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Exploring the Isoptics of Fermat Curves in the Affine Plane Using DGS and CAS

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Abstract Isoptic curves of plane curves are a live domain of study, mostly for closed, smooth, strictly convex curves. A technology-rich environment allows for a two-fold development: dynamical geometry systems enable us to perform experiments and to derive conjectures, and computer algebra systems (CAS) are the appropriate environments for an algebraic approach for determining isoptics, with its Gröbner bases packages, amongst others. Closed Fermat curves in the affine plane present a specific problem: the variables of the involved polynomials represent the coordinates of points on the Fermat curve, which avoid the dense set of points with two rational coordinates (excepted 4 "trivial" points). Therefore, automated computation has to be performed over the field $\mathbb{R}$ of the real numbers, with which the Gröbner bases packages do not work. Instead, other packages implemented in CASs have to be used. First, we present a study of the orthoptics of closed Fermat curves of even order. Then we proceed to an algebraic study of these curves using a CAS. The generalization to angles other than 90° is performed afterwards, with an algebraic approach using support functions, then using numerical methods.

Keywords Isoptic curves · Fermat curves · Automated methods · Conputer algebra system · Dynamic geometry system

Mathematics Subject Classification 53A04 · 53A15 · 51M15 · 65–04

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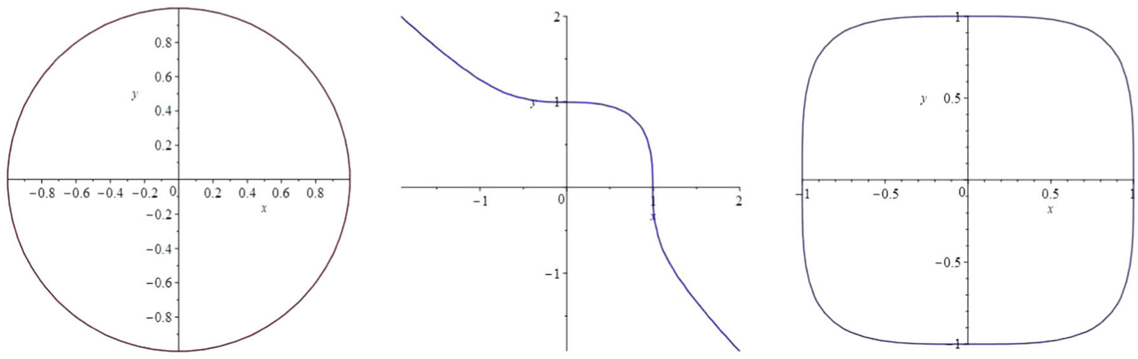


(a) $k = 2$ **(b)** $k = 3$ **(c)** $k = 4$

Fig. 1 Fermat curves

1 Introduction

Let a plane curve $\mathscr{C}$ and an angle θ be given. If it exists, the geometric locus of points through which passes a pair of tangents to $\mathscr{C}$ making an angle equal to θ is called an θ-*isoptic curve* of $\mathscr{C}$ and is denoted by $\mathbf{Opt}(\mathscr{C}, \theta)$. The name comes from the fact that from points on this geometric locus the curve $\mathscr{C}$ is seen under a fixed angle equal to θ. We will call θ the *isoptic angle*.

The study of isoptic curves has been an active field of research for a long time, both for strictly convex curves and for open curves. The following important result is proven in [3]:

Theorem 1 *If $\mathscr{C}$ is a convex closed curve, then for any angle θ, its θ-isoptic is a closed and periodic curve with period 2π.*

Illustrations of this theorem are given in [6,7,9,12]. In [6,7] work followed an algebraic path:

- Translate the question into equations;
- Transform the equations into polynomial equations;
- Solve the equations/systems of equations using computations of Gröbner bases. This generally yields a parametric representation of the desired isoptic curve;
- Implicitize the parametric presentation using elimination ideals.

Throughout this paper, we call *Fermat curves* the curves in the affine plane whose equation is of the form $x^n + y^n = k$, where n is a non-negative integer and $k > 0$, and denote them by $\mathscr{F}_n$. WLOG, we will work with $k = 1$, as changing the value of k does not influence the topology of the curves and the existence of isoptics.

A Fermat curve $\mathscr{F}_n$ is a closed convex curve when the parameter n is even, and is open otherwise, see Fig. 1. In this paper we primarily work with closed Fermat curves only, with an extension towards curves that we will call *generalized Fermat curves* in Sect. 5.

For even n, the given curve is a smooth, closed, strictly convex curve. Therefore, it defines a partition of the plane: through an interior point passes no tangent to $\mathscr{F}_n$, through a point on $\mathscr{F}_n$ there exists a unique tangent, and through an exterior point passes a pair of tangents (see Fig. 2, obtained with the **Tangent** command of the GeoGebra software).

Finding the tangents to $\mathscr{F}_n$ through a given exterior point requires the solution of a system of nonlinear equations. The mathematical situation here is similar to [8]. Nevertheless, we had to use a different approach in order to derive implicit equations for isoptics.

In this paper we will present several approaches:

- The algebraic approach presented above for the special case of orthoptics.
- Automated methods for determining and plotting an orthoptic (Sect. 2).

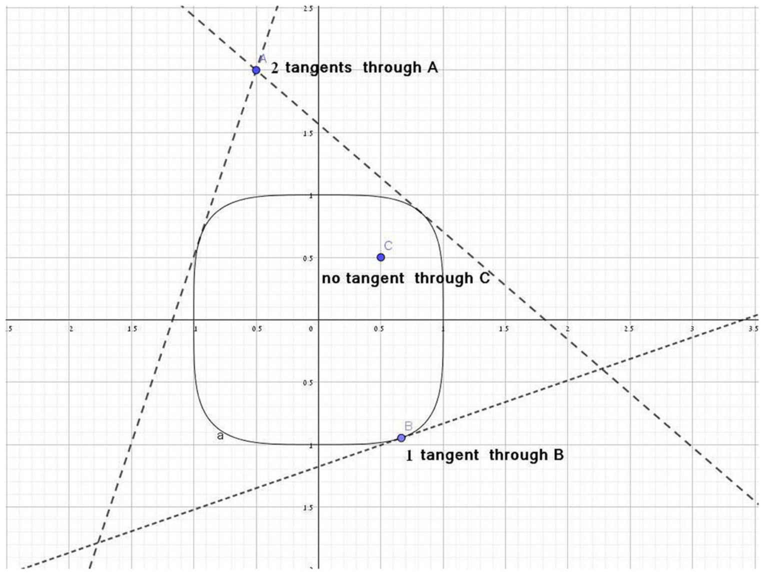


Fig. 2 Tangents to a closed Fermat curve

- A construction based on the usage of support functions, which will yield both parametric equations and implicit equations in the general case (Sect. 4). The parametrization used here is different from the parametrization in the previous case.
- A 2-dimensional, 3rd order numerical approach which quickly yields a good visualization of the isoptic curves (Sect. 5), see three examples in Fig. 19.

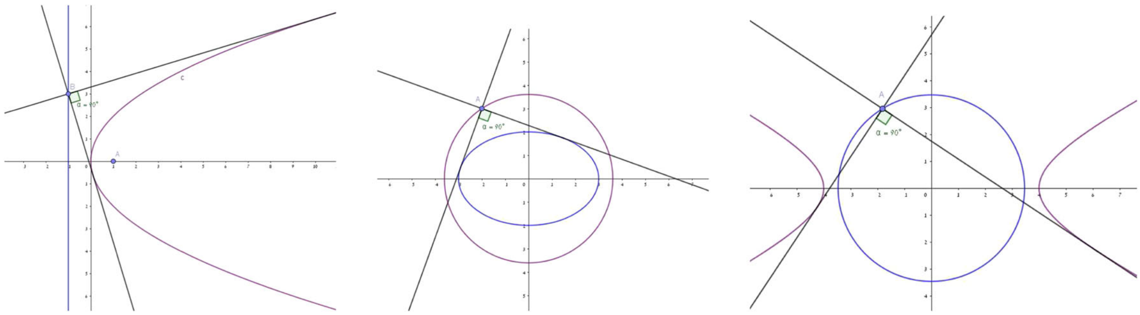
When possible, a central tool we use is the automatic solution of nonlinear systems of polynomial equations, based on algorithms for the computation of Gröbner bases. The algorithms are described in [1,4]. The first step consists in translating the given data into polynomial equations. The results are obtained in polynomial form. Sometimes a mixed treatment is useful, both algebraic and analytic. The dynamic features of the CAS or of the DGS (e.g., a slider bar to move of a point using an ad-hoc button, etc.) have a central role in the experimental work.

In Sect. 2, we use an approach explained in [5] for isoptics of an astroid. The symmetries of the given curve are a central feature and generalization to arbitrary angles for θ-isoptics is not obvious. We should mention that astroids are non-convex and have cusps, whereas Fermat are convex and smooth, so here the work is somehow easier. Actually, it is possible here to derive a parametrization describing the entire Fermat curve. This will be done in Sect. 4, using support functions. This method can be applied for a large set of curves, as explained in [3]. Note that for a general presentation of parametrizations, we refer to [11]. In Sect. 5, we present a numerical approach to the same question. The fact that the isoptics constructed are identical to those constructed by an analytic-algebraic approach in previous sections validates the numerical algorithms developed here. They are applied in an ongoing work for the determination of isoptics of curves for which algebraic methods either fail or require heavy CAS computations.

In this work, we used two Computer Algebra Systems (Maple 2019 and Mathematica 12), according to the software packages available on our different campuses. For dynamical experimentation, we use GeoGebra, a dynamical geometry system (DGS).

