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Locus Computation in Dynamic Geometry Environment

Jiří Blažek · Pavel Pech

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Abstract The article is focused on investigation of geometric loci by current dynamic geometry software. On a few problems we illustrate ability of actual software to determine an unknown locus and its equation in terms of given properties using GeoGebra commands `Locus` and `LocusEquation`. After the use of these commands we present how computer arrives at the searched locus and how to analyze the cases when we get some extra components apart from the locus, either due to degenerate instances of the construction or due to Zariski closure of an algebraic set. We also demonstrate the ways how to attain new results by experiments, which would be hardly accessible without computers.

Keywords Dynamic geometry · Locus · Locus equation

Mathematics Subject Classification 97G40 · 14H50 · 68U05

1 Introduction

Geometric locus is a set of points satisfying some conditions. Searching for loci belongs, in our opinion, to difficult parts of mathematics school curricula all over the world. New technologies considerably facilitate searching for loci not only to students, but also to mathematicians.

Command `Locus` belongs to standard dynamic geometry systems (DGS) developed by Cabri [10] and The geometers sketchpad [8] teams in the eighties of the 20th century. Nowadays we encounter computer algebra systems in the frame of DGS. Recently new commands `Prove` or `LocusEquation`, which are based on the theory of automated theorem proving, appeared.

For application of the command `Locus` we need two points. The first one is a mover, the point which usually moves along a certain object. The second one—a tracer—is somehow dependent on the mover and draws the sought trajectory. When searching for the locus we sometimes obtain, besides the real locus, "unwanted" spurious parts

J. Blažek · P. Pech (✉)
Faculty of Education, University of South Bohemia, České Budějovice, Czech Republic
e-mail: pech@pf.jcu.cz

J. Blažek
e-mail: blazej02@pf.jcu.cz

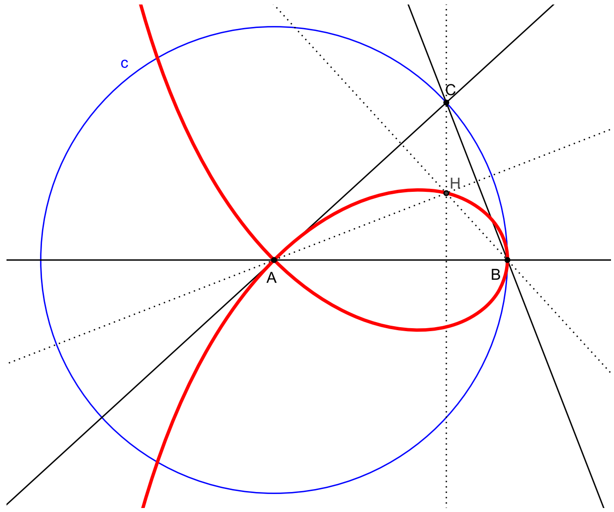

Fig. 1 The locus of H is the strophoid

usually representing undefined cases, which should be removed. In the Problem 1 we meet such a case and show how to treat it.

In the section Command `LocusEquation` it is shown that the command `Locus` cannot be thoughtlessly applied to all problems. Some problems need the use of a more advanced command `LocusEquation`, recently implemented into GeoGebra version 5, see [2,3,7].

This command represents a new approach in searching for loci. Unlike the method which firstly appeared in Cabri II plus [10], which is based on a random selection of several locus points and the computation of the best approaching polynomial curve (up to degree six) to this set of points [1,12], this method belongs to automated discovery [11], a part of automated theorem proving [5]. It is based on elimination of variables in a system of algebraic equations describing the locus. It returns an implicit equation of a curve. It is well known that this output is the Zariski closure of a projection on the space of local coordinates [6]. This often leads to the fact that instead of a real locus we get the smallest variety which contains, besides the locus, also some extraneous objects not pertaining to the locus itself.

Commands `AreParallel` or `AreConcyclic`, containing Boolean expressions, are tested in two examples resulting in plane curves. After application of GeoGebra commands it is shown, how computer arrives at the results. We analyze procedures leading to the locus and its equation including the points which do not pertain to the locus—the fact that we are not able to find out by the current GeoGebra commands. Besides the locus equation other locus properties are investigated—e.g. various kinds of loci or decomposabilities.

By searching for loci we apply algorithms based on Gröbner bases method using software CoCoA [4].

