



Presentation of the Book The Gröbner Cover

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Abstract The aim of this paper is to present the research book “The Gröbner Cover” Montes (The Gröbner Cover, ACM-series, Springer, Berlin, 2019) recently published. This book is divided into two parts, one theoretical and one focusing on applications, and offers a complete description of the Canonical Gröbner Cover, to the author’s best knowledge, the most accurate algebraic method for discussing parametric polynomial systems. It also includes applications to the automatic deduction of geometric theorems, loci computation and envelopes. The theoretical part is a self-contained exposition on the theory of Parametric Gröbner Systems and Bases. It begins with Weispfenning introduction of Comprehensive Gröbner Systems (CGS), a fundamental contribution made in 1992, and provides a complete description of the Canonical Gröbner Cover (GC), which includes a canonical discussion of a set of parametric polynomial equations developed in Montes, Wibmer (J Symb Comput 45:1391–1425, 2010). In turn, the application part selects three problems for which the Gröbner Cover offers valuable new perspectives. The automatic deduction of geometric theorems (ADGT) becomes fully automatic and straightforward using GC, representing a major improvement on all previous methods. In terms of loci and envelope computation, GC makes it possible to introduce a taxonomy of the components and automatically compute it. The book also generalizes the definition of the envelope of a family of hyper-surfaces, and provides algorithms for its computation, as well as for discussing how to determine the real envelope. All the algorithms described in the book have also been included in the Singular software library `grobcov.lib` implemented by the author and H. Schönemann, the book serving also as User Manual for the library.

Keywords Gröbner Cover · Parametric polynomial systems · Canonical discussion · Automatic discovery of geometric theorems · Geometric locus computation · Geometric envelope computation

Mathematics Subject Classification 08-02 Research exposition · 13F20 Polynomial rings and ideals · 11C08 Polynomials

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1 Introduction

The index of the book “The Gröbner Cover” [1] is the following

Preface

1. Preliminaries

Part 1. Theory

2. Constructible sets

3. Comprehensive Gröbner Systems

4. I -regular functions on a locally closed set

5. The Canonical Gröbner Cover

Part 2. Applications

6. Automatic Deduction of Geometric Theorems

7. Geometric Loci

8. Geometric Envelopes

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2 The Gröbner Cover

The existence of the Gröbner Cover of a parametric polynomial $I \subset K[\mathbf{a}][\mathbf{x}]$ ideal is a consequence of Wibmer’s Theorem. It was described in (2010) by Montes and Wibmer [2].

Wibmer’s theorem establishes that for an *homogeneous parametric polynomial ideal* $I \subset K[\mathbf{a}][\mathbf{x}]$ ideal,

- There exists a canonical partition of the parameter space into locally closed segments $S_i = \mathbf{V}(\mathfrak{p}_i) \setminus \mathbf{V}(\mathfrak{q}_i)$;
- Each segment accepts a set of I -regular functions B_i that specializes to the reduced Gröbner basis for every point $\mathbf{a} \in S_i$,
- Preserving the set of leading power products (lpph_i) of the basis,
- And each segment S_i having a different set of lpph_i .

The set of triplets $(\text{lpph}_i, B_i, S_i)$ constitutes the *Gröbner Cover* of the homogeneous ideal I .

For a *non-homogeneous parametric ideal* we proceed as follows:

- Homogenize the ideal,
- Determine its Gröbner Cover,
- And dehomogenize.

The result is the canonical *Gröbner Cover* of the non-homogeneous ideal. It can contain segments with the same set of lpph_i , but deriving from different sets of lpph_i of the homogenized ideal, it is still canonic.

2.1 Example: The Singular Points of a Conic

We give an example to show the reduced number of segments that GC generates, and the canonical character of the output. We consider a conic

$$F := x^2 + 2cxy + by^2 + 2dx + 2ey + f = 0$$

and determine their singular points, depending on the parameter values, by considering the system $[F, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}]$. We can see that the result represents the canonical classification of a conic by its singular points. We consider the ideal

$$\begin{cases} x^2 + 2cxy + by^2 + 2dx + 2ey + f = 0 \\ 2x + 2cy + 2d = 0 \\ 2cx + 2by + 2e = 0. \end{cases}$$

and apply the `grobcov` command of the SINGULAR “grobcov.lib” library [3,4]. We use the option “showhom”, 1 in order to show the `lpphi` that provides information about the behaviour of the singularities at infinity, which is significant for this problem. The algorithm uses the auxiliary variable `@t` for homogeneizing. The Gröbner Cover determines four distinct segments, depending on the parameter values. The result can be summarized in the following table:

