b.\ \bigwedge \overrightarrow{a \in \mu}\ \&\ b \in \overline{\vartheta}. \Rightarrow \circledast_2(\vec{a}, [b]_j) \notin \beta.$$

1.2 To prove that (fm) holds, a suitable ϑ has to be found, or rather "constructed." Let us define a set x as $\{ b \colon \forall a.\ \bigwedge \overrightarrow{a \in \mu} \Rightarrow \circledast_2(\vec{a}, [b]_j) \notin \beta \}$, where each μ is an η or an $\overline{\eta}$ (from the second assumption) depending on whether Q'_2 requires an ideal or filter in that argument place. The shape of the definition makes it clear that $Q''_2(\vec{\mu}, [x]_j, [\beta]_{z'+1})$, (where we use $''$ to indicate that we do not claim that x is an ideal). The set x is a co-cone. To see this, let us assume that $b \in x$ and $c \leq b$. $\circledast_2$ is monotone in its jth argument place, that is, $\circledast_2(\vec{a}, [c]_j) \leq \circledast_2(\vec{a}, [b]_j)$. Then using the definition of x, $\forall a.\ \bigwedge \overrightarrow{a \in \mu} \Rightarrow \circledast_2(\vec{a}, [c]_j) \notin \beta$, hence $c \in x$.

Now let us assume that $d_1, d_2 \notin x$. Then there are some c's and e's such that $c \in \mu$ and $e \in \mu$, for each c, e and μ as appropriate. Also, $\circledast_2(\vec{c}, [d_1]_j) \in \beta$ as well as $\circledast_2(\vec{e}, [d_2]_j) \in \beta$. Taking meets and joins of the c's and e's to get a's when the μ's are filters and ideals (i.e., setting $a = c \curlywedge\!\!\curlyvee e$), we get $\circledast_2(\vec{a}, [d_1]_j) \in \beta$ and $\circledast_2(\vec{a}, [d_2]_j) \in \beta$. β is a prime filter, and so $\circledast_2(\vec{a}, [d_1 \wedge d_2]_j) \in \beta$, that is, $d_1 \wedge d_2 \notin x$. This means that we have that $\overline{R}'_2(\vec{\eta}, [\overline{x}]_j, [\beta]_{z'+1})$, where the prime indicates that the arguments are filters though, perhaps, some of them are not prime.

1.3 Next we show that $\overline{x}$ stands in the R'_1 relation with the α's and ε. Let us suppose that for some a's, $a \pitchfork \alpha$ and $b \in \overline{x}$, but $\circledast_1(\vec{a}, [b]_i) \notin \varepsilon$. Using the definition of x, we get that there are c's such that $c \pitchfork \eta$ for the η's, whereas $\circledast_2(\vec{c}, [b]_j) \in \beta$. Then—by the second assumption—$\circledast_2(\vec{c}, [\circledast_1(\vec{a}, [b]_i)]_j) \notin \gamma$. However, in $\mathfrak{A}_{\mathfrak{M}}$ the inequation (M) is true, which means that $\circledast_1(\vec{a}, [\circledast_2(\vec{c}, [b]_j)]_i) \notin \gamma$, since γ is upward closed. Then—by the first assumption—$\circledast_2(\vec{c}, [b]_j) \notin \beta$, because the other possibilities in the disjunction are excluded by the $a \pitchfork \alpha$'s. Having finished with the reduction, we have that $R'_1(\vec{\alpha}, [\overline{x}]_i, [\varepsilon]_{z+1})$.

1.4 We showed that x is prime (in step **1.2**), however, we have not proven x to be an ideal, which means that $\overline{x}$ is not known to be a prime filter. The preceding step though provides us with a sufficient condition for a squeeze lemma, once we will have observed that R_1 (hence R'_1) is antitone in its ith argument place. [Cf. (f4) in definition 3.2.] We start with the definition of a set that we will call E.

$$E = \{\, F \colon \overline{x} \subseteq F \wedge \forall a.\ \bigwedge \overrightarrow{a \pitchfork \alpha}\ \&\ b \in F. \Rightarrow \circledast_1(\vec{a}, [b]_i) \in \varepsilon \,\}.$$

E is clearly nonempty, because $\overline{x} \in E$. Obviously, $\langle E, \subseteq \rangle$ is a poset, furthermore, chains of elements of E have an upper bound in E. By Zorn's lemma, there is a maximal element in E; let us say ϑ is such an element. Having the definitions of R'_1 and E mingled, we obtain that $R'_1(\vec{\alpha}, [\vartheta]_i, [\varepsilon]_{z+1})$.

