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# Visualizations of the Square of Opposition

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**Abstract.** In logic, diagrams have been used for a very long time. Nevertheless philosophers and logicians are not quite clear about the logical status of diagrammatical representations. Fact is that there is a close relationship between particular visual (resp. graphical) properties of diagrams and logical properties. This is why the representation of the four categorical propositions by different diagram systems allows a deeper insight into the relations of the logical square. In this paper I want to give some examples.

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## 1. Introduction

In traditional logic the relationships between the four categorical propositions – *All S are P* (short form *SaP*), *No S is P* (short form *SeP*), *Some S are P* (short form *SiP*), and *Some S are not P* (short form *SoP*) – are represented in the so-called 'square of opposition' (Figure 1).

The *contrary relation* between $SaP$ and $SeP$ means that these propositions cannot both be true but can both be false. The *subcontrary relation* between $SiP$ and $SoP$ means that $SiP$ and $SoP$ cannot both be false but may both be true. The *contradictory relation* between $SaP$ and $SoP$ (resp. $SeP$ and $SiP$) means that one of them is true if and only if the other is false. The *subaltern relation* between $SiP$ and $SaP$ and between $SoP$ and $SeP$ means that if $SaP$ is true then $SiP$ is true and if $SeP$ is true then $SoP$ is true.

Since the modern interpretation of the four categorical forms has been accepted – $\forall x(Sx \to Px)$ for $SaP$, $\forall x(Sx \to \neg Px)$ for $SeP$, $\exists x(Sx \wedge Px)$ for $SiP$, and $\exists x(Sx \wedge \neg Px)$ for $SoP$ – the square has lost much of its interest because only the contradictory relationships are valid under this interpretation. But this lack of interest is not justified because the other relationships are valid under the

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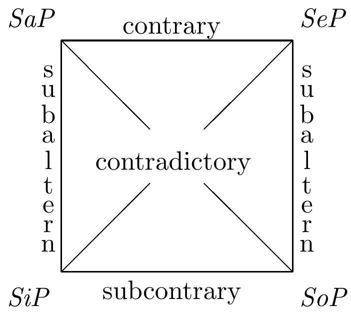



FIGURE 1. Relations of the square of opposition.

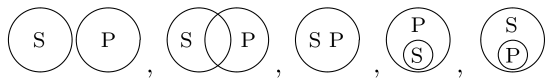



FIGURE 2. Gergonne relations.

condition of existential import, and the square as a whole shows a specific structure – i.e., the structure of the system of traditional logic – which is interesting as such but also for the research in cognitive science and automated reasoning. The representation of the four categorical propositions by different diagram systems allows a deeper insight into this structure.

## 2. Euler diagrams and Venn diagrams

Euler diagrams represent the relationships between the extensions of terms by help of the relationships between the areas of circles, [3, 7]. With this graphical semantic one can depict five different relationships (the so-called *Gergonne relations*, [10], (Figure 2)) which can be interpreted as state-descriptions. Therefore Euler diagrams represent the four categorical propositions by the help of these five relations.

The distribution of Euler diagrams in the square of opposition reflects some features of the relationships between the categorical forms (Figure 3):

1. The subcontrary relationship becomes obvious by the fact that on the one hand $SiP$ and $SoP$ can partly be represented by the same diagrams and that on the other hand $SiP$ and $SoP$ together include every possible case, so that at least one diagram of them must be constructible.
2. The subaltern relation is appropriate to the fact that the two diagrams representing $SaP$ form a subset of the diagrams representing $SiP$ and that the diagram for $SeP$ is also one of the diagrams for $SoP$.

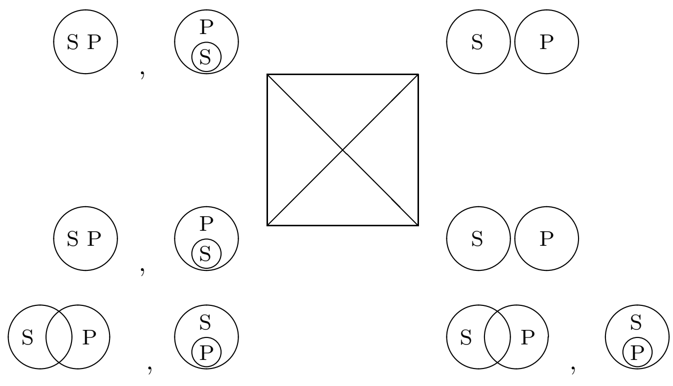



FIGURE 3. Square of opposition represented by Euler diagrams.

