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A Galois Connection

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Abstract. The connection presented in this paper mirror-links two metamathematical structures, the finitary closure operators, and the compact consistency properties, in such a way that a specification of one structure induces a provably equivalent specification of the other.

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1. Introduction

The present paper is a sequel to an earlier paper of mine (see the References). That earlier paper deals with both Galois (i.e., antitone) as well as isotone connections between various metamathematical structures including, among others, finitary closure operators and compact consistency properties. Incidentally, any connection between these two particular structures appears to be a Galois connection rather than an isotone connection.

Also, no special attention is given in the earlier paper to the problem of how to make a particular connection descriptively effective or constructive in some desired sense of these words. And that is simply because the approach followed in that paper is perhaps too general to help discern between, say, the "finitary" and the "infinitary".

In what follows we take the above remarks into account when we re-focus attention on the two structures, the finitary closure operators and the compact consistency properties, and describe a new connection between them. Clearly, the description of this new connection, like the description of any other connection, requires the sets of the respective objects it connects together to be "immersed" into, and described within, a language with a fixed "inner" structure. That is because it is in this way that an individual sentence can be formally recognized as

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either a simple (i.e., basic) sentence or as a compound sentence (i.e., a sentence made up of simpler sentences with the help of some specified logical constants). For the sake of brevity only two logical constants, the sentential negation $\neg$ and the sentential conditional $\rightarrow$ are employed below. But, of course, a similar approach also applies to other standard logical constants. This means that the choice of negation $\neg$ and conditional $\rightarrow$ is more of the matter of convenience than that of substance.

Another purpose of the present paper is to use the new connection to help identify (and possibly also understand better) the very concept of consistency when it is considered as an intrinsically primitive concept, independently from, and contrary to, the idea of consistency as a derivative concept, whose meaning is being persistently determined by special definitions within a foreign infrastructure of either proof theory or model theory. By keeping this figurative way of speech, such a connection can help carve the body of consistency out of its traditional closure-operatic trunk and present it as an autonomous statuary.

More generally, such a connection functions as a two-way vehicle, which mirror-links the two metamathematical structures in such a way that their respective theories, which are or at least can be understood as both-ways independent, are provably equivalent to each other. Consequently, specification of one of these theories induces a "linked" specification of the other. And in this way, by assuming the knowledge of its respective counterpart, we can try and determine what can be called a classical consistency theory or a classical theory of closure operators.

A similar approach can be tried and applied, *mutatis mutandis*, to at least some of the conceptualizations of consistency outside the classical area. It is because this connection can also open ways to mirror-imaging various "non-classical" closure-operatic structures on their respective consistency-based counterparts or vice versa.

To make the present paper self-contained and readable independently of the above mentioned earlier paper of mine we begin by a summary description of the necessary background.

2. The necessary background

Let $(P, \leq)$ be a poset, i.e., a set P with a binary relation $\leq$ of partial ordering. A mapping f from a poset $(P, \leq)$ to a poset $(Q, \leq)$ is called *antitone* iff $x \leq y$ implies that $fy \leq fx$, for any $x, y \in P$. Given a mapping f from $(P, \leq)$ to $(Q, \leq)$ and a mapping g from $(Q, \leq)$ to $(P, \leq)$, we say that the pair (f, g) is *antitone* just in case f and g are both antitone. Finally, we say that the pair (f, g) is a *weak connection* between $(P, \leq)$ and $(Q, \leq)$ iff both $x \leq gfx$ and $y \leq fgy$, for any $x \in P$ and $y \in Q$. In this context we refer to f and g as weak connectors of the respective posets. The antitone weak connections are what we know from algebra under the name of (weak) Galois connections. Moving now to a brief summary discussion of the antitone connections, the following simple but useful fact is well known (Cf. [1] and [2]).

(1) If (f, g) is an antitone weak connection between $(P, \leq)$ and $(Q, \leq)$ then $fgfx = fx$ and $gfgy = gy$ for any $x \in P$ and $y \in Q$.

Also known is the following fact due to J. Schmidt (Cf. [3]).

(2) The following two conditions are equivalent. (i) (f, g) is an antitone weak connection between $(P, \leq)$ and $(Q, \leq)$; (ii) f is a mapping from $(P, \leq)$ to $(Q, \leq)$ and g a mapping from $(Q, \leq)$ to $(P, \leq)$ such that $x \leq gy$ is equivalent to $y \leq fx$ for any $x \in P$ and $y \in Q$.