2 Orthoptic Curve of a Fermat Curve of Even Exponent

Let a curve $\mathscr{C}$ be given in the affine plane. The *orthoptic curve* of $\mathscr{C}$, if it exists, is the geometric locus of points in the plane from which the curve $\mathscr{C}$ is seen under a right angle, i.e. through which passes a pair of perpendicular



(a) Parabola **(b)** Ellipse **(c)** Hyperbola

Fig. 3 Orthoptic curves of conic sections

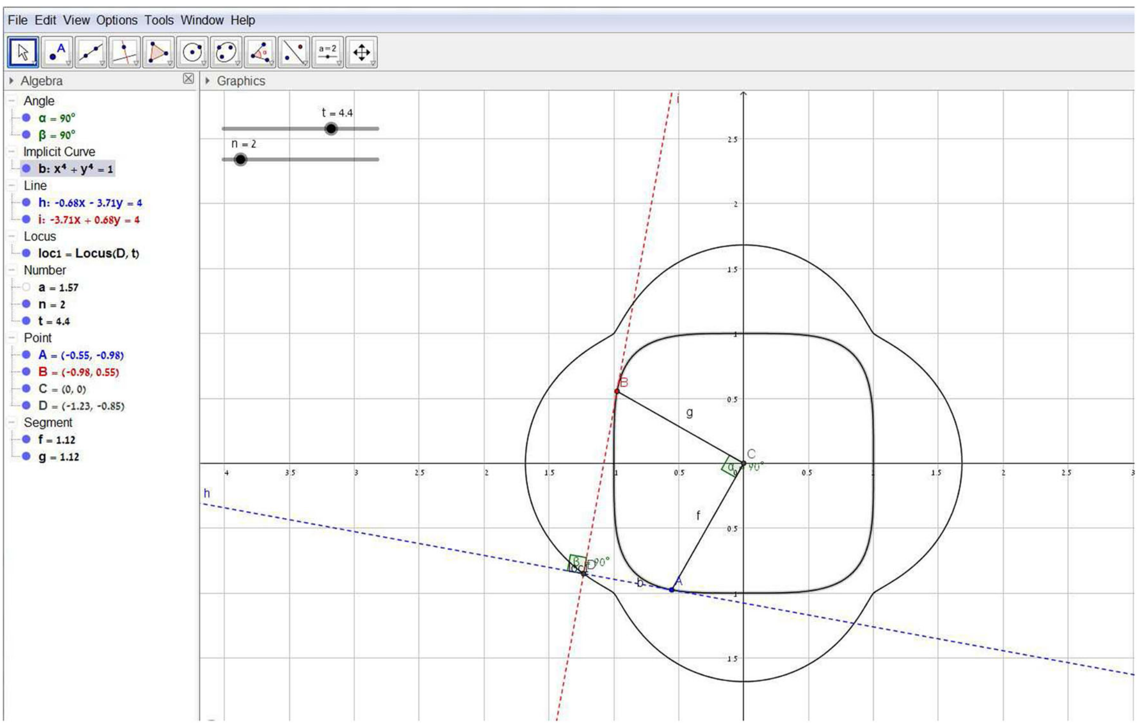


Fig. 4 Orthoptic curve of the 4th degree Fermat curve

tangents. The best known occurrences of orthoptics are the directrix of a parabola, and the director circle either of an ellipse or of a hyperbola (when it exists); see [6] and Fig. 3.

In this section we look for the orthoptic curve of a closed Fermat curve $\mathscr{F}_n$. The DGS GeoGebra has a **Locus** command enabling an automated construction of a geometric locus. Figure 4 shows an automated construction of the orthoptic $\mathscr{F}_4$, using the **Locus** command of GeoGebra.[1]

Note the following properties:

- The orthoptic curve of $\mathscr{F}_4$ is not convex.

[1] GeoGebra has a companion command **LocusEquation** to compute an implicit equation of the desired locus. Because of the algorithms in use, it does not give an answer in our case. This issue has been discussed with one of the developers. For a general discussion of these command, see [2].

- The orthoptic curve of $\mathscr{F}_4$ presents the same symmetries as $\mathscr{F}_4$, namely:
  1. There are 4 axial symmetries (about the coordinate axes and the angle bisectors of the coordinate axes).
  2. The curve is invariant by a counterclockwise rotation (and consequently, also by clockwise rotation) around the origin by an angle of 90°.

Experiments with other positive integer values of n show the same properties. Note also that these orthoptic curves provide examples of Theorem 1.

In order to find an algebraic representation of the orthoptic curve of $\mathscr{F}_n$, we need a parametrization of $\mathscr{F}_n$, from which a general representation of the tangents to $\mathscr{F}_n$ can be derived. Then we use the invariance of the curve by the counterclockwise rotation around the origin by an angle of 90°. We denote this rotation by $\mathscr{R}$; its matrix is $M_{\mathscr{R}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Recall that, over a field of characteristic not dividing n, in particular over the field $\mathbb{R}$, the curve $\mathscr{F}_n$ has no rational parametrization (see [10], p.7). Therefore we will use a non-rational parametric representation. Consider the following equations:

$$\begin{cases} x(t) = \cos^{\frac{2}{n}} t \\ y(t) = \sin^{\frac{2}{n}} t \end{cases} \tag{1}$$

for $t \in \left[0, \frac{\pi}{2}\right]$. They are a non-rational parametric representation of the arc of $\mathscr{F}_n$ in the first quadrant (denoted later quadrant I).

Successive multiplications by the matrix $M_{\mathscr{R}}$ yield parametrizations for the arcs of $\mathscr{F}_n$ in the other quadrants. A "global" parametrization can be obtained but may be hard to manipulate, with absolute values, etc. Therefore we will use the symmetries of the curves and the successive rotations we mentioned above.

In a first step, we need parametrizations of the arcs of $\mathscr{F}_n$ in quadrants I and II. For quadrant I, we use Eq.'(1) with $t \in [0, \pi/2]$. For quadrant II, we will use the following parametrization:

$$\begin{cases} x(t) = -\sin^{\frac{2}{n}} t \\ y(t) = \cos^{\frac{2}{n}} t \end{cases} \tag{2}$$

for $t \in [0, \pi/2]$. Formulas (2) are obtained by multiplication of Formulas (1) by the matrix $M_{\mathscr{R}}$. We could obtain the same formulas using a reflection.

Let A_t be a point of the arc of $\mathscr{F}_n$ in quadrant I, corresponding to the value t of the parameter. A direction vector $\overrightarrow{V_t} = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix}$ for the tangent to $\mathscr{F}_n$ at A_t is given by:

$$\overrightarrow{V}_t = \begin{pmatrix} v_1(t) \\ v_2(t) \end{pmatrix} = \begin{pmatrix} \frac{d}{dt}\left(\cos^{\frac{2}{n}} t\right) \\ \frac{d}{dt}\left(\sin^{\frac{2}{n}} t\right) \end{pmatrix} = \begin{pmatrix} -\frac{2\cos^{\frac{2}{n}} t \cdot \sin t}{n \cos t} \\ \frac{2\sin^{\frac{2}{n}} t \cos t}{n \sin t} \end{pmatrix}. \tag{3}$$

Therefore, an equation of the tangent to $\mathscr{F}_n$ at A_t, i.e., the line through A_t with direction vector $\overrightarrow{V}_t$ is given by

$$\begin{vmatrix} x - \cos^{\frac{2}{n}} t & y - \sin^{\frac{2}{n}} t \\ -\frac{2\cos^{\frac{2}{n}} t \cdot \sin t}{n \cos t} & \frac{2\sin^{\frac{2}{n}} t \cos t}{n \sin t} \end{vmatrix} = 0. \tag{4}$$

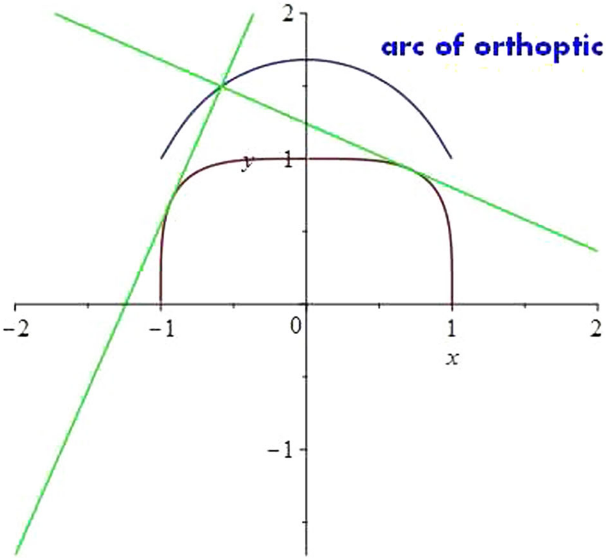


Fig. 5 First quarter of the orthoptic of $\mathscr{F}_4$

Let $B_t = \mathscr{R}(A_t)$. The point B_t is a general point on the arc of the Fermat curve $\mathscr{F}_n$ in the second quadrant. By the same method used above, we find an equation for the tangent to that arc at B_t. Cartesian equations of the tangents to $\mathscr{F}_n$ at A_t and at B_t are displayed later, in Formulas (5) and (6).

Because of the rotational symmetry, the tangents to $\mathscr{F}_n$ at A_t and at B_t are perpendicular. They intersect at a point P_t, and the geometric locus of P_t when $t \in \left[0, \frac{\pi}{2}\right]$ is a part (actually one quarter) of the orthoptic we are looking for. Figure 5 shows this arc of the orthoptic of $\mathscr{F}_4$.