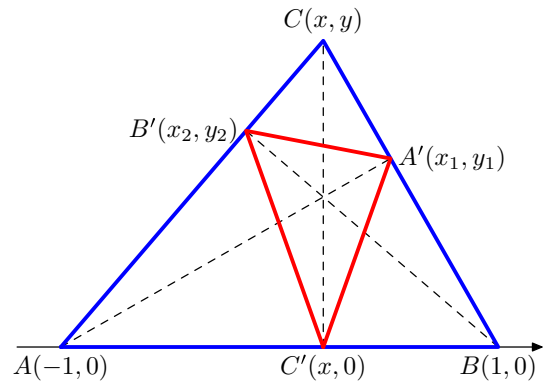
1. Generic case: <code>lpp = {1}</code> Basis: <code>{1}</code> Segment: $\mathbb{C}^5 \setminus \mathbf{V}(bd^2 - bf + c^2f - 2cde + e^2)$ Description: Non-degenerate conic without singular points. <code>lpph=1</code>
2. <code>lpp = {y, x}</code> Basis: $\left\{ y - \frac{d^2 - f}{cd - e}, x - \frac{cf - de}{cd - e} \right\}$ Segment: $\mathbf{V}(bd^2 - bf + c^2f - 2cde + e^2) \setminus \mathbf{V}(cd - e, b - c^2)$ Description: A unique singular point: two intersecting lines. <code>lpph=y,x</code>
3. <code>lpp = {x}</code> Basis: <code>{x + cy + d}</code> Segment: $\mathbf{V}(d^2 - f, cf - de, cd - e, b - c^2)$ Description: Singular double line. <code>lpph=x</code>
4. <code>lpp = {1}</code> Basis: <code>{1}</code> . Segment: $= \mathbf{V}(cd - e, b - c^2) \setminus \mathbf{V}(d^2 - f, cf - de, cd - e, b - c^2)$ Description: Special case without singular points: two parallel lines. <code>lpph=@t,x</code>

The above table gives a precise and complete discussion, classifying the conics by their singular points into 4 segments given as the difference between two varieties. The default option determines the P-representation (PREP) of the segment, which in this example is almost the same as the C-representation (CREP), because the segments have one single component with one single hole. (Using option “rep”, 1 we would obtain the CREP). The `lpph` represents the set of `lpp` of the homogenized ideal that uses the technical auxiliary homogenizing variable `@t`.

Even if segments 1 and 4 have the same basis `{1}`, showing that the system has no singular points, they correspond to two different types of conics. In segment 1 there is no singular point even considering the points at infinity, as shown by `lpph=1`, so that the homogenized ideal does not have any solution either. Thus the conic has no singular point at infinity.

For segment 4 the basis is also 1, showing that the conic has no singular points. But `lpph=@t,x` shows that at infinity it has a solution. Substituting the values of $b = c^2$ and $e = cd$, determined by the segment, the variety becomes

$$\begin{aligned} V &= \mathbf{V}(x^2 + 2cxy + c^2y^2 + 2dx + 2cdy + f) \\ &= \mathbf{V}\left((x + cy + d + \sqrt{d^2 - f})(x + cy + d - \sqrt{d^2 - f})\right) \end{aligned}$$

Fig. 1 Orthic triangle

$$= \mathbf{V} \left(x + cy + d + \sqrt{d^2 - f} \right) \cup \mathbf{V} \left(x + cy + d - \sqrt{d^2 - f} \right),$$

showing that, for $d^2 - f \geq 0$, it consists of two real parallel lines. The two lines intersect at infinity in the projective space, thus the conic has a singular point at infinity. This is the reason why segments 1 and 4 are separated in the Gröbner Cover, for which the different segments correspond to different lpph. (The segments are lpph-segments).

In the Application part, the book selects three problems for which the Gröbner Cover offers valuable new perspectives: Automatic Deduction of Geometric Theorems, Geometric Loci and Geometric Envelopes.

3 Automatic Deduction of Geometric Theorems

Consider a geometric proposition of the form

$$(H \wedge \neg H_1) \Rightarrow (T \wedge \neg T_1).$$

The book provides an *automatic algorithm* (ADGT) for obtaining *supplementary conditions* for transforming the proposition into a theorem. Consider as example the orthic triangle.

Let $A(-1, 0)$, $B(1, 0)$, $C(x, y)$ be the vertices of a triangle, and let $A'(x_1, y_1)$, $B'(x_2, y_2)$, $C'(x, 0)$ be the feet of the heights. The *orthic triangle* of ABC is $A'B'C'$. See Fig. 1.

Associated to the orthic triangle, consider the following ideals of hypothesis and thesis.

