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The Depth of Compositions

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Abstract In this paper we consider compositions of n as bargraphs. The depth of a cell inside this graphical representation is the minimum number of horizontal and/or vertical unit steps that are needed to exit to the outside. The depth of the composition is the maximum depth over all cells of the composition. We use finite automata to study the generating function for the number of compositions having a depth of at least r.

Keywords Composition · Automaton · Generating function

Mathematics Subject Classification Primary 05A05 · 05A15; Secondary 05A16

1 Introduction

A composition $\sigma = \sigma_1 \cdots \sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is $|\sigma| = \sigma_1 + \cdots + \sigma_m = n$. For instance, the compositions of 3 are 3, 21, 12 and 111. The number of summands, namely m, is called the number of parts of the composition and n is called the weight of σ. It is well known that the number of compositions of n is given by 2^{n-1} and the number of compositions of n with m parts is given by $\binom{n-1}{m-1}$. Compositions have been studied extensively in recent years (for example, see [4]).

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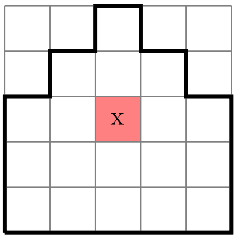

Fig. 1 The composition 34,543

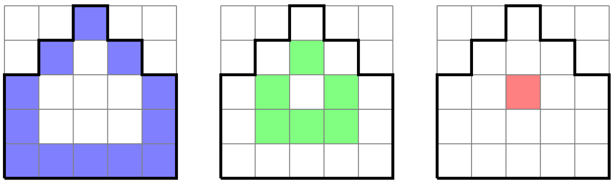

Fig. 2 The 3 consecutive "peelings" of the inner site-perimeter of the composition 34543

Compositions can be represented as bargraphs where the lower edge lies on a horizontal axis (for instance, see [6,10,11]). They are drawn on a regular planar lattice grid and are made up of square cells. So a composition is uniquely defined by the size of each part. The size of the composition is the total number of cells in the representing bargraph. We will study the depth of compositions, which are presented as bargraphs. Let σ be any composition and let c be any cell of σ. The depth of the cell c, denoted by $depth(c)$, is the minimum number of horizontal or vertical steps to exit σ starting from c. The depth of σ is defined as $depth(\sigma) = \max_c depth(c)$, where the maximum is over all cells c of σ. For example, the cell that is indicated by x in Fig. 1 of the composition $\sigma = 34543$, needs three steps to exit σ. Moreover, $depth(\sigma) = 3$ since all other cells require fewer than three steps to exit. An alternative conception of the depth of a composition is given by the size of its Durfee square, see [2].

Equivalently, the depth of a composition can be understood as the number of times the inner site-perimeter can be recursively removed or "peeled" from the composition until nothing remains, as shown below. The inner site-perimeter is defined as all cells in the composition whose depth is one, see [3,7]. Each successive inner site-perimeter is coloured in Fig. 2.

The aim of this paper is to study the generating function for the number of compositions having a depth of at least r. To achieve our goals, we use finite automata technique (for such technique in enumerative combinatorics, for instance, we refer the reader to [1,5,8,9,12]).

2 Main results

We say that the composition $\sigma = \sigma_1\sigma_2\cdots\sigma_s$ contains the composition $\tau = \tau_1\tau_2\cdots\tau_m$ if there exist j such that $0 \le j \le s-m$ and $\sigma_{j+i} \ge \tau_i$ for all $i = 1, 2, \ldots, m$. For instance, the composition 4534 contains 234 but does not contain 144. For all $r \ge 2$, define

$$\tau^{(r)} := r(r+1)\cdots(2r-2)(2r-1)(2r-2)\cdots(r+1)r.$$

In Fig. 1, the illustration is for $r = 3$.

From the definitions, we can state the following fact.

Proposition 1 *A composition σ satisfies $depth(\sigma) \ge r$ if and only if it contains $\tau^{(r)}$.*

In order to enumerate the compositions that avoid (i.e., do not contain) $\tau^{(r)}$, we need the following notation and definitions. A *prefix* of a composition σ is a sub-composition σ' (possible empty) such that there exists a nonempty sub-composition σ'' with $\sigma = \sigma'\sigma''$. Let $\mathcal{V}_r$ be the set of all prefixes of $\tau^{(r)}$. For example, $\mathcal{V}_3 = \{\epsilon, 3, 34, 345, 3454\}$ when $\tau^{(3)} = 34543$ and ϵ is the empty composition.