None of the inequalities $x \leq gfx$ and $y \leq fgy$, appearing in the definition of a weak connection (f, g), can be strengthened to equality simply on the basis of the definition of antitone weak connectionship. Yet, practical considerations justify the separation of this subclass from all antitone weak connections. We say that (f, g) is a *strong connection* between $(P, \leq)$ and $(Q, \leq)$, to be called from now on simply a connection between $(P, \leq)$ and $(Q, \leq)$, iff $gfx = x$ and $fgy = y$ for any $x \in P$ and $y \in Q$. Here f and g are referred to as strong connectors or simply as connectors of their respective posets. Furthermore, we have the following two facts involving these concepts (they are implicit in [1]).

(3) If (f, g) is an antitone weak connection between posets $(P, \leq)$ and $(Q, \leq)$ then the following three conditions are pairwise equivalent. (i) f is a connector of (f, g), i.e., $gfx = x$; (ii) g is onto; (iii) f is one-to-one.

(4) If (f, g) is an antitone connection between $(P, \leq)$ and $(Q, \leq)$ then f is one-to-one and g is a reverse of f.

3. Basic notation and the necessary definitions

We proceed now to the deployment of details for the mentioned connection involving two metamathematical structures. For convenience rather than for substance we shift, from now on, to the usual set-theoretic terminology and symbolic notation. In particular, symbols "$\subseteq$", "$\cup$" and "$\cap$" stand for the relation of set-inclusion, the operation of set-union and that of set-intersection, respectively. We denote by S the set of all sentences of a fixed language, and we use letters $X, Y, Z, \ldots$ and $A, B, C, \ldots$, with or without indices, to denote subsets of S and members of S, respectively. And we only assume of S that it is a non-empty set of sentences of some specified structure. The usual symbolic expression 2^Z denotes, of course, the powerset of Z, i.e., the set of all subsets of Z. We also write 2_Z to denote the class of all subsets of S which extend (or are supersets of) Z. The symbol $\emptyset$ stands for the empty set. For brevity, expressions of the kind of $\phi(X \cup \{A_1, A_2, \ldots, A_n\})$ are being rendered throughout the rest of the paper simply as $\phi(X, A_1, A_2, \ldots, A_n)$. In some parts of the paper to follow we also use "$X \in fin(Y)$", an extra piece of our symbolic notation, to stand for the fact that X is a finite subset of Y.

We begin by the definition of a finitary closure (¬)-operator *cn* of the kind worked with in this paper. We say that, for any $X, Y \in S$, (i) *cn* is *reflexive* iff $X \subseteq cn(X)$, (ii) *cn* is *monotonic* iff $X \subseteq Y$ implies that $cn(X) \subseteq cn(Y)$, and (iii) *cn* is idempotent iff $cn(cn(X)) \subseteq cn(X)$. An operator *cn* is called a closure operator iff the *cn* satisfies conditions (i)–(iii). Next we say that (iv) *cn* is finitary iff $cn(X) \subseteq \cup\{cn(Y) : Y \in fin(X)\}$. The above definitions are all due to A. Tarski (Cf. [5–7]).

To write down two remaining conditions involving operator *cn* we must use negation symbol ¬ with which to make $\neg A$, i.e., the negation of an arbitrary sentence A in S. We say that, for any $A \in S$ and any $X \subseteq S$, (v) *cn* is (¬)-analytic iff $cn(A, \neg A) = S$, and (vi) *cn* has the (¬)-cut property iff $cn(X, A) \cap cn(X, \neg A) \subseteq cn(X)$.

We call an operator *cn* a (¬)-operator iff it satisfies conditions (v) and (vi).

Now to the definition of *cons* as a compact consistency (¬)-property which runs as follows. We say that, for any $A \in S$ and any $X, Y \in S$, (i) *cons* is *non-trivial* iff $S \notin cons$, and (ii) *cons* is *hereditary* iff $X \in cons \cap 2_Y$ implies that $Y \in cons$. We call *cons* a *consistency* property iff it satisfies conditions (i)–(ii); (iii) *cons* is *compact* iff the fact that $X \notin cons$ implies that $Y \notin cons$ for some $Y \in fin(X)$, (iv) *cons* is (¬)-*analytic* iff $\{A, \neg A\} \notin cons$, and (v) *cons* has the (¬)-*extension* property iff $X \in cons$ implies that $X \cup \{A\} \in cons$ or $X \cup \neg\{A\} \in cons$.

We call a consistency property *cons* a (¬)-property iff *cons* satisfies conditions (iv) and (v).

Given these definitions involving cons we can now state and prove a lemma being, in fact, a *cons*-variant of the well-known Lindenbaum extension lemma originally stated in application to closure operators (Cf. [5–7]). In the formulation of this lemma we make use of an auxiliary concept of a regular *cons*. By definition, *cons* is regular iff the fact that $X \in cons$ implies that there is $Z \in cons \cap 2_X$ such that $Z' = Z$ for any $Z' \in cons \cap 2_Z$.