Hypothesis H: The triangle $A'B'C'$ is the orthic triangle of ABC

$$H = -yx_1 + (x - 1)y_1 + y, (x - 1)(x_1 + 1) + yy_1, \\ -yx_2 + (x + 1)y_2 - y, (x + 1)(x_2 - 1) + yy_2$$

Hypothesis H₁: The triangle ABC is *degenerate* (we shall deny it):

$$H_1 = y$$

Thesis T: The orthic triangle $A'B'C'$ is *isosceles* ($\overline{A'C'} = \overline{B'C'}$):

$$T = (x_1 - x)^2 + y_1^2 - (x_2 - x)^2 - y_2^2$$

Thesis T₁: The orthic triangle $A'B'C'$ is *degenerate* (points aligned, we shall deny it):

$$T_1 = x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1$$

We consider now two propositions.

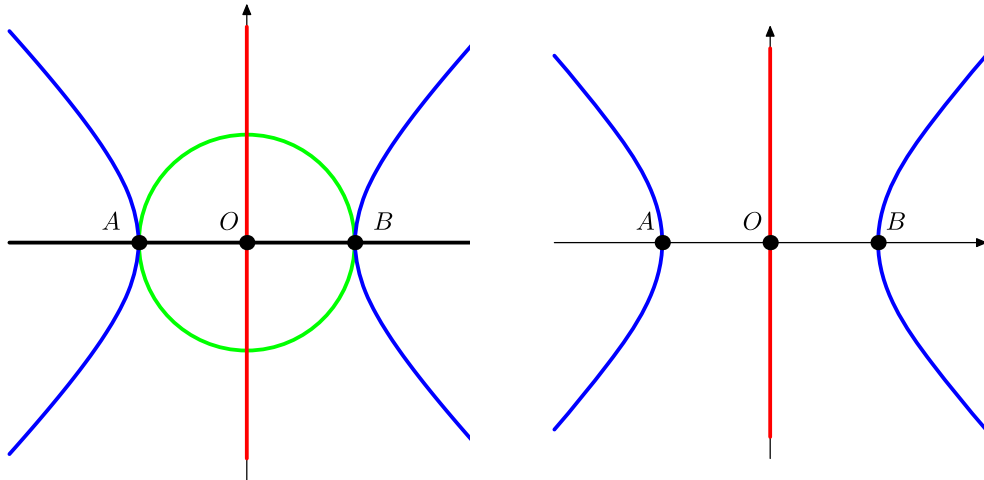


Fig. 2 Left: Complementary conditions for Proposition 3.1. Right: Complementary conditions for Proposition 3.2

Proposition 3.1 $H \Rightarrow T$. *The orthic triangle is isosceles.*

Calling sequence: $ADGT(H, T, 1, 1)$; Result:

$$(x, y) \in S_1 \cup S_2 \cup S_3$$

$$S_1 = \mathbf{V}(x^2 - y^2 - 1) \setminus \mathbf{V}(y^2 + 1, x)$$

$$S_2 = \mathbf{V}(x^2 + y^2 - 1)$$

$$S_3 = \mathbf{V}(x) \setminus \mathbf{V}(y^2 + 1, x)$$

We obtain three components, except complex points that are irrelevant, for the supplementary conditions for Proposition 3.1 to become a theorem (See Fig. 2, left). But part of the supplementary conditions correspond to degenerate triangles ABC or degenerate orthic triangles $A'B'C'$. Thus we consider a new proposition, in which we eliminate degenerate triangles.

Proposition 3.2 $H \wedge \neg H_1 \Rightarrow T \wedge \neg T_1$. *The orthic triangle $A'B'C'$ of a non-degenerate triangle ABC is isosceles but non-degenerate.*

Calling sequence: $ADGT(H, T, H_1, T_1)$; Result:

$$(x, y) \in (S_1 \setminus (A \cup B)) \cup (S_3 \setminus O)$$

The result is that the points of the x -axis, which correspond to degenerate ABC are eliminated from the supplementary conditions, as well as the component S_2 which corresponds to degenerate $A'B'C'$. (See Fig. 2, right).

4 Geometric Loci

Loci problems are usually considered in 2d or at most 3d, and are obtained by elimination techniques with unprecise definition (See [5]).

Using the Gröbner Cover we are able to

- Generalize them for n -dimensional space,
- Precise their irreducible components,
- And assign a *taxonomy* to every component.

A locus problem has

- A *tracer point* $T(\mathbf{x}) = T(x_1, \dots, x_n)$, corresponding to the coordinates \mathbf{x} of the locus points.
- A set of *auxiliary variables* $\mathbf{u} = (u_1, \dots, u_m)$ containing eventually a *mover point* $M(w_1, \dots, w_n)$. Alternatively we can consider as mover variables the set of all auxiliary variables u . Doing so, the taxonomy becomes more precise.
- An *ideal* $F[\mathbf{x}, \mathbf{u}]$ expected to have $n - 1$ degrees of freedom.