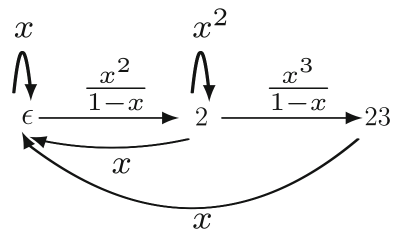

Fig. 3 The graph G_2

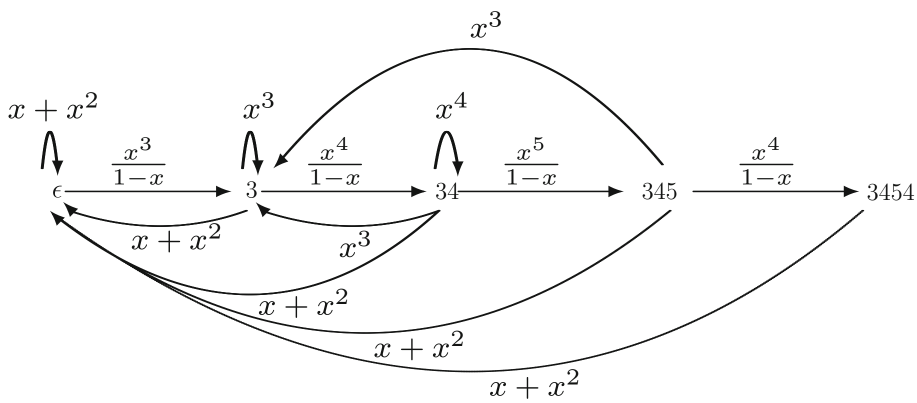

Fig. 4 The graph G_3

For given $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ and $k \in \mathbb{N}$, we define the reduction $red(\sigma k)$ of σk to be $\sigma' = \sigma'_1\sigma'_2\cdots\sigma'_{m'}$ if $\sigma' \in \mathcal{V}_r$, m' is maximal and $\sigma_{m+1-m'+j} \geq \sigma'_j$ for all $j = 1, 2, \ldots, m'$, where $\sigma_{m+1} = k$. Otherwise, we say the reduction of σk is not defined. For instance, if $r = 3$ then $red(245) = 34$ and $red(613445) = 345$.

Now let us define a directed graph (finite automata, for example, see [1,5]) G_r on the vertices $\mathcal{V}_r$ with edges between $\sigma = \sigma_1\sigma_2\cdots\sigma_m \in \mathcal{V}_r$ and $red(\sigma k)$ with weight $x^{|\sigma k|-|red(\sigma k)|}$, where $red(\sigma k)$ is defined. For simplicity, if there are two edges between σ and σ' with weights w, w' then we just write as one edge with weight $w + w'$.

Example 1 Let $r = 2$ ($\tau^{(2)} = 232$). The graph G_r has three vertices $\epsilon, 2, 23$. Note that

- $red(\epsilon k) = \epsilon$ for $k = 1$ and $red(\epsilon k) = 2$ for all $k \geq 2$. Thus, there is an edge between ϵ and ϵ with weight x and there is an edge between ϵ and 2 with weight $x^2 + x^3 + \cdots = \frac{x^2}{1-x}$. This corresponds to the generating function for adding a part k of size 2 or more.
- $red(21) = \epsilon$, $red(22) = 2$ and $red(2k) = 23$ for all $k \geq 3$. So there is an edge between 2 and ϵ with weight x, there is an edge between 2 and 2 with weight x^2, and there is an edge between 2 and 23 with weight $x^3/(1-x)$.
- $red(231) = \epsilon$ and for $k > 1$ $red(23k)$ is not defined. So there is an edge between 23 and ϵ with weight x.

Summarizing this information, we obtain the graph G_2 as presented in Fig. 3.

Example 2 Let $r = 3$ ($\tau^{(2)} = 34543$). The graph G_r on its five vertices $\epsilon, 3, 34, 345, 3454$ is presented in Fig. 4.