The computations for this section have been performed using the Maple software. We present here briefly the code we used, but only part of the output is displayed for the reader to see how the output looks.

```
> restart: with(plots): with(LinearAlgebra):
> #parametrization of the arc in quadrant I
> a1 := cos(t)^(2/n); a2 := sin(t)^(2/n);
```

$$a1 := (\cos(t))^{2n^{-1}}$$

$$a2 := (\sin(t))^{2n^{-1}}$$

```
> # direction vector of a tangent
> va1 := diff(a1, t);  va2 := diff(a2, t);
```

$$va1 := -2\,\frac{\sin(t)}{n\cos(t)}\,(\cos(t))^{2n^{-1}}$$

$$va2 := 2\,\frac{\cos(t)}{n\sin(t)}\,(\sin(t))^{2n^{-1}}$$

```
> # Implicit equation of a tangent
> La := Matrix(2, 2, [[x-a1, y-a2], [va1, va2]]);
> eq_tgt_at_At := Determinant(La):
> eq_tgt_at_At := numer(\%)
```

The output, i.e., an equation for the tangent to F_n at A_t, is:

$$-2\sin^{\frac{2}{n}} t\ \cos^2 t\ \cos^{\frac{2}{n}} t - 2\ \cos^{\frac{2}{n}} t\ \sin^2 t\ \sin^{\frac{2}{n}} t + 2y\ \cos^{\frac{2}{n}} t\ \sin^2 t + 2x\ \sin^{\frac{2}{n}} t\ \cos^2 t = 0,$$

i.e., after simplification:

$$-2\sin^{\frac{2}{n}} t\ \cos^{\frac{2}{n}} t + 2y\ \cos^{\frac{2}{n}} t\ \sin^2 t + 2x\ \sin^{\frac{2}{n}} t\ \cos^2 t = 0. \tag{5}$$

Now in quadrant II:

```
> # parametrization of arc in 2nd quadrant
> b1 := -sin(t)^(2/n): b2 := cos(t)^(2/n):
> # direction vector to the tangent
> vb1 := diff(b1, t);vb2 := diff(b2, t);
```

$$vb1 := -2\,\frac{\cos(t)}{n\sin(t)}\,(\sin(t))^{2\,n^{-1}}$$

$$vb2 := -2\,\frac{\sin(t)}{n\cos(t)}\,(\cos(t))^{2\,n^{-1}}$$

```
> # computation of equation of a tangent
> Lb := Matrix(2, 2, [[x-b1, y-b2], [vb1, vb2]]):
> eq_tgt_at_Bt := Determinant(Lb):
> eq_tgt_at_Bt := numer(%);
```

The output, i.e., an equation for the tangent to $\mathscr{F}_n$ at B_t, is:

$$-2\cos^{\frac{2}{n}} t\ \sin^2 t\ \sin^{\frac{2}{n}} t - 2\sin^{\frac{2}{n}} t\ \cos^2 t\ \cos^{\frac{2}{n}} t + 2y\ \sin^{\frac{2}{n}} t\ \cos^2 t - 2x\ \cos^{\frac{2}{n}} t\ \sin^2 t = 0,$$

i.e., after simplification:

$$-2\cos^{\frac{2}{n}} t\ \sin^{\frac{2}{n}} t + 2y\ \sin^{\frac{2}{n}} t\ \cos^2 t - 2x\ \cos^{\frac{2}{n}} t\ \sin^2 t = 0 \tag{6}$$

```
> # point of intersection of 2 perpendicular tangents
> orthop:= solve({eq_tgt_at_At=0, eq_tgt_at_Bt=0},{x, y})
```

The answer is given by the following formulas (after simplification):

$$x = \frac{\cos^{\frac{2(n-1)}{n}} t - \sin^{\frac{2(n-1)}{n}} t}{\cos^{\frac{n}{2}} t + \sin^{\frac{n}{2}} t} \tag{7}$$

$$y = \frac{\cos^{\frac{2(n-1)}{n}} t + \sin^{\frac{2(n-1)}{n}} t}{\cos^{\frac{n}{2}} t + \sin^{\frac{n}{2}} t} \tag{8}$$

In our Maple code, the variables x and y are denoted by $xorth := orthop[1]$ and $yorth := orthop[2]$, respectively. Equations (7) and (8) are transformed into rational-radical expressions using the following, well-known formulas:

$$\begin{cases} \cos t = \frac{1-t^2}{1+t^2} \\ \sin t = \frac{2t}{1+t^2}. \end{cases} \tag{9}$$

After simplification, we obtain:

$$xorth_{12} = \frac{t^{-\frac{2}{n}} \cdot (t^2+1)^{\frac{(n-2)^2}{2n}} \cdot \left((t^2-1)^2 \cdot (2t)^{\frac{2}{n}} - 4t^2 \cdot (1-t^2)^{\frac{2}{n}}\right) \cdot (2\ (1-t^2))^{-\frac{2}{n}}}{(1-t^2)^{\frac{n}{2}} + (2t)^{\frac{n}{2}}} \tag{10}$$

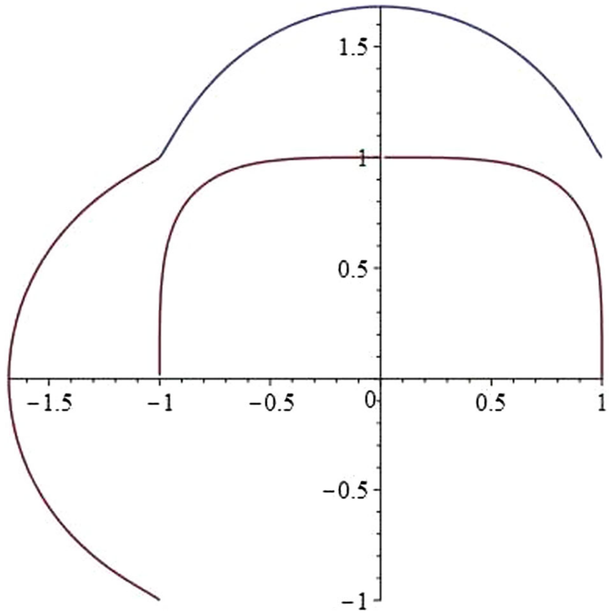


Fig. 6 Second step of construction of the orthoptic

$$yorth_{12} = \frac{t^{-\frac{2}{n}} \cdot (t^2+1)^{\frac{(n-2)^2}{2n}} \cdot \left((t^2-1)^2 \cdot (2t)^{\frac{2}{n}} + 4t^2 \cdot (1-t^2)^{\frac{2}{n}}\right) \cdot (2\,(1-t^2))^{-\frac{2}{n}}}{(1-t^2)^{\frac{n}{2}} + (2t)^{\frac{n}{2}}} \tag{11}$$

In order to have the whole orthoptic of the Fermat curve $\mathscr{F}_n$, we compute parametric presentations of the three other arcs, using the matrix $M_{\mathscr{R}}$ defined above. For example, the arc of the orthoptic in quadrants II and III has the following parametric representation:

$$xorth_{23} = -yorth_{12}$$
$$yorth_{23} = xorth_{12}$$

We iterate the computation for the two remaining arcs of the orthoptic and obtain Fig. 6.

Putting it all together, we have a complete plot of the orthoptic, displayed in Fig. 7 (note that it is identical to Fig. 4, but computed and plotted with a different software tool). The internal curve is $\mathscr{F}_4$, the external one is its orthoptic.

3 The Behaviour of the Fermat Curve and Its Orthoptic When *n* (an Even Positive Integer) Increases Towards Infinity

Different substitutions for n yield the orthoptic curves of Fermat curves of other degrees, as in Fig. 8.

For increasing n towards ∞, the Fermat curve looks more and more like the unit square, whose vertices have respective coordinates $(1, 1)$, $(-1, 1)$, $(-1, -1)$, $(1, -1)$. If we compute the limit at infinity of the expressions in Eq. (10), we have:

$$X := \lim_{n\to\infty} xorth = \frac{t^8 - 4t^6 - 10t^4 - 4t^2 + 1}{t^8 - 4t^6 + 22t^4 - 4t^2 + 1}$$
$$Y := \lim_{n\to\infty} yorth = \frac{t^8 + 4t^6 + 6t^4 + 4t^2 + 1}{t^8 - 4t^6 + 22t^4 - 4t^2 + 1}$$

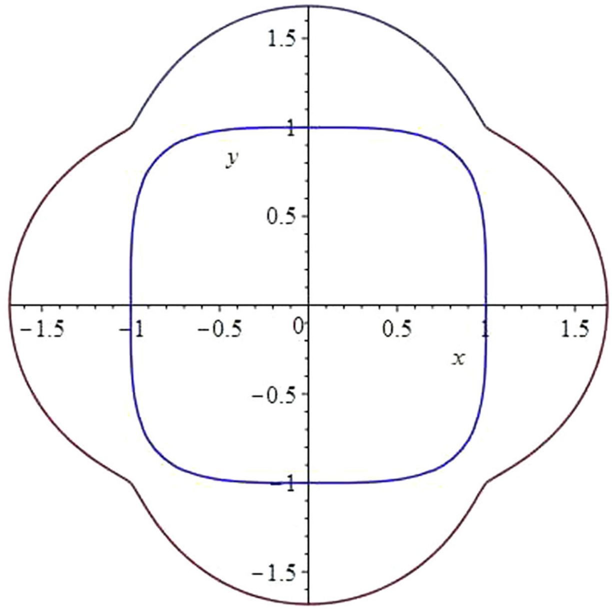
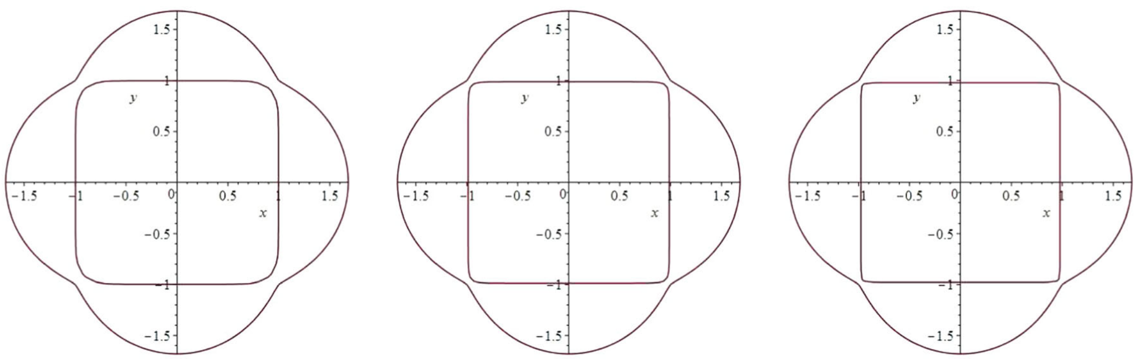