2 Command `Locus`

In this section we demonstrate the use of the command `Locus`. It can be applied to problems, where one needs a point to trace a locus path—the tracer, which is somehow dependent on another point—the mover which moves along a certain object. The command `Locus` is very simple and useful.

But sometimes we encounter the case, where together with the locus also some extraneous components occur due to degenerate instances of the construction. In the Problem 1 we meet such an extra component and propose how to remove it.

Problem 1 *Let ABC be a triangle with a side AB and a vertex C on a circle c centered at A and radius $|AB|$. Determine the locus of the orthocenter H of ABC when C moves along c.*

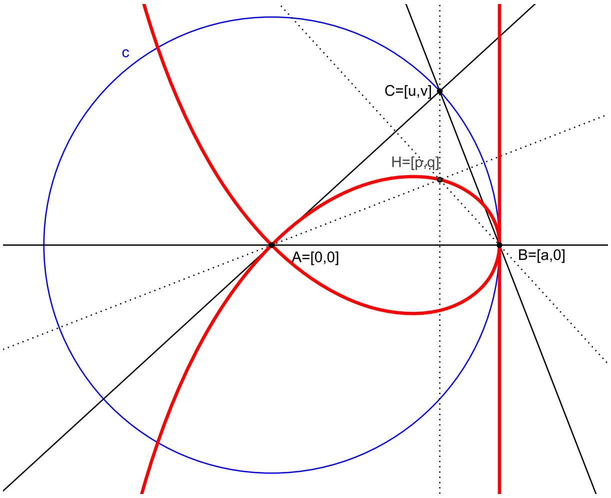

Fig. 2 The locus is the strophoid and the line

Using the command `Locus`, first clicking on the tracer H and then on the mover C, we get the curve of the *third* degree, a strophoid [13], see Fig. 1.
How does the computer arrive at it?

Let us introduce a rectangular system of coordinates such that $A = [0, 0]$, $B = [a, 0]$, $C = [u, v]$ and $H = [p, q]$. Then:

$HC \perp AB \Leftrightarrow h_1 : p - u = 0,$
$HA \perp BC \Leftrightarrow h_2 : p(u - a) + qv = 0,$
$C \in c \Leftrightarrow h_3 : u^2 + v^2 - a^2 = 0.$

Elimination of u, v in the ideal $I = (h_1, h_2, h_3)$, using graded reverse lexicographical order of variables a, p, q, u, v,

```
Use R::=Q[a,p,q,u,v];
I:=Ideal(p-u,p(u-a)+vq,u^2+v^2-a^2);
Elim(u..v,I);
```

gives the elimination ideal consisting of the polynomial of the *fourth* degree which decomposes into the strophoid and a line

$$(p^3 - ap^2 + aq^2 + pq^2)(p - a) = 0, \tag{1}$$

see Fig. 2. Why does the line appear in the locus?

The problem arises when C arrives at B, i.e. when $u = a$, $v = 0$, and the line BC is not defined. Then the system $h_1 = 0$, $h_2 = 0$, $h_3 = 0$ transforms into the relation

$p - a = 0$

representing the line.

To avoid this "unwanted" component, we add the condition $B \neq C$ or equivalently $((u - a)^2 + v^2)t - 1 = 0$, where t is a slack variable, into the ideal above. Then we get the only equation

```
Use R::=Q[a,p,q,u,v,t];
J:=Ideal(p-u,p(u-a)+vq,u^2+v^2-a^2,((u-a)^2+v^2)t-1);
Elim(u..t,J);
```

$\kappa : p^3 - ap^2 + aq^2 + pq^2 = 0.$

Similarly, to detect the case when the line $p-a = 0$ appears, suppose that the lines $p-u = 0$ and $p(u-a)+vq = 0$ coincide or are not defined. This is fulfilled for $u = a$ and $v = 0$. Then it suffices to suppose $(u - a)t - 1 = 0$ or $vt - 1 = 0$.

It remains to verify whether each point of the found curve is really the orthocenter of some triangle ABC, (this important step in the process of determination of the locus is often ignored).
Opposite implication:

Let $H \in \kappa$ and suppose that $B \neq C$, when the orthocenter is not uniquely defined. We ask whether the vertex C lies in the circumcircle of ABC for a given orthocenter H. Adding the condition $q \neq 0$ to the respective ideal,

```
Use R::=[Q[a,p,q,u,v,t,s];
L:=Ideal(p-u,p(u-a)+vq,p^3-ap^2+aq^2+pq^2,((u-a)^2+v^2)t-1,
q(u^2+v^2-a^2)s-1);
NF(1,L);
```

we get `NF(1,L)=0.`

Conclusion: The locus is the strophoid without the point $B = [a, 0]$.