To state our next observation, we define a weight of a path in the graph G_r to be the product of the weights of all the edges. For example, the weight of the path $\epsilon, 3, 3, 34, 345, \epsilon, 3$ is

$$\frac{x^3}{1-x}\cdot x^3\cdot\frac{x^4}{1-x}\cdot\frac{x^5}{1-x}\cdot(x+x^2)\cdot\frac{x^3}{1-x} = \frac{x^{19}(1+x)}{(1-x)^4}.$$

Proposition 2 *Set $r \geq 2$. Then the generating function for the number of compositions of n with exactly m parts that avoid $\tau^{(r)}$ is given by the total weight of all paths of length m starting from ϵ in G_r.*

Proof Let $\sigma = \sigma_1\sigma_2\cdots\sigma_m$ be any composition that avoids $\tau^{(r)}$ with exactly m parts. From the definition of red, we find that the path corresponding to σ in G_r is given by

$$\epsilon \underset{\sigma_1}{\rightarrow} \sigma^{(1)} \underset{\sigma_2}{\rightarrow} \sigma^{(2)} \cdots \underset{\sigma_m}{\rightarrow} \sigma^{(m)},$$

where $\sigma^{(j)} = red(\sigma^{(j-1)}\sigma_j)$ with $\sigma^{(0)} = \epsilon$. On the other hand, for each $\sigma^{(j)}$, $j = 0, 1, \ldots, m$, we can use the graph G_r and the above path to define unique elements σ_j, $j = 1, 2, \ldots, m$ such that $\sigma^{(j)} = red(\sigma^{(j-1)}\sigma_j)$. Thus if σ follows from a path (as described above), then σ is a composition that avoids $\tau^{(r)}$ with m parts. Thus, for each composition that avoids $\tau^{(r)}$ there exists a unique path in G_r with weight $x^{|\sigma|}$. □

Let $\mathbf{A}_r$ be the adjacency matrix of the directed graph G_r. By Proposition 2, we have that the generating function for the number of compositions with exactly m parts that avoid $\tau^{(r)}$ is given by $w^T\mathbf{A}_r^m v$, where $w = (1, 0, 0, \ldots, 0)^T$ and $v = (1, 1, \ldots, 1)^T$ are vectors with $|\mathcal{V}_r| = 2r - 1$ coordinates. Thus, the generating function for the number of compositions that avoid $\tau^{(r)}$ is given by, see [4]

$$\sum_{m\geq 0} w^T\mathbf{A}_r^m v = w^T(I - \mathbf{A}_r)^{-1}v.$$

Hence, we obtain our main result,

Theorem 1 *Let $r \geq 2$, the generating function for the number of compositions of n that avoid $\tau^{(r)}$ is given by $w^T(I - \mathbf{A}_r)^{-1}v$, where I is the unit matrix and $w = (1, 0, \ldots, 0)^T$ and $v = (1, 1, \ldots, 1)^T$ are vectors of $2r - 1$ coordinates. Moreover, this generating function is a rational function in x.*

Example 3 From Examples 1 and 2, we obtain

$$\mathbf{A}_2 = \begin{pmatrix} x & \frac{x^2}{1-x} & 0 \\ x & x^2 & \frac{x^3}{1-x} \\ x & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}_3 = \begin{pmatrix} x+x^2 & \frac{x^3}{1-x} & 0 & 0 & 0 \\ x+x^2 & x^3 & \frac{x^4}{1-x} & 0 & 0 \\ x+x^2 & x^3 & x^4 & \frac{x^5}{1-x} & 0 \\ x+x^2 & x^3 & 0 & 0 & \frac{x^4}{1-x} \\ x+x^2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence, the generating function for the number of compositions of n that avoid $\tau^{(2)} = 232$ is given by

$$\frac{(1-x)^2 + x^3(1-x) + x^5}{(1-2x)(1-x) + x^3(1-2x) + x^5 - x^6},$$

and the generating function for the number of compositions of n that avoid $\tau^{(3)} = 34{,}543$ is given by

$$\frac{(1-x)^4 + x^5(1-x)^3 + x^9(1-x)^2 + x^{13} - x^{14} + x^{16}}{(1-2x)(1-x)^3 + x^5(1-2x)(1-x)^2 + x^9(1-2x)(1-x) + x^{13}(1-2x) + x^{16} - x^{17} - x^{18}}.$$

Actually, from the definition of the graph G_r, we can state an explicit formula for the matrix $\mathbf{A}_r$.