1.5 Let $d_1 \vee d_2$ be an element of ϑ and let us assume for a reduction that neither d_1 nor d_2 is in ϑ. Then there have to be some c's and e's that are (or are not) elements of the α's (as appropriate), and there are $f_1, f_2 \in \vartheta$ such that $\circledast_1(\vec{c}, [d_1 \wedge f_1]_i) \notin \varepsilon$ and $\circledast_1(\vec{e}, [d_2 \wedge f_2]_i) \notin \varepsilon$. However, the membership in the α's (or the lack thereof) is in harmony with the tonicity of $\circledast_1$. For example, if $c_k, e_k \notin \alpha_k$, then $\circledast_1$ is antitone in its kth argument place, $\circledast_1(\vec{c}, [c_k \vee e_k]_k, [d_1 \wedge f_1]_i) \notin \varepsilon$ and $\circledast_1(\vec{e}, [c_k \vee e_k]_k, [d_2 \wedge f_2]_i) \notin \varepsilon$, as well as $c_k \vee e_k \notin \alpha_k$ due to α_k's primeness. ϑ is a filter, hence $f_1 \wedge f_2 = f \in \vartheta$. Then by $\circledast_1 \colon \vec{\updownarrow}, [\uparrow]_i$, it follows that $\circledast_1(\vec{c}, [d_1 \wedge f]_i) \notin \varepsilon$ and $\circledast_1(\vec{e}, [d_2 \wedge f]_i) \notin \varepsilon$ too. After we have melded the c's and the e's, as well as the f's, we have that $\circledast_1(\overrightarrow{c \curlywedge\!\!\curlyvee e}, [d_1 \wedge f]_i) \notin \varepsilon$ and $\circledast_1(\overrightarrow{c \curlywedge\!\!\curlyvee e}, [d_2 \wedge f]_i) \notin \varepsilon$. ε is prime, thus $\circledast_1(\overrightarrow{c \curlywedge\!\!\curlyvee e}, [d_1 \wedge f]_i) \vee \circledast_1(\overrightarrow{c \curlywedge\!\!\curlyvee e}, [d_2 \wedge f]_i) \notin \varepsilon$. Two distributions lead to $\circledast_1(\overrightarrow{c \curlywedge\!\!\curlyvee e}, [(d_1 \vee d_2) \wedge f]_i) \notin \varepsilon$. The apparent contradiction means that ϑ is a prime ideal, that is, $R_1(\vec{\alpha}, [\vartheta]_i, [\varepsilon]_{z+1})$. Finally, we note that $\bar{R}_2$ is increasing in its jth argument place [cf. (f4) in Definition 3.2], hence $\bar{R}_2(\vec{\eta}, [\vartheta]_j, [\beta]_{z'+1})$ is immediate from step **1.2**.

We note that $\mathfrak{F}_\mathfrak{c} \vDash$ (fm) may be thought to suffice for the claim that the canonical structure of an $\mathfrak{A}_\mathfrak{M}$ gaggle is a frame for that gaggle, because we already know that the canonical frame of a bounded distributive lattice is a Priestley space, and we also know that the additional operations do not interfere with the properties of the canonical frame of the bounded distributive lattice. In other words, the additional operations lead to "independent" additions in the structure. □

Now we can prove the following completeness theorem.

Theorem 3.7. *An $\mathfrak{A}_\mathfrak{M}$ gaggle is* isomorphic *to a concrete gaggle of sets defined on the canonical frame of $\mathfrak{A}_\mathfrak{M}$.*