Lemma 3.1. *Each compact consistency property is regular.*

Proof. Clearly, a compact consistency property is a property of finite character in the sense of general topology. Using this fact we apply the well-known-Teichmuller–Tukey theorem (Cf. [8] and [9]) to get the Lemma. □

Clearly, Lemma 3.1 can also be proved without the Teichmuller–Tukey theorem, i.e., by a direct use of the above stated definition of a compact consistency (¬)-property. But a direct proof is omitted here just to safe space.

4. The Galois connection $(cons[cn], cn[cons])$

We begin by stating two definitions, pivotal to the case of the Galois connection between finitary closure (¬)-operators and compact consistency (¬)-properties. These definitions run as follows.

The definition of $cons[cn]$ in terms of a finitary closure $(\neg)$-operator cn, i.e., $X \in cons[cn]$ if and only if $cn(X) \neq S$.

The definition of $cn[cons]$ in terms of a compact consistency $(\neg)$-property $cons$, i.e., if $A \in cn[cons](X)$ if and only if $\{A, B\} \in cons$ for any B such that $X \cup \{B\} \in cons$.

These pivotal definitions, it will be seen, describe two important mappings. The first mapping sends each finitary closure $(\neg)$-operator cn over S to a compact consistency $(\neg)$-property $cons[cn]$ over the same S. The second mapping sends each compact consistency $(\neg)$-property cons to a finitary closure $(\neg)$-operator $cn[cons]$. This is so because we have two lemmas to state and prove as below.

Lemma 4.1. *If cn is a finitary closure $(\neg)$-operator then (i) $cons[cn]$ is a compact consistency $(\neg)$-property; (ii) $cn[cons[cn]] = cn$; and (iii) $cons[cn]$ is antitone.*

Proof. Case (i). The fact that $cons[cn]$ is non-trivial and hereditary follows directly from the definition of $cons[cn]$ in terms of cn and the fact that, by the hypothesis, cn is reflexive and monotonic. This proves that $cons[cn]$ is a consistency property.

Proof that $cons[cn]$ is compact. If $X \notin cons[cn]$, i.e., $A \in cn(X)$ for any $A \in S$ then there is $Y \in fin(X)$ such that $A \in cn(Y)$ for any A, i.e., that $cn(Y) = S$ because, by the hypothesis, cn is finitary. Using the definition of $cons[cn]$ we conclude that $Y \notin cons[cn]$ for some $Y \in fin(X)$ which means that $cons[cn]$ is compact. This ends the proof.

Proof that $cons[cn]$ is $(\neg)$-analytic. If, contrary to the fact, $cons[cn]$ is not $(\neg)$-analytic then, by the definition of $cons[cn]$, $cn(A, \neg A) = S$, i.e., that cn is not $(\neg)$-analytic, contrary to the hypothesis. This ends the proof.

Proof that $cons[cn]$ has the $(\neg)$-extension property. Suppose that (1) $X \in con[cn]$ and that, contrary to the fact, (2) $X \cup A \notin cons[cn]$ and $X \cup \{\neg A\} \notin cons[cn]$. Using the definition of $cons[cn]$ we infer from (2) that (3) $cn(X, A) = S = cn(X, \neg A)$, i.e., that $cn(X, A) \cap cn(X, \neg A) = S$. It follows that (4) $cn(X) = S$ because, by the hypothesis, cn has the $(\neg)$-cut property. Using the definition of $cons[cn]$ we infer from (4) that $X \notin cons[cn]$, contrary to step (1). This ends the proof.

Case (ii). Proof that $cn[cons[cn]] \subseteq cn$. Suppose that (1) $A \in cn[cons[cn]](X)$ and that (2) $A \notin cn(X)$. By (1) and the definition of $cn[cons[cn]]$ in terms of $cons[cn]$ and, then, by the definition of $cons[cn]$ in terms of cn we can infer that (3) $cn(A, B) \neq S$ for any B such that $cn(X, B) \neq S$. Our hypothesis implies that (4) $cn(X, A) \cap cn(X, \neg A) \subseteq cn(X)$. Steps (2) and (4) imply that (5) $A \notin cn(X, \neg A)$, i.e., that $cn(X, \neg A) \neq S$ because cn is reflexive. But by step (3) it follows that $cn(X, \neg A) \neq S$ implies that $cn(A, \neg A) \neq S$. Hence by (5) we infer that $cn(A, \neg A) \neq S$, contrary to the hypothesis that cn is $(\neg)$-analytic. This proves that $cn[cons[cn]] \subseteq cn$.