Proposition 3 *Let $r \geq 2$ and $\mathbf{A}_r = (a_{ij})_{1\leq i,j\leq 2r-1}$. Then*

$$a_{ij} = \begin{cases} \sum_{k=1}^{r-1} x^k, & j = 1, \\ \frac{x^{r-1+i}}{1-x}, & j = i+1, i = 1, 2, \ldots, r, \\ \frac{x^{3r-1-i}}{1-x}, & j = i+1, i = r+1, r+2, \ldots, 2r-2, \\ x^{r-2+j}, & 2 \leq j \leq r, j \leq i \leq 2r-j, \\ 0, & \textit{otherwise}. \end{cases}$$

Define $f_s = x+x^2+\cdots+x^s$ and define δ_X to be 1 if X holds and 0 otherwise. Let $\mathbf{L}_r = (\ell_{ij})_{1\le i,j\le 2r-1}$ be a lower triangular matrix and $\mathbf{U}_r = (u_{ij})_{1\le i,j\le 2r-1}$ be a upper triangular matrix such that $\ell_{11} = \cdots = \ell_{(2r-1)(2r-1)} = 1$, $u_{11} = 1 - f_{r-1}$, $u_{ij} = 0$ for all $j \ge i+2$, and

$$\ell_{ij} = -\frac{f_{r-1}}{1-f_{r-1}} \frac{x^{\binom{r+j-1}{2}-\binom{r}{2}}}{(1-x)^{j-1}\prod_{s=2}^{j} u_{ss}} - \sum_{k=0}^{j-2} \frac{x^{\binom{r+j-1}{2}-\binom{r+j-2-k}{2}}\delta_{i\le 2r-j+k}}{(1-x)^k \prod_{s=j-k}^{j} u_{ss}},$$
$$\text{for } j+1 \le i \le 2r-1,\ 2 \le j \le r+1, \tag{2.1}$$

$$\ell_{ij} = \frac{x^{\binom{2r-1}{2}-\binom{3r-j}{2}}}{(1-x)^{j-r-1}\prod_{k=r+2}^{j} u_{kk}} \ell_{i(r+1)}, \quad \text{for } j+1 \le i \le 2r-1,\ r+2 \le j \le 2r-2, \tag{2.2}$$

$$u_{i(i+1)} = \begin{cases} -\frac{x^{r-1+i}}{1-x}, & i = 1,2,\ldots,r, \\ -\frac{x^{3r-1-i}}{1-x}, & i = r+1, r+2, \ldots, 2r-2. \end{cases} \tag{2.3}$$

$$u_{ii} = \begin{cases} 1 - x^{r-2+i} + \ell_{i(i-1)}\frac{x^{r-2+i}}{1-x}, & i = 2,3,\ldots,r, \\ 1 + \ell_{i(i-1)}\frac{x^{3r-i}}{1-x}, & i = r+1,\ldots,2r-1. \end{cases} \tag{2.4}$$

For instance,

$$\mathbf{L}_2 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{x}{1-x} & 1 & 0 \\ -\frac{x}{1-x} & \frac{x^3}{x^4-x^3+2x-1} & 1 \end{pmatrix}$$

and

$$\mathbf{U}_2 = \begin{pmatrix} 1-x & -\frac{x^2}{1-x} & 0 \\ 0 & \frac{-x^4+x^3-2x+1}{(1-x)^2} & -\frac{x^3}{1-x} \\ 0 & 0 & \frac{-x^6+x^5-2x^4+x^3+2x^2-3x+1}{(-x^4+x^3-2x+1)(1-x)} \end{pmatrix}.$$

Theorem 2 *For all $r \ge 2$, $I - \mathbf{A}_r = \mathbf{L}_r\mathbf{U}_r$, where I is the unit matrix.*

Proof Define $b_{ij} = \sum_{k=1}^{2r-1} \ell_{ik}u_{kj}$. By Proposition 3, we have to show that $b_{ij} = \delta_{i=j} - a_{ij}$. Clearly, by (2.1)–(2.4), we have $b_{11} = \ell_{11}u_{11} = 1 - f_{r-1} = 1 - a_{11}$, $b_{12} = \ell_{11}u_{12} = -a_{12}$, and $b_{1j} = \ell_{11}u_{1j} = 0 = a_{1j}$ for all $j = 3,4,\ldots,2r-1$. Moreover, $b_{i1} = \ell_{i1}u_{11} = -f_{r-1} = -a_{i1}$, for all $i = 2,3,\ldots,2r-1$.

