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Mathematics in Computer Science

On a Class of Central Configurations in the Planar 3n-Body Problem

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Abstract The existence of a central configuration of $2n$ bodies located on two concentric regular n-gons with the polygons which are homotetic or similar with an angle equal to $\frac{\pi}{n}$ and the masses on the same polygon, are equal, has proved by Elmabsout (C R Acad Sci 312(5):467–472, 1991). Moreover, the existence of a planar central configuration which consists of $3n$ bodies, also situated on two regular polygons, the interior n-gon with equal masses and the exterior $2n$-gon with masses on the $2n$-gon alternating, has shown by author. Following Smale (Invent Math 11:45-64, 1970), we reduce this problem to one, concerning the critical points of some effective-type potential. Using computer assisted methods of proof we show the existence of ten classes of such critical points which corresponds to ten classes of central configurations in the planar six-body problem.

Keywords N-body problem · Central configuration · Critical points · Degenerate central configuration

Mathematics Subject Classification Primary 70F10; Secondary 70F15

1 Introduction

One of the main aims concerning central configurations in the N-body problem is to determine parameters of the bodies for which such configuration exists. The well known list of classical central configurations of Euler and Lagrange [1] has been completed by Elmabsout [2] and Grebenikov [3]. They have added configurations consisting of $2n$ equal point-masses, located at the vertices of two regular, concentric n-gons. They also have proved that such configuration exists if and only if these two polygons are homothetic or differ by an angle of π/n. Recently, the existence of the bifurcations in central configurations became an important problem studied by many researchers. The authors which have dealt with bifurcations in the n-body problem are Sekiguchi [4], Lei and Santoprete [5]. They have analyzed a highly symmetrical *rosette configuration* of $(2n+1)$ point-masses; Sekiguchi has considered $2n$ point-masses, each with mass m and situated at the vertices of two different coplanar and concentric regular n-gons, whilst Lei and Santoprete have analyzed $2n$ point-masses, n particles of which with mass m_1 each are located at the vertices of a n-gon and the rest of n particles with mass m_2 lie at the vertices of another n-gon. Both

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n-gons are regular, concentric and rotated of an angle $\frac{\pi}{n}$. Finally, the mass m_0 lies at the center of masses of such configuration. The second configuration corresponds to those ones considered by Elmabsout and Grebenikov. Sekiguchi has established that for $n \geq 3$ there is a critical mass μ_c such for $\mu < \mu_c$, there exist three central configurations, and only one for $\mu \geq \mu_c$. Lei and Santoprete, in their article, have shown that, if $n \geq 3$, then there exists a degenerate central configuration. The authors describe a bifurcation scenario near this configuration.
In this paper, we deal with a new family of planar twisted configurations in N-body problem, more precisely, we study a configuration which contains $3n$ point-masses; we assume that n equal point-masses $m_1, m_2, \ldots, m_n$ are located at the vertices of a n-gon, whereas the other $2n$ point-masses with $m'_1, m'_2, \ldots, m'_{2n}$ are located at the vertices of a $2n$-gon, both polygons being concentric. The existence of such configuration has been shown numerically by Grebenikov [6] and for $n = 2$ it was established rigorously by Siluszyk [7].
The paper is organized as follows. In the Sect. 2 we recall general definitions, theorems and equations for central configurations, i.e. we recall important relations between the homographic solutions, relative equilibria and critical points in the N-body problem. In the Sect. 3 we present the main theorem. Here, we investigate the critical points of the function IU^2. We show that there are ten different critical points which correspond to ten classes of central configurations.

2 Equations of Central Configurations: Theoretical Background

The equations of the motion of N point-masses under the influence of the Newton's gravitational law can be written as (see, e.g., [1])

$$m_i \cdot \ddot{q}_i = -\frac{\partial U}{\partial q_i}, \quad i = 1, \ldots, N, \tag{2.1}$$

where $U = \sum_{1 \leq i < j \leq N} \frac{m_i\, m_j}{|q_i - q_j|}$ is the Newton potential. In order to avoid collisions we assume that the potential is defined on R^{2N} except the collision set $\Delta = \left\{q : \ q_i = q_j, \ i \neq j\right\}$. Here, $q_i \in R^2 - \Delta$ is the radius-vector of the position in the plane of the ith body. We also consider the barycentric frame of reference, so we assume that the center of mass is located at the origin, i.e. $\sum_{i=1}^{N} m_i q_i = 0$.

Definition 2.1 [1] We call $q_i(t)$, $(i = 1, 2, \ldots, N)$ a homographic solution of the N-body problem if there exist a function $s(t) > 0$ and an orthogonal matrix-function $\Omega_2(t) = \begin{pmatrix} cos\,\omega(t) & -sin\,\omega(t) \\ sin\,\omega(t) & cos\,\omega(t) \end{pmatrix}$ such that for each $i = 1, \ldots, N$ and t

$$q_i(t) = s(t) \cdot \Omega_2(t) \cdot q_i^0. \tag{2.2}$$

This means that the configuration of the bodies remains similar to itself at all times t; here q_i^0 denotes q_i at some initial instant t_0.