Proof that $cn \subseteq cn[cons[cn]]$. Suppose that (1) $A \in cn(X)$ and that (2) $A \notin cn[cons[cn]](X)$. By (2) and the definition of $cn[cons[cn]]$ in terms of $cons[cn]$ there is B such that (3) $X \cup \{B\} \in cons[cn]$ and that (4) $\{A, B\} \notin cons[cn]$. By (3) and the definition of $cons[cn]$ we conclude that (5) $cn(X, B) \neq S$ and hence by (1) we infer that (6) $cn(X, A, B) \neq S$, i.e., that $X \cup \{A, B\} \in cons[cn]$. By the hypothesis *cons* is hereditary so it follows from (6) that $\{A, B\} \in cons[cn]$ contrary to (4). This ends the proof.

Case (iii). Proof that $cons[cn]$ is antitone. Suppose that (1) $cn_1 \subseteq cn_2$ and that (2) $X \in cons[cn_2]$. By (2) and the definition of $cons[cn]$ we infer that (3) $cn_2(X) \neq S$. By (1) and (3), $cn_1(X) \neq S$. Hence, by the definition of $cons[cn_1]$, $X \in cons[cn_1]$. This ends the proof. This completes the proof of Lemma 4.1. □

Lemma 4.2. *If cons is a compact consistency* (¬)*-property then (i)* $cn[cons]$ *is a finitary closure* (¬)*-operator; (ii)* $cons[cn[cons]] = cons$*; and (iii)* $cn[cons]$ *is antitone.*

Proof. Case (i). Proof that $cn[cons]$ is reflexive. If (1) $A \in X$ and (2) $A \notin cn[cons](X)$ then by the definition of $cn[cons]$ there is B such that (3) $X \cup \{B\} \in cons$ and that (4) $\{A, B\} \notin cons$. Steps (1) and (3) imply that (5) $X \cup \{A, B\} \in cons$. From (5) and the fact that cons is hereditary we finally get that $\{A, B\} \in cons$, contrary to (4). This ends the proof.

Proof that $cn[cons]$ is monotonic. Suppose that (1) $X \subseteq Y$, that (2) $A \in cn[cons](X)$ and that (3) $A \notin cn[cons](Y)$. By (3) and the definition of $cn[cons]$ there is B such that (4) $Y \cup \{B\} \in cons$ and that (5) $\{A, B\} \notin cons$. Steps (2) and (5) imply that (6) $X \cup \{B\} \notin cons$. Hence by (1) and the fact that *cons* is hereditary we conclude that $Y \cup \{B\} \notin cons$, contrary to (4). This ends the proof.

Proof that $cn[cons]$ is idempotent. We begin by first showing that (♯) $X \in cons$ implies that $cn[cons](X) \in cons$. To prove this step suppose that (1) $X \in cons$ and that (2) $cn[cons](X) \notin cons$. By the hypothesis *cons* is compact. Hence by (2) there is Y such that (3) $Y \in fin(cn[cons](X))$ and that (4) $Y \notin cons$. By (3) there exist n and $A_1, \ldots, A_n$ such that (5) $Y = \{A_1, \ldots, A_n\}$ and that (6) $A_1, \ldots, A_n \in cn[cons](X)$. By Lemma 3.1 and step (1) there is Z such that (7) $Z \in 2_X$, that (8) $Z \in cons$ and that (9) $U = Z$ for any $U \in cons \cap 2_Z$. By (6) and the definition of $cn[cons]$ (10) $\{A_1, B_1\} \in cons$ for any B_1 such that $Z \cup \{B_1\} \in cons, \ldots, \{A_n, B_n\} \in cons$ for any B_n such that $Z \cup B_n \in cons$. By taking each of $B_1, \ldots, B_n$ to be $\neg A_1, \ldots, \neg A_n$, respectively, we infer that (11) $Z \cup \{A_1\} \in cons$ implies that $\{A_1, \neg A_1\} \in cons, \ldots, Z \cup \{A_n\} \in cons$ implies that $\{A_n, \neg A_n\} \in cons$. By the hypothesis cons is (¬)-analytic. Hence by (11) we can conclude that (12) $Z \cup \{\neg A_1\} \notin cons, \ldots, Z \cup \{\neg A_n\} \notin cons$. By the hypothesis *cons* has the (¬)-extension property. Hence by (12) we can conclude that $Z \cup \{A_1\} \in cons, \ldots, Z \cup \{A_n\} \in cons$. Steps (5), (9) and (13) imply that (14) $\{A_1, \ldots, A_n\} = Y \subseteq Z$. By the hypothesis cons is hereditary. Hence by (8) and (14) we can conclude that (15) $Y \in cons$, contrary to (4). This proves our auxiliary statement (♯).