Now let us show that $b_{ii} = 1 - a_{ii}$: note that $b_{ii} = u_{ii} + \ell_{i(i-1)}u_{(i-1)i}$, so by (2.1)–(2.4), we have $b_{ii} = 1 - a_{ii}$, as required.

Now we show that $b_{ij} = -a_{ij}$ for all $2 \le i < j \le 2r-1$: note that $b_{ij} = \ell_{i(j-1)}u_{(j-1)j} + \ell_{ij}u_{jj}$, so by (2.1)–(2.4), we have $b_{i(i+1)} = u_{i(i+1)} = -a_{i(i+1)}$ and $b_{ij} = 0 = -a_{ij}$ for all $j \ge i+2$.

Thus, it remains to show that $b_{ij} = -a_{ij}$ for all $2 \le j < i \le 2r-1$. We show only the cases $2 \le j \le r$ and $j+1 \le i \le 2r-1$ since the other cases are similar. By (2.1)–(2.4) and $b_{ij} = \ell_{i(j-1)}u_{(j-1)j} + \ell_{ij}u_{jj}$, we see that

$$\begin{aligned} b_{ij} &= \left(\frac{f_{r-1}}{1-f_{r-1}} \frac{x^{\binom{r+j-2}{2}-\binom{r}{2}}}{(1-x)^{j-2}\prod_{s=2}^{j-1} u_{ss}} + \sum_{k=0}^{j-3} \frac{x^{\binom{r+j-2}{2}-\binom{r+j-3-k}{2}}\delta_{i\le 2r-j+k+1}}{(1-x)^k \prod_{s=j-1-k}^{j-1} u_{ss}} \right) \frac{x^{r+j-2}}{1-x} \\ &\quad - \left(\frac{f_{r-1}}{1-f_{r-1}} \frac{x^{\binom{r+j-1}{2}+\binom{r}{2}}}{(1-x)^{j-1}\prod_{s=2}^{j} u_{ss}} + \sum_{k=0}^{j-2} \frac{x^{\binom{r+j-1}{2}-\binom{r+j-2-k}{2}}\delta_{i\le 2r-j+k}}{(1-x)^k \prod_{s=j-k}^{j} u_{ss}} \right) u_{jj} \\ &= \frac{f_{r-1}}{1-f_{r-1}} \frac{x^{\binom{r+j-1}{2}-\binom{r}{2}}}{(1-x)^{j-1}\prod_{s=2}^{j-1} u_{ss}} + \sum_{k=1}^{j-2} \frac{x^{\binom{r+j-1}{2}-\binom{r+j-2-k}{2}}\delta_{i\le 2r-j+k}}{(1-x)^k \prod_{s=j-k}^{j-1} u_{ss}} \end{aligned}$$

$$- \frac{f_{r-1}}{1-f_{r-1}} \frac{x^{\binom{r+j-1}{2}+\binom{r}{2}}}{(1-x)^{j-1}\prod_{s=2}^{j-1} u_{ss}} - \sum_{k=0}^{j-2} \frac{x^{\binom{r+j-1}{2}-\binom{r+j-2-k}{2}} \delta_{i\le 2r-j+k}}{(1-x)^k \prod_{s=j-k}^{j-1} u_{ss}}$$
$$= -x^{\binom{r+j-1}{2}-\binom{r+j-2}{2}} = -x^{r+j-2} = -a_{ij},$$

as required. □

Moreover, by using similar techniques (but more complicated) as in the Proof of Theorem 2, we can state the following explicit formulas for u_{ss} and $\ell_{s(s-1)}$.