LOCUS allows to define and determine the *taxonomy* of its components.

Consider the solution of the system:

$$\mathbf{V}(F) = \{(\mathbf{x}, \mathbf{u}) \in \mathbb{C}^{n+m} : \forall f \in F, f(\mathbf{x}, \mathbf{u}) = 0\}.$$

Denote π_1, π_2 the *projections* onto the \mathbf{x} and the \mathbf{u} spaces respectively:

$$\begin{array}{ll} \pi_1 : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^n & \pi_2 : \mathbb{C}^{n+m} \rightarrow \mathbb{C}^m \\ (\mathbf{x}, \mathbf{u}) \mapsto \mathbf{x} & (\mathbf{x}, \mathbf{u}) \mapsto \mathbf{u} \end{array}$$

and the *anti-image* $\mathcal{A}(\mathbf{x})$ of a locus point \mathbf{x} on the \mathbf{u} space, and $\mathcal{A}_m(\mathbf{x})$ on the \mathbf{w} space of the mover variables.

$$\begin{array}{ll} \mathcal{A} : \mathbb{C}^n \rightarrow \mathbb{C}^m & \mathcal{A}_m : \mathbb{C}^n \rightarrow \mathbb{C}^{n'} \\ \mathbf{x} \mapsto \pi_2(\mathbf{V}(F) \cap \pi_1^{-1}(\mathbf{x})) & \mathbf{x} \mapsto \pi_3(\mathcal{A}(\mathbf{x})) \end{array}$$

We can give a formal generic definition of locus in algebraic terms.

Definition 4.1 (Algebraic locus) The locus L associated to the parametric polynomial system $F(\mathbf{x}, \mathbf{u})$, is the set

$$L = \pi_1(\mathbf{V}(F)) \subset \mathbb{C}^n.$$

Definition 4.2 (Normal and non-normal locus) We define two class of locus points $\mathbf{x} \in \mathbb{C}^n$:

$$\text{Normal point } \mathbf{x} \text{ if } \dim(\mathcal{A}_m(\mathbf{x})) = 0.$$

$$\text{Non-normal point } \mathbf{x} \text{ if } \dim(\mathcal{A}_m(\mathbf{x})) > 0$$

The set of all normal points is called the *normal locus* and the set of all non-normal points is called the *non-normal locus*.

Proposition 4.3 The normal locus L_{nor} and the non-normal locus L_{nonor} are disjoint constructible sets. We have $L = L_{\text{nor}} \oplus L_{\text{nonor}}$, and each of both subsets can be decomposed into irreducible components

$$C_i = \mathbf{V}(\mathfrak{p}_i) \setminus \left(\bigcup_j \mathbf{V}(\mathfrak{p}_{ij}) \right) \quad (4.1)$$

where all the \mathfrak{p}_i and all the \mathfrak{p}_{ij} are prime ideals, being

$$L_{\text{nor}} = \bigcup_j C_{\text{nor},j}, \quad L_{\text{nonor}} = \bigcup_j C_{\text{nonor},j}$$

We can assign a *taxonomy* to each component of a locus

Definition 4.4 (“Normal” and “Special” components) Let C be a component of the *normal locus* and let \mathbf{w} be the subset of the auxiliary variables \mathbf{u} representing the mover coordinates (if they exist), or the last n' auxiliary variables \mathbf{u} conveniently chosen. A component C of the normal locus is “*Normal*” if $\dim(C) = \dim(\mathcal{A}_m(C))$. But it can happen that $\dim(C) > \dim(\mathcal{A}_m(C))$, and then the component is “*Special*”.

Definition 4.5 (“Degenerate” and “Accumulation” components) A component of the *non-normal locus* is “*Degenerate*” if its dimension is $n - 1$. If its dimension is smaller than $n - 1$, then it is an “*Accumulation*” component.

Example 4.6 Richard Serra surfaces.

Richard Serra is an artist that has constructed very nice mathematical sculptures, and particularly in the Guggenheim museum of Bilbao. They are generated by two curves located in two parallel planes in the 3-dimensional space, generating a surface. The surface is obtained in the following way:

Consider a parabola $y_1 = x_1^2$ on the plane $z_1 = -1$ (*floor*) and a parabola $x_2 = y_2^2$ on the plane $z_2 = 1$ (*ceiling*):

Determine the locus formed by the lines relying the points of both parabolas having parallel tangents. The tangent vectors are respectively $(-2x_1, 1, 0)$ and $(1, -2y_2, 0)$. The condition of being parallel is $4x_1y_2 - 1 = 0$. The system is:

$$\begin{aligned} F = & x_1^2 - y_1, z_1 + 1, \\ & y_2^2 - x_2, z_2 - 1, \\ & 4x_1y_2 - 1, \\ & x - x_1 - \lambda(x_2 - x_1), y - y_1 - \lambda(y_2 - y_1), z - z_1 - \lambda(z_2 - z_1) \end{aligned}$$

