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Descriptive Proximities. Properties and Interplay Between Classical Proximities and Overlap

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Abstract The theory of descriptive nearness is usually adopted when dealing with subsets that share some common properties, even when the subsets are not spatially close. Set description arises from the use of probe functions to define feature vectors that describe a set; nearness is given by proximities. A probe on a nonempty set X is an n-dimensional, real-valued function that maps each member of X to its description. We establish a connection between relations on an object space X and relations on the corresponding feature space. In this paper, the starting point is what is known as $\mathcal{P}_\Phi$ proximity (two sets are $\mathcal{P}_\Phi$-near or Φ-*descriptively near* if and only if their Φ-descriptions intersect). We extend, elucidate and explain the connection between overlap and strong proximity in a theoretical approach to a more visual form of proximity called *descriptive proximity*, which leads to a number of applications. Descriptive proximities are considered on two different levels: weaker or stronger than the $\mathcal{P}_\Phi$ proximity. We analyze the properties and interplay between descriptions on the one hand and classical proximities and overlap relations on the other hand. Axioms and results for a descriptive Lodato strong proximity relation are given. A common descriptive proximity is an Efremovič proximity, whose underlying topology is R_0 (symmetry axiom) and Alexandroff-Hopf. For every description Φ, any Čech, Lodato or EF Φ-descriptive proximity is at the same time a Čech, Lodato or EF-proximity, respectively. But, the converse fails. A detailed practical application is given in terms of the construction of Efremovič descriptive proximity planograms, which complements recent

Dedicated to the Memory of Som Naimpally.

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operations research work on the allocation of shelf space in visual merchandising. Specific instances of applications of descriptive proximity are also cited.

Keywords Descriptive proximity · Descriptive nearness · Overlap · Probe functions · Proximity

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1 Introduction

This article carries forward recent work on proximities [1–7]. Pivotal in this paper is the notion of a probe used to represent descriptions and proximities. A *probe* on a nonempty set X is a real-valued function $\Phi : X \to \mathbb{R}^n$, where $\Phi(x) = (\phi_1(x), \ldots, \phi_n(x))$ and each ϕ_i represents the measurement of a particular feature of an object $x \in X$ [8] (see also [9]). So, $\Phi(x)$ is a feature vector containing numbers representing feature values extracted from x. And a probe $\Phi(x)$ is also called *description* or *codification* of x. We establish a connection between relations on the object space X and relations on the feature space $\Phi(X)$. Usually, probe functions describe or codify physical features and act like "sensors" in extracting characteristic feature values from the objects. The theory of descriptive nearness [10] is usually adopted when dealing with subsets that share some common properties even though the subsets are not spatially close.

We talk about *non-abstract points* when points have locations and features that can be measured. The description-based theory is particularly relevant when we want to focus on some distinguishing characteristics of sets of non-abstract points. For example, if we take a picture element x in a digital image, we can consider grey-level intensity or colour of x. Of course, nearness or apartness depends essentially on the selected features that are compared.

Each pair of ovals in Fig. 1a, b contain circular-shaped coloured segments. Each segment in the ovals corresponds to an equivalence class, where all pixels in the class have matching descriptions, i.e., pixels with matching colours. For the ovals in Fig. 1a, b, we observe that the sets are not*spatially near,* but they can be considered near viewed in terms of colour intensities. Again, for example, the ovals in Fig. 1c, d contain segments that correspond to equivalence classes containing pixels with matching greyscale intensities. The ovals in Fig. 1c, d are descriptively near sets, since the equivalence classes contain matching greylevels. Moreover, we can also tell if they are *more or less near.* In the sequel, we will express these ideas of resemblance in mathematical terms.

The first fusion of description with proximity occurs in [1, §3], by introducing the notion of the *descriptive intersection* of two nonempty sets:

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \Phi(A), \Phi(x) \in \Phi(B)\},$$

and by declaring two sets *descriptively near* if and only if their descriptive intersection is nonempty or, equivalently, if and only if their descriptions intersect. The introduction of descriptive intersection led to new forms of proximity (see, *e.g.*, [11, §4.3, pp. 84–85], [1, §3, p. 90]). That is the first step in passing from the classical spatial proximity

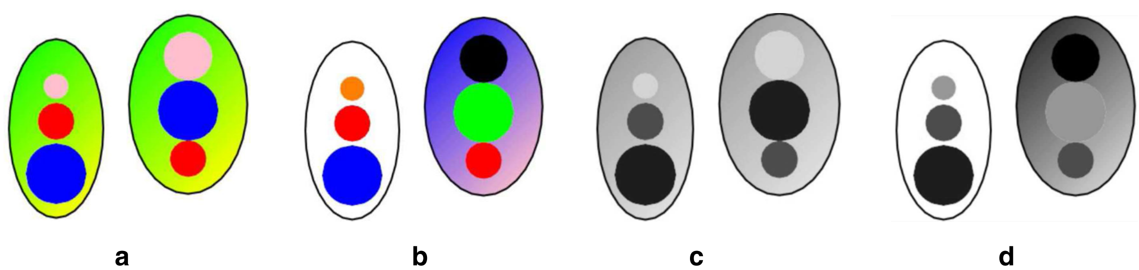


Fig. 1 Descriptively near sets via colour or greysale intensity. **a** Descriptively very near colour sets, **b** descriptively minimally near colour sets, **c** descriptively very near greyscale sets, **d** descriptively minimally near greyscale sets

to the more visual descriptive proximity. The new point of view is a really different approach to proximity which has a broad spectrum of applications. The $\mathcal{P}_\Phi$ proximity, two sets are $\mathcal{P}_\Phi$-near or Φ-*descriptively near* if and only if their Φ-descriptions intersect, is the Φ-pullback of the set-intersection. By replacing the set-intersection with the descriptive intersection, we construct a theoretical approach to the more visual form of proximity, namely, descriptive proximity (denoted by δ_Φ). The notion and the formal structure of descriptive proximity had already been introduced in [8] and used in [10,12–19], but lacked a more general vision of the connections and relations arising from descriptive proximity. The focus in the previous papers was on applied descriptive proximity. We organize descriptive proximities in two different levels: weaker ($A\delta_\Phi B \Rightarrow A \underset{\Phi}{\cap} B \neq \varnothing$) or stronger ($A \underset{\Phi}{\cap} B \neq \varnothing \Rightarrow A\delta_\Phi B$) than the $\mathcal{P}_\Phi$ proximity.

Since the set-intersection can be analyzed from the following two different perspectives: as the finest classical proximity, the discrete proximity, but also as the weakest overlapping relation (also called a *connection relation* [20, §1.2, p. 10]), we exhibit significant examples of descriptive proximities weaker than the $\mathcal{P}_\Phi$ proximity by following two different options: the proximal approach and the overlapping approach. In the sequel we will denote weaker descriptive proximities by $\overset{\Bbbw}{\delta}_\Phi$, while stronger ones by $\overset{\wedge\!\!\wedge}{\delta}_\Phi$.