Proposition 4 *Let* $r \ge 2$. *Then* $u_{jj} = \frac{\alpha_{j+1}}{(1-x)\alpha_j}$ *for all* $j = 1, 2, \ldots, 2r-1$, *where*

$$\alpha_i = x^{\binom{r+i-1}{2}-\binom{r+1}{2}}(1-f_{r-1}) + \sum_{k=0}^{i-3} x^{\binom{r+i-1}{2}-\binom{r+i-1-k}{2}}(1-2x)(1-x)^{i-3-k}, \quad 2 \le i \le r+1,$$

$$\alpha_{r+i} = (1-2x)(1-x)^{r+i-3} + \sum_{k=1}^{i} x^{2r(2k-1)-k^2-k+1}(1-2x)(1-x)^{r+i-2k-2}$$
$$+ \sum_{k=1}^{i-1} x^{4rk-k^2-2k}(1-2x)(1-x)^{r+i-2k-3}$$
$$+ \sum_{k=1}^{r-i-2} x^{2r(2i-1+k)-i(i+1+k)-k(k+1)/2-1}(1-2x)(1-x)^{r-2-i-k}$$
$$+ x^{r(3r+4i-7)/2+1-i(i+1)/2}(1-f_{r-1}), \quad r+2 \le i \le 2r.$$

Moreover,

$$\ell_{i(i-1)} = \left(\frac{\alpha_{i+1}}{(1-x)\alpha_i} - 1 + x^{r-2+i}\right)\frac{1-x}{x^{r-2+i}}, \quad 2 \le i \le r,$$
$$\ell_{i(i-1)} = \left(\frac{\alpha_{i+1}}{(1-x)\alpha_i} - 1\right)\frac{1-x}{x^{3r-i}}, \quad r+1 \le i \le 2r-1.$$

Theorem 3 *Set* $r \ge 2$. *Let* $x = (x_1, \ldots, x_{2r-1})^T$ *and* $w = (1, 1, \ldots, 1)^T$ *be two vectors with* $2r-1$ *coordinates. The solution of the system of equation* $(I - \mathbf{A}_r)x = w$ *is given by*

$$x_i = (1-x)\sum_{k=i}^{2r-1} \frac{y_k \alpha_i}{\alpha_{k+1}} \prod_{s=i}^{k-1}\left(x^{r-1+s}\delta_{s\le r} + x^{3r-1-s}\delta_{s\ge r+1}\right),$$

where $y_i = 1 - \sum_{j=1}^{i-1} \ell_{ij} y_j$ *for all* $i = 1, 2, \ldots, 2r-1$.

Proof By Theorem 2, we have that $I - \mathbf{A}_r = \mathbf{L}_r\mathbf{U}_r$. Let $\mathbf{L}_r y = w$ and $\mathbf{U}_r x = y$. Then, for all $i = 1, 2, \ldots, 2r-1$, we have $y_i = 1 - \sum_{j=1}^{i-1} \ell_{ij} y_j$ and $x_i = \frac{y_i}{u_{ii}} - \frac{u_{i(i+1)}}{u_{ii}} x_{i+1}$ with $z_{2r} = 0$. Thus, by induction on i, we see that

$$x_i = \sum_{k=i}^{2r-1} (-1)^{k-i} y_k \frac{\prod_{s=i}^{k-1} u_{s(s+1)}}{\prod_{s=i}^{k} u_{ss}},$$

which, by Proposition 4 and (2.3), completes the proof. □

By Theorems 1–3 and the fact that $(1-x)\alpha_1 = 1$, we obtain the following result.

Theorem 4 *The generating function for the number of compositions of n that avoid* $\tau^{(r)}$ *is given by*

$$\sum_{k=1}^{2r-1} \frac{y_k}{\alpha_{k+1}} \prod_{s=1}^{k-1} \left(x^{r-1+s}\delta_{s\le r} + x^{3r-1-s}\delta_{s\ge r+1} \right),$$

where $y_i = 1 - \sum_{j=1}^{i-1} \ell_{ij} y_j$ *for all* $i = 1, 2, \ldots, 2r-1$ *and* ℓ_{ij} *and* u_{ij} *are given by* (2.1)–(2.4).