Remark 1 Nowadays, using the command `LocusEquation`, we still obtain the Eq. (1) of the fourth degree. In the future it will be possible to get rid of such cases automatically, using the Gröbner Cover algorithm [1].

3 Command `LocusEquation`

The command `Locus` cannot be applied to every locus. Problems presented further are of this case. To solve them we have to use a more advanced command `LocusEquation`, recently implemented into GeoGebra version 5.

Before using the command `LocusEquation` we have to construct in GeoGebra a geometric diagram describing the locus.

After constructing the diagram we apply the command `LocusEquation` with two parameters. The first one is the thesis T (which must be a Boolean expression), the second one is a free point P whose locus we are looking for. The result of `LocusEquation` [T,P] produces the set V such that "if T is true then $P \in V$". Boolean expressions in the form of commands `AreParallel` and `AreConcyclic` are tested in the following two problems.

Problem 2 *Let $ABCD$ be a quadrilateral and K, L, M, N the feet of the perpendiculars from a point P to the lines AB, BC, CD, DA. Determine the locus of P such that the lines KN and LM are parallel.*

It is obvious that in this case the command `Locus` cannot be applied. However the command `LocusEquation` solves the problem.
Procedure determining the locus is following:

1. First construct a geometric diagram
   - Draw a quadrilateral $ABCD$.
   - Choose an arbitrary point P.
   - Construct the feet K, L, M, N of the perpendiculars from P to the lines AB, BC, CD and DA.
   - Denote $m = KN$ and $n = LM$.
2. Enter the command `LocusEquation[AreParallel[m,n],P]`.

Besides the graph of the searched locus one also gets its equation in given rectangular coordinates

$(x - 1)^2 + (y - 3)^2 = 10.$

It suggests that the locus is the circle, Fig. 3.

We will try to formulate following questions which can not be answered by the current GeoGebra:

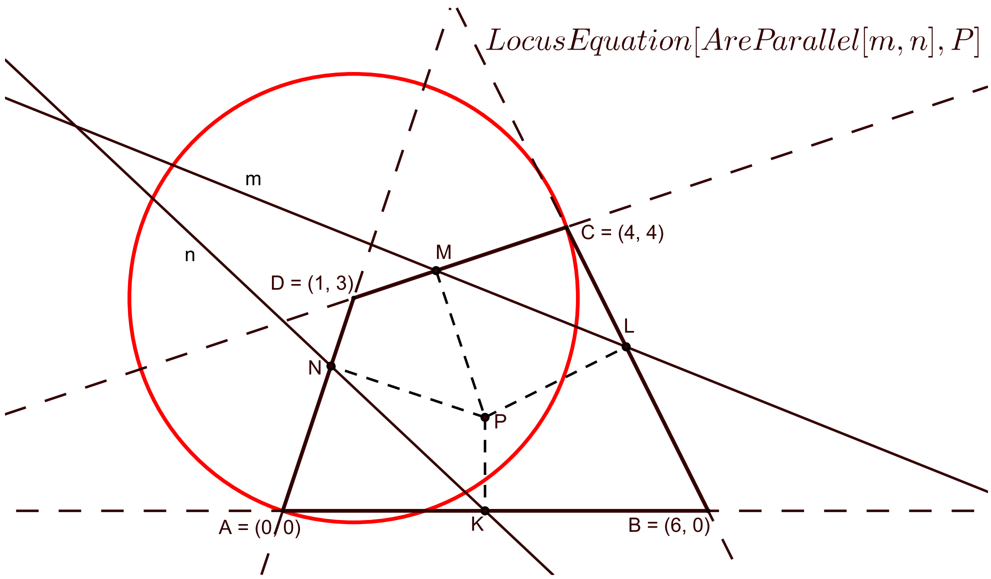

Fig. 3 On the screen the circle appears

- Is the solution the *entire* circle?
- Is the solution always a circle (and not another curve)?
- If the solution is not a circle, which positions of the vertices *A, B, C, D* does it happen for?

Which way does the computer proceed?

Let us show how to arrive at the solution using the theory of automated discovery [11].