In analogy with the classical case, we can introduce for any descriptive proximity, a "closure operator" associating to any subset A the set of points $cl(A)$ which are near to it. In the weak case we obtain a Kuratowski operator inducing a natural underlying topology. Unfortunately, in the strong case that operator has a bad behaviour. It is not extensive. In other words we can find points which are not near to themselves or "dislocated". But, by adding to $cl(A)$ the subset A, then we have a Kuratowski operator and consequently an underlying topology.

The main contribution of this paper is the refinement and extension of the original form of descriptive proximity.

2 Preliminaries

In this section, we take as our starting point the work on proximities by Naimpally [21] and Di Concilio [22] recalling the simplest example of proximities, namely, Lodato proximities [23–25], which guarantee the existence of a natural underlying topology.

Definition 2.1 (*Lodato*) Let X be a nonempty set. A *Lodato proximity* δ is a relation on 2^X, the collection of all subsets of X, which satisfies the following properties for all subsets A, B, C of X:

P_0): $A\ \delta\ B \Rightarrow A \neq \varnothing$ and $B \neq \varnothing$
P_1): $A\ \delta\ B \Leftrightarrow B\ \delta\ A$
P_2): $A \cap B \neq \varnothing \Rightarrow A\ \delta\ B$
P_3): $A\ \delta\ (B \cup C) \Leftrightarrow A\ \delta\ B$ or $A\ \delta\ C$
P_4): $(A\ \delta\ B$ and $\{b\}\ \delta\ C$ for each $b \in B) \Rightarrow A\ \delta\ C$

Further, δ is *separated* if

P_5): $\{x\}\ \delta\ \{y\} \Rightarrow x = y$.

When we write $A\ \delta\ B$, we read "A is near to B", while when we write $A\ \not\delta B$ we read "A is far from B". A relation δ which satisfies only $P_0) - P_3)$ is called a *Čech* [26] or *basic proximity*.

With any basic proximity one can associate a closure operator, cl_δ, by defining as closure of any subset A of X :

$$\mathrm{cl}_\delta A = \{x \in X : \{x\}\ \delta\ A\}.$$

Definition 2.2 An *EF-proximity* [27,28] is a relation on 2^X which satisfies P_0) through P_3) and in addition the property:

$$(EF)\ \ A\ \not\delta B \Rightarrow \exists E \subset X \text{ such that } A\ \not\delta E \text{ and } B\ \not\delta X \setminus E.$$

Since the EF-property is stronger than the Lodato property, every EF-proximity is indeed a Lodato proximity.

The following remarkable properties reveal the potentialities of Lodato proximity. When δ is a Lodato proximity, then:

Property 1: The associated closure operator cl_δ is a Kuratowski operator [29,30], i.e.:

1. $\mathrm{cl}_\delta\varnothing = \varnothing$ ($\varnothing$-preservation),
2. $A \subseteq \mathrm{cl}_\delta(A),\ \forall A \subset X$ (extensivity),
3. $\mathrm{cl}_\delta(A \cup B) = \mathrm{cl}_\delta A \cup \mathrm{cl}_\delta B,\ \forall A, B \subset X$ ($\bigcup$-distributivity),
4. $\mathrm{cl}_\delta(\mathrm{cl}_\delta A) = \mathrm{cl}_\delta A,\ \forall A \subset X$ (idempotency).

Hence, every Lodato proximity space (X, δ) determines an associated topology $\tau(\delta)$ whose closed sets are just the subsets which agree with their own closures.

Furthermore:

Property 2: For each subsets A, B: $A\ \delta\ B$ iff $\mathrm{cl}_\delta A\ \delta\ \mathrm{cl}_\delta B$, or, in other words, two sets are near iff their closures are near.

If (X, τ) is a topological space, we say that it admits a compatible Lodato proximity, provided there is a Lodato proximity δ on X such that $\tau = \tau(\delta)$. A question arises when a topological space has a compatible Lodato proximity. This happens iff the space satisfies the R_0-*separation property*, i.e. $x \in \mathrm{cl}\{y\} \Leftrightarrow y \in \mathrm{cl}\{x\}$. In fact, every R_0 topological space (X, τ) admits as a compatible Lodato proximity, δ_0, given by:

$$A\ \delta_0\ B \Leftrightarrow \mathrm{cl}A \cap \mathrm{cl}B \neq \varnothing \text{ (fine Lodato proximity [21]).}$$

In other words, A, B are δ_0-near iff they are *spatially near.*

On the other hand, a topological space has a compatible EF-proximity if and only if it is a completely regular topological space [22,31]. Recall that a topological space is completely regular if and only if, whenever A is a closed set and $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$ [31].

Property 3: Let T_1, T_2 denote separation spaces. Any Lodato T_1 (EF + T_2) proximity becomes *spatial* by a T_1 (T_2) compactification procedure. More specifically, any Lodato $T_1(EF + T_2)$ proximity space (X, δ) can be densely embedded in a compact $T_1(T_2)$ space $\gamma(X)$ so that two subsets A, B of X are δ-near iff their closures in $\gamma(X)$ intersect. For an introduction to the $T_1 = T_0 + R_0$ Frechet-Riesz space (called an accumulation [Häufungspunkte] T_1 space [32, §4.2]) and Hausdorff T_2 space, see [33, §1.4] and how a proximity can be embedded in a separation space, see [21, §2].

2.1 Examples

• *Discrete proximity on a nonempty set*: From a spatial point of view, discrete proximity appears as a generalization of the set-intersection, (see P_3 in Definition 2.1). The set-intersection is an EF-proximity, named the *discrete proximity*, just giving rise to the discrete topology, [§2.1][22]:

$$A\ \delta\ B \Leftrightarrow\ A \cap B \neq \varnothing.$$

• *Metric proximity*: A pivotal EF-proximity is the *metric proximity* δ_d associated with a metric space (X, d) defined by considering the gap between two sets in a metric space ($d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$ or ∞ if A or B is empty) and by putting:

$$A\ \delta_d\ B \Leftrightarrow d(A, B) = 0.$$

That is, A and B are δ_d-near iff they *either intersect or are asymptotic*: for each natural number n there is a point a_n in A and a point b_n in B such that $d(a_n, b_n) < \frac{1}{n}$.