3. The contrary relationship becomes obvious by the fact that on the one hand $SaP$ and $SeP$ have no diagram in common and that on the other hand $SaP$ and $SeP$ together do not include every possible case, so that at least one diagram must be constructible without representing them.
4. The contradictory relationship can be recognized by the fact that each of the diagonally opposite diagrams represents exactly the five possible relationships. The fact that at least one of these five relationships must be the case means that one diagram per diagonal must be constructible. If thus the diagrams for $SaP$ cannot be constructed, at least a diagram for $SoP$ must be constructible. The same is true the other way round, as well as regarding the second diagonal.

This distribution is also the reason for the specific amounts of (semantic) information these diagrams assign to the categorical propositions determined according to the semantic information measure $cont(x)$ which was developed by Bar-Hillel and Carnap and gives the share of excluded possibilities [1]: $SeP = 4/5 = 0.8$, $SaP = 3/5 = 0.6$, $SoP = 2/5 = 0.4$, and $SiP = 1/5 = 0.2$.

Interestingly enough a look at another diagram system shows that the amount of information depends not only on the interpretation (existential import, intensional versus extensional, etc.) but also on the geometrical (resp. topological) constraints of the medium. The Euler system uses five logical relations which are isomorphic to the following point-set models: (i) points(s) $\cap$ points(p) $= \emptyset$, (ii) points(s) $\cap$ points(p) $\neq \emptyset$, (iii) points(s) $=$ points(p), (iv) points(s) $\subset$ points(p), and (v) points(p) $\subset$ points(s). The point-set approach makes it impossible to define relations that are based on the distinction of particular parts of the point-sets such as the *interior* and the *boundary*. In the 1990s Max Egenhofer and his colleagues developed a diagram system which uses such parts to represent binary *topological* (instead of *logical*) relations [5, 6]. The result is a graphical extension of the Euler system. For this new system a change from strong convex to weak convex forms

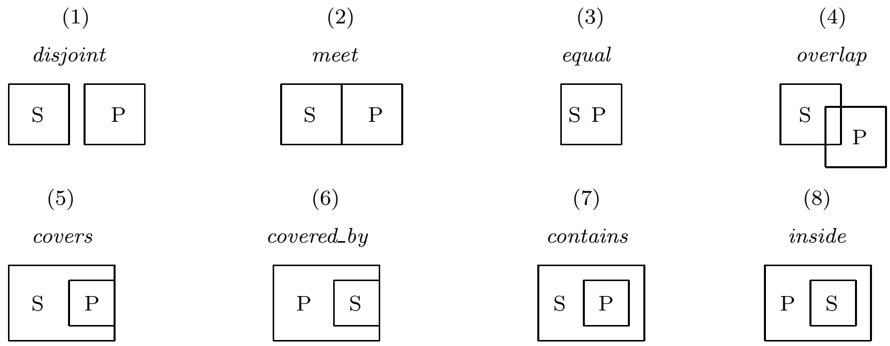



FIGURE 4. Representation of eight topological relations.

is necessary (the first diagram system with weak convex forms was developed by Johann Maass in 1793; [4]).

The definitions of these eight so-called *scene descriptions* (instead of *state-descriptions*) are (Figure 4):

(1) disjoint(s,p) := interior(s) $\cap$ interior(p) = $\emptyset$ and boundary(s) $\cap$ boundary(p) = $\emptyset$
(2) meet(s,p) := interior(s) $\cap$ interior(p) = $\emptyset$ and boundary(s) $\cap$ boundary(p) $\neq \emptyset$
(3) equal(s,p) := interior(s) = interior(p)
(4) overlap(s,p) := interior(s) $\cap$ interior(p) $\neq \emptyset$
(5) covers(s,p) := interior(p) $\subset$ interior(s) and boundary(s) $\cap$ boundary(p) $\neq \emptyset$
(6) covered_by(s,p) := interior(s) $\subset$ interior(p) and boundary(s) $\cap$ boundary(p) $\neq \emptyset$
(7) contains(s,p) := interior(p) $\subset$ interior(s) and boundary(s) $\cap$ boundary(p) = $\emptyset$
(8) inside(s,p) := interior(s) $\subset$ interior(p) and boundary(s) $\cap$ boundary(p) = $\emptyset$