Let $A = [0, 0]$, $B = [a, 0]$, $C = [u, v]$, $D = [w, z]$ $P = [p, q]$, $K = [k, 0]$, $L = [l_1, l_2]$, $M = [m_1, m_2]$ $N = [n_1, n_2]$. Suppose that the lines KN and LM are parallel, see Fig. 4. Then:

$$PK \perp AB \Leftrightarrow h_1 := p - k = 0,$$
$$L \in BC \Leftrightarrow h_2 := ul_2 + av - al_2 - vl_1 = 0,$$
$$PL \perp BC \Leftrightarrow h_3 := (p - l_1)(u - a) + (q - l_2)v = 0,$$
$$M \in CD \Leftrightarrow h_4 := um_2 + zm_1 + vw - wm_2 - uz - vm_1 = 0,$$
$$PM \perp CD \Leftrightarrow h_5 := (p - m_1)(w - u) + (q - m_2)(z - v) = 0,$$
$$N \in DA \Leftrightarrow h_6 := wn_2 - zn_1 = 0,$$
$$PN \perp DA \Leftrightarrow h_7 := (p - n_1)w + (q - n_2)z = 0,$$
$$KN \parallel LM \Leftrightarrow h_8 := (l_1 - m_1)n_2 - (l_2 - m_2)(n_1 - k) = 0.$$

After eliminating variables $k, l_1, l_2, m_1, m_2, n_1, n_2$ in the system $h_1 = 0, h_2 = 0, \ldots, h_8 = 0$, we get

$$z(av - vw - az + uz)S = 0, \tag{2}$$

where

$$S = (p^2 + q^2)(avw - 2uvw + vw^2 - auz + u^2z - v^2z + vz^2)$$
$$+p(u^2vw + v^3w - avw^2 + au^2z - u^3z + av^2z - uv^2z - avz^2)$$
$$-q(au^2w - u^3w + av^2w - uv^2w - auw^2 + u^2w^2 + v^2w^2 - u^2vz - v^3z - auz^2 + u^2z^2 + v^2z^2).$$

In (2) we suppose that $z \neq 0$ and $av - vw - az + uz \neq 0$, otherwise the quadrilateral $ABCD$ degenerates. Thus (2) implies the locus equation of $P = [p, q]$

$$S = 0.$$

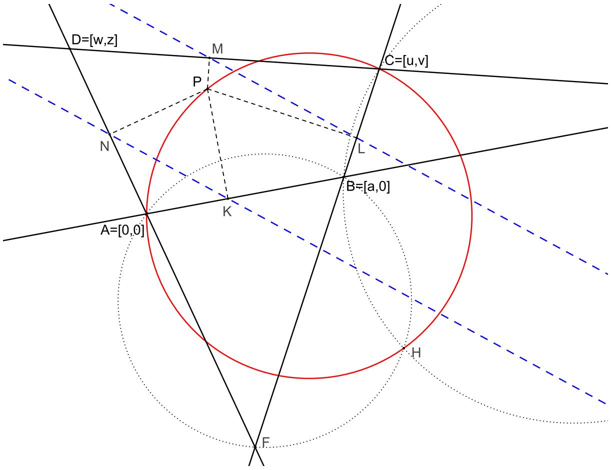

Fig. 4 The locus of P is the circumcircle of ABC

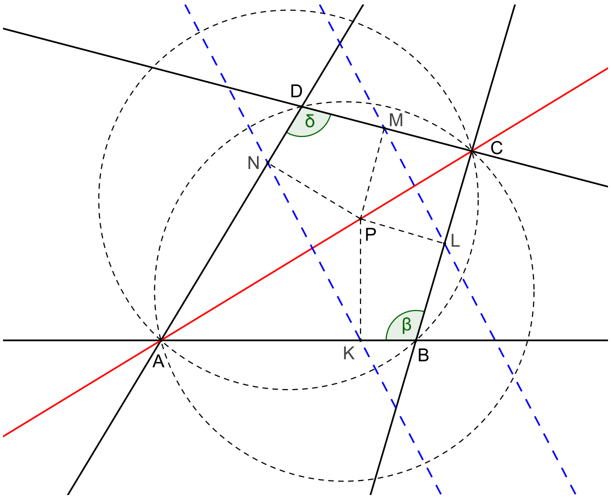

Fig. 5 If the angles by B and D are equal then the locus of P is diagonal AC

Denote the coefficient at $p^2 + q^2$ by T, i.e.