• *Fine Lodato proximity*: The *fine Lodato proximity* δ_0, a spatial proximity, on a topological space is defined as follows:

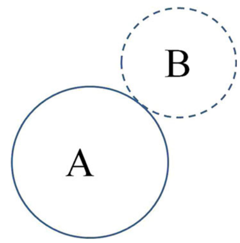


Fig. 2 Comparison

$$A\ \delta_0\ B \Leftrightarrow \mathrm{cl}A \cap \mathrm{cl}B \neq \varnothing.$$

The proximity δ_0 is the finest Lodato proximity compatible with a given topology.

Consider $\mathbb{R}^2$ endowed with the Euclidean topology and the sets in Fig. 2. The set A is a closed disk while B is an open disk. Touching on their boundaries, they are near in the fine Lodato proximity associated with the Euclidean topology but, not overlapping, they are far in the discrete proximity.

- *Functionally indistinguishable proximity*: δ_F on a completely regular space [22, §2.1, p. 94].

$$A\ \not\delta_F\ B \Leftrightarrow \text{There is a continuous function} f : X \to [0,1] :\ f(A) = 0,\ f(B) = 1.$$

The functionally indistinguishable proximity on a completely regular space X is an EF-proximity, which is further the finest EF-proximity compatible with X. Moreover, δ_F coincides with the fine Lodato proximity if and only if X is normal.

2.2 Strong Inclusion

Any proximity δ on X induces a binary relation over the powerset 2^X, usually denoted as $\ll_\delta$ and named the *natural strong inclusion associated with* δ, by declaring that A is *strongly included* in B, $A \ll_\delta B$, when A is far from the complement of B, i.e., $A\ \not\delta X \setminus B$ [22]. In terms of strong inclusion associated with an EF-proximity δ, the *Efremovič property* for δ can be formulated as the betweenness property:

$$\text{(EF2)}\quad \text{If} A \ll_\delta B, \text{then there exists some} C \text{such that} A \ll_\delta\ C \ll_\delta\ B.$$

We conclude by emphasizing that a topological structure is based on the nearness between points and sets and a function between topological spaces is continuous provided it preserves nearness between points and sets, while a function between two proximity spaces is *proximally continuous*, provided it preserves nearness between sets. Of course, any proximally continuous function is continuous with respect to the underlying topologies.

3 Original Form: Descriptive Intersection and $\mathcal{P}_\Phi$-Proximity

Peters [1, §3] made the first fusion of description with proximity, so passing from the classical spatial proximity to the recent more visual descriptive proximity which has a number of applications [6,10,11,16,20,34,35].

The starting idea is that two sets are near when the feature-values differences are so small so that they can be considered indistinguishable. The notion of descriptive intersection, playing a role similar to set-intersection in the classical case, is crucial in our recent project to approach new forms of descriptive proximities. The mixture of description with proximity reveals an advantageous augmentation.

The *descriptive intersection* or Φ-*intersection* is defined by

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B : \Phi(x) \in \Phi(A),\ \Phi(x) \in \Phi(B)\}.$$

The descriptive intersection of two sets A, B is nonempty provided there is at least one element in A with a description that matches the description of at least one element in B. The sets A, B could not share any point in common but they could have a nonempty descriptive intersection.

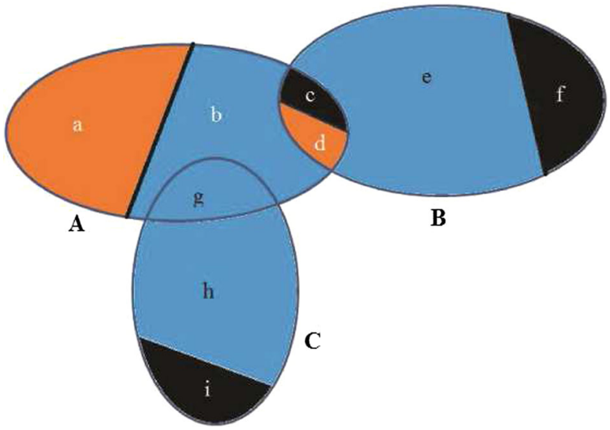


Fig. 3 Descriptive intersection

Example 3.1 Let X be $\mathbb{R}^2$ and $\Phi : \mathbb{R}^2 \to \mathbb{R}^3$ be the probe that associates with each point its RGB-colour. In Fig. 3, consider sets A, B, C and their subsets $a, b, c, d, e, f, g, h, i$. Observe that $\Phi(A) \cap \Phi(B)$ is given by colours black and orange. So, $\Phi^{-1}(\Phi(A) \cap \Phi(B)) = a \cup c \cup d \cup f \cup i$ is done of all points coloured black or orange, while descriptive intersection is obtained by taking, among these points, those belonging to $A \cup B$: $A \underset{\Phi}{\cap} B = \Phi^{-1}(\Phi(A) \cap \Phi(B)) \cap (A \cup B) = a \cup c \cup d \cup f$. Also, $B \underset{\Phi}{\cap} C = c \cup f \cup i$, but $B \cap C = \varnothing$. □

The first natural descriptive proximity, which we denote as $\mathcal{P}_\Phi$, declares two nonempty sets *descriptively near* if and only if their descriptions intersect. Or, in other words: Let X be a nonempty set, A and B be nonempty subsets of X, and $\Phi : X \to \mathbb{R}^n$ be a probe, then:

$A \, \mathcal{P}_\Phi \, B \Leftrightarrow \Phi(A) \cap \Phi(B) \neq \varnothing.$

Namely, $\mathcal{P}_\Phi$ proximity is the Φ-pullback of the discrete proximity. It is easily seen that we can rewrite the previous definition by using Φ-saturation of sets.

Remark 3.2 Recall that a set A is called *Φ-saturated* if and only if $\Phi^{-1}(\Phi(A)) = A$. Furthermore, the complement of a Φ-saturated set is in its turn Φ-saturated. □

Remark 3.3 Recall that a topological space has the *Alexandroff property* iff any intersection of open sets is in turn open [32].

Proposition 3.4 *Let X be a nonempty set, A a subset of X, and $\Phi : X \to \mathbb{R}^n$ a probe. Then, A is closed in the topology induced by $\mathcal{P}_\Phi$, $\tau(\mathcal{P}_\Phi)$, if and only if it is Φ-saturated. Moreover, whenever X is not a singleton and Φ is not constant, then $\tau(\mathcal{P}_\Phi)$ is disconnected.*

Proof By observing that the $\tau(\mathcal{P}_\Phi)$-closure of any subset A of X coincides with $\Phi^{-1}(\Phi(A))$, we simply derive the first result. As a consequence of the previous remark, in the case X is not a singleton and Φ is not constant, the $\tau(\mathcal{P}_\Phi)$- closure of any point of X and its complement give a disconnection of X. □

Theorem 3.5 *The $\mathcal{P}_\Phi$ proximity is an Efremovič proximity, whose underlying topology is R_0 and Alexandroff. Furthermore, $\mathcal{P}_\Phi$ is T_0, then T_2, if and only if the probe Φ is injective.*

Proof P_0) through P_3) in Definition 2.1 follow immediately from properties of the usual operations between sets and Φ-pullback. The EF-property comes from:

$A \neg\mathcal{P}_\Phi \, B \;\Leftrightarrow\; \Phi(A) \;\cap\; \Phi(B) = \varnothing \;\Leftrightarrow\; \Phi^{-1}(\Phi(A)) \cap \Phi^{-1}(\Phi(B)) = \varnothing$

that implies:

$\Phi^{-1}(\Phi(A)) \neg \mathcal{P}_\Phi B$ and $A \neg \mathcal{P}_\Phi \Phi^{-1}(\Phi(B))$.