Now to complete the proof that $cn[cons]$ is idempotent suppose that, for any A and any X, (1) $A \in cn[cons](cn[cons])(X)$ and (2) $A \notin cn[cons](X)$. Step (2) and the definition of $cn[cons]$ imply that there is B such that (3) $Y \cup \{B\} \in cons$ and that (4) $\{A, B\} \notin cons$. By Lemma 3.1 it follows from (3) that there is Y such that (5) $X \cup \{B\} \subseteq Y$, that (6) $Y \in cons$, and that for any Z, (7)

$Z \in \mathit{cons} \cap 2_Y$ implies that $Z = Y$. Step (5) implies that (8) $B \in Y$. By our auxiliary statement (♯), step (6) implies that (9) $\mathit{cn}[\mathit{cons}](Y) \in \mathit{cons}$. From (7) and (9) it follows that (10) $\mathit{cn}[\mathit{cons}](Y) = Y$. On the other hand, steps (1) and (5) imply that (11) $A \in \mathit{cn}[\mathit{cons}](\mathit{cn}[\mathit{cons}])(Y)$ because, by the hypothesis, $\mathit{cn}[\mathit{cons}]$ is reflexive and monotonic. Step (11) and the definition of $\mathit{cn}[\mathit{cons}]$ imply that (12) $\mathit{cn}[\mathit{cons}](Y) \cup \{B\} \in \mathit{cons}$ implies that $\{A, B\} \in \mathit{cons}$. From (4) and (12) we conclude that (13) $\mathit{cn}[\mathit{cons}](Y) \cup \{B\} \notin \mathit{cons}$. Steps (10) and (13) imply that (14) $Y \notin \mathit{cons}$ contrary to (6). This completes the proof that $\mathit{cn}[\mathit{cons}]$ is idempotent.

Proof that $\mathit{cn}[\mathit{cons}]$ is finitary. Suppose that (1) $A \in \mathit{cn}[\mathit{cons}](X)$ and, contrary to the fact, that (2) $A \notin \mathit{cn}[\mathit{cons}](Y)$ for any $Y \in fin(X)$. By (1) and the definition of $\mathit{cn}[\mathit{cons}]$ (3) $\{A, B\} \in \mathit{cons}$ for any B such that $X \cup \{B\} \in \mathit{cons}$. By step (3), of course, (4) $X \cup \{\neg A\} \in \mathit{cons}$ implies that $\{A, \neg A\} \in \mathit{cons}$. But by the hypothesis cons is ($\neg$)-analytic so by (4) we can conclude that (5) $X \cup \{\neg A\} \notin \mathit{cons}$. By the hypothesis *cons* is compact so by (5) there is Z such that (6) $Z \in fin(X, \neg A)$, and that (7) $Z \notin \mathit{cons}$. It follows from (6) that (8) $Z - \{\neg A\} \in fin(X)$ so taking the "Y" to be "$Z - \{\neg A\}$" we can conclude from step (2) that (9) $A \notin \mathit{cn}[\mathit{cons}](Z - \{\neg A\})$. From (9) and the definition of $\mathit{cn}[\mathit{cons}]$ there is C such that (10) $Z - \{\neg A\} \cup \{C\} \in \mathit{cons}$, and that (11) $\{A, C\} \notin \mathit{cons}$. Clearly, $(Z - \{\neg A\}) \cup \{\neg A\} = Z$. Combining this with the hypothesis that both *cons* is hereditary and has the ($\neg$)-extension property, we conclude from steps (10) and (11) that $Z \in \mathit{cons}$, contrary to (7). This proves that $\mathit{cn}[\mathit{cons}]$ is finitary.

Proof that $\mathit{cn}[\mathit{cons}]$ is ($\neg$)-analytic. Suppose, contrary to the fact, that there is A such that (1) $\mathit{cn}[\mathit{cons}](A, \neg A) \neq S$, i.e., that there is B such that $B \notin \mathit{cn}[\mathit{cons}](A, \neg A)$. By the definition of $\mathit{cn}[\mathit{cons}]$ this implies that there is C such that (2) $\{A, \neg A, C\} \in \mathit{cons}$, and that (3) $\{B, C\} \notin \mathit{cons}$. It follows from (2) that $\{A, \neg A\} \in \mathit{cons}$, i.e., that *cons* is not ($\neg$)-analytic, contrary to the hypothesis. This ends the proof.