Proof. We omit most of the details. However, we prove that h (defined as before) commutes with $\circledast_1$ (in $\mathfrak{A}_\mathfrak{M}$) and with $\circledast_1$ (defined in the model on $\mathfrak{F}_\mathfrak{c}$).
1 Let α be an element of $h(\circledast_1(\vec{a}, [b]_i))$, that is, $\circledast_1(\vec{a}, [b]_i) \in \alpha$ by the definition of h. In the argument places other than i (if there are any), $\circledast_1$ may be antitone or monotone. If $[\uparrow]_k$, then we let ε_k to be $[a_k)$, otherwise, $\varepsilon_k = (a_k]$. The definition of the uniform accessibility relation Q_1' gives that $Q_1'(\vec{\varepsilon}, [\,[b)\,]_i, [\alpha]_{z+1})$. Let E, a set of sequences of filters and ideals be defined as

$$E = \left\{ \langle \vec{\eta}, [\beta]_i \rangle \colon \bigwedge_{1=k, i \neq k}^{z} \varepsilon_k \subseteq \eta_k \;\&\; [b) \subseteq \beta \;\&\; Q_1'(\vec{\eta}, [\beta]_i, [\alpha]_{z+1}) \right\}.$$

Obviously, $E \neq \emptyset$. E is partially ordered by pointwise inclusion, and the existence of an upper bound of a chain of elements of E is assured by pointwise unions. The conditions for an application of Zorn's lemma are met, thus we may denote by $\langle \vec{\eta}', [\beta']_i \rangle$ a maximal element of E. The elements may be shown to be prime similarly as we showed ϑ to be prime in the proof of lemma 3.6. (We omit the details here.) That is, $Q_1(\vec{\eta}', [\beta']_i, [\alpha]_{z+1})$, and then due to the relationship between the uniform accessibility relation and R_1, we have $R_1(\vec{\mu}, [\beta']_i, [\alpha]_{z+1})$, where μ's are the η''s if those are filters, otherwise, $\mu = -\eta'$. $a \pitchfork \mu$'s and $b \in \beta'$ yield $\mu \pitchfork ha$'s and $\beta' \in hb$, by the definition of h. The formula $\exists \vec{\mu}, \beta.\, R_1(\vec{\mu}, [\beta]_i, [\alpha]_{z+1}) \;\&\; \bigwedge \overrightarrow{\mu \pitchfork ha} \;\&\; \beta \in hb$ gathers the pieces together, which means that obviously, $\alpha \in \circledast_1(\overrightarrow{ha}, [hb]_i)$.

For the converse we start with assuming $\alpha \in \circledast_1(\overrightarrow{ha}, [hb]_i)$. From the assumption, we get by (v3), $\mu \pitchfork ha$ for each a, as well as $\beta \in hb$. After an application of the definition of h, we have the antecedent of the implication on the right-hand side in (a3); and so by detachment, $\circledast_1(\vec{a}, [b]_i) \in \alpha$. By an application of the definition of h, $\alpha \in h(\circledast_1(\vec{a}, [b]_i))$, as desired. □

The next theorem mirrors the previous theorem in the class of structures.

Theorem 3.8. *A structure for $\mathfrak{A}_\mathfrak{M}$ is* homeomorphic *and* relationally isomorphic *to a concrete structure for $\mathfrak{A}_\mathfrak{M}$ that is built out of sets.*

Proof. The idea of the proof is to take as situations sets of elements of the gaggle that comprise clopen cones on the frame. (We omit most of the details.)

We show though that f (as previously defined) is an order preserving map. If $\alpha \leq \beta$, then $f\alpha \subseteq f\beta$, because $\gamma \in O$ (where $O \in \mathfrak{C}(\mathfrak{O})^{\uparrow}$) iff $O \in f\gamma$. That is, $O \in f\alpha$ implies $\alpha \in O$, which implies $\beta \in O$, because O is a cone and $\alpha \leq \beta$. Then $O \in f\beta$.

The other direction of the equivalence may be shown to hold as follows. If $\alpha \nleq \beta$, then there is a clopen cone O such that $\alpha \in O$ without $\beta \in O$—because the structure's topology is totally order disconnected. $O \in f\alpha$ and $O \notin f\beta$ suffices for $f\alpha \nsubseteq f\beta$. $\square$

To further expand the correspondence between the super gaggles $\mathfrak{A}_{\mathfrak{M}}$ and their structures, we delineate the sort of admissible maps between structures.

Definition 3.9. The *category of* $\mathfrak{A}_{\mathfrak{M}}$ *gaggles* comprises algebras that satisfy the defining equations for these super gaggles (as objects) together with homomorphisms (as maps).

The *category of structures for* $\mathfrak{A}_{\mathfrak{M}}$ *gaggles* consists of frames as defined in 3.2 (as objects) together with frame morphisms (as maps) which satisfy the following conditions. (Let ψ be a function from $\mathfrak{F}$ into $\mathfrak{F}'$.)