There are two limiting types of homographic solutions. If $\Omega_2(t)$ is the identity matrix (the configuration is dilating without rotation) then this solution is named homothetic, i.e. $q_i(t) = s(t) \cdot q_i^0$, $(i = 1, 2, \ldots, N)$, or if the configuration is rotating without dilatation, i.e. if $s(t) = 1$ for all $t \in R$. Such motion is the simplest possible motion in the plane, when the whole system of N point-masses rotates as a rigid body about its center of mass. In general, the rotation velocity ω may depend on t. The following result is taken from [1] (see also [8,9]):

Theorem 2.2 [1] *A homographic solution $q(t)$ is a relative equilibrium if and only if it is planar and the configuration rotates with constant angular velocity.*

Wintner gives also the following criterion:

Theorem 2.3 [1] *A set of N material points (q_i, m_i), $(i = 1, \ldots, N)$ forms a central configuration if and only if at each such point the following equalities hold true*

$$\frac{\partial U}{\partial q_i} = \sigma \frac{\partial I}{\partial q_i}, \ (i = 1, \ldots, N), \tag{2.3}$$

where the multiplier σ is a function that does not depend on i, and $I = \sum_{i=1}^{N} m_i |q_i|^2$ is the polar moment of inertia.

Remark 2.4 It is worth noticing that many authors take the equation (2.3) as definition of a central configuration (see, e.g., [8]).

Remark 2.5 Since I is a (first) integral of the equation (2.1), we can interpret (2.3) as the equation for a conditional extremum of the potential U under the condition $I = const.$, (say, $I = 1$).

It is more convenient to identify central configurations with some usual critical points of an appropriately chosen function. Doing that we follow [8]. More precisely, both functions U and I are homogeneous of the degrees -1 and 2, respectively. Applying Euler's formula for homogeneous functions to U and I, we have

$$\sum_{i=1}^{N} q_i \frac{\partial U}{\partial q_i} = -U,$$
$$\sum_{i=1}^{N} q_i \frac{\partial I}{\partial q_i} = 2I.$$

Multiplying both sides of (2.3) by q_i and summing, one obtains the following equality

$$-U = 2\sigma I,$$

which implies $\sigma = -\frac{U}{2I}$. If one substitutes this value of σ in the equations (2.3), one obtains that $I \frac{\partial U}{\partial q_i} = -\frac{1}{2} U \frac{\partial I}{\partial q_i}$, $(i = 1, \ldots, N)$. After multiplying both sides of the last equalities by U we obtain the following result

Theorem 2.6 [8] *The configuration of point-masses $\{(q_i, m_i) : i = 1, \ldots, N\}$ represents a central configuration if and only if $(q_1, q_2, \ldots, q_N) \in R^{2N}$ is a critical point of the function IU^2.*

In what follows, we are concerned with a class of central configurations in the six-body problem by identifying them with the corresponding critical points of the function IU^2. Such configuration consists of N-point-masses (q_i, m_i), $(i = 1, \ldots, N)$ situated at the vertices of p regular polygons. In the article [10], Elmabsout has suggested that a relative equilibrium configuration exists when the bodies are located at the vertices of p regular polygons centered at a given mass m_0, with the bodies on the same polygon having equal masses. We denote this type of configurations by (p, pn), where p denotes the number of polygons and n is the number of points on each polygon, so that the total number N of point-masses equals pn. The existence of a central configuration of the type $(2, 2n)$ has been established by Elmabsout and Grebenikov (see, e.g., [2,3]). This configuration was named by Sekiguchi, Lei and Santoprete as *symmetric rosette configuration*. Central configurations of the type (p, pn) represent a particular case of the general type (p, N), where different masses on the same polygon are allowed. For example, the configuration of the type (2, 1) means Newton's 2-body problem. For (3, 1) we obtain Euler's central configuration or the configuration as a collinear case. For $N = 4$ we get the following cases: (1, 4) as a square with four equal masses at the vertices [1], (2, 2) as a rhombus with two bodies having equal masses and located at the vertices of two segments [11], (4, 1) as the collinear case. We add here also the case (1, 3) as an equilateral triangle with mass $m_0 \neq 0$ located at its center of gravity ([3,10,12]). An interesting configuration of the type (p, N) can be (2, 8), where eight point-masses are situated at the vertices of an equilateral triangle and a pentagon, which are concentric, (see, for example [13]).