Proof that $\mathit{cn}[\mathit{cons}]$ has the ($\neg$)-cut property. Suppose, contrary to the fact, that there is B such that (1) $B \in \mathit{cn}[\mathit{cons}](X, A)$, that (2) $B \in \mathit{cn}[\mathit{cons}](X, \neg A)$, and that (3) $B \notin \mathit{cn}[\mathit{cons}](X)$. By (3) and the definition of $\mathit{cn}[\mathit{cons}]$ there is C such that (4) $X \cup \{C\} \in \mathit{cons}$ and that (5) $\{B, C\} \notin \mathit{cons}$. By (2) and the definition of $\mathit{cn}[\mathit{cons}]$, (6) $X \cup \{\neg A, C\} \in \mathit{cons}$ implies that $\{B, C\} \in \mathit{cons}$. Steps (5) and (6) imply that (7) $X \cup \{\neg A, C\} \notin \mathit{cons}$. By (1) and the definition of $\mathit{cn}[\mathit{cons}]$, (8) $X \cup \{A, C\} \in \mathit{cons}$ implies that $\{B, C\} \in \mathit{cons}$. Steps (5) and (8) imply that (9) $X \cup \{A, C\} \notin \mathit{cons}$. By the hypothesis *cons* has the ($\neg$)-extension property. Hence by (7) and (9) we conclude that (10) $X \cup \{C\} \notin \mathit{cons}$, contrary to (4). This ends the proof.

Case (ii). Proof that $\mathit{cons}[\mathit{cn}[\mathit{cons}]] \subseteq \mathit{cons}$. If $X \in \mathit{cons}[\mathit{cn}[\mathit{cons}]]$ then by the definition of $\mathit{cons}[\mathit{cn}[\mathit{cons}]]$ in terms of $\mathit{cn}[\mathit{cons}]$ we infer that $\mathit{cn}[\mathit{cons}](X) \neq S$, i.e., that $A \notin \mathit{cn}[\mathit{cons}](X)$ for some A. Hence by the definition of $\mathit{cn}[\mathit{cons}]$ there is B such that $X \cup \{B\} \in \mathit{cons}$ and $\{A, B\} \notin \mathit{cons}$. By the hypothesis *cons* is hereditary. Hence $X \in \mathit{cons}$. This ends the proof.

Proof that $\mathit{cons} \subseteq \mathit{cons}[\mathit{cn}[\mathit{cons}]]$. Suppose that (1) $X \in \mathit{cons}$ and that (2) $X \notin \mathit{cons}[\mathit{cn}[\mathit{cons}]]$. From (2) and the definition of $\mathit{cons}[\mathit{cn}[\mathit{cons}]]$ it follows that

(3) $cn[cons](X) = S$. By Lemma 3.1 and step (1) there is Y such that (4) $Y \in 2_X$, that (5) $Y \in cons$ and that (6) $Z = Y$ for any $Z \in cons \cap 2_Y$. We shall prove now that the following step holds true. (7) $cn[cons](X) \subseteq Y$. Suppose that (7.1) $A \in cn[cons](X)$ and that (7.2) $A \notin Y$. From (7.1) and the definition of $cn[cons]$ we infer that (7.3) $\{A, B\} \in cons$ for any B such that $Y \cup \{B\} \in cons$. Given step (7.3) we can conclude that, in particular, (7.4) $Y \cup \{\neg A\} \notin cons$ implies that $\{A, \neg A\} \in cons$. By the hypothesis $cons$ is $(\neg)$-analytic so step (7.4) implies that (7.5) $Y \cup \{\neg A\} \notin cons$. Given that $cons$ has the $(\neg)$-extension property, it follows from (5) and (7.5) that (7.6) $Y \cup \{A\} \in cons$. Steps (6) and (7.6) imply that $A \in Y$, contrary to (7.2). This proves step (7). From (3) and (7) it follows that $Y = S$. Since cons is non-trivial we get from (8) that $Y \notin cons$, contrary to (5). This ends the proof.

Case (iii). Proof that $cn[cons]$ is antitone. In this proof we use the following auxiliary statement: (♯♯) For any A and X, $X \cup \{\neg A\} \notin cons$ implies that $A \in cn[cons](X)$. Its proof is as follows. Suppose that (1) $X \cup \{\neg A\} \notin cons$ and that (2) $A \notin cn[cons](X)$. Step (2) and the definition of $cn[cons]$ imply that for some B (3) $X \cup \{B\} \in cons$ and that (4) $\{A, B\} \notin cons$. Step (1), on the other hand, implies that (5) $X \cup \{\neg A, B\} \notin cons$ because $cons$ is hereditary. Steps (3) and (5) imply that (6) $X \cup \{A, B\} \in cons$ because $cons$ has the $(\neg)$-extension property. Step (6), in turn, implies that (7) $\{A, B\} \in cons$ because $cons$ is hereditary. Step (7) is contrary to step (4) and this fact concludes the proof of statement (♯♯).