Applying `locus` we have the following calling sequence:

```
> LIB "grobcov.lib";
> ring R = (0, x, y, z), (lambda, x2, y2, z2, x1, y1, z1), lp;
> ideal F = x1^2 - y1, y2^2 - x2, 4x1y2 - 1, z1 + 1, z2 - 1,
    x - x1 - lambda(x2 - x1), y - y1 - lambda(y2 - y1),
    z - z1 - lambda(z2 - z1);
> locus(F);
```

and obtain the result

$$\begin{aligned} S = & \mathbf{V}(2048x^3z + 2048x^3 - 4096x^2y^2 + 1152xyz^2 - 1152xy \\ & - 2048y^3z + 2048y^3 + 27z^4 - 54z^2 + 27) \\ & \setminus (\mathbf{V}(z + 1, y) \cup \mathbf{V}(z - 1, x)) \end{aligned}$$

$T = \text{Special}$

$$A_m = \mathbf{V}(z_1 + 1, x_1^2 - y_1)$$

We highlight two questions.

- (1) `locus` obtains the equation of the surface and characterizes it as “Special” component. The “Special” taxonomy is the consequence of the 2-dimensional locus surface being generated by the parabola $\mathbf{V}(z_1 + 1, x_1^2 - y_1)$ in the mover variables (x_1, y_1, z_1) which has dimension 1 (see Fig. 3). If we use instead as mover variables the set of all auxiliary variables $(\lambda, x_2, y_2, z_2, x_1, y_1, z_1)$, the taxonomy becomes “Normal”, because the anti-image of C corresponds then to the parabolas and free value of λ , and so $\dim(C) = \dim(A_m(C)) = 2$.
- (2) `locus` also determines that the lines $\mathbf{V}(z + 1, x)$ and $\mathbf{V}(z - 1, x)$ do not form part of the affine locus. The reason is that each of these lines, included in the top of the envelope, would be generated by the tangent to one of the parabolas at the vertex, whose corresponding point in the second parabola is at infinity but not at any real point of them.

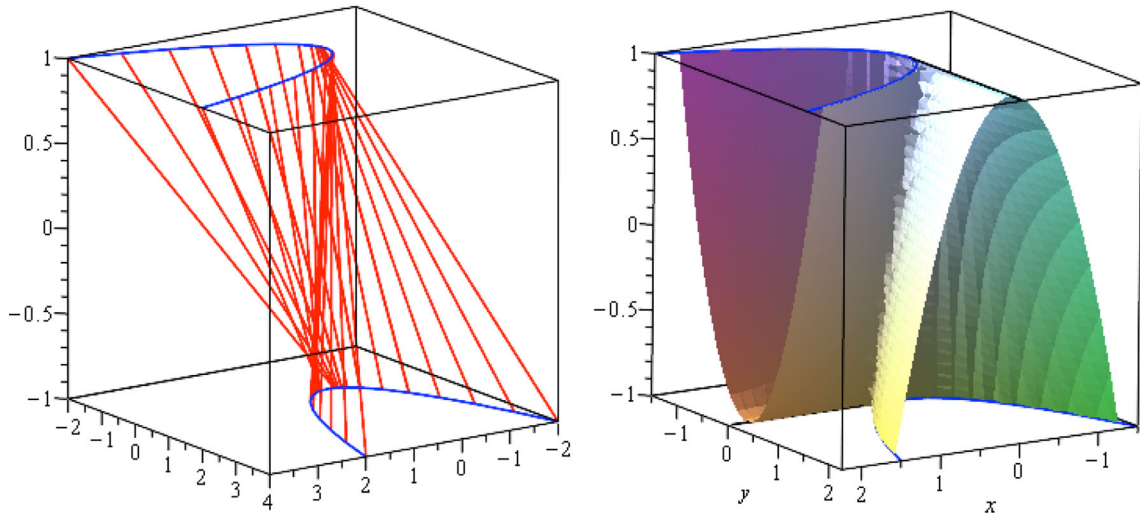


Fig. 3 Richard Serra surface

5 Envelopes

Usually the definition of *Envelope* concerns a family of curves or a family of surfaces with $n - 1$ degrees of freedom (Fig. 3). We generalize the problem to an n dimensional space with higher degrees of freedom. We have the following definitions and theorems.