$T = avw - 2uvw + vw^2 - auz + u^2z - v^2z + vz^2$.

If $T \neq 0$, then (2) is the circle, passing through the points A, C and the Miquel point H [9].
If $T = 0$, then (2) is the line passing through the vertices A, C.

Realize that for $P \in AC$ the angles by B and D are equal, see Fig. 5.

Opposite implication.

Does every point $P = [p, q]$ satisfying $S = 0$ have the required property $KN \parallel LM$? To prove it, suppose that P obeys $S = 0$. We will show that then $KN \parallel LM$, i.e. $h_8 = 0$. Using the command Normal Form NF in CoCoA, we enter

```
Use R::=Q[a,u,v,w,z,k,l[1..2],m[1..2],n[1..2],p,q,t];
J:=Ideal(h1,h2,h3,h4,h5,h6,h7,S,z(av-vw-az+uz)h8*t-1);
NF(1,J);
```

and get $NF \neq 0$. The answer is negative.
Adding the conditions $P \neq A$ and $P \neq C$, i.e. $(p^2 + q^2)((p - u)^2 + (q - v)^2)s - 1 = 0$, into the ideal J,

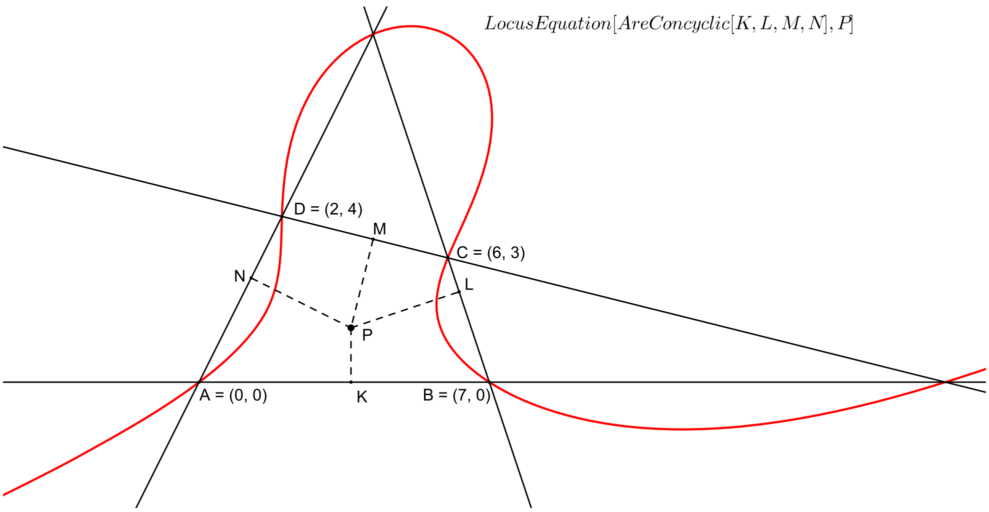

Fig. 6 Computer displays the cubic curve

```
Use R::=Q[a,u,v,w,z,k,l[1..2],m[1..2],n[1..2],p,q,t,s];
K:=Ideal(h1,h2,h3,h4,h5,h6,h7,S,
(p^2+q^2)((p-u)^2+(q-v)^2)s-1,z(av-vw-az+uz)h8*t-1);
NF(1,K);
```

we get `NF(1,K)=0`. This implies that $h_8 = 0$.
Realize that if $P = A$ or $P = C$, then $K = N$ or $L = M$, and the lines KN or LM are not defined.

Conclusion:

- If the angles by B and D of a quadrilateral $ABCD$ are distinct then the locus is the circle through the points A, C and the Miquel point H, without A and C.
- If the angles by B and D are equal then the locus is the line AC without the points A and C.

Problem 3 *Determine the locus of the point P such that the feet K, L, M, N of the perpendiculars from P to the side-lines of a given quadrilateral $ABCD$ are concyclic.*

Procedure determining the locus is following:

1. First construct a geometric diagram
   - Draw a quadrilateral $ABCD$.
   - Choose an arbitrary point P.
   - Construct feet K, L, M, N of perpendiculars from P to the lines AB, BC, CD and DA.
2. Enter the command `LocusEquation[AreConcyclic[K,L,M,N],P]`.