Fig. 7 Orthoptic curve of the 4th degree Fermat curve



(a) $n = 8$ **(b)** $n = 16$ **(c)** $n = 24$

Fig. 8 Three Fermat curves and their orthoptics

We transform this parametric presentation into polynomials. Consider the polynomials $P_1(X, Y) = X \cdot (t^8 - 4t^6 + 22t^4 - 4t^2 + 1) - (t^8 - 4t^6 - 10t^4 - 4t^2 + 1)$, and $P_2(X, Y) = Y \cdot (t^8 - 4t^6 + 22t^4 - 4t^2 + 1) - (t^8 + 4t^6 + 6t^4 + 4t^2 + 1)$. These polynomials generate an ideal $K = \langle P_1, P_2 \rangle$. Now we use the **PolynomialIdeals** package and the **EliminationIdeal** of Maple. The code of the session is as follows:

```
> with(PolynomialIdeals);
> P1:=x*denom(xorth)-numer(xorth):
> P2:=y*denom(yorth)-numer(yorth):
> K := <P1, P2>:
> KE := EliminationIdeal(K, {x, y});
```

The output is $KE = \langle x^2 + y^2 - 2y \rangle$, yielding the equation $x^2 + y^2 - 2y = 0$ which for us describes the half-circle passing through the upper vertices of the unit square defined previously. This is not a surprise: if two points A and B are given, the geometric locus of points H, such that the angle $\angle AHB$ is a right angle, is a circle whose diameter is AB (with the exception of the two points A and B, as in these cases the triangle ABH is not well-defined). Here

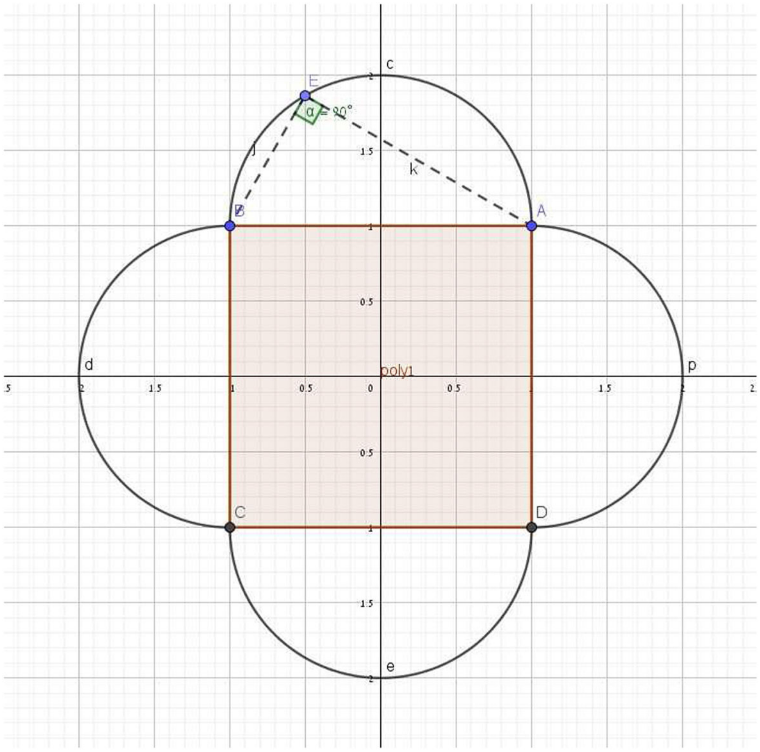


Fig. 9 Limiting case of the orthoptics of Fermat curves

we obtain half a circle only, outside of the unit square, as it is the limiting case of curved arcs which are outside the Fermat curves. Figure 9 shows the 4 half circles which are obtained by this process.

4 An Approach Using Support Functions

In previous sections, we used an approach to parametrization as shown in [5]; the curve has been divided into arcs and each arc had its own parametric presentation. We now derive a global parametrization, using gradients.

4.1 The Parametrization

Let N be an even positive integer and let $F(x, y) = x^n + y^n - 1$. Then grad $F = n\left[x^{n-1}, y^{n-1}\right]$. For a point (u, v) on the curve, an outward normal unit vector is $\left[\frac{u^{n-1}}{\sqrt{u^{2n-2}+v^{2n-2}}}, \frac{v^{n-1}}{\sqrt{u^{2n-2}+v^{2n-2}}}\right]$. See Fig. 10.

Therefore, we obtain:

$$\frac{u^{n-1}}{\sqrt{u^{2n-2}+v^{2n-2}}} = \cos t,$$
$$\frac{v^{n-1}}{\sqrt{u^{2n-2}+v^{2n-2}}} = \sin t.$$

Squaring both sides of the first equality and performing simple calculations we get:

$$v = u \tan^{\frac{1}{n-1}} t.$$

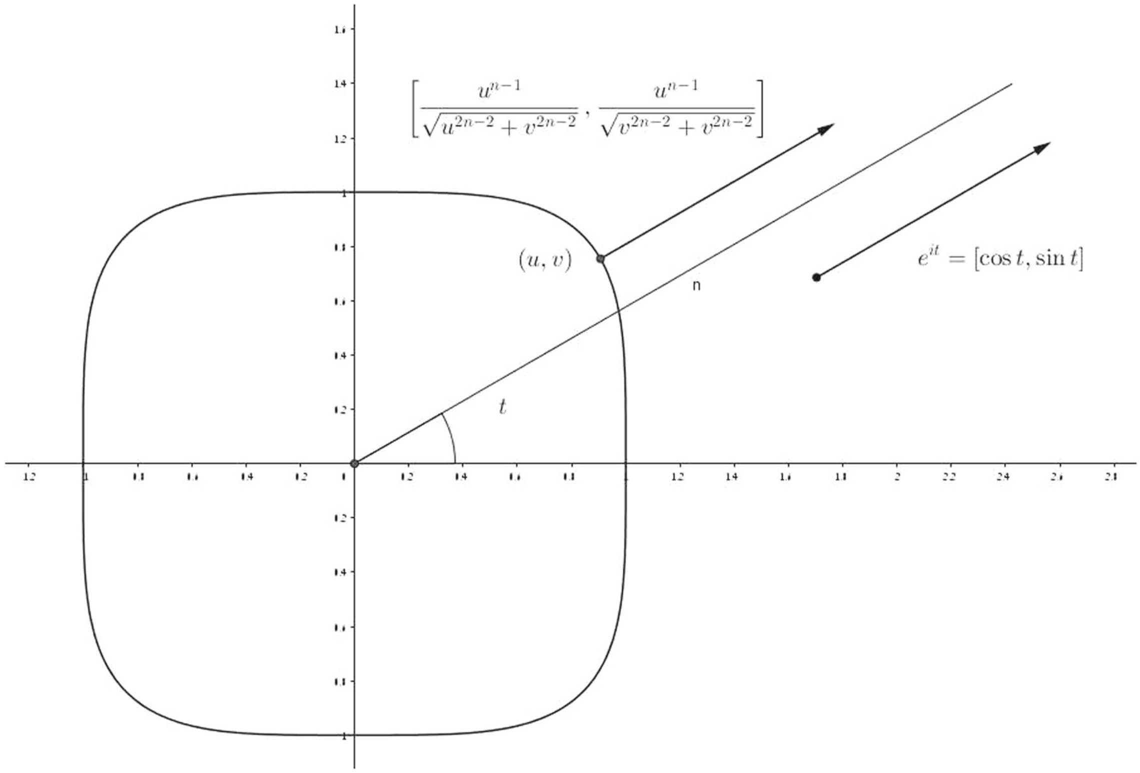


Fig. 10 Construction of a parametrization of the Fermat curve $x^4 + y^4 = 1$

Since

$$1 = u^n + v^n = u^n + u^n \tan^{\frac{n}{n-1}} t = \left(1 + \tan^{\frac{n}{n-1}}\right) u^n$$

we obtain the following parametrization of the Fermat curve $\mathscr{F}_n$:

$$\begin{cases} u = \dfrac{\cos^{\frac{1}{n-1}} t}{\sqrt[n]{\sin^{\frac{1}{n-1}} t + \cos^{\frac{1}{n-1}} t}} \\ v = \dfrac{\sin^{\frac{1}{n-1}} t}{\sqrt[n]{\sin^{\frac{1}{n-1}} t + \cos^{\frac{1}{n-1}} t}} \end{cases} \tag{12}$$

for $t \in [0, 2\pi]$.

Remark 1 Note that this parametrization is valid for any quadrant and that we use the following property of roots of odd order n: $\sqrt[n]{-x} = -\sqrt[n]{x}$.