The symmetry of $\mathcal{P}_\Phi$ entails the R_0-property:

$$y \in cl_{\mathcal{P}_\Phi}(x) \Leftrightarrow y\ \mathcal{P}_\Phi\ x \Leftrightarrow x\ \mathcal{P}_\Phi\ y \Leftrightarrow x \in cl_{\mathcal{P}_\Phi}(y).$$

Alexandroff property is obtained by observing that the union of saturated sets is in its turn saturated.
In the trivial case Φ is injective, then:

$$A\ \mathcal{P}_\Phi\ B \Leftrightarrow \Phi(A) \cap \Phi(B) \neq \varnothing \Leftrightarrow A \cap B \neq \varnothing.$$

So, $\mathcal{P}_\Phi$ is the discrete proximity. Giving rise to the discrete topology, $\mathcal{P}_\Phi$ is T_2. Conversely, suppose Φ is not injective. Then there exist two distinct points having the same image. Thus, having the same closure they cannot be T_0 separated. □

If we consider the relation on X given by $x \mathscr{R}_\Phi y \Leftrightarrow \Phi(x) = \Phi(y)$, then we have an equivalence relation whose classes are of type $[x] = \Phi^{-1}(\Phi(x))$, where $x \in X$.

- *Two subsets A and B of X are $\mathcal{P}_\Phi$-near if and only if they intersect a same class of the partition induced by $\mathscr{R}_\Phi$.*

4 General Forms of Descriptive Proximities

This section introduces two forms of descriptive proximity. The $\mathcal{P}_\Phi$ proximity is a link between nearness or overlapping of descriptions in the codomain $\mathbb{R}^n$ with relations on pairs of subsets on the domain of codification. But $\mathcal{P}_\Phi$ proximity might be considered in some cases too strong or in some other ones too weak. So, by relaxing or stressing $\mathcal{P}_\Phi$, we obtain general forms of descriptive proximities, that can work better than it in particular settings. Since, from a spatial point of view, classical proximity is a generalization of the set-intersection, in our treatment we choose $\mathcal{P}_\Phi$ proximity as the unique separation element between two different broad classes of descriptive proximities. If we entrust the descriptive intersection with the same role of the set-intersection in the classical case we get the following two options: descriptive intersection versus descriptive proximity, i.e.,

First option: weaker form:

$$A \underset{\Phi}{\cap} B \neq \varnothing \Rightarrow A \ \overset{\mathbb{W}}{\delta_\Phi}\ B$$

Second option: stronger form:

$$A \ \overset{\wedge\!\!\!\wedge}{\delta_\Phi}\ B \Rightarrow A \underset{\Phi}{\cap} B \neq \varnothing.$$

4.1 Weaker Form

This is the case in which two sets having nonempty descriptive intersection are descriptively near:

$$A \underset{\Phi}{\cap} B \neq \varnothing \Rightarrow A \ \overset{\mathbb{W}}{\delta_\Phi}\ B.$$

Definition 4.1 Let X be a nonempty set, $A,\ B,\ C$ be subsets of X, and $\Phi : X \to \mathbb{R}^n$ be a probe. The relation $\overset{\mathbb{W}}{\delta_\Phi}$ on $\mathscr{P}(X)$, the powerset of X, is a *Čech Φ-descriptive proximity* iff the following properties hold:

D_0): $A \ \overset{\mathbb{W}}{\delta_\Phi}\ B \Rightarrow A \neq \varnothing$ and $B \neq \varnothing$
D_1): $A \ \overset{\mathbb{W}}{\delta_\Phi}\ B \Leftrightarrow B \ \overset{\mathbb{W}}{\delta_\Phi}\ A$

D_2): $A \underset{\Phi}{\cap} B \neq \varnothing \Rightarrow A \ \overset{\Cup}{\delta}_{\Phi} \ B$

D_3): $A \ \overset{\Cup}{\delta}_{\Phi} \ (B \cup C) \Leftrightarrow A \ \overset{\Cup}{\delta}_{\Phi} \ B$ or $A \ \overset{\Cup}{\delta}_{\Phi} \ C$
If, additionally:

D_4): $(A \ \overset{\Cup}{\delta}_{\Phi} \ B$ and $\{b\} \ \overset{\Cup}{\delta}_{\Phi} \ C$ for each $b \in B) \Rightarrow A \ \overset{\Cup}{\delta}_{\Phi} \ C$ holds, then $\overset{\Cup}{\delta}_{\Phi}$ is a *Lodato* Φ*-descriptive proximity* [10, §4.15.2, p. 155].

Furthermore, if the following property holds:

$$EF_{\Phi}) \ \ A \ \overset{\Cup}{\not\delta}_{\Phi} \ B \Rightarrow \exists E \subset X \text{ such that } A \ \overset{\Cup}{\not\delta}_{\Phi} \ E \text{ and } B \ \overset{\Cup}{\not\delta}_{\Phi} \ X \setminus E,$$

then $\overset{\Cup}{\delta}_{\Phi}$ is an *EF* Φ*-descriptive proximity.*

We explicitly observe that descriptive axioms D_0) through D_4) plus EF_{Φ}) are formally the same as in the classical definition with the set-intersection replaced by the Φ-intersection. And, also, that:

Proposition 4.2 *For every description* Φ*, any Čech, Lodato or EF* Φ*-descriptive proximity* $\overset{\Cup}{\delta}_{\Phi}$ *is at the same time a Čech, Lodato or EF-proximity, respectively. But, the converse fails.*

Proof It is enough to observe that $A \cap B \subseteq A \underset{\Phi}{\cap} B$. So, if $A \cap B \neq \varnothing$, then $A \underset{\Phi}{\cap} B \neq \varnothing$ in its turn. Hence, by D_2) it follows that $A \cap B \neq \varnothing$ implies $A \ \overset{\Cup}{\delta}_{\Phi} \ B$. Consequently, $\overset{\Cup}{\delta}_{\Phi}$ satisfies property P_2) in 2.1. The converse fails. We produce a counterexample as follows. Let S be a proper subset of $\Phi(X)$. Say that A is δ-near B iff they intersect or their codifications intersect in a point of S. To prove that δ does not satisfy D_2), it is enough to exhibit two subsets A and B which don't intersect, whose codifications do intersect but outside of S. □