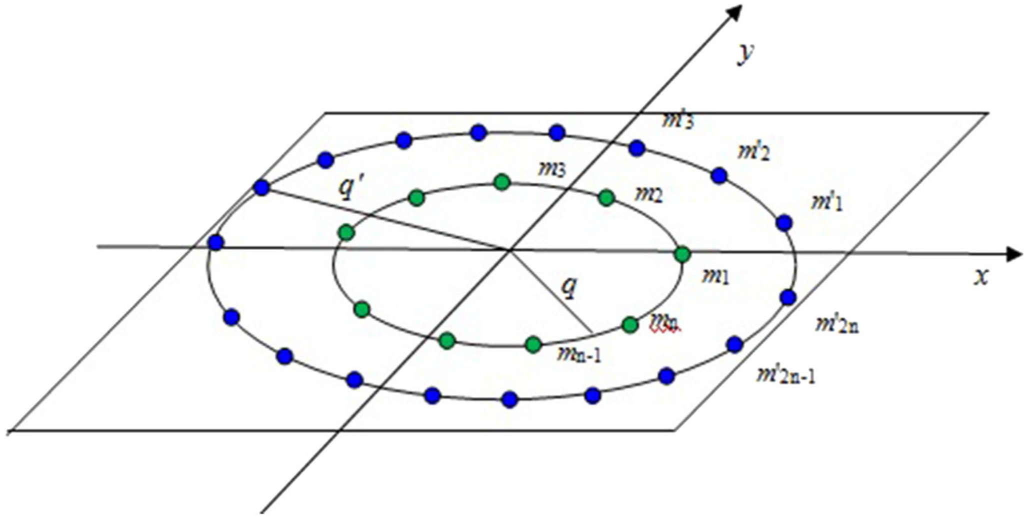

Fig. 1 Configuration $(2, 3n)$

In our paper we deal with a special class of central configurations of the type $(2, 3n)$, consisting of $3n$ bodies located at the vertices of two regular concentric polygons: the "interior" one of the radius q and with n equal point-masses, and other $2n$ bodies located at the vertices of the second regular $2n$-gon of the radius, say, $q^{'}$, with point-masses of two categories: one half, i.e., n masses, all equal to, say m_2, and another half, i.e. n masses, all equal to, say, m_3. Moreover, the point-masses on the "$2n$-gon" alternate: a point of mass m_2 is followed by a point of mass m_3, and so on (see Fig. 1). It is worth noticing [8], that if $\{(q_i, m_i),\ i = 1, 2, \ldots, N\}$ is a central configuration, then instead of studying the Eq. (2.1) we can consider the point-masses which satisfy the following condition:

$$-\frac{U}{2I} q_i = -\sum_{j=1}^{N} m_j \frac{q_i - q_j}{\left|q_i - q_j\right|^3}, \quad i = 1, \ldots, N \tag{2.4}$$

Due to symmetry of the problem, the potential U does not depend on the coordinates of the bodies, but only on the number of bodies, their masses, and the distances q and q'. More precisely, for a configuration of the type $(2, 3n)$ with any natural n and reals $m_1 > 0, q, q' > 0$, we have (see [7]) that the potential U in the planar N-body problem with a central configuration of the type $(2, 3n)$ is reducible to the following form:

$$U = \frac{n}{4} A \left(\frac{m_1^2}{q} + \frac{m_2^2}{q'} + \frac{m_3^2}{q'} \right) + \frac{n}{q} m_1 m_2 B + \frac{n}{q} m_1 m_3 C + \frac{n}{2q'} m_2 m_3 D, \tag{2.5}$$

where the coefficients A, B, C, D are given by:

$$\begin{cases} A = \displaystyle\sum_{j=1}^{n-1} \frac{1}{\left|\sin \frac{\pi j}{n}\right|}, \\ B = \displaystyle\sum_{j=1}^{n} \frac{q}{\sqrt{q^2 + q'^2 - 2qq' \cos(\frac{2\pi j}{n} + \frac{\pi}{n})}}, \\ C = \displaystyle\sum_{j=1}^{n} \frac{q}{\sqrt{q^2 + q'^2 - 2qq' \cos \frac{2\pi j}{n}}}, \\ D = \displaystyle\sum_{j=1}^{n} \frac{1}{\left|\sin(\frac{2\pi j}{n} + \frac{\pi}{2n})\right|}. \end{cases} \tag{2.6}$$

In the same paper [7] the author has established the following result:

Theorem 2.7 [7] *Given the natural n and real positive numbers $m_1, m_2, m_3, q, q' > 0$, a central configuration of type $(2, 3n)$ exists if and only if*

$$\begin{cases} m_2 = \dfrac{r^2}{(\kappa_1-\kappa_6)(\kappa_1-4r^3(\kappa_2+\kappa_4)+4r^4(\kappa_3+\kappa_5)+\kappa_6)}(-16r^5(\kappa_2-\kappa_4)\kappa_5 \\ \quad +16r^4(\kappa_2\kappa_4-\kappa_4^2+\kappa_5(\kappa_3-\kappa_5))-16r^3\kappa_4(\kappa_3-\kappa_5) \\ \quad -r(-\kappa_1^2-4\kappa_4\kappa_6+\kappa_1(4\kappa_2+\kappa_6))+4\kappa_1\kappa_3-4\kappa_5\kappa_6)m_1, \\ m_3 = \dfrac{r^2}{(\kappa_1-\kappa_6)(\kappa_1-4r^3(\kappa_2+\kappa_4)+4r^4(\kappa_3+\kappa_5)+\kappa_6)}(16r^5\kappa_3(\kappa_2-\kappa_4) \\ \quad -16r^4(\kappa 2(\kappa 2-\kappa_4)+\kappa_3(\kappa_3-\kappa_5))+16r^3\kappa_2(\kappa_3-\kappa_5) \\ \quad +r(\kappa_1(\kappa_1-4\kappa_4-\kappa_6)+4\kappa_2\kappa_6)+4(\kappa_1\kappa_5-\kappa_3\kappa_6))m_1, \end{cases} \tag{2.7}$$