4.2 The Support Function for the Fermat Curve

Let us recall how to parameterize the strictly convex curve $\mathscr{C}$ by its support function. We take a coordinate system with origin O in the interior of $\mathscr{C}$. Let $p(t)$, $t \in [0, 2\pi]$, be the distance from O to the tangent line $l(t)$ of $\mathscr{C}$ perpendicular to the vector $e^{it} = \cos t + i \sin t$ (Fig. 11). It is well-known that $p(t)$ is of class C^2 and that the

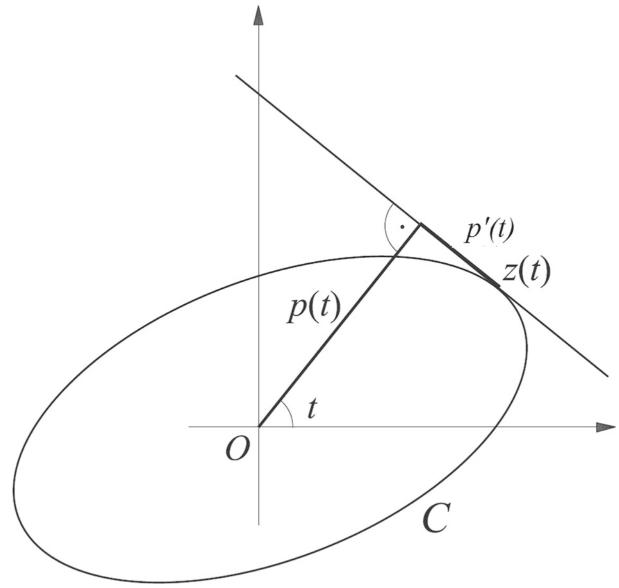


Fig. 11 Parametrization of the strictly convex curve by its support function $p(t)$

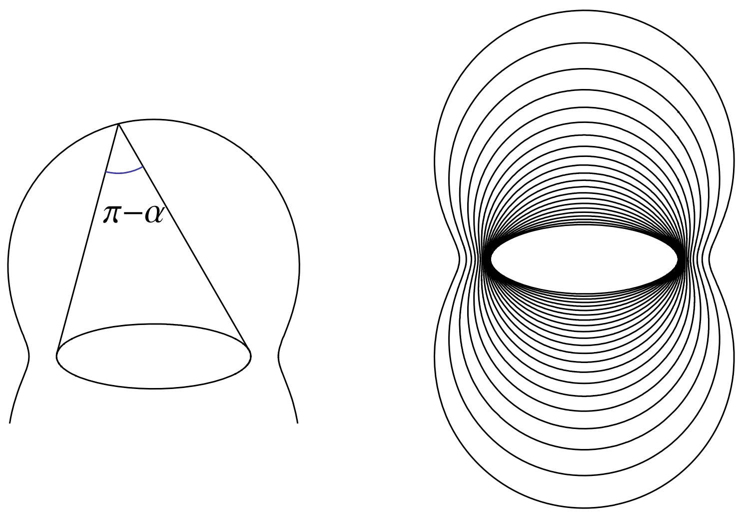


Fig. 12 On the left α-isoptic of an ellipse. On the right a family of the isoptics for an ellipse

parametrization of $\mathscr{C}$ in terms of $p(t)$ is given by the formula

$$z(t) = p(t)e^{it} + p'(t)ie^{it}, \tag{13}$$

where $ie^{it} = -\sin t + i\cos t$. Note that the support function p can be extended to a periodic function on $\mathbb{R}$ with period 2π.

Now we recall the notion of isoptics. Note that we take another angle for the definition of an α-isoptic. In fact our equation here is given for the angle $\pi - \alpha$. It is more convenient for this way of computing.

Definition 1 Let $\mathscr{C}_\alpha$ be a locus of apices of a fixed angle $\pi - \alpha$, where $\alpha \in (0, \pi)$, formed by two support lines of the oval $\mathscr{C}$. The curve C_α will be called an α-*isoptic* of $\mathscr{C}$. □

Examples are shown in Fig. 12.

It is convenient and easy to parameterize the α-isoptic $\mathscr{C}_\alpha$ with the aid of the support function of the curve $\mathscr{C}$, so that the equation of $\mathscr{C}_\alpha$ has the form

$$z_\alpha(t) = p(t)e^{it} + \left(-p(t)\cot\alpha + \frac{1}{\sin\alpha}p(t+\alpha)\right)ie^{it}, \tag{14}$$

where $e^{it} = [\cos t, \sin t]$ and $ie^{it} = [-\sin t, \cos t]$.

Now, we derive the support function for the Fermat curve $\mathscr{F}_n : x^n + y^n = k$. By (q, r) we denote the intersection point of the tangent line at the point $(u, v) \in \mathscr{F}_n$ with the straight line with the direction coefficient equal to $\tan t$ passing through the origin of the coordinate system. Thus we have to solve the following system of equations:

$$\begin{cases} x\cos t + y\sin t = u\cos t + v\sin t \\ -x\sin t + y\cos t = 0. \end{cases}$$

Hence

$$\begin{cases} q = (u\cos t + v\sin t)\cos t \\ r = (u\cos t + v\sin t)\sin t. \end{cases}$$

From the above equations and formula (12) the distance $p(t)$ between (q, r) and the origin is given by

$$p(t) = u\ \cos\ t + v\ \sin\ t = \left(\sin^{\frac{n}{n-1}} t + \cos^{\frac{n}{n-1}} t\right)^{\frac{n-1}{n}},$$

where $t \in \mathbb{R}$. Thus, using the formula (13), we get the following parametrization of the Fermat curve $\mathscr{F}_n$:

$$\begin{aligned} z_n(t) &= p(t)e^{it} + p'(t)ie^{it} \\ &= \left(\sin^{\frac{n}{n-1}} t + \cos^{\frac{n}{n-1}} t\right)^{\frac{n-1}{n}} e^{it} \\ &\quad + \left(\sin^{\frac{n}{n-1}} t + \cos^{\frac{n}{n-1}} t\right)^{-\frac{1}{n}} \left(\cos t \sin^{\frac{1}{n-1}} t - \cos^{\frac{1}{n-1}} t \sin t\right) ie^{it}, \end{aligned} \tag{15}$$

where $t \in \mathbb{R}$.

Using Eq. (14), an α-isoptic of $\mathscr{F}_4 \colon x^4 + y^4 = 1$ has the following parametric presentation:

$$\begin{aligned} z_{4,\alpha}(t) = \Bigg(&\cos t \left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}} \\ &- \sin t \Bigg(-\cot(t+\alpha)\left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}} \\ &+ \csc(\alpha)\left(\cos(t+\alpha)\cos^{\frac{1}{3}}(t+\alpha) + \sin(t+\alpha)\sin^{\frac{1}{3}}(t+\alpha)\right)^{\frac{3}{4}}\Bigg), \\ &\sin t \left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}} \\ &+ \cos t \Bigg(-\cot(t+\alpha)\left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}} + \\ &+ \csc(\alpha)\left(\cos(t+\alpha)\cos^{\frac{1}{3}}(t+\alpha) + \sin(t+\alpha)\sin^{\frac{1}{3}}(t+\alpha)\right)^{\frac{3}{4}}\Bigg)\Bigg). \end{aligned}$$

The curve $\mathscr{F}_4$ and a support line are shown in Fig. 13.

In particular, the orthoptic of $\mathscr{F}_4$ has the following equation:

$$z_{4,\frac{\pi}{2}}(t) = \left(\cos t \left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}} - \sin t \left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}},\right.$$

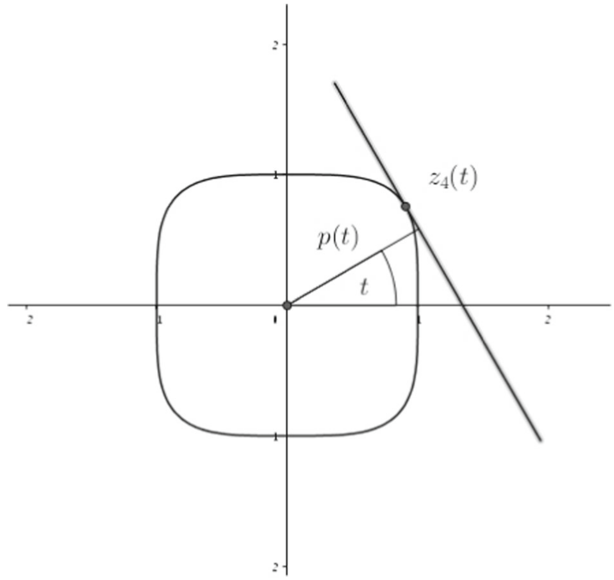


Fig. 13 Parametrization of the Fermat curve $\mathscr{F}_4 \colon x^4 + y^4 = 1$ by means of its support function

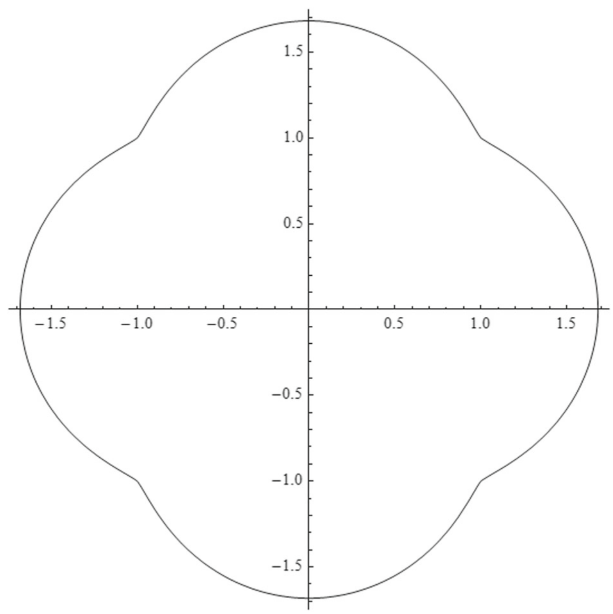


Fig. 14 Orthoptic of the Fermat curve $\mathscr{F}_{40}$

$$\cos t \left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}} + \sin t \left(\cos t \cos^{\frac{1}{3}} t + \sin t \sin^{\frac{1}{3}} t\right)^{\frac{3}{4}}\Bigg)$$

and it is illustrated in Fig. 14.

In the next section we will derive an implicit equation of the orthoptic of the Fermat curve.

Using the derived formula we present in Fig. 15 a family of isoptics of the curve $\mathscr{F}_{40} \colon x^{40} + y^{40} = 1$.

4.3 Orthoptic of the Fermat Curve

In this subsection, we derive the equation of the orthoptic for the Fermat curve $\mathscr{F}_n : \; x^n + y^n = 1$ for any even positive integer n (Fig. 16). First, we note that, if a point (u, v) lies on this curve then the point $(-v, u)$ belongs to this curve, too. We note also that gradients at these points are orthogonal.