4.2 The Underlying Topology in the Weaker Case

As in the classical case, for any descriptive proximity $\overset{\Cup}{\delta}_{\Phi}$ and for each subset A in X we define the $\overset{\Cup}{\delta}_{\Phi}$*-descriptive closure* of A as:

$$Cl_{\overset{\Cup}{\delta}_{\Phi}}(A) =: \{x \in X : x \ \overset{\Cup}{\delta}_{\Phi} \ A\}$$

Corollary 4.3 *Whenever* $\overset{\Cup}{\delta}_{\Phi}$ *is a Lodato* Φ*-descriptive proximity, then the closure operator* $Cl_{\overset{\Cup}{\delta}_{\Phi}}$ *is a Kuratowski operator. Moreover:* $A \ \overset{\Cup}{\delta}_{\Phi} \ B \Leftrightarrow Cl_{\overset{\Cup}{\delta}_{\Phi}} A \ \overset{\Cup}{\delta}_{\Phi} \ Cl_{\overset{\Cup}{\delta}_{\Phi}} B$.

Proof It is an immediate consequence of the Proposition 4.2. □

5 Approaches: Proximity and Overlap

The $\mathcal{P}_{\Phi}$ proximity is the Φ-pull back of the set-intersection. The set-intersection can be considered in two different aspects. It is the finest proximity on one side and the weakest overlap relation on the other side. So, to construct significant examples of descriptive proximities weaker than the $\mathcal{P}_{\Phi}$ proximity we have two possible approaches: the proximal approach, which arises when looking at the the set-intersection as a proximity; the overlap approach, when looking at the set-intersection as an overlap relation [22,36,37]. In this context, by *overlapping* we mean a special intersection, that is intersection of some *extension* of descriptions.

5.1 Proximity Approach

Let X be a nonempty set, A and B be subsets of X, $\Phi : X \to \mathbb{R}^n$ be a probe and δ be a proximity on $\Phi(X) \subseteq \mathbb{R}^n$. Then, if we define δ_Φ as follows:

$$A \ \delta_\Phi \ B \Leftrightarrow \Phi(A) \ \delta \ \Phi(B)$$

we get a descriptive proximity weaker than the $\mathcal{P}_\Phi$ proximity. Of course, the prototype is the $\mathcal{P}_\Phi$ proximity when $\mathbb{R}^n$ is equipped with the discrete proximity. In this case, as we have seen in the previous paragraph, the $Cl_{\mathcal{P}_\Phi}(A)$ is the Φ-preimage of $\Phi(A)$.

Absorbing and transferring their own similar properties to the other one, the descriptive proximity δ_Φ and the standard proximity δ are very close to each other.

Theorem 5.1 *Let X be a nonempty set, $\Phi : X \to \mathbb{R}^n$ be a probe and δ be a proximity on $\Phi(X)$. For each description Φ, δ_Φ is a Čech, Lodato or EF Φ- descriptive proximity iff the proximity δ is a Čech, Lodato or EF proximity.*

Proof By the definitions of both kinds of proximities, properties can be transferred from one side to the other one by simply using properties of images and pull-back. Equivalences between P_0 and D_0 and between P_1 and D_1 are easy and, as well, P_2) implies D_2). The vice versa follows from $A \underset{\Phi}{\cap} B \neq \emptyset$ that exactly means $\Phi(A) \cap \Phi(B) \neq \emptyset$. And two sets that intersect are near in every proximity. To prove the equivalence between D_3 and P_3 we can limit to consider Φ-saturated sets. In fact, from the definition of δ_Φ (page 99), it follows that $A \not\delta_\Phi B$ iff $\Phi^{-1}(\Phi(A)) \not\delta_\Phi \Phi^{-1}(\Phi(B))$. So now we prove that, if we consider Φ-saturated sets, then $A \ll_{\delta_\Phi} B$ is equivalent to $\Phi(A) \ll_\delta \Phi(B)$. Actually, when B is Φ-saturated, i.e. $\Phi^{-1}(\Phi(B)) = B$, then $A \ll_{\delta_\Phi} B$ entails $\Phi^{-1}(\Phi(A)) \ll_{\delta_\Phi} B$. Further, since $\Phi(X) \setminus \Phi(B) \subseteq \Phi(X \setminus B)$ for each $B \subseteq X$, it happens that $A \ll_{\delta_\Phi} B$ implies $\Phi(A) \ll_\delta \Phi(B)$. The vice versa is also true when B is Φ-saturated. In fact, in this case $\Phi(X \setminus B) = \Phi(X) \setminus \Phi(B)$. By the premises we can limit to consider Φ-saturated sets. By this observation it follows easily that P_3 is equivalent to D_3.

Finally, suppose δ_Φ is EF. Let $\Phi(A) \ll_\delta \Phi(B)$. That is equivalent to $A \ll_{\delta_\Phi} B$. But, then there exists $C \subset X$ so that $A \ll_{\delta_\Phi} C \subseteq \Phi^{-1}(\Phi(C)) \ll_{\delta_\Phi} B$. Thus, $\Phi(A) \ll_\delta \Phi(C) \ll_\delta \Phi(B)$. And, consequently, δ is EF in its turn. The vice versa can be acquired following a similar procedure. □

Observe that, given a proximity δ on $\mathbb{R}^n$, $\overset{\mathbb{W}}{\delta}_\Phi$ is the coarsest proximity on X for which the probe Φ is proximally continuous, i.e. $A \ \overset{\mathbb{W}}{\delta}_\Phi \ B \Rightarrow \Phi(A) \ \delta \ \Phi(B)$ [11, §1.7, p. 16].

Another significant example of descriptive proximity is the fine Lodato descriptive proximity associated with a given topology. When $\mathbb{R}^n$ is equipped with a topology τ, *the fine Lodato descriptive proximity* $\delta^0_\Phi(\tau)$ associated with τ is obtained by putting:

$$A \ \delta^0_\Phi(\tau) \ B \Leftrightarrow Cl_\tau(\Phi(A)) \cap Cl_\tau(\Phi(B)) \neq \emptyset,$$

where Cl_τ represents the τ-closure operator.