(m1) ψ is continuous,

(m2) $\alpha \leq \beta$ implies $\psi\alpha \leq \psi\beta$,

(m3) $R_1(\vec{\alpha}, [\beta]_i, [\gamma]_{z+1}) \Rightarrow R'_1(\overrightarrow{\psi\alpha}, [\psi\beta]_i, [\psi\gamma]_{z+1})$,

(m4) $\bar{R}_2(\vec{\alpha}, [\beta]_j, [\gamma]_{z'+1}) \Rightarrow \bar{R}'_2(\overrightarrow{\psi\alpha}, [\psi\beta]_j, [\psi\gamma]_{z'+1})$,

(m5) $R'_1(\vec{\alpha}, [\beta]_i, [\psi\gamma]_{z+1}) \Rightarrow \exists\vec{\varepsilon}, \delta.\, R_1(\vec{\varepsilon}, [\delta]_i, [\gamma]_{z+1})\ \&\ \bigwedge \overrightarrow{\alpha \lesseqqgtr \psi\varepsilon}\ \&\ \beta \leq \psi\delta$,

(m6) $\bar{R}'_2(\vec{\alpha}, [\beta]_j, [\psi\gamma]_{z'+1}) \Rightarrow \exists\vec{\varepsilon}, \delta.\, \bar{R}_2(\vec{\varepsilon}, [\delta]_j, [\gamma]_{z'+1})\ \&\ \bigwedge \overrightarrow{\alpha \lesseqqgtr \psi\varepsilon}\ \&\ \beta \geq \psi\delta$,

where $\lesseqqgtr$ is $\geq$ or $\leq$ depending on the tonicity type of the accessibility relation.

Lemma 3.10. Homomorphisms *between* $\mathfrak{A}_{\mathfrak{M}}$ *gaggles* commute *with* h (*that is an isomorphism by theorem* 3.7). Frame morphisms *between structures for* $\mathfrak{A}_{\mathfrak{M}}$ *gaggles* commute *with* f (*that is a homeomorphism and an isomorphism by theorem* 2.11).

Proof. The steps of this proof are straightforward, therefore, we include here only the proof of the first claim of the lemma.

Let $\alpha \in h\varphi a$ where α is a prime filter and φ is a homomorphism from an $\mathfrak{A}_{\mathfrak{M}}$ gaggle into an $\mathfrak{A}_{\mathfrak{M}}$ gaggle. Due to the definition of h, this obtains exactly when $\varphi a \in \alpha$. That is, $a \in \varphi^{-1}[\alpha]$, by the definition of inverse image. However, $\varphi^{-1}[\alpha]$ may be shown to be a prime filter itself, hence by the definition of h, $\varphi^{-1}[\alpha] \in ha$. The latter holds iff $\alpha \in \varphi^{-1-1}[ha]$. $\square$

The quintessential result about the super gaggles we considered in this paper is the content of the next theorem, that in turn, relies on the previous lemmas and theorems proven in this section.

Theorem 3.11. *The class* $\mathfrak{A}_{\mathfrak{M}}$ *of gaggles and the class of structures for* $\mathfrak{A}_{\mathfrak{M}}$ *gaggles are* dual categories, *and the canonical constructions between them are* functors.

Proof. The proof of this theorem is a combination of the previous theorems and their proofs. Additionally, it should be noted that certain identity functions exist and certain functions compose. The two categories turn out to be each other's duals, because given a φ and a ψ as above, φ^{-1} and ψ^{-1} are a frame morphism and a homomorphism, respectively, between objects that are obtained via the canonical constructions. □

4. Conclusions

In this paper, we have investigated *symmetric gaggles*, that are a special and well-motivated subclass of *super gaggles*. We first looked at a residuated groupoid and its dual placed into the context of a distributive lattice. After proving a series of lemmas and theorems about symmetric gaggles that include a certain interaction between the two families of operations expressed by *distribution like* inequations, we abstracted out the "essence" of the symmetric fusion–fission gaggles into super gaggles in which certain multiplicative and additive operations interact in a distribution like fashion. We hope to further develop these ideas and results about super gaggles, that algebraize logics comprising a variety of connectives.