The representations of the categorical propositions show the relationships of the logical square in the same way like Euler diagrams but their amount of information is different – partly greater, partly smaller (because from the point-set view (1) and (2), (5) and (7), and (6) and (8) are equivalent; Figure 5): $SeP = 6/8 = 0.75$, $SaP = 5/8 = 0.625$, $SoP = 3/8 = 0.375$, and $SiP = 2/8 = 0.25$.

Euler diagrams cannot be used without difficulties to test the validity of a syllogism. This is why several improvements of this diagram system have been developed [2]. However a lot of these improvements lost the visual evidence Euler diagrams have. For example, the diagrams of John Neville Keynes [12] are more complicated than Euler's because one has to delete the dotted lines or transform them into continuous lines to represent a relation (except in the case of $SeP$). This is why these diagrams do not directly 'show' the characteristics of the relationships between the categorical forms like Euler diagrams do (Figure 6).

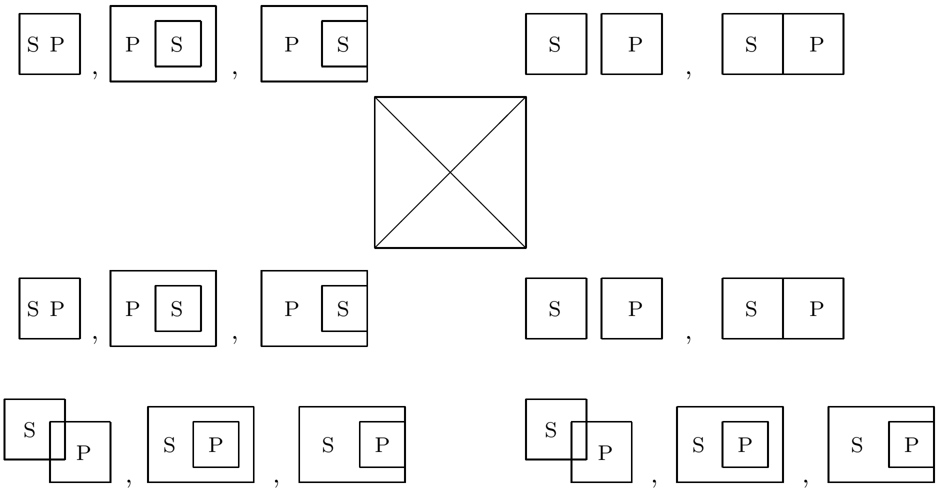



Figure 5. Square of opposition represented by topological relations.

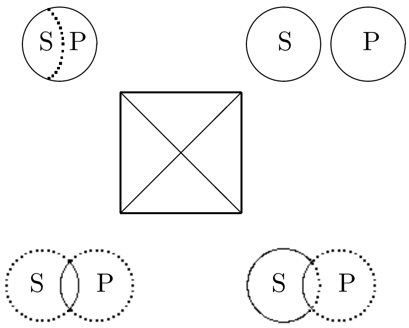
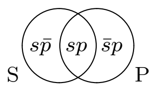



Figure 6. Square of opposition represented by Keynes diagrams.

The most well-known improvement of Euler's diagrams is the system of John Venn [15,17]. Venn diagrams represent categorical propositions by two overlapping circles which build four regions (the region outside the circles, i.e., $\bar{s}\bar{p}$ has not to be taken into account here):



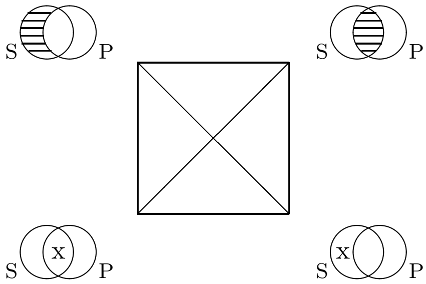
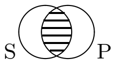
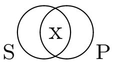
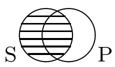
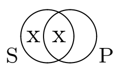


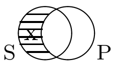
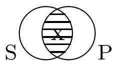

Figure 7. Square of opposition represented by Venn diagrams.