Theorem 5.2 *The fine Lodato descriptive proximity $\delta^0_\Phi(\tau)$ is the finest one among all "general" Lodato descriptive proximities whose underlying topology on X is the same as $\delta^0_\Phi(\tau)$.*

Proof The proof follows analogous steps as in the classical case by using the results in Corollary 4.3 [21]. □

5.2 Overlapping Approach

Suppose that for any subset A of X a specific enlargement, $e(\Phi(A))$, i.e., $\Phi(A) \subseteq e(\Phi(A))$, of $\Phi(A)$ in $\mathbb{R}^n$ can be associated with A and moreover:

$(*)$ For each pair $A, B \subseteq X : e(\Phi(A)) \cup e(\Phi(B)) = e(\Phi(A) \cup \Phi(B))$ (*additivity*) and, also:

$$(**)\ \ \text{For each } x \in X \text{ and } A \subseteq X,\ \ e(\Phi(x)) \cap e(\Phi(A)) \neq \varnothing \Rightarrow e(\Phi(x)) \subseteq e(\Phi(A)).$$

In this context, by *overlapping* we mean a special intersection, that is intersection of some *extension* of descriptions.

Then, if we put:

$$A \overset{e}{\delta_\Phi} B \text{ iff } e(\Phi(A)) \cap e(\Phi(B)) \neq \varnothing,$$

we have:

Proposition 5.3 *The relation* $\overset{e}{\delta_\Phi}$ *is a* Φ*-descriptive Lodato proximity.*

Proof Properties D_0, D_1 are immediate. D_3 comes from additivity. D_2 follows from extensivity and finally, property $(**)$ entails D_4. □

When choosing as $\epsilon > 0$ as level of approximation and as enlargement for any subset A of $\mathbb{R}^n$ the ϵ-enlargement $S_\epsilon(A) = \cup\{S_\epsilon(x) : x \in A\}$, we have a peculiar case in the overlapping approach where properties D_0) through D_3) are satisfied, but, in general, Lodato property is not. Recall that the ϵ-enlargement of a point x is just the ball of diameter ϵ around x. So arguing, the condition $(**)$ cannot be removed in general. But, in the case where $\Phi(X)$ is equipped with an ultrametric, then the ϵ-enlargement operator satisfies also the property $(**)$. As a consequence, the relation $\overset{e}{\delta_\Phi}$ is a descriptive proximity for each codification Φ. As is well known, ultrametrics are characterized as metrics for which if two balls intersect, then one of them has to be contained in the other one and then coincide when the balls have the same radius. In general, an *ultrametric* is a metric which satisfies the strengthened (max) version of the triangle inequality, i.e., for a map $d : X \times X \longrightarrow \mathbb{R}$, for all $x, y, z \in X$,

$d(x, z) \leq max\,\{d(x, y), d(y, z)\}$ strengthened triangle inequality.

For the correspondence between dated, compact, rooted trees and ultrametrics, see Böcker and Dress [38].

It is not possible to remove the $(*)$-condition of additivity as the following geometric example, involving the affine structure of $\mathbb{R}^n$, proves:

$A \overset{conv}{\delta_\Phi} B \Leftrightarrow \text{conv}(\Phi(A)) \cap \text{conv}(\Phi(B)) \neq \varnothing,$

where $\text{conv}(\Phi(A))$ = minimal convex set containing $\Phi(A)$. The above relation $\overset{conv}{\delta_\Phi}$ verifies the properties D_0, D_1, D_2, D_4 but only one way in D_3. This is due to the fact that, in general, the union of two convex sets is not convex. This is a special case where the enlargement of any point reduces just to the same point.

5.3 Partial Matches

When requiring $A\ \mathcal{P}_\Phi\ B$, we look at the match of the entire feature vectors on points of A and B. But, it can be useful to consider a fixed part of the vector of feature values. In this way descriptive nearness of sets can be established on a partial match of descriptions. To achieve this result, we introduce:

Example 5.4 (α_Φ) Let X be a nonempty set, A and B be subsets of X, $\Phi : X \to \mathbb{R}^n$ be a probe, and $\Phi_i = p_i \circ \Phi$, $i \in \{1, \ldots, n\}$, where p_i is the i-th natural projection. We define:

$A\ \alpha_\Phi\ B \Leftrightarrow \Phi_i(A) \cap \Phi_i(B) \neq \varnothing, \quad \forall i = 1, \ldots, n$

The nearness relation α_Φ is an EF Φ-descriptive proximity intersection of the $\mathcal{P}_\Phi$ proximities associated with the descriptions $p_i \circ \Phi$, $i \in \{1, \ldots, n\}$. Observe that $\mathcal{P}_\Phi$ is stronger than α_Φ, that is $A\mathcal{P}_\Phi B \Rightarrow A\alpha_\Phi B$.

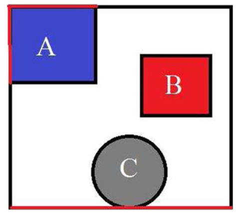


Fig. 4 $\mathcal{P}_\Phi \Rightarrow \alpha_\Phi$

Example 5.5 Let X be $[0,1]^2 \subset \mathbb{R}^2$ endowed with the Euclidean topology and $\Phi : X \to \mathbb{R}^4$, where $(\Phi_1, \Phi_2, \Phi_3)(x)$ represents the RGB-colour of x and $\Phi_4(x)$ equals 1 if x is on the boundary of X, while $\Phi_4(x)$ equals 0 otherwise. In Fig. 4 $A\mathcal{P}_\Phi C$ but $A\neg\mathcal{P}_\Phi B$ because there is no point with the same colour and the same position with respect to the boundary of X. Instead, $A\alpha_\Phi B$ because there are points in A and B that have the same colour and also points that have the same position with respect to the boundary of X.

Remark 5.6 The topology associated with α_Φ is defined by the Kuratowski operator Cl_{α_Φ}:

$$x \in Cl_{\alpha_\Phi}(A) \Leftrightarrow x \in \bigcap_{i=1,\ldots,n} \Phi_i^{-1}(\Phi_i(A))$$

□

Example 5.7 (β_Φ) Further, we generalize α_Φ by composing the probe Φ with the projection π_m forgetting the last $n-m$ coordinates:

$$\pi_m \circ \Phi : x \longrightarrow (\pi_m \circ \Phi)(x) = (\phi_1(x), \ldots, \phi_m(x)).$$

we have:

$$A \ \beta_\Phi \ B \Leftrightarrow \pi_m(\Phi(A)) \cap \pi_m(\Phi(B)) \neq \varnothing.$$

The nearness relation β_Φ is an EF Φ-descriptive proximity being the $\mathcal{P}_\Phi$ proximity associated with $\pi_m \circ \Phi$.
A third kind of descriptive nearness defined by probes and intersection is given as follows.

Example 5.8 (γ_Φ) Let X be a nonempty set, A and B be subsets of X, and $\Phi : X \to \mathbb{R}^n$ be a probe. We define:

$$A \ \gamma_\Phi \ B \Leftrightarrow \exists i \in \{1, \ldots, n\} : \ \Phi_i(A) \cap \Phi_i(B) \neq \varnothing.$$

Observe that β_Φ is stronger than γ_Φ, that is: $A\beta_\Phi B \Rightarrow A\gamma_\Phi B$. Note that the relation γ_Φ does not verify the Lodato axiom D_4) (see Definition 4.1). We illustrate this by the following example based on Fig. 5.