Definition 5.1 (Family of hyper-surfaces) We say that

$$F(\mathbf{x}, \mathbf{u}) = F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$$

and the independent *restrictions* $C = \{g_1, \dots, g_s\}$ (with $s < m$)

$$\begin{cases} g_1(u_1, \dots, u_m) = 0 \\ \dots \\ g_s(u_1, \dots, u_m) = 0 \end{cases}$$

represent a *family of hyper-surfaces* of \mathbb{C}^n , if F depends at least on 1 *parameter* u . Thus, if the g_i are independent $d = \dim(C) = m - s \geq 1$.

Definition 5.2 (*Envelope*) Given a family of hyper-surfaces $F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$ with m parameters \mathbf{u} , constrained by the $s < m$ independent equations $C = \langle g_1(\mathbf{u}), \dots, g_s(\mathbf{u}) \rangle$, let

$$J = \frac{\partial(F, g_1, \dots, g_s)}{\partial(u_1, \dots, u_m)}; J_c = \{\text{minors}(J) \text{ of order } (s + 1) \times (s + 1)\}.$$

The set of $\binom{m}{s+1}$ equations J_c , imply that the *rank of the Jacobian* is less than or equal than s . Consider the ideal $S = \langle F, C, J_c \rangle$,

The *algebraic envelope* of F, C , if it exists, is

$$L = \text{locus}(S).$$

Theorem 5.3 (Associated Tangent element) *Let* $F(x_1, \dots, x_n; u_1, \dots, u_m) = 0$ *be a family of hyper-surfaces with* m *parameters* \mathbf{u} , *constrained by the* $s < m$ *equations* $C = \langle g_1(\mathbf{u}), \dots, g_s(\mathbf{u}) \rangle$.

Let E be the envelope, and $E_i = \mathbf{V}(p_i(\mathbf{x})) \setminus \mathbf{V}(q_i(\mathbf{x}))$ the C -representation of a “Normal” standard $(n - 1)$ -dimensional component of E corresponding to a hyper-surface, and $\mathbf{x}^{(0)} \in E_i$ be a regular point of $p_i(\mathbf{x}) = 0$.

Then there exists one Associated Tangent hyper-surface $F(\mathbf{x}, \mathbf{u}^{(0)})$ of the family F (or more than one) that passes at point $\mathbf{x}^{(0)}$ (i.e. $F(\mathbf{x}^{(0)}, \mathbf{u}^{(0)}) = 0$) and is tangent to $\mathbf{V}(p_i(\mathbf{x}))$ at point $\mathbf{x}^{(0)}$.

The SINGULAR `grobcov.lib` library has incorporated the following algorithms and commands related to envelopes:

1. `envelop`: Determines the envelope components and their taxonomies.
2. `AssocTanToEnv`: Determines the associated tangent element of the family passing at a regular point of a “Normal” component.
3. `FamElemsAtEnvComPoints`: Determines all the elements of the family passing at a point of the envelope.
4. `discrim`: Determines the discriminant with respect to a variable, when an equation is of degree 2.

Example 5.4 Consider the family of spheres of radius 1, centered at point (x_1, y_1, z_1) of a sphere of center $(0, 0, t)$ and radius \sqrt{t} . We have the following family and restrictions:

$$F = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 - 1,$$

$$C = \langle x_1^2 + y_1^2 + (z_1 - t)^2 - t \rangle.$$

Applying `envelop(F, C)` we obtain 2 components:

$$E_1 = \mathbf{V}(16x^6 + 48x^4y^2 + 16x^4z^2 - 32x^4z - 56x^4 + 48x^2y^4 + 32x^2y^2z^2$$

$$- 64x^2y^2z - 112x^2y^2 - 32x^2z^3 - 24x^2z^2 + 29x^2 + 16y^6$$

$$+ 16y^4z^2 - 32y^4z - 56y^4 - 32y^2z^3 - 24y^2z^2 + 29y^2$$

$$+ 16z^4 - 24z^3 - 15z^2 + 38z - 15) \setminus \mathbf{V}(z - 1, x^2 + y^2)$$

Normal

$$E_2 = \mathbf{V}(z - 1, y, x)$$

Accumulation

The envelope has a “Normal” component E_1 , and an “Accumulation” point. As the restriction C has degree 2 in t , we can compute its discriminant with respect to t

$$\Delta_t = 4z_1 - 1 - 4x_1^2 - 4y_1^2,$$

which divides the space into two regions. On the region $\Delta_t(x_1, y_1, z_1) \geq 0$ there can exist centers of family spheres, but they cannot exist on the region $\Delta_t(x_1, y_1, z_1) < 0$. These regions also separate both parts of the envelope, E_{ext} , for $\Delta_t(x_1, y_1, z_1) < 0$ and E_{int} for $\Delta_t(x_1, y_1, z_1) > 0$ (Fig. 4).