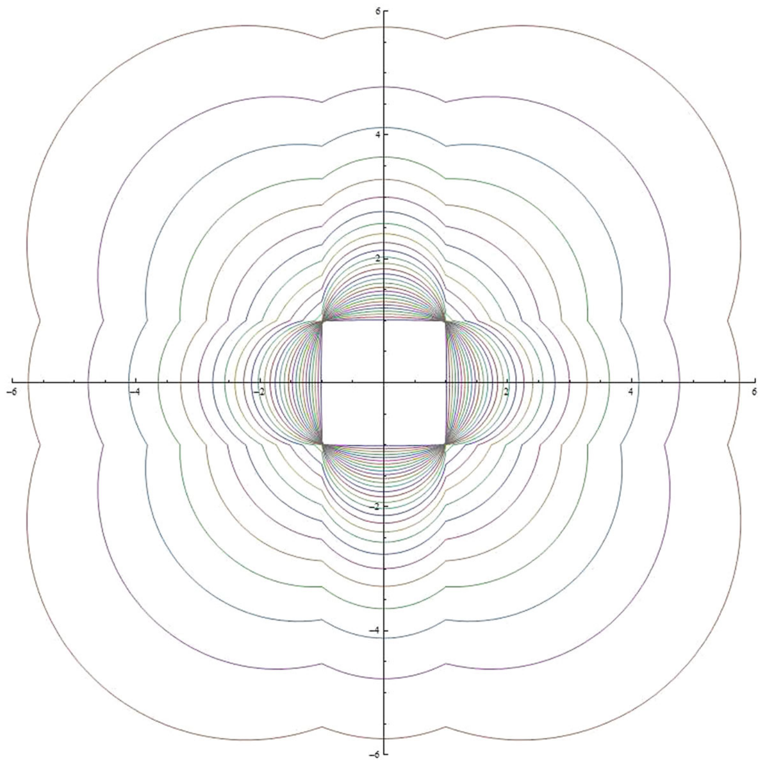


Fig. 15 Family of isoptics for various angles of the Fermat curve $\mathscr{F}_{40}$

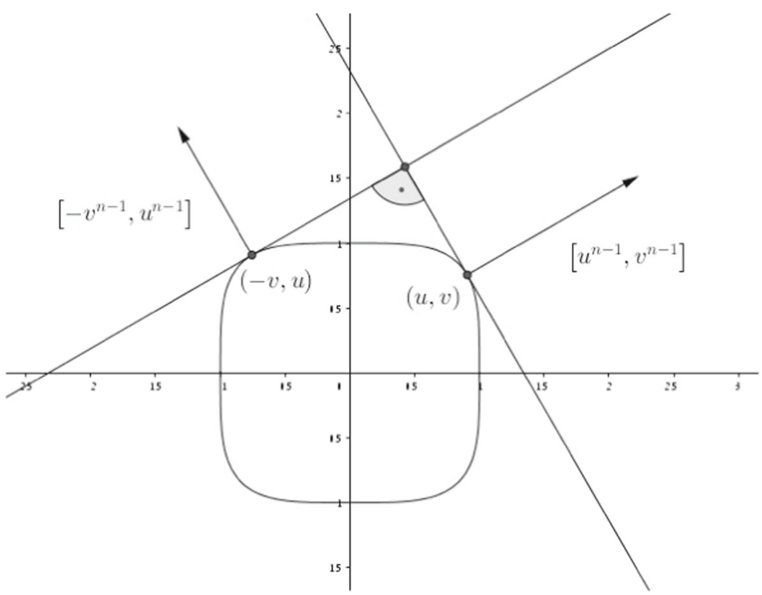


Fig. 16 Construction of the orthoptic of the Fermat curve $x^4 + y^4 = 1$

The tangent line at the point (u, v) is given by the following equation:

$$u^{n-1}(x-u)+v^{n-1}(y-v)=0. \tag{16}$$

So we have

$u^{n-1}x+v^{n-1}y=1.$

Thus, in order to derive the equation of the orthoptic, we solve the following system of equations of tangents at these points:

$$\begin{cases} u^{n-1}x+v^{n-1}y=1 \\ -v^{n-1}x+u^{n-1}y=1 \end{cases} \tag{17}$$

and we obtain the following system:

$$\begin{cases} x=\dfrac{u^{n-1}-v^{n-1}}{u^{2n-2}+v^{2n-2}}, \\ y=\dfrac{u^{n-1}+v^{n-1}}{u^{2n-2}+v^{2n-2}}, \\ 1=u^n+v^n. \end{cases} \tag{18}$$

Dividing sidewise the first two equations in (18), we get:

$$\frac{y}{x}=\frac{u^{n-1}+v^{n-1}}{u^{n-1}-v^{n-1}}, \tag{19}$$

and next

$$v^{n-1}=\frac{y-x}{x+y}u^{n-1}. \tag{20}$$

Substituting this result into the first equation of the system (18), we obtain:

$$u^{n-1}=\frac{x+y}{x^2+y^2}, \tag{21}$$

and similarly, we obtain:

$$v^{n-1}=\frac{y-x}{x^2+y^2}. \tag{22}$$

Finally, we derive the desired equation of the orthoptic:

$$1=u^n+v^n=\left(\frac{x+y}{x^2+y^2}\right)^{\frac{n}{n-1}}+\left(\frac{y-x}{x^2+y^2}\right)^{\frac{n}{n-1}}, \tag{23}$$

which can be simplified in the following form:

$$(x+y)^{\frac{n}{n-1}}+(y-x)^{\frac{n}{n-1}}=(x^2+y^2)^{\frac{n}{n-1}}. \tag{24}$$

Proposition 1 *For any even, non-negative integer n, the orthoptic curve* $\boldsymbol{Opt}(\mathscr{F}_n, 90)$ *is smooth.* □

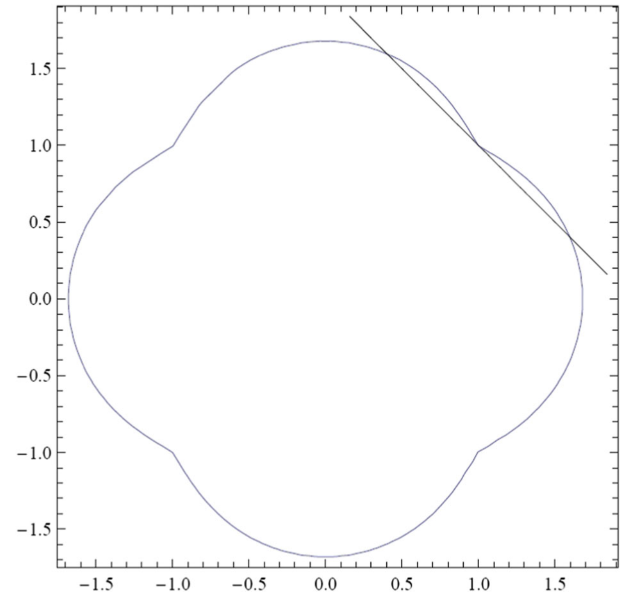


Fig. 17 The orthoptic of the Fermat curve $x^4 + y^4 = 1$ and its tangent at a "corner"

Proof Denote

$$F_{opt}(x, y) = (x + y)^{\frac{n}{n-1}} + (y - x)^{\frac{n}{n-1}} - (x^2 + y^2)^{\frac{n}{n-1}}. \tag{25}$$

and compute the gradient of F_{opt}:

$$\overrightarrow{\text{grad}} F_{opt}(x, y) = \begin{pmatrix} \frac{n-1}{n}\left(-2x(x^2 + y^2)^{-\frac{1}{n}} + (x + y)^{-\frac{1}{n}} - (y - x)^{-\frac{1}{n}}\right) \\ -\frac{n-1}{n}\left(2y(x^2 + y^2)^{-\frac{1}{n}} - (x + y)^{-\frac{1}{n}} - (y - x)^{-\frac{1}{n}}\right) \end{pmatrix}. \tag{26}$$

The gradient vanishes at two points, namely the origin and the point whose coordinates are $\left(\frac{1}{2}e^{-\frac{\ln 2}{n-1}}, \frac{1}{2}e^{-\frac{\ln 2}{n-1}}\right)$. None of these points belongs to the curve $\mathbf{Opt}(\mathscr{F}_n, 90)$. As the gradient is non zero at every point on $\mathbf{Opt}(\mathscr{F}_n, 90)$, this curve is smooth. □

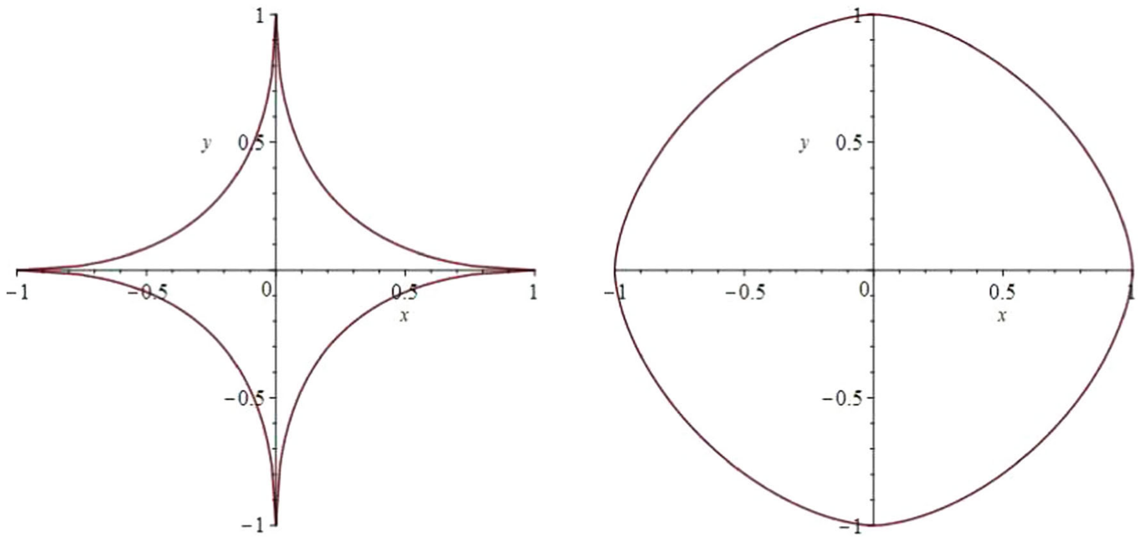
As an example, we display in Fig. 17 the orthoptic $\mathbf{Opt}(\mathscr{F}_4, 90)$ and its tangent at the point $A(1, 1)$. By substitution, it is easily to verify that A belongs to $\mathbf{Opt}(\mathscr{F}_4, 90)$ and that the the gradient at A has coordinates $\left(-\frac{4}{3} \cdot 2^{1/3}, -\frac{4}{3} \cdot 2^{1/3}\right)$. Thus, the tangent vector is $\left(\frac{4}{3} \cdot 2^{1/3}, -\frac{4}{3} \cdot 2^{1/3}\right)$. A parametric presentation for the tangent to $\mathbf{Opt}(\mathscr{F}_4, 90)$ at A is:

$$(x, y) = (1, 1) + t\ \left(\frac{4}{3} \cdot 2^{1/3}, -\frac{4}{3} \cdot 2^{1/3}\right). \tag{27}$$