Example 5.9 Let $A = \{a, b\}$, $C = \{c, d\}$, $B = \{e, f, g\}$. In this figure we have $\Phi_1(A) = \{1\}$, $\Phi_2(A) = \{2, 3\}$, $\Phi_1(C) = \{2, 4\}$, $\Phi_2(C) = \{4, 1\}$, $\Phi_1(B) = \{1, 2, 5\}$. So $A\gamma_\Phi B$ because $\Phi_1(A) \cap \Phi_1(B) \neq \varnothing$, and for each $x \in B$ $x \ \gamma_\Phi \ C$. But $A \ \not\gamma_\Phi C$ because $\Phi_1(A) \cap \Phi_1(C) = \varnothing$ and $\Phi_2(A) \cap \Phi_2(C) = \varnothing$. In other words γ_Φ is not a Lodato proximity.

5.4 Second Option: Stronger Form

This is the case in which two sets that are descriptively near have a nonempty descriptive intersection: $A \ \overset{\wedge\!\!\wedge\!\!\wedge}{\delta}_\Phi \ B \Rightarrow A \underset{\Phi}{\cap} B \neq \varnothing$.

Definition 5.10 Let X be a nonempty set, $A, \ B, \ C$ be subsets of X and $\Phi : X \to \mathbb{R}^n$ be a probe. The relation $\overset{\wedge\!\!\wedge\!\!\wedge}{\delta}_\Phi$ on $\mathscr{P}(X)$ is a Φ-descriptive Lodato strong proximity iff the following properties hold:

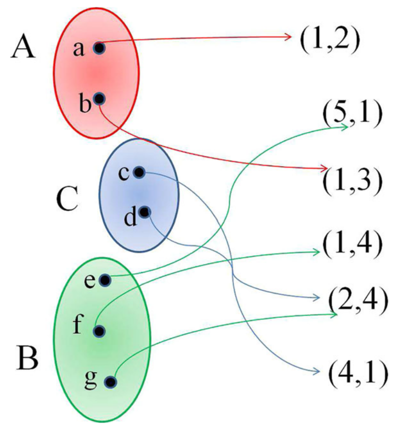


Fig. 5 γ_Φ is not a Lodato proximity

(S_0): $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} B \Rightarrow A \neq \varnothing$ and $B \neq \varnothing$
(S_1): $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} B \Leftrightarrow B \overset{\wedge\!\!\wedge}{\delta}_{\Phi} A$
(S_2): $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} B \Rightarrow A \underset{\Phi}{\cap} B \neq \varnothing$
(S_3): $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} (B \cup C) \Leftrightarrow A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} B$ or $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} C$
(S_4): $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} B$ and $\{b\} \overset{\wedge\!\!\wedge}{\delta}_{\Phi} C$ for each $b \in B \Rightarrow A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} C$.

As an example, when we can distinguish a significant subset $S \subseteq \Phi(X)$, we can put: $A \overset{\wedge\!\!\wedge}{\delta}_{\Phi} B$ if and only if $\Phi(A)$ shares some common point with $\Phi(B)$ belonging to S, thereby obtaining a strong Φ-descriptive proximity. This means that A and B are Φ-descriptively strongly near iff $\Phi(A)$ and $\Phi(B)$ intersect on S. Looking at the domain, A and B are Φ-descriptively strongly near when they intersect at the same element $\Phi^{-1}(s)$ with $s \in S$.

This case looks particularly interesting when $\mathbb{R}^n$ is covered by a grid and S is done from the vertices of the given grid, where the essential information in the description can be assumed to be concentrated.

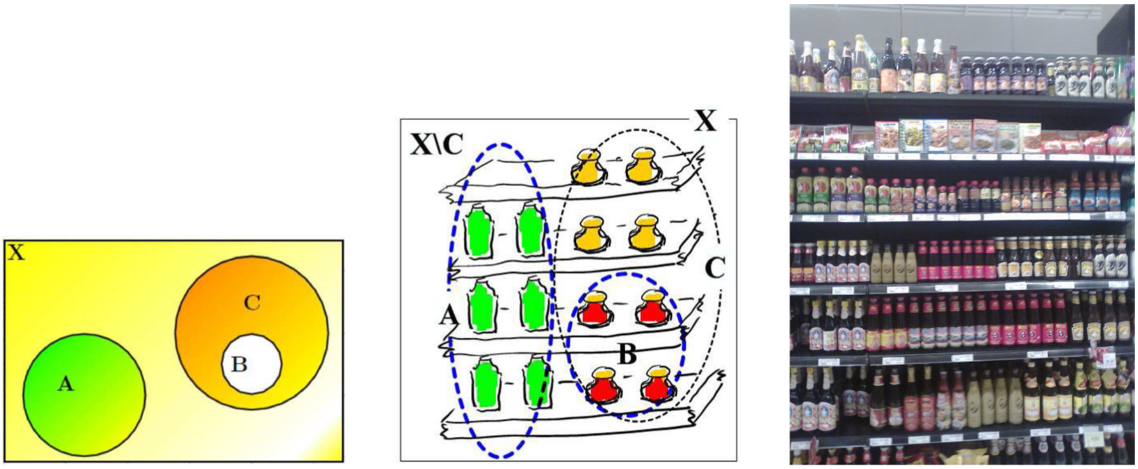
5.5 The Underlying Topology in the Stronger Case

As in the weak case, for any strong Φ-descriptive proximity $\overset{\wedge\!\!\wedge}{\delta}_{\Phi}$ and for each subset A in X we can define:

$$cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(A) =: \{x \in X : x \overset{\wedge\!\!\wedge}{\delta}_{\Phi} A\}.$$

In contrast with the weak case, the cl operator has a bad behaviour. There are points x "dislocated" with respect to their own $cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(x)$, i.e. not belonging to $cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(x)$. But, if we add to $cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(A)$ also the set A, then the operator $A \to A \cup cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(A)$ can be easily proved to be a Kuratowski operator, so inducing a topology underlying to the strong Φ-descriptive proximity $\overset{\wedge\!\!\wedge}{\delta}_{\Phi}$.