where $r = \frac{q'}{q}$, $s = 1, \ldots, 6$, and

$$\begin{cases} \kappa_1 = \displaystyle\sum_{j=1}^{n} \frac{1}{\left|\sin \dfrac{\pi(k-j)}{n}\right|}, \\ \kappa_2 = \displaystyle\sum_{j=1}^{n} \frac{1}{\left(1+r^2-2r\cos\left(\frac{2\pi(j-k)}{n}+\frac{\pi}{n}\right)\right)^{\frac{3}{2}}}, \\ \kappa_3 = \displaystyle\sum_{j=1}^{n} \frac{\cos\left(\frac{2\pi(j-k)}{n}+\frac{\pi}{n}\right)}{\left(1+r^2-2r\cos\left(\frac{2\pi(j-k)}{n}+\frac{\pi}{n}\right)\right)^{\frac{3}{2}}}, \\ \kappa_4 = \displaystyle\sum_{j=1}^{n} \frac{1}{(1+r^2-2r\cos(\frac{2\pi(j-k)}{n}+\frac{2\pi}{n}))^{\frac{3}{2}}}, \\ \kappa_5 = \displaystyle\sum_{j=1}^{n} \frac{\cos\left(\frac{2\pi(j-k)}{n}+\frac{2\pi}{n}\right)}{\left(1+r^2-2r\cos\left(\frac{2\pi(j-k)}{n}+\frac{2\pi}{n}\right)\right)^{\frac{3}{2}}}, \\ \kappa_6 = \displaystyle\sum_{j=1}^{n} \frac{1}{\left|\sin\left(\frac{2\pi(j-k)}{2n}+\frac{\pi}{2n}\right)\right|}. \end{cases} \tag{2.8}$$

Easily seen, that the sums on the right hand side do not depend on k. The masses m_2, m_3 in the Theorem 2.4 were obtained from the Eq. (2.4) where, instead of the expression $\frac{U}{2I}$ we have taken the value ω^2, with ω as the angular velocity of the rotation around the Oz axis of the orthonormal frame $Oxyz$. We describe the motion of N point-masses in these coordinates by the differential equations:

$$\begin{cases} \dfrac{d^2x_i}{dt^2} - 2\omega\dfrac{dy_i}{dt} = \omega^2 x_i - \displaystyle\sum_{j=1, j\neq i}^{N} m_j \frac{x_i - x_j}{|q_i - q_j|^3}, \\ \dfrac{d^2y_i}{dt^2} + 2\omega\dfrac{dx_i}{dt} = \omega^2 y_i - \displaystyle\sum_{j=1, j\neq i}^{N} m_j \frac{y_i - y_j}{|q_i - q_j|^3}, \\ \qquad \dfrac{d^2z_i}{dt^2} = -\displaystyle\sum_{j=1, j\neq i}^{N} m_j \frac{z_i - z_j}{|q_i - q_j|^3}. \end{cases} \tag{2.9}$$

Denoting $q_i = (x_i, y_i)$ $(i = 1, 2, ...N)$, the Eq. (2.4) of a central configuration can be rewritten as follows:

$$\omega^2 q_i = \sum_{j=1, j\neq i}^{N} m_j \frac{q_i - q_j}{\left|q_i - q_j\right|^3}. \tag{2.10}$$

Taking into the account the relations (2.7) and (2.8), the following equation of the central configuration we are looking for:

$$\omega^2 = \frac{m_1}{r\, q^3}(r\,\kappa_2 - \kappa_3) + \frac{m_2}{4r^3q^3}\kappa_1 + \frac{m_3}{4r^3q^3}\kappa_6. \tag{2.11}$$

3 The Case of Six-Body Problem

In this section we are looking for a configuration of six point-masses, where two of them with equal masses m_1 are situated at the ends of a segment having length $2q$, while other two pairs of equal masses m_2 and m_3, respectively, are located at the vertices of a square whose side is $q'\sqrt{2}$. Moreover, the vertices of this square are on the axis of symmetry of the segment.