In Fig. 5 left is represented a section at $y = 0$ of the envelope, the separating paraboloid, and a set of family spheres with centers in the paraboloid. We observe that these spheres are tangent to both parts E_{ext} and E_{int} of the envelope. In Fig. 5 right are also represented circles of the family centered everywhere.

Considering a point $P = (x, y, z)$ of the envelope component E_1 , `AssocTanToEnv(F, C, E_1)` gives:

$$t = -\frac{12x^4 + 24x^2y^2 - 4x^2z^2 - 28x^2z - 31x^2 + 12y^4 - 4y^2z^2 - 28y^2z - 31y^2 - 16z^4 - 28z^3 - 3z^2 + 10z + 10}{16z^3 + 24z^2 + 12z - 25}$$

$$z_1 = -\frac{24x^4 + 48x^2y^2 - 8x^2z^2 - 56x^2z - 62x^2 + 24y^4 - 8y^2z^2 - 56y^2z - 62y^2 - 32z^4 - 40z^3 + 18z^2 + 32z - 5}{32z^3 + 48z^2 + 24z - 50}$$

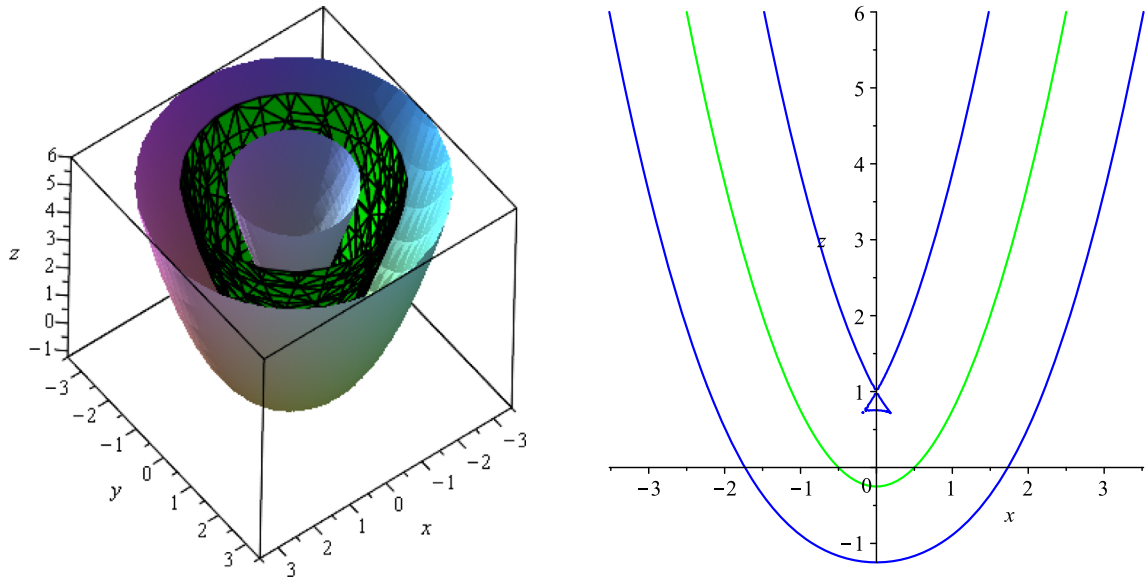


Fig. 4 Envelope and surface $\Delta_t = 0$ 2d-Section

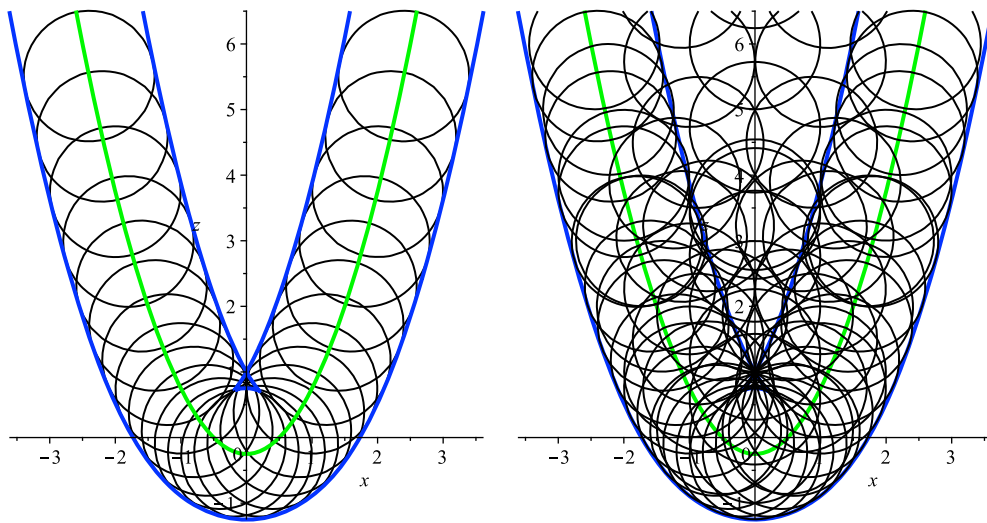


Fig. 5 Section for $y = 0$

$$y_1 = \frac{6x^4yz + 32x^4y + 32x^2y^3z + 64x^2y^3 + 16x^2yz^3 - 16x^2yz^2 - 100x^2yz - 116x^2y + 16y^5z + 32y^5}{32z^4 + 16z^3 - 24z^2 - 74z + 50}$$