In the previous example, $cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(x)$ is the empty-set when $\Phi(x)$ is not in S, while $cl_{\overset{\wedge\!\!\wedge}{\delta}_{\Phi}}(x) = \Phi^{-1}(\Phi(x))$ when x belongs to S. Moreover, any dislocated point is isolated in the underlying topology which is disconnected from the set of dislocated points and its complement.



a **b** **c**

Fig. 6 Sample EF-relationships. **a** EF relation, **b** EF display, **c** Thai display

6 Application: Descriptive Proximity in Visual Merchandising

This section briefly introduces an application of descriptive proximity in visual merchandising. It is common for retailers to maintain planograms for the arrangement of the displays of their merchandise. A *planogram* is detailed product-level map of a store layout [39]. A planogram provides retailers with a blueprint for how merchandise are arranged visually on shelves to facilitate high visibility and price point options.

Example 6.1 (Visual Descriptive EF-Proximity Relationships) The use of descriptive EF-proximity in visual merchandising was introduced in [11, §3.3, pp. 75–78]. Let $A, C \subset X$, $B \subset C$ and let C^c be the compliment of C. A descriptive EF proximity (denoted by $\not\delta_{\Phi}$) has the following property:

$\boldsymbol{A} \not\delta_{\Phi} \boldsymbol{B} \Leftrightarrow \mathbf{A} \not\delta_{\Phi} \mathbf{C}$ and $\mathbf{B} \not\delta_{\Phi} \mathbf{C^c}$.

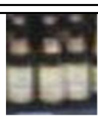
A representation of this descriptive EF proximity relation is shown in Fig. 6a. The import of an EF-proximity relation is extended rather handily to visual displays of products in a supermarket (see, *e.g.*, Fig. 6c). The sets of bottles that have an underlying EF-proximity to each other is shown conceptually in the sets in Fig. 6b. The basic idea with this application of topology is to extend the normal practice in the vertical and horizontal arrangements of similar products with a consideration of the topological structure that results when remote sets are also taken into account, representing the relations between these remote sets with an EF-proximity. □

The proposed descriptive proximity approach to visual merchandising holds promise, since it leads to highly accurate measurements of the similarities between the visual content of displayed products. The high acuity of descriptive proximities of visual products derives from the use of probe function-based product descriptions to determine the descriptive proximities between pairs of sets of products. These descriptive proximities are then carried over in the construction of planograms. Further, the descriptive proximity-based approach to planograms complements recent operations research work on shelf space allocation, where the focus is on the shelf spatial requirements of various products (see, e.g., [40]).

Example 6.2 (Descriptive EF Proximity Relationships in Planograms) A partial view of the allocation of space on Thai shop shelves based on the EF description proximity of sets of bottles is shown in Fig. 7. Let X be a set of bottles in a shelf space endowed with an EF descriptive proximity relation and A, B, C, E be subsets of X. To begin constructing a planogram based on the appearance of the bottles, we consider descriptions of the members of these sets derived from two probe functions

Fig. 7 EF-descriptive proximity-based visual display

Table 1 Partial EF-descriptive proximity-based planogram

Bottles A	Bottles B	Colour Probe φ_1	Height Probe φ_2	Proximity	Arrangement
		$\varphi_1(A) = red$, $\varphi_1(B) = white$	$\varphi_2(A) = avg.$, $\varphi_2(B) = tall$	$A \not\delta_{\Phi} B$	$\Phi(A) \neq \Phi(B) \Rightarrow$ bottles spatially separated
Bottles A	Bottles C	φ_1 values	φ_2 values	Proximity	Arrangement
		$\varphi_1(A) = red$, $\varphi_1(C) = yellow$	$\varphi_2(A) = avg.$, $\varphi_2(C) = avg.$	$A \not\delta_{\Phi} C$	$\Phi(A) \neq \Phi(C) \Rightarrow$ bottles spatially separated
Bottles B	Bottles C	φ_1 values	φ_2 values	Proximity	Arrangement
		$\varphi_1(B) = white$, $\varphi_1(C) = yellow$	$\varphi_2(B) = tall$, $\varphi_2(C) = avg.$	$B \not\delta_{\Phi} C$	$\Phi(B) \neq \Phi(C) \Rightarrow$ bottles spatially separated
Bottles A	Bottles E	φ_1 values	φ_2 values	Proximity	Arrangement
		$\varphi_1(A) = red$, $\varphi_1(E) = red$	$\varphi_2(A) = avg.$, $\varphi_2(E) = avg.$	$A \delta_{\Phi} E$	$\Phi(A) = \Phi(E) \Rightarrow$ bottles spatially close

$$\varphi_1 : X \longrightarrow \{red, yellow, white\} \text{ (\textbf{colour probe})},$$
$$\varphi_2 : X \longrightarrow \{avg., tall\} \text{ (\textbf{height probe})}.$$

For simplicity in this example, the codomain of a probe function is a set of nominal (instead of numerical) values. For example, the description of a bottle a in A is the feature vector $\Phi(a)$, defined by

$$\Phi(a) = (\varphi_1(a), \varphi_2(a)) = (red, avg.) \text{ (\textbf{feature vector describes } } \boldsymbol{a \in A}\text{)}.$$

So $\Phi(A) = \bigcup_{a \in A} \Phi(a)$. The components of the feature vectors for the set A as well as the sets B, C, E in Fig. 7 are given in the partial planogram in Table 1. The basic approach is to separate sets of bottles that are descriptively remote, i.e., when the sets do not have matching descriptions. Sets with non-matching descriptions have unequal feature vectors. So, for example,

$$\begin{aligned}
&\Phi(A) \neq \Phi(B), \text{ since}\\
(\varphi_1(A), \varphi_2(A)) &\neq (\varphi_1(B), \varphi_2(B)), \text{ i.e.}\\
(red, avg.) &\neq (white, tall) \text{ (unequal feature vectors), so that}\\
&A \not\delta_\Phi B, \text{ resulting in}\\
&A \text{ being separated from } B \text{ on the shelves in Fig. 7.}
\end{aligned}$$

This approach to constructing a planogram is easily operationalized in real-time using a scanner or robotic vision system programmed to probe feature values, compare descriptions of sets of shelf items and allocate shelf space based on the nearness or remoteness of the descriptions. □

Specific applications of descriptive proximity besides visual merchandising include: (i) proximitizing sets of physical objects [10, §3.7, p. 119], (ii) description of qubits in an approach to solving the problem of quantum entanglement [17, §3], (iii) proximal nearness of Voronoï regions in friendship networks [13], and (iv) descriptive proximity of nerve complexes recently introduced in [15] leading to the study proximal object shapes [18, 19].

7 Conclusions

In this article, the basic notion of descriptive proximity is extended and elucidated in terms of a number of different descriptive proximities viewed in the context of classical spatial proximity. Descriptive intersection is a fundamental structure underlying the descriptive proximities considered here. With the introduction of descriptive proximity, it is a straightforward next step to consider nonempty sets that are spatially remote and yet descriptively close. A paradigm for descriptive proximities is the visitor to a museum who observes the visual closeness of paintings, which can be either side-by-side or in different locations.

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