Below we give necessary and sufficient conditions for such a configuration to exist in terms of critical points of the function IU^2. In this case the Newton potential from (2.5) and the polar moment of inertia for all values $q, r > 0, r \neq 1$ and $m_1 > 0$ take the forms:

$$U = \frac{m_1^2}{q}\left(\left(\frac{1}{2} + \frac{\mu^2+\nu^2}{2r}\right) + 4\frac{\mu}{\sqrt{1+r^2}} + 2\nu\left(\frac{1}{\sqrt{(r-1)^2}} + \frac{1}{\sqrt{(r+1)^2}}\right) + 2\sqrt{2}\frac{\mu\,\nu}{r}\right), \tag{3.1}$$

and

$$I = 2m_1q^2(1 + r^2(\mu+\nu)), \tag{3.2}$$

respectively, where $\mu = \frac{m_2}{m_1}$ and $\nu = \frac{m_3}{m_1}$. Both function U and I depend on variables m_1, μ, ν, r, q. U^2 gets the expression $U^2 = \frac{m_1^4}{q^2}(*)^2$, where $(*)$ is remaining part of the formulas (3.1), moreover the fact that $\frac{\partial I}{\partial \mu} \neq 0$, $\frac{\partial I}{\partial \nu} \neq 0$, $\forall r > 0, r \neq 1$ means that $I\,U^2$ is the function only of r. The function U has a pole at $r = 1$, so we divide the domain $r > 0$ into two intervals and then we can obtain the following theorem

Theorem 3.1 *Consider the configuration, consisting of six mass-points, two of which of the common mass m_1 being located at the ends of the segment of the length $2q$, and other two pairs of equal masses m_2 and m_3, respectively, being located at the vertices of a square whose side is $q'\sqrt{2}$ (the segment and the square have the same axis of symmetry, see Fig. 2). There exist ten classes of central configurations, corresponding to the following values of the parameter r:*

1. *There are six distinct central configurations corresponding, to the following values of the parameter r from the interval (0;1)*
   $r_1 = 0.072255500985496 \pm \epsilon_1$, $r_2 = 0.187400673520093 \pm \epsilon_2$, $r_3 = 0.425321919060473 \pm \epsilon_3$, $r_4 = 0.455866590203477 \pm \epsilon_4$, $r_5 = 0.460152407511265 \pm \epsilon_5$, $r_6 = 0.580685227755403 \pm \epsilon_6$,

2. *There exist four central configurations for* $r_7 = 3.514134674303432 \pm \epsilon_7$,
   $r_8 = 4.077744508561068 \pm \epsilon_8$, $r_9 = 4.949764128835926 \pm \epsilon_9$, $r_{10} = 5.153295885139827 \pm \epsilon_{10}$.

Here, $\epsilon_1, \ldots, \epsilon_{10} \in \left\{10^{-16}, 9\cdot 10^{-16}\right\}$.

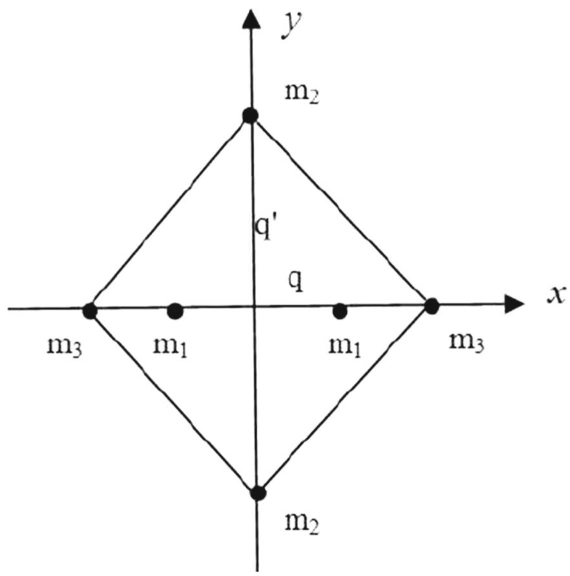

Fig. 2 A configuration of the type (2, 6)

Proof Consider the function IU^2. Denote $g : R^+ - \{1\} \longrightarrow R$, $g(r) \equiv IU^2(r)$, where

$$g(r) = \frac{1}{2}m_1^5(1 + r^2(\mu + \nu)) \left(1 + 4\left(\frac{2\mu}{\sqrt{r^2+1}} + \nu\left(\frac{1}{r+1} + \frac{1}{\sqrt{(r-1)^2}}\right)\right) + \frac{\mu^2 + 4\sqrt{2}\mu\nu + \nu^2}{r}\right)^2. \tag{3.3}$$

Introducing the masses $\mu = \mu(r)$ and $\nu = \nu(r)$ (see Theorem [7]), the equation for the zeros of the gradient of the function IU^2 can be reduced to the zeros of the derivative g'(r) , where

$$\begin{aligned} g'(r) = \frac{1}{2}m_1^5 &\left(1 + 4\left(\frac{2\mu}{\sqrt{r^2+1}} + \nu\left(\frac{1}{r+1} + \frac{1}{\sqrt{(r-1)^2}}\right)\right) + \frac{\mu^2 + 4\sqrt{2}\mu\nu + \nu^2}{r}\right) \\ &\times \left((2r(\mu+\nu) + r^2(\mu'_r + \nu'_r))\left(1 + 4\left(\frac{2\mu}{\sqrt{r^2+1}} + \nu\left(\frac{1}{r+1} + \frac{1}{\sqrt{(r-1)^2}}\right)\right)\right.\right. \\ &+ \left.\frac{\mu^2 + 4\sqrt{2}\mu\nu + \nu^2}{r}\right) + 2(1 + r^2(\mu+\nu))\left(\frac{8(\mu'_r(r^2+1) - \mu r)}{(r^2+1)\sqrt{r^2+1}} + 4\nu'_r\left(\frac{1}{\sqrt{(r-1)^2}} + \frac{1}{r+1}\right)\right. \\ &+ \left.\left. 4\nu\left(-\frac{2}{r-1} - \frac{1}{(r+1)^2}\right) + \frac{r(2\mu\mu'_r + 4\sqrt{2}(\mu'_r\nu + \mu\nu'_r) + 2\nu\nu'_r) - (\mu^2 + 4\sqrt{2}\mu\nu + \nu^2)}{r^2}\right)\right). \end{aligned} \tag{3.4}$$