5 Isoptic Curves of a Fermat Curve of Even Exponent: A Numerical Approach

Definition 2 We call a generalized Fermat curve (GFC) any curve in the affine plane given by an equation of the form $|x|^n + |y|^n = 1$, where n is a positive, non-integer number.

Figure 18 shows two examples. For $0 < n < 1$, the curve is neither smooth nor convex. The GFC with $n = \frac{2}{3}$ is called an *astroid* and its isoptics are studied in [5]. For a non integer $n > 1$, the curve is convex, but not smooth.



(a) $n = \frac{1}{2}$ **(b)** $n = \frac{5}{3}$

Fig. 18 Two generalized Fermat curves

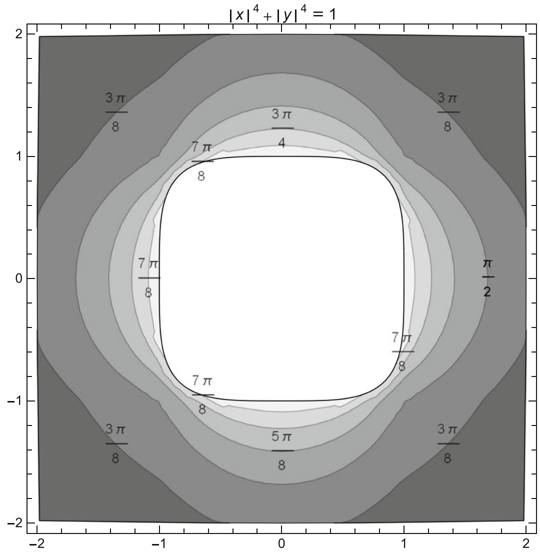
We wish now to study θ-isoptic curves of these GFC, for $\theta \neq \frac{\pi}{2}$. Here the rotational symmetry still exists and will be useful, but not enough for finding a parametric representation of either the GFC, or of the isoptic curve, using the method of Sect. 2. Therefore we turn to numerical methods.

In order to generate the isoptic curves numerically, we mapped out a mesh outside of the Fermat curve, and calculated the isoptic angle, θ_{isoptic}, for each mesh point. Initially we used an equi-spaced Cartesian mesh, but eventually moved to a polar one. The polar mesh was also initially equi-spaced (in terms of r and θ), but also afforded us the ability to more easily add mesh points at specific angles (e.g., the corners—see Sect. 5.1), in order to obtain higher resolution. Note that due to obvious symmetries, we only calculated for $0 \leq \theta \leq \frac{\pi}{4}$, and copied over as appropriate.

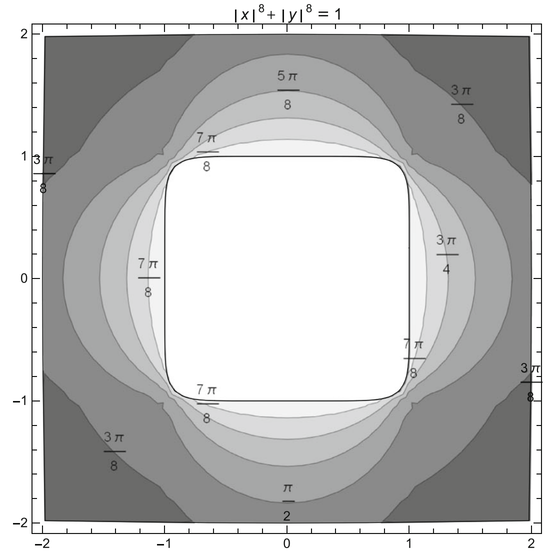
For the present section, all of our calculations were performed in *Mathematica*. We found the function `NSolve[]` (for numerical solutions of linear and nonlinear equations), to be more numerically stable than `FindInstance[]` (finding instances of solutions), for handling the system of nonlinear equations, required to find the two tangent points on the Fermat curves. However, the former function nonetheless had to be handled with kit gloves. Here are some of the restrictions of using `NSolve[]` for this problem:

1. We could not include the absolute value function, `Abs[]`, in the equations (needed for the generalized Fermat curve—see below).
2. The (otherwise) very helpful function `PossibleZeroQ[]`, for checking numbers which are very nearly zero (for the given accuracy of the calculations), was similarly not usable in the equations.
3. We needed to increase the `WorkingPrecision` to 20 digits.
4. The results needed to be wrapped in the function `N[]` to produce numerical values. This was both in order to get rid of machine errors, as well as a *huge* speedup for subsequent calculations of `VectorAngle[]`, with the mesh point as the vertex, connected with the two tangent points.
5. Still, with a high value of n, and near the Fermat curves, the number of tangent points produced could be any of: 0, 1, 2, 3 or more.
6. Therefore, we automated the process so that:

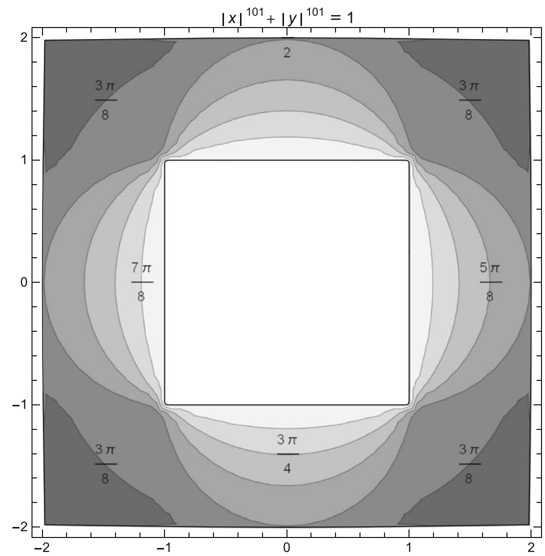
    a. if fewer than two points were generated, the geometry was analyzed in order to uncover the missing point(s), and



(a) $\mathscr{F}_4$



(b) $\mathscr{F}_8$



(c) $\mathscr{F}_{101}$

Fig. 19 Three Fermat curves and their isoptic curves

b. if more than two points were generated, the duplicates were removed.

5. Finally, infrequently, `NSolve[]` did not converge at all.

Once the isoptic angles were calculated for all of the mesh points, the next numerical challenge was to generate the isoptic curves.

For the Fermat curves themselves, it was enough to use either `RegionPlot[]` or `ContourPlot[]`. For the isoptic curves, we opted on `ListContourPlot[]`. The advantages over the simpler `ContourPlot[]`, are that:

1. the mesh need not be Cartesian,

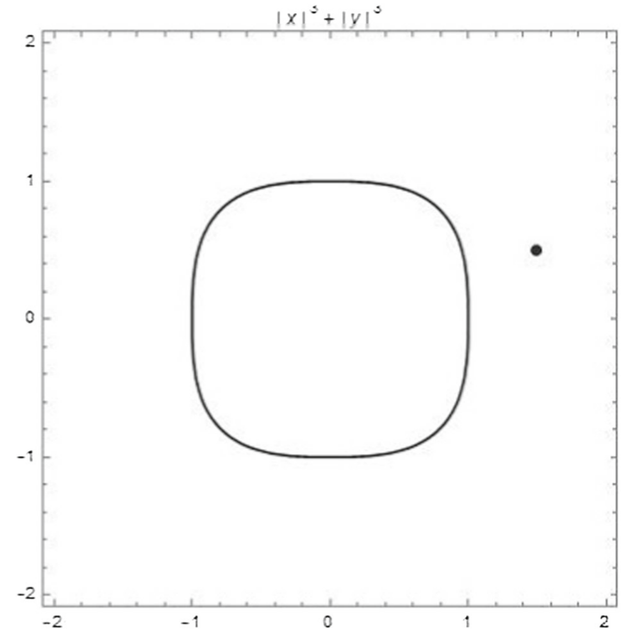
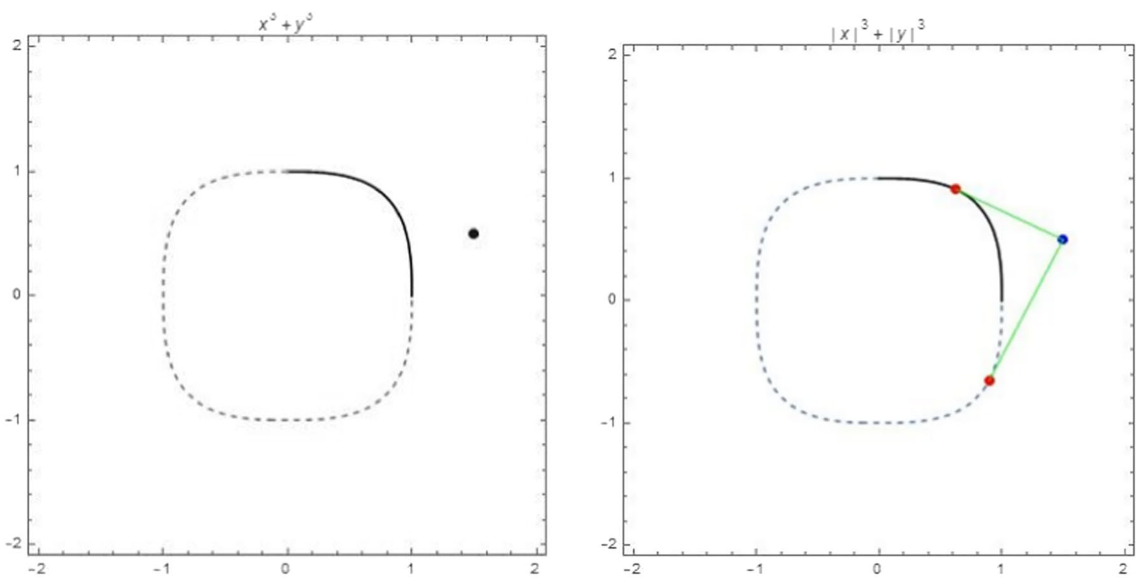