   The algebraic curve of the third degree

$$x^3 - 3x^2y + xy^2 - 3y^3 - 25x^2 + 28xy + 31y^2 + 126x - 168y = 0$$

is displayed, see Fig. 6.
Following questions may arise:

- Do we always get a cubic curve?
- If a curve is not cubic, which shape of $ABCD$ does it happen for?

– Is it possible to obtain decomposable curves?

Let us try to answer at least some of these questions.
First of all, let us analyze the way the computer arrives to this result.

Let us choose a coordinate system such that $A = [0, 0]$, $B = [a, 0]$, $C = [u, v]$, $D = [w, z]$ and $P = [p, q]$, and denote $K = [k_1, k_2]$, $L = [l_1, l_2]$, $M = [m_1, m_2]$ and $M = [n_1, n_2]$ the feet of the perpendiculars from P to the lines AB, BC, CD and DA. Then

$$
\begin{aligned}
&K \in AB \Leftrightarrow h_1 := k_2 = 0,\\
&L \in BC \Leftrightarrow h_2 := vl_1 + al_2 - av - ul_2 = 0,\\
&M \in CD \Leftrightarrow h_3 := vm_1 + uz + wm_2 - vw - zm_1 - um_2 = 0,\\
&N \in DA \Leftrightarrow h_4 := zn_1 - wn_2 = 0,\\
&PK \perp AB \Leftrightarrow h_5 := p - k_1 = 0,\\
&PL \perp BC \Leftrightarrow h_6 := (p - l_1)(u - a) + (q - l_2)v = 0,\\
&PM \perp CD \Leftrightarrow h_7 := (p - m_1)(w - u) + (q - m_2)(z - v) = 0,\\
&PN \perp DA \Leftrightarrow h_8 := (p - n_1)w + (q - n_2)z = 0,\\
&K, L, M, N \text{ are concyclic } \Leftrightarrow h_9 := \begin{vmatrix} k_1^2, & k_1, & 0, & 1 \\ l_1^2 + l_2^2, & l_1, & l_2, & 1 \\ m_1^2 + m_2^2, & m_1, & m_2, & 1 \\ n_1^2 + n_2^2, & n_1, & n_2, & 1 \end{vmatrix} = 0.
\end{aligned}
$$

The elimination of $k_1, k_2, l_1, l_2, m_1, m_2$ and n_1, n_2 in the system $h_1 = 0, h_2 = 0, \ldots, h_9 = 0$ gives

$$zv(-vw + uz)(av - vw - az + uz)S = 0,$$

where

$$
\begin{aligned}
S := {} & (v - z)p^3 + p^2q(a - u + w) + (v - z)pq^2 + (a - u + w)q^3 + (az - av - vw + uz)p^2 - 2awpq \\
& + (uz - av - vw - az)q^2 + a(vw - uz)p + a(uw + vz)q.
\end{aligned}
$$

Assuming that no three points of A, B, C, D are collinear, we get the locus equation

$$S = 0. \tag{3}$$

Let us explore the special case when the coefficients at the cubic terms in (3) vanish. It happens if

$$v - z = 0 \quad \text{and} \quad a - u + w = 0.$$

This implies $A = [0, 0]$, $B = [a, 0]$, $C = [u, v]$, $D = [u - a, v]$, and the quadrilateral is a parallelogram. Then instead of (3) one obtains

$$H := vp^2 + 2(a - u)pq - vq^2 - avp + (u^2 - au + v^2)q = 0. \tag{4}$$

If

$$u^2 - 2au + v^2 \neq 0$$

then (4) is the equilateral hyperbola, Fig. 7.
If

$$u^2 - 2au + v^2 = 0, \tag{5}$$

then (4) decomposes into two mutually orthogonal lines.