Mathematica [14] is a tool that can help us to calculate a derivative of the function IU^2 moreover it helps to solve non algebraic equations such as $(IU^2(r))'_r = 0$. Presentation of the function $g'(r)$ we can see by using powerful facilities of *Mathematica* as the built-in functions **Manipulate** and **Animate**. Both these functions make possibility to vary the parameters r and m_1, and to observe the behavior of the function $g'(r)$. Thus we can understand better the simulated system of parameters. This can be done with the function **Manipulate**, which allows for an interactive manipulation of the value of m_1 (see, Fig. 3), i.e., Out[12]=

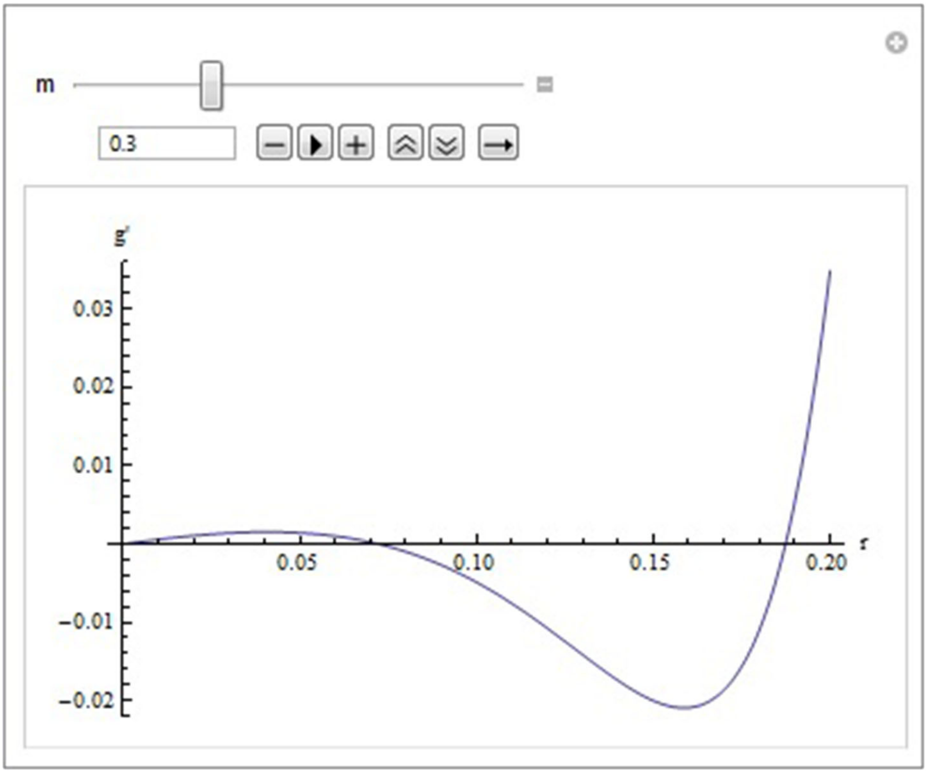

Fig. 3 The functions $g^{'}(r)$ for $r \in (0, 0.2)$ and $m_1 = 0.3$

```
Manipulate[Plot[g', {r, 0.00001, 0.2}, AxesLabel → {"r", "g'"}, AxesOrigin → {0, 0}], {m, 0.0000001, 1}]
```

In the procedure **Manipulate**, instead of the mass m_1 we take m.

I. Assume that $r \in (0, 1)$. Then $lim_{r\to 0^+} g'(r) = 0$, $lim_{r\to 1^-} g'(r) = \infty$. Numerical calculations show, that for $m_1 > 0$ and $0 < r < 0.5$ the function $g'(r)$ has minimum and maximum (see Fig. 4a, c, e and 4b, d, f), whereas for $0.5 < r < 1$ the function $g'(r)$ is increasing. At this moment, there doesn't exist the analytical methods to solve the equation $g'(r) = 0$, where $g'(r)$ is complicated irregular function. These solutions can be obtained only by using numerical methods. In the *Mathematica* we can apply the procedure **FindRoot**, which searches a numerical root of the function, starting from the point $r = r_0$, i.e., where after checking precision of our calculations we have

```
In[32]:= FindRoot[∂r g == 0, {r, 0.07}, WorkingPrecision → 15]

Out[32]= {r → 0.0722555009854966}

In[33]:= ∂r g /. {r → 0.0722555009854966`}

Out[33]= 1.87315 × 10^-35
```

Here, we give a set of solution of the equation $g'(r) = 0$, i.e.,

$$0.0722555009854966 \pm \epsilon_1, 0.187400673520093 \pm \epsilon_2, 0.425321919060473 \pm \epsilon_3,$$
$$0.455866590203477 \pm \epsilon_4, 0.460152407511265 \pm \epsilon_5, 0.580685227755403 \pm \epsilon_6.$$

The number of zeros of the function $g'(r)$ do not depend on m_1, so the same results we have obtained for values of the parameter $m_1 > 0$.