Shading a region means that in this area nothing exists. For example, the diagram   means that there is no $S$ which is $P$. The symbol ‘x’ marks nonempty regions, e.g.,   means that there exists at least one thing which is $S$ and $P$. The blankness of an area represents a lack of information.

These diagrams show (if one knows how to read them) the existential import of the categorical propositions under the modern interpretation (Figure 7). In order to see the relationships of the square one should put two diagrams each on top of the other. The six new diagrams constructed in this way show the following relationships:

1. *Contrary relationship:*   . This diagram shows that if $SaP$ and $SeP$ are both true then things which are $S$ cannot exist.

2. *Subcontrary relationship:*   . This diagram shows that if things which are $S$ exist, then $SiP$ or $SoP$ (or both) has to be true.

3. *Contradictory relationships:*   and   . These diagrams show the reason why $SaP$ and $SoP$ (resp. $SeP$ and $SiP$) are contradictory: The proposition $SaP$ means that nothing exists which is $S$ and not $P$ whereas $SiP$ says that at least one such thing exists (similarly for $SeP$ and $SoP$ with regard to things which are $S$ and $P$).

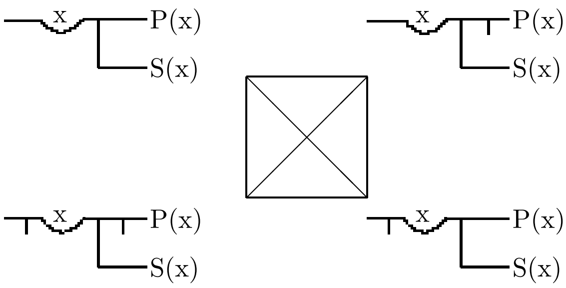
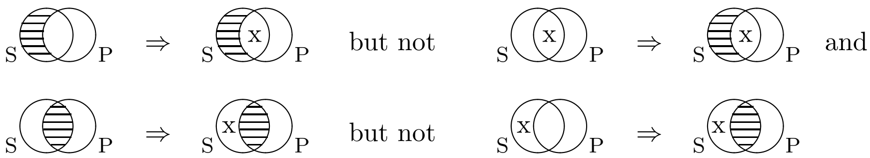



FIGURE 8. Square of opposition represented by Begriffsschrift.

4. *Subaltern relationships:*

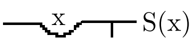


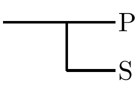

These diagrams show – existential import for $S$ presupposed – that one can turn a diagram which displays one empty $S$-section ($s\bar{p}$ in the case of $SaP$ resp. $sp$ in the case of $SeP$) into a diagram that indicates the other $S$-section as nonempty, but not vice versa.

## 3. Begriffsschrift and existential graphs

In his 'Begriffsschrift' (concept-script) Frege represents the conditional with two connected lines, the universal quantifier with a concavity, the negation with a little stroke, and several variables with letters [8, 16]. For example, 'Everything is S' is represented as: 


'It is not the case that everything is S' is represented as: 


'Everything is not S' is represented as: 


And 'If S then P' is represented as: 


In the 'Begriffsschrift' Frege also represents the four categorical propositions by his concept-script in the square of opposition (Figure 8).

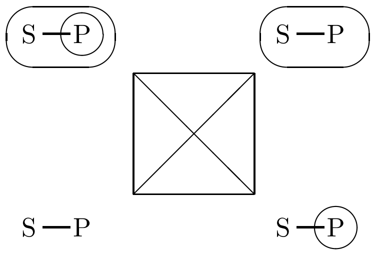
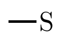


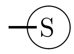

Figure 9. Square of opposition represented by Existential Graphs.

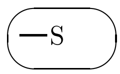
These four diagrammatical signs differ from each other only through the position of the negation stroke (resp. strokes). Regarding this stroke one can differentiate between an 'outer' and an 'inner' negation. In terms of traditional logic one can say that the first negates the proposition whereas the second negates a predicate (and the fact that $S(x)$ is never negated may be seen as corresponding to the existential import of $S$). Regarding to the position of the negation stroke the Begriffsschrift items show the features of the following relationships:

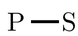
1. To contradict a categorical proposition means to negate the proposition as a whole (nota bene $\neg\neg X \rightarrow X$).
2. Categorical propositions are contrary/ subcontrary to each other if their predicate term is negated – in affirmative (i.e., not negated) propositions = contrary, in negative propositions = subcontrary (nota bene $\neg\neg X \rightarrow X$).