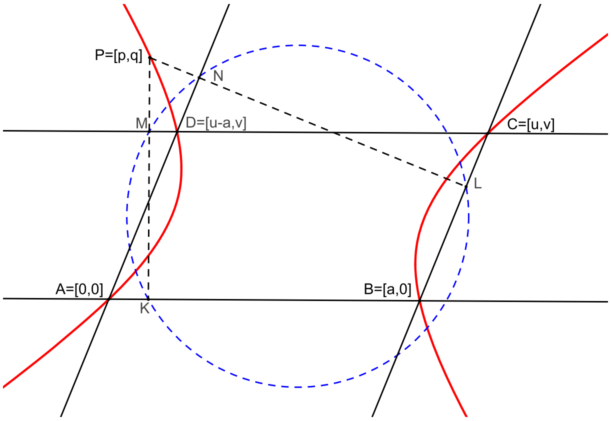

Fig. 7 If $u^2 - 2au + v^2 \neq 0$ then the locus is the equilateral hyperbola

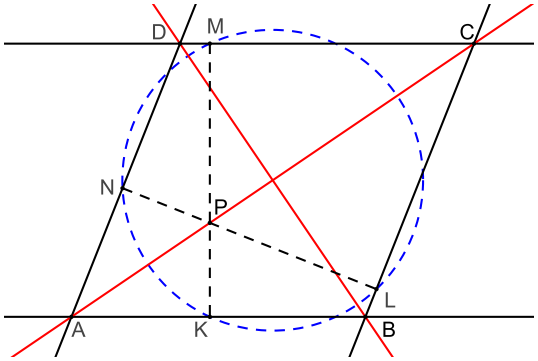

Fig. 8 If $ABCD$ is a rhombus then the locus of P consists of its diagonals

The condition (5) together with $z = v$ and $w = u - a$ lead to $|AB| = |BC| = |CD| = |DA|$, and $ABCD$ is a rhombus, see Fig. 8.

Now let us turn our attention to the opposite implication:

Let P be a point of the curve given by (4). The question is whether P fulfills given conditions, i.e. that the feet K, L, M, N are concyclic. Consider the ideal $I = (h_1, h_2, \ldots, h_8, H, h_9t - 1)$. Then the normal

```
Use R::=Q[a,u,v,w,z,p,q,k[1..2],l[1..2],m[1..2],n[1..2],t];
I:=Ideal(h1,h2,h3,h4,h5,h6,h7,h8,H,h9*t-1);
NF(1,I);
```

form `NF(1,I)=1`. Elimination of variables $k_1, k_2, l_1, l_2, m_1, m_2, n_1, n_2, t$ in I gives the elimination ideal `Ideal`$(-a, -u^2 - v^2)$. We suppose $a \neq 0$, otherwise the quadrilateral degenerates. Putting $at - 1 = 0$ into the ideal I, the respective normal form is equal to 0.

Conclusion:

- If $ABCD$ is a parallelogram which is not a rhombus, then the locus of P is the equilateral hyperbola.
- If $ABCD$ is a rhombus then the locus of P consists of two lines—the diagonals of the rhombus.

The problem just solved inspires us to the following consideration. There exists a parabola with the focal point at the Miquel point such that the line joining the feet of the perpendiculars is its tangent at the vertex. From the properties of a parabola it follows that these four lines are tangents of the parabola. This immediately brings us to the question: What is the locus of focal points of remaining conics (ellipse, hyperbola), to which the four lines are

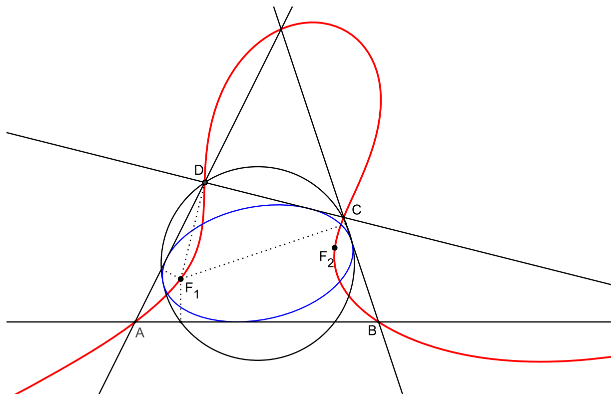

Fig. 9 The foci of a conic lie on the cubic (3)

tangents? The solution is based on property inherent to any conic. The feet of the perpendiculars from the focus to four tangents of a conic lie on the principal circle of the conic, see Fig. 9.

4 Conclusions

In the text the use of GeoGebra commands `Locus` and `LocusEquation` is described. These commands are applied to concrete non-trivial problems to obtain their graph of the locus including the locus equation. The analysis how the computer arrives at the solution leads to additional information (e.g. which part of the found curve really belongs to the locus, or existence of special cases) which the current GeoGebra commands are not able to give in fullness. Getting this additional information in an automated way, without human interaction, is a difficult problem.

Despite this, we hope that in the next versions of GeoGebra commands we will get more reliable information about loci.

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