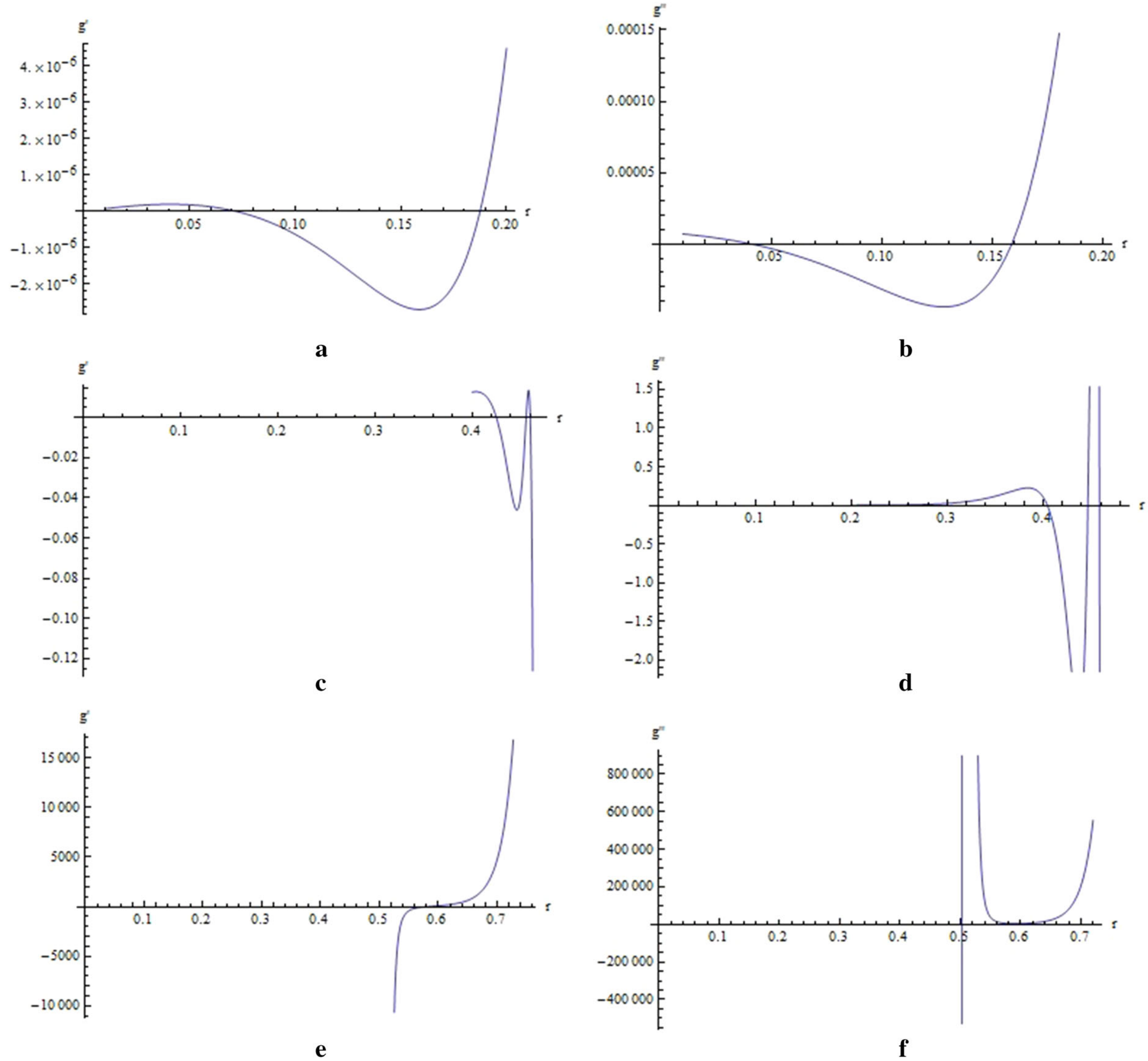

Fig. 4 The functions $g^{'}(r)$ and $g^{''}(r)$ for $r \in (0, 1)$ and $m_1 = 0.05$