Now to the proof that $cn[cons]$ is antitone. Suppose that (1) $cons_1 \subseteq cons_2$, that (2) $A \in cn[cons_2](X)$ and that (3) $A \notin cn[cons_1](X)$. The auxiliary statement (♯♯) and step (3) imply that (4) $X \cup \{\neg A\} \in cons_1$. Steps (1) and (4) imply that (5) $X \cup \{\neg A\} \in cons_2$. Step (2) and the definition of $cn[cons]$ imply that (6) $X \cup \{\neg A\} \in cons_2$ implies that $\{A, \neg A\} \in cons_2$. It follows from (5) and (6) that $\{A, \neg A\} \in cons_2$, i.e., that $cons_2$ is not $(\neg)$-analytic, contrary to the hypothesis, and this proves that $cn[cons]$ is antitone. This ends the proof of Lemma 4.2. □

The result which Lemmas 4.1 and 4.2 yield can now be summarised as follows.

Theorem 4.3. *The pair of mappings* $(cons[cn], cn[cons])$ *is a Galois connection between finitary* $(\neg)$*-closure operators and compact* $(\neg)$*-consistency properties.*

5. An upgrade of the Galois connection $(cons[cn], cn[cons])$

The $(\neg, \rightarrow)$-case. To upgrade the necessary definitions and notations we begin by enriching the language and add the second sentential constant, that of conditional $\rightarrow$.

The case of a finitary closure $(\neg, \rightarrow)$-operator cn. We say that cn is $(\rightarrow)$-analytic iff the fact that $A \rightarrow B \in cn(X)$ implies that $B \in cn(X, A)$. And we say that cn $(\rightarrow)$-synthetic iff the fact that $B \in cn(X, A)$ implies that $A \rightarrow B \in cn(X)$. A cn is said to have the $(\rightarrow)$-cut property iff $cn(X, A) \cap cn(X, A \rightarrow B) \subseteq cn(X)$. We call a closure operator cn a $(\rightarrow)$-operator iff the cn is $(\rightarrow)$-analytic, $(\rightarrow)$-synthetic and has the $(\rightarrow)$-cut property. Finally, we call a closure operator cn a

$(\neg, \to)$-operator iff it is a finitary closure operator which is both a $(\neg)$-operator and a $(\to)$-operator.

The case of a *cons* as a compact consistency $(\neg, \to)$-property. We say that *cons* is $(\to)$-*analytic* iff the fact that $X \cup \{A, A \to B\} \in cons$ implies that $X \cup \{A, B\} \in cons$. We define a *cons* as $(\to)$-*synthetic* iff the fact that $X \cup \{A, B\} \in cons$ implies that $X \cup \{A, A \to B\} \in cons$. And a *cons* is said to have the $(\to)$-extension property iff the fact that $X \in cons$ implies that $X \cup \{A\} \in cons$ or $X \cup \{A \to B\} \in cons$. A compact consistency *cons* is called a $(\to)$-property iff it is $(\to)$-analytic, $(\to)$-synthetic and has the $(\to)$-extension property. Finally, we define a compact consistency property as a $(\neg, \to)$-property iff it is both a $(\neg)$-property and a $(\neg)$-property.

Lemma 5.1. *If cn is a finitary closure* $(\neg, \to)$*-operator then cons*$[cn]$ *is a compact consistency* $(\neg, \to)$*-property.*

Proof. Proof that $cons[cn]$ is $(\to)$-analytic. Suppose that (1) $X \cup \{A, A \to B\} \in cons[cn]$ and, contrary to the fact, that (2) $X \cup \{A, B\} \notin cons[cn]$. Step (2) and the definition of $cons[cn]$ imply that (3) $cn(X, A, B) = S$ while step (1) and the same definition imply that (4) $cn(X, A, A \to B) \neq S$. By the hypothesis, cn is reflexive so $A \to B \in cn(X, A \to B)$. This, in turn, implies that $B \in cn(X, A, A \to B)$ because, by the hypothesis, cn is $(\to)$-analytic. It follows from here that (5) $cn(X, A, B) \subseteq cn(X, A, A \to B)$. By steps (4) and (5) we can now conclude that $cn(X, A, B) \neq S$, contrary to step (3). This ends the proof of our statement.