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References

[1] Allwein, G., Dunn, J.M.: Kripke models for linear logic. J. Symbolic Logic **58**, 514–545 (1993)

[2] Anderson, A.R., Belnap, N.D., Dunn, J.M.: Entailment. The Logic of Relevance and Necessity, vol. II. Princeton University Press, Princeton (1992)

[3] Bimbó, K.: Semantics for structurally free logics $LC+$. Logic J. IGPL **9**, 525–539 (2001)

[4] Bimbó, K.: Relevance logics. In: Dale, J. (ed.) Philosophy of Logic. In: Gabbay, D., Thagard, P., Woods, J. (eds.) Handbook of the Philosophy of Science, vol. 5, pp. 723–789. Elsevier (North-Holland), Amsterdam (2007)

[5] Bimbó, K.: Dual gaggle semantics for entailment. Notre Dame J. Formal Logic **50**(1), 23–41 (2009)

[6] Bimbó, K., Dunn, J.M.: Two extensions of the structurally free logic LC. Logic J. IGPL **6**, 403–424 (1998)

[7] Bimbó, K., Dunn, J.M.: Generalized Galois logics. Relational Semantics of Nonclassical Logical Calculi. CSLI Lecture Notes, vol. 188, CSLI Publications, Stanford (2008)

[8] Carnielli, W., Coniglio, M.E.: Logic, combining. Stanford. In: Zalta, E. (ed.) Encyclopedia of Philosophy. URL: plato.stanford.edu/entries/logic-combining (2007)

[9] Church, A.: Introduction to Mathematical Logic, Revised and Enlarged. Princeton University Press, Princeton (1996)

[10] Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, 2nd edn. Cambridge University Press, Cambridge (2002)

[11] Dunn, J.M.: Relevance logic and entailment. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. 3, 1st edn, pp. 117–229. D. Reidel, Dordrecht (1986)

[12] Dunn, J.M.: Gaggle theory: an abstraction of Galois connections and residuation with applications to negation, implication, and various logical operators. Logics. In: van Eijck, J. (ed.) AI: European Workshop JELIA '90, Lecture Notes in Computer Science, vol. 478, pp. 31–51. Springer, Berlin (1991)

[13] Dunn, J.M.: Partial-gaggles applied to logics with restricted structural rules. In: Došen, K., Schroeder-Heister, P. (eds.) Substructural Logics, pp. 63–108. Clarendon, Oxford (1993)

[14] Dunn, J.M.: Gaggle theory applied to intuitionistic, modal and relevance logic. In: Max, I., Stelzner, W. (eds.) Logik und Mathematik. Frege–Kolloquium Jena, pp. 335–368. W. de Gruyter, Berlin (1995)

[15] Dunn, J.M., Hardegree, G.M.: Algebraic Methods in Philosophical Logic. Oxford Logic Guides, vol. 41, Oxford University Press, Oxford (2001)

[16] Dunn, J.M., Meyer, R.K.: Combinators and structurally free logic. Logic J. IGPL **5**, 505–537 (1997)

[17] Dunn, J.M., Restall, G.: Relevance logic. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, vol. 6, 2nd edn, pp. 1–128. Kluwer, Amsterdam (2002)

[18] Grishin, V.N.: On a generalization of the Ajdukiewicz–Lambek system. In: Mikhaĭlov, A.I. (ed.) Investigations into Nonclassical Logics and Formal Systems, pp. 315–334. Nauka, Moscow (1983)

[19] Lambek, J.: The mathematics of sentence structure. Am. Math. Mon. **65**, 154–169 (1958)

[20] Lambek, J.: On the calculus of syntactic types. In: Jacobson, R. (ed.) Structure of Language and its Mathematical Aspects, pp. 166–178. American Mathematical Society, Providence (1961)

[21] Meyer, R.K., Routley, R.: Algebraic analysis of entailment I. Logique Anal. **15**, 407–428 (1972)

[22] Moortgat, M.: Symmetries in natural language syntax and semantics: the Lambek–Grishin calculus. In: Leivant, D., de Queiroz, R. (eds.) WOLLIC 2007, Lecture Notes in Computer Science, vol. 4576, pp. 264–284. Springer, Heidelberg (2007)

[23] Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bull. Lond. Math. Soc. **2**, 186–190 (1970)

[24] Restall, G.: An Introduction to Substructural Logics. Routledge, London (2000)

[25] Stone, M.H.: Topological representations of distributive lattices and Brouwerian logics. Časopis pro pěstování matematiky a fysiky, Čast matematická **67**, 1–25 (1937–1938)

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