II. Assume that $r \in (1, \infty)$. We claim that there exist at most four critical points of the function $g(r)$ which correspond to four critical points of the function IU^2. Easily seen that $lim_{r \to 1^+} g'(r) = \infty$, and $lim_{r \to \infty} g'(r) = \infty$. Independently of m_1 we can see (Fig. 5g, h) that the number of critical points is equal to 4. Numerical analysis of the Fig. 5 gives two local minima and one local maximum, note them $r^*_{min} = 3.714499977675438 \pm \delta_1$, $r^{**}_{min} = 5.06001856651343 \pm \delta_2$ and $r^{\#}_{max} = 4.456337830721095 \pm \delta_3$, respectively, here $\delta_1, \delta_2, \delta_3 \in \left\{10^{-16}, 9 \cdot 10^{-16}\right\}$. Studying the derivative of the function $g'(r)$ it follows that for $1 < r < r^*_{min}$, $g'(r)$ is monotonically decreasing, whereas if $r^*_{min} < r < r^{\#}_{max}$, $g'(r)$ is a monotonically increasing and $g'(r^*_{min}) < 0$, whilst $g'(r)$ is monotonically decreasing on the interval $r^{\#}_{max} < r < r^{**}_{min}$, and increasing for $r^{**}_{min} < r < +\infty$. Consequently, the function $g'(r)$ has four zeros in $(1, \infty)$ i.e.,

$$\{3.514134674303432 \pm \epsilon_7, 4.077744508561068 \pm \epsilon_8, 4.949764128835926 \pm \epsilon_9, 5.153295885139827 \pm \epsilon_{10}\}.$$

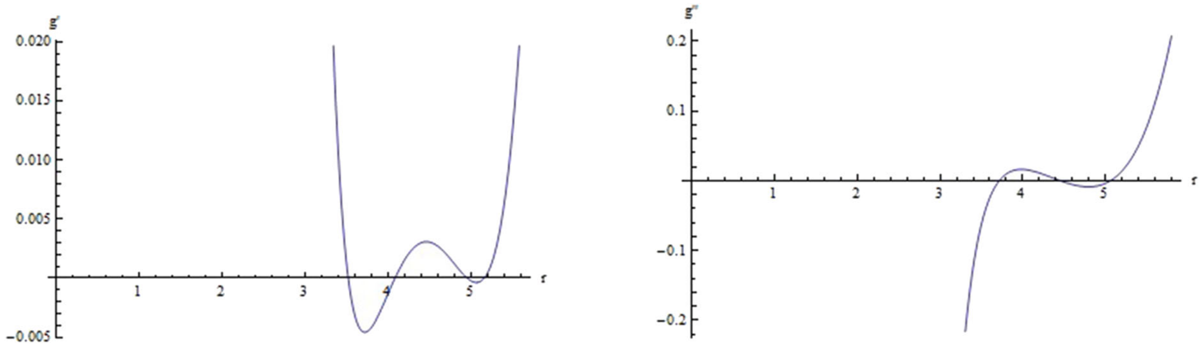

Fig. 5 The functions $g'(r)$ and $g''(r)$ for $r \in (1, \infty)$ and $m = 0.05$

Table 1 The values of r, $g'(r)$ and $g''(r)$ for $m_1 = 0, 0001$

r	$g'(r)$	$g''(r)$
$0.0722555009854966 \pm \epsilon_1$	$1.87315 \cdot 10^{-35}$	$-4.45959 \cdot 10^{-19}$
$0.187400673520093 \pm \epsilon_2$	$-1.16033 \cdot 10^{-32}$	$7.72416 \cdot 10^{-18}$
$0.425321919060473 \pm \epsilon_3$	$7.88861 \cdot 10^{-30}$	$-5.11395 \cdot 10^{-14}$
$0.455866590203477 \pm \epsilon_4$	$-3.51536 \cdot 10^{-29}$	$2.6649 \cdot 10^{-13}$
$0.460152407511265 \pm \epsilon_5$	$9.82308 \cdot 10^{-29}$	$-5.99538 \cdot 10^{-13}$
$0.580685227755403 \pm \epsilon_6$	$-4.43479 \cdot 10^{-25}$	$1.65081 \cdot 10^{-10}$
$3.514134674303432 \pm \epsilon_7$	$-5.51073 \cdot 10^{-26}$	$-1.77279 \cdot 10^{-15}$
$4.077744508561068 \pm \epsilon_8$	$7.51883 \cdot 10^{-31}$	$4.83647 \cdot 10^{-16}$
$4.949764128835926 \pm \epsilon_9$	$-5.48505 \cdot 10^{-31}$	$-1.91338 \cdot 10^{-16}$
$5.153295885139827 \pm \epsilon_{10}$	$-2.19532 \cdot 10^{-28}$	$2.68523 \cdot 10^{-16}$

We call a relative equilubrium degenerate, if the corresponding critical point of the function $I\,U^2$ is degenerate. From numerical analysis we can put in evidence some degenerated critical points which are degenerated (see, Figs. 4, 5 and the Table 1).

4 Conclusion

Presented here the relative equilibrium of the type $(2, 3n)$ belongs to the general model of the type $(2, N)$ of the N-body problem in the barycentric system. We reduce the problem of finding central configurations to the corresponding problem concerning the critical points of an appropriately chosen function. In the case of the six-body problem we prove the existence of ten central configurations, depending on the proportion between the radii of the concentric regular polygons, on which the central configuration is located.

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