Proof that $cons[cn]$ is $(\to)$-synthetic. If (1) $X \cup \{A, B\} \in cons[cn]$ then by the definition of $cons[cn]$ it follows that (2) $cn(X, A, B) \neq S$. We will show now that (3) $cn(X, A, A \to B) \subseteq cn(X, A, B)$. Indeed, given that cn is reflexive we infer that (3.1) $B \in cn(X, A, B)$. This step implies that (3.2) $A \to B \subseteq cn(X, B)$ because cn is $(\to)$-synthetic. But (3.2) gives directly that (3.3) $cn(X, A, A \to B) \subseteq cn(X, A, B)$. This proves step (3). Now steps (2) and (3) imply that (4) $cn(X, A, A \to B) \neq S$. Hence by the definition of $cons[cn]$ we infer that (5) $X \cup \{A, A \to B\} \in cons[cn]$, and this completes the proof.

Proof that $cons[cn]$ has the $(\to)$-extension property. Indeed, if (1) $X \in cons[cn]$ then (2) $cn(X) \neq S$. Step (2) implies that (3) $cn(X, A) \cap cn(X, A \to B) \neq S$ because, by the hypothesis, cn has the $(\to)$-cut property. Step (3) and the definition of $cons[cn]$ implies that (4) $X \cup \{A\} \in cons[cn]$ or $X \cup \{A \to B\} \in cons[cn]$, and this completes our proof. This completes the proof of Lemma 4. ☐

Lemma 5.2. *If cons is a compact consistency* $(\neg, \to)$*-property then cn*$[cons]$ *is a finitary closure* $(\neg, \to)$*-operator.*

Proof. Proof that $cn[cons]$ is $(\to)$-analytic. Suppose that (1) $A \to B \in cn[cons](X)$ and that (2) $B \notin cn[cons](X, A)$. By (2) and the definition of $cn[cons]$ there is C such that (3) $X \cup \{A, C\} \in cons$ and (4) $\{B, C\} \notin cons$. On the other hand, step (1) implies that $A \to B \in cn[cons](X, A)$. Hence using the definition of $cn[cons]$ we infer that (5) $X \cup \{A, C\} \in cons$ implies that $\{A \to B, C\} \in cons$. By (3) and (5) we can conclude that (6) $\{A \to B, C\} \in cons$. But cons is $(\to)$-analytic so we can draw from (6) that (7) $\{A, B, C\} \in cons$. Since cons is hereditary, step (7) implies that $\{B, C\} \in cons$, contrary to (4). This proves our statement.

Proof that cn[cons] is $(\to)$-synthetic. Suppose that (1) $B \in cn[cons](X, A)$ and (2) $A \to B \notin cn[cons](X)$. By (2) and the definition of *cn*[*cons*] there is C such that (3) $X \cup \{C\} \in cons$ and (4) $\{A \to B, C\} \notin cons$. Step (4) implies that (5) $\{A, B, C\} \notin cons$ because *cons* is $(\to)$-synthetic. Using step (3) we infer that (6) $X \cup \{A, C\} \in cons$ or $X \cup \{A \to B, C\} \in cons$ because cons has the $(\to)$-extension property. Using steps (4) and (6) we conclude that (7) $X \cup \{A, C\} \in cons$. Now steps (1) and (7) combined with the definition of *cn*[*cons*] imply that (8) $\{B, C\} \in cons$. Steps (5) and (8) imply that (9) $\{A \to B, B, C\} \in cons$ because cons has the $(\to)$-extension property. Given that *cons* is hereditary we infer from step (9) that $\{A \to B, C\} \in cons$, contrary to (4). This ends the proof.

Proof that *cn*[*cons*] has the $(\to)$-cut property. Suppose that (1) $C \in cn[cons](X, A)$, that (2) $C \in cn[cons](X, A \to B)$ and that (3) $C \notin cn[cons](X)$. By (3) and the definition of *cn*[*cons*] there is D such that (4) $X \cup \{D\} \in cons$ and that (5) $\{C, D\} \notin cons$. Using (2) and (5) we infer that (6) $X \cup \{D, A \to B\} \notin cons$. On the other hand, by steps (1) and (5) combined with the definition of *cn*[*cons*], we infer that (7) $X \cup \{D, A\} \notin cons$. Given that cons has $(\to)$-cut we infer from steps (6) and (7) that $X \cup \{D\} \notin cons$, contrary to (4), and this proves our statement. This completes the proof of Lemma 5.2. □

Using Lemmas 5.1 and 5.2 we get the following upgrading of Theorem 4.3.

Theorem 5.3. *The pair of mappings* $(cons[cn], cn[cons])$ *is a Galois connection between finitary closure* $(\neg, \to)$*-operators and compact consistency* $(\neg, \to)$*-properties.*

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