The 'Existential Graphs' of Charles Sanders Peirce consist of three elements: predicate symbols, so-called 'lines of identity', and so-called 'cuts' [13, p. 4.418–4.4584], [14]. The 'line of identity' is a solid line which denotes the existence of something. A 'cut' is a closed curve and represents the negation of everything that is written inside it. The following diagrams are examples of primitive Existential Graphs:

'There exists something that is S': 


'There exists something that is not S': 


'It is not the case that something is S': 


'There exists an P which is S': 


Thus Existential Graphs can also be used to represent the four categorical propositions (Figure 9).

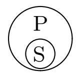
The translations of Existential Graphs and Frege's Begriffsschrift into linguistic formulas show that these systems are quite similar. The natural way to translate Existential Graphs into linguistic formulas is to use only predicate symbols, the existential quantifier, and a symbol for negation. The diagrams therefore correspond to conjunctive normal forms:

$$\begin{aligned} SaP &= \neg\exists x(Sx \land \neg Px)\\ SeP &= \neg\exists x(Sx \land Px)\\ SiP &= \ \ \exists x(Sx \land Px)\\ SoP &= \ \ \exists x(Sx \land \neg Px) \end{aligned}$$

And the adequate translation of the Begriffsschrift is:

$$\begin{aligned} SaP &= \ \ \forall x(Sx \to Px)\\ SeP &= \ \ \forall x(Sx \to \neg Px)\\ SiP &= \neg\forall x(Sx \to \neg Px)\\ SoP &= \neg\forall x(Sx \to Px) \end{aligned}$$

Because of the laws $\forall xPx \leftrightarrow \neg\exists x\neg Px$ and $(P \to S) \leftrightarrow \neg(P \land \neg S)$ Existential Graphs visualize the contradictory relationship like the Begriffsschrift as 'outer negation' and the contrary/subcontrary relationships as 'inner negation'. Interestingly enough both – Frege and Peirce – reject the difference made in traditional logic between the negation of a proposition and the negation of a predicate, resp. 'negative judgments' and 'infinite judgments' [9, p. 355], [13, p. 2.380]. Besides, neither Frege nor Peirce uses the negation-sign analog to ordinary language (where only $SoP$ and $SeP$ are negative). Frege's aim was a logical system that was capable of representing arithmetical propositions and inferences. Therefore his Begriffsschrift should be a 'formula language of *pure* thought', not of *actual* thought [8]. Peirce instead took his Existential Graphs to reveal the fundamental operations of reasoning. With the same intention the psychologist Johnson-Laird developed in the 1980s the so-called 'mental models' [11]. This diagrammatic system has a sign for *diversity* to represent the negative propositions $SeP$ and $SoP$ (i.e., a line which separates the S's from the P's). Since mental models are isomorphic to Euler diagrams they visualize all relationships of the logical square analog to them (Figure 10): the letters in brackets are *possible letters* in that sense that, e.g., s p = 

and $\begin{matrix} & p \\ s & p \end{matrix}$ = 

are two models of $SiP$ (mental models have to read off line by line).

## 4. Conclusion

It is argued that the representation of the four categorical propositions by different diagram systems allows a deeper insight into the relations of the logical square. With Euler diagrams one can 'see' that these relations are based on state-descriptions which are implied by the categorical propositions. Venn diagrams

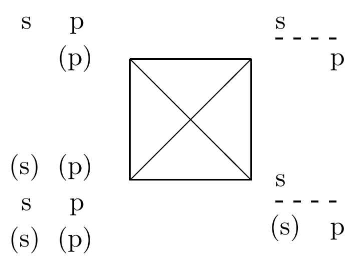



Figure 10. Square of opposition represented by mental models.

show the role that existential import plays. Frege's 'Begriffsschrift' and Peirce's 'Existential Graphs' visualize in different ways that the relation of contradiction on the one hand and the contrary and subcontrary relations on the other hand can be differentiated by two kinds of negation which was in traditional logic called 'negation of the judgement' and 'negation of the predicate'.

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