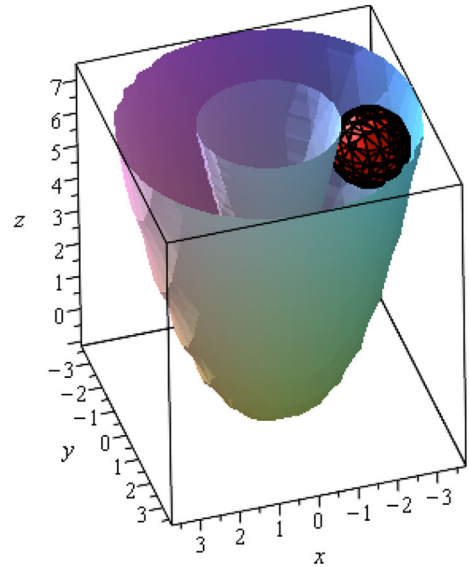
$$- \frac{16y^3z^3 - 16y^3z^2 - 100y^3z - 116y^3 - 48yz^4 - 68yz^3}{32z^4 + 16z^3 - 24z^2 - 74z + 50}$$

$$x_1 = \frac{16x^5z + 32x^5 + 32x^3y^2z + 64x^3y^2 + 16x^3z^3 - 16x^3z^2 - 100x^3z - 116x^3 + 16xy^4z + 32xy^4}{32z^4 + 16z^3 - 24z^2 - 74z + 50}$$

$$- \frac{16xy^2z^3 - 16xy^2z^2 - 100xy^2z - 116xy^2 - 48xz^4 - 68xz^3 - 36xz^2 + 36xz + 35x}{32z^4 + 16z^3 - 24z^2 - 74z + 50}$$

determining the values of (t, x_1, y_1, z_1) of the parameters of the associated tangent sphere to E_1 at $P(x, y, z)$.

Fig. 6 Envelope E_1 and the associated tangent sphere at point P_1



For example, at point $P_1 = (-3.519505319, 0, 6)$, the associated tangent sphere has center $(-2.538358523, 0., 6.193264030)$. It is represented in red in Fig. 6. This sphere is also the associated tangent sphere of the family at point $P_2 = (-1.557188274, 0, 6.386408905)$.

We can observe two disjoint *real* parts of the component E_1 . In order to obtain the true *real* envelope, we determine all family elements passing at a point of E_1 , calling

FAMELEMSATENVCOMP POINTS(F, C, E_1), we obtain:

$$\begin{aligned} \text{lpp} &= \{z_1^2, t^2\} \\ B &= \{z_1^2 - 2zz_1 + y_1^2 - 2yy_1 + x_1^2 - 2xx_1 + x^2 + y^2 + z^2 - 1, \\ &\quad t^2 - 2tz_1 - t + 2zz_1 + 2yy_1 + 2xx_1 - x^2 - y^2 - z^2 + 1\}, \end{aligned}$$

and compute the *discriminant* of both polynomials. After some manipulation we obtain:

$$\begin{aligned} \text{discrim}(B[1], z_1) &= -(y - y_1)^2 + (x - x_1)^2 + (z - z_1)^2 - 1) + (z - z_1)^2 = (z - z_1)^2, \\ \text{discrim}(B[2], t) &= (y - y_1)^2 + (x - x_1)^2 + (z - z_1)^2 - 1) - \left(x_1^2 + y_1^2 - z_1 - \frac{1}{4}\right) \\ &= -\left(x_1^2 + y_1^2 - z_1 - \frac{1}{4}\right) \end{aligned}$$

The first discriminant is always positive, but the second is only positive when

$$z_1 \geq x_1^2 + y_1^2 - \frac{1}{4}.$$

Computing the envelope of the set of spheres F with the restriction of having their centers in the separating paraboloid $z_1 = x_1^2 + y_1^2 - \frac{1}{4}$, we obtain the same result as for the initial problem. This indicates that the real envelope is the external part of E_1 , i.e.,

$$\text{Real envelope} = E_{\text{ext}} = \left\{ (x, y, z) \in E_1 : z < x^2 + y^2 - \frac{1}{4} \right\}.$$

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