Theorem 4 has already been illustrated for $r = 2$ and 3 in Example 3. Here we illustrate Theorem 4 for $r = 4$ and 5.

Example 4 The generating function for the number of compositions that avoid $\tau^{(4)} = 4{,}567{,}654$ is $\frac{s(x)}{t(x)}$ where

$$\begin{aligned} s(x) = {} & 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6 + x^7 - 5x^8 + 10x^9 - 10x^{10} \\ & + 5x^{11} - x^{12} + x^{13} - 4x^{14} + 6x^{15} - 4x^{16} + x^{17} + x^{19} - 3x^{20} + 3x^{21} \\ & - x^{22} + x^{24} - 2x^{25} + x^{26} + x^{29} - x^{30} + x^{33}, \end{aligned}$$

and

$$\begin{aligned} t(x) = {} & 1 - 7x + 20x^2 - 30x^3 + 25x^4 - 11x^5 + 2x^6 + x^7 - 6x^8 + 14x^9 - 16x^{10} \\ & + 9x^{11} - 2x^{12} + x^{13} - 5x^{14} + 9x^{15} - 7x^{16} + 2x^{17} + x^{19} - 4x^{20} + 5x^{21} \\ & - 2x^{22} + x^{24} - 3x^{25} + 2x^{26} + x^{29} - 2x^{30} + x^{33} - x^{34} - x^{35} - x^{36}, \end{aligned}$$

Thus as a series expansion, the generating function for the number of compositions of depth $r \ge 4$ is

$$\frac{1-x}{1-2x} - \frac{s(x)}{t(x)} = x^{37} + 9x^{38} + 47x^{39} + 187x^{40} + \cdots.$$

For example the nine compositions of 38 that contain 4,567,654 are: 14,567,654, 5,567,654, 4,667,654, 4,577,654, 4,567,754, 4,568,654, 4,567,664, 4,567,655 and 45,676,541. Similarly:

Example 5 For $r = 5$, the generating function for the number of compositions that avoid $\tau^{(5)} = 5{,}678{,}765$ is $\frac{u(x)}{v(x)}$ where

$$\begin{aligned} u(x) = {} & 1 - 8x + 28x^2 - 56x^3 + 70x^4 - 56x^5 + 28x^6 - 8x^7 + x^8 + x^9 - 7x^{10} + 21x^{11} \\ & - 35x^{12} + 35x^{13} - 21x^{14} + 7x^{15} - x^{16} + x^{17} - 6x^{18} + 15x^{19} - 20x^{20} + 15x^{21} \\ & - 6x^{22} + x^{23} + x^{25} - 5x^{26} + 10x^{27} - 10x^{28} + 5x^{29} - x^{30} + x^{32} - 4x^{33} + 6x^{34} \\ & - 4x^{35} + x^{36} + x^{39} - 3x^{40} + 3x^{41} - x^{42} + x^{45} - 2x^{46} + x^{47} + x^{51} - x^{52} + x^{56}, \end{aligned}$$

and

$$\begin{aligned} v(x) = {} & 1 - 9x + 35x^2 - 77x^3 + 105x^4 - 91x^5 + 49x^6 - 15x^7 + 2x^8 + x^9 - 8x^{10} + 27x^{11} \\ & - 50x^{12} + 55x^{13} - 36x^{14} + 13x^{15} - 2x^{16} + x^{17} - 7x^{18} + 20x^{19} - 30x^{20} + 25x^{21} \\ & - 11x^{22} + 2x^{23} + x^{25} - 6x^{26} + 14x^{27} - 16x^{28} + 9x^{29} - 2x^{30} + x^{32} - 5x^{33} + 9x^{34} \\ & - 7x^{35} + 2x^{36} + x^{39} - 4x^{40} + 5x^{41} - 2x^{42} + x^{45} - 3x^{46} + 2x^{47} + x^{51} - 2x^{52} + x^{56} \\ & - x^{57} - x^{58} - x^{59} - x^{60}. \end{aligned}$$

As before the series expansion for generating function for the number of compositions of depth $r \geq 5$ is

$$\frac{1-x}{1-2x} - \frac{u(x)}{v(x)} = x^{61} + 11x^{62} + 68x^{63} + 312x^{64} + 1186x^{65} + \cdots .$$

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