Fig. 20 Generalized Fermat curve $\mathscr{F}_3$



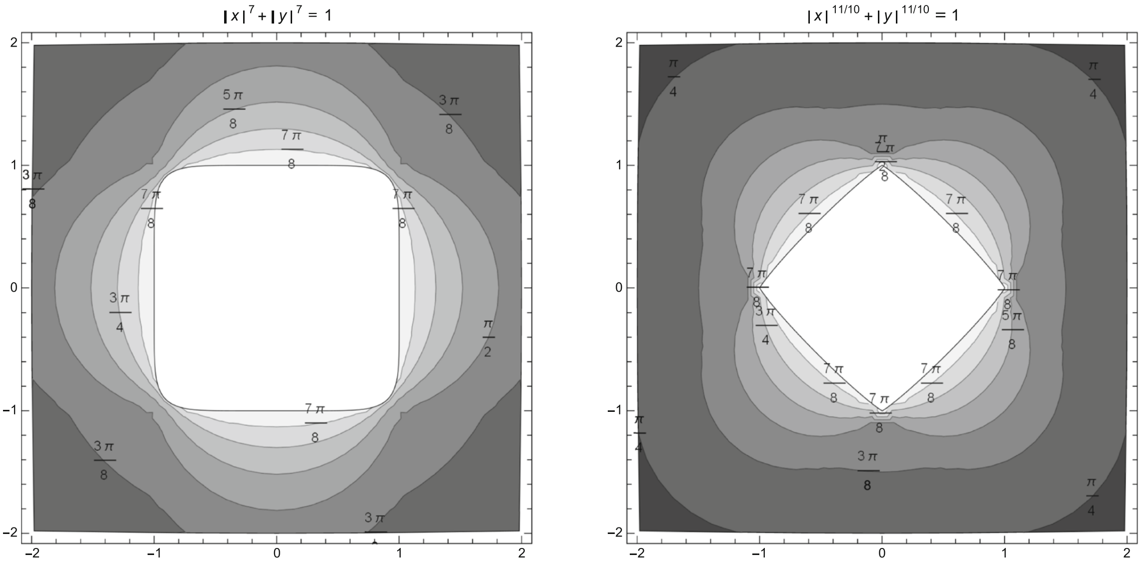
(a) composite plot **(b)** tangents of external point

Fig. 21 Generalized $\mathscr{F}_3$

2. we could add points for higher resolution, wherever we wanted (e.g., at the corners[2]), and
3. in the rare cases that `NSolve[]` did not converge, we could simply skip those mesh points.

We used the options of 2D, third-order interpolation, as well as "`Quality`" (vs. speed) for the `PerformanceGoal`. Figure 19 shows a number of isoptic curves, for each of three Fermat curves, with n values of 4, 8 and 101. (Below

[2] The *corners* are the points of intersection of the Fermat curve, for $n > 2$, with the lines whose equations are $y = x$ and $y = -x$, respectively. For $1 < n < 2$, the corners occur along the x and y axes.



(a) $\mathscr{F}_7$ (b) $\mathscr{F}_{\frac{11}{10}}$

Fig. 22 Generalized Fermat curves with isoptics curves

we explain the validity of this last, odd value.) We note that as n grows, $\mathscr{F}_n$ approaches the unit square. For $n = 101$, the x and y values of the corners (in absolute values) are $\sqrt[101]{\frac{1}{2}} \approx 0.993$.

5.1 Exploring Numerically, Fermat Curves of Other Exponents

What if n is an odd integer? What if it is a positive, non-integer number?

We already saw in Fig. 1 (in Sect. 1) that for odd n, the Fermat curve is not closed. The only convex part of the curve is in the first quadrant.

Regarding (positive) non-integer values of n, it was pointed out in Sect. 1 that for $n < 1$, the Fermat curve is not convex. The case of the astroid, i.e., for $n = \frac{2}{3}$, has been studied in [5]. There, the usage of a Dynamic Geometry System revealed an isoptic behaviour very different from what we saw in the present paper. Now, we only concern ourselves here with non-integer values of $n > 1$. For these values, the Fermat curve is only defined in the first quadrant.

In order to handle *all* values of $n > 1$, we define the generalized Fermat curve. From a mathematical standpoint, this can be accomplished with the absolute value: $|x|^n + |y|^n = 1$. We see an example of this in Fig. 20 for $n = 3$, with a random external point. However, we need to be more clever to define this curve using *Mathematica*, since, as mentioned previously, `NSolve[]` cannot include the `Abs[]` function.

For the first quadrant, the non-generalized Fermat curve always exists, and we combine it with appropriate mirror images through the innate symmetries (see Fig. 20).

The composite graph can then be used for finding the tangents for any given external point (see Fig. 21).

With this new definition, we can now allow for n to take on the values of odd integers, or any non-integer larger then 1 (this restriction, as pointed out above, being due to the requirement of convexity). Figure 22 demonstrates two examples of the isoptic curves for $n = 7$ and $n = \frac{11}{10}$.

Note that for $n > 2$ the (rounded) "corners" are top-right, top-left, etc., whereas for $n < 2$ the (rounded) "corners" are top-center, left, etc. (For $n = 2$, $\mathscr{F}_2$ is the simple unit circle, and all external isoptic curves are concentric circles.)

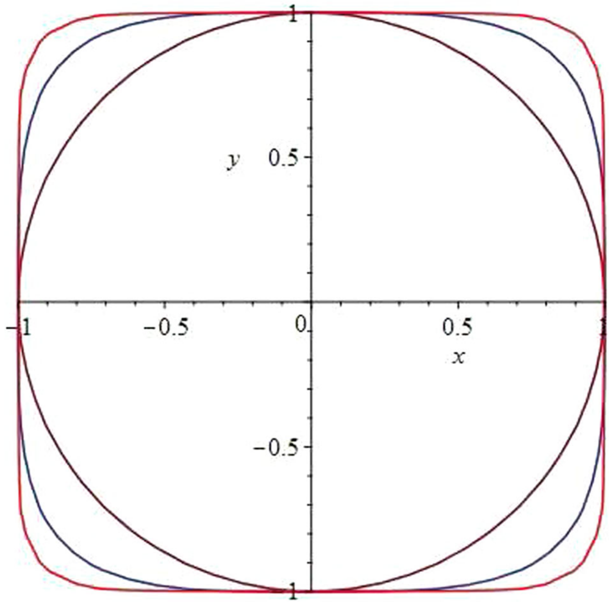


Fig. 23 Three Fermat curves for $n = 2, 4, 8$

Fig. 24 A Fermat isoptic tiling

6 Conclusions and Directions for Further Work

When even n increases and tends to infinity, the curve looks more and more like the square in which all the closed Fermat curves are inscribed. The upper-right "corner" K_n of the curve $\mathscr{F}_n$ has coordinates $(2^{-1/n}, 2^{-1/n})$. See Fig. 23, which shows the Fermat curves for $n = 2, 4, 8$.

Let $\mathscr{C}$ be a smooth plane curve. Denote by $\overrightarrow{T}$ a unit vector of the tangent to $\mathscr{C}$ at some point M. As $\overrightarrow{T}$ is a unit vector, only its direction may change when the point M moves along the curve. The *curvature* of $\mathscr{C}$ is the rate at which $\overrightarrow{T}$ turns per unit of length along $\mathscr{C}$; its is denoted by the letter κ. If a curve is given by an equation of the form $y = f(x)$, then $\kappa = \frac{|f''(x)|}{(1+f'(x)^2)^{\frac{3}{2}}}$ (for more details, see [13] p. 936 sq.).

Intuitively, the curvature at the upper-right corner of $\mathscr{F}_n$ increases with n; by the symmetries mentioned previously, the same remark is valid at the other corners. At this point, the curvature κ_n of $\mathscr{F}_n$ is given by:

$$\kappa_n = (n-1) \cdot 2^{\frac{2-n}{2n}}. \tag{28}$$

Note that the curve $\mathscr{F}_n$ is its own 180°-isoptic curve. At K_n the curvature is a positive number and the 180-isoptic is convex. An open question is as follows: the examples in this section and in Sect. 2 showed isoptic curves which are concave at their intersection point with the line whose equation is $y = x$. What is the limiting value of the angle where the isoptic curves change from convex to concave?

As to the numerical approach, in lieu of analytically finding appropriate families of curves which match the isoptics, we can perform nonlinear fitting of the (orthogonal or polar) mesh of θ values, to various (families of) functions. Optimistically, this will give us insight as to which are the appropriate families of curves we seek.

Finally, we wish to share with the reader a surprise that the first author experienced on a visit to Budapest. Figure 24 shows part of the floor at the entrance of a building, which he entered.

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