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# Point-Free Geometries: Proximities and Quasi-Metrics

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**Abstract** In the Euclidean geometry points are the primitive entities. Point-based spatial construction is dominant but apparently, from a constructive point of view and a naïve knowledge of space, the region-based spatial theory is more quoted, as recent and past literature strongly suggest. The point-free geometry refers directly to sets, the *spatial regions*, and *relations between regions* rather than referring to points and sets of points. One of the approaches to point-free geometry proposes as primitives the concepts of region and quasi-metric, a non-symmetric distance between regions, yielding a natural notion of diameter of a region that, under suitable conditions, makes it possible to reconstruct the canonical model. The intended canonical model is the hyperspace of the non-empty regularly closed subsets of a metric space equipped with the Hausdorff excess. The canonical model can be enriched by adding more qualitative structure involving a distinguished family of *bounded regions* and a group of *similitudes* preserving bounded regions, so producing a metric geometry in which shape is relevant. The main purpose of this article is to highlight the role of nearness and emphasize the proximity aspects taking part in the construction by quasi-metrics of point-free geometries.

**Keywords** Nearness · Quasi-metric · Point-free geometry · Qualitative structure · Region · Similitudes

**Mathematics Subject Classification (2010)** Primary 54B20; Secondary 54E05

## 1 Introduction

The main purpose of this article is to highlight the role of nearness and emphasize the proximity aspects taking part in the construction by quasi-metrics of point-free metric geometries, not only the Euclidean one but all metric geometries in which the notion of betweenness, metric convexity, external convexity and other basic geometric properties make sense [3]. Euclidean geometry has points as primitive entities. Point-based spatial construction is dominant but apparently, from a constructive point of view and a naïve knowledge of space, the region-based spatial theory is more quoted, as recent and past literature strongly suggest. The point-free geometry refers directly to sets, the *spatial regions*, and *relations between regions* rather than referring to points and sets of points. At the beginning [5,15,16] a region-based theory of geometry of space alternative to Euclidean geometry, which is point-based,

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corresponded more to a logical-philosophical exigency rather than to a mathematical query. But then region-based theories of space have been revealed useful in applications [1,2,4,17], especially in Computer Science and, in particular, in qualitative spatial representation and reasoning, QSTR, [10]. In common life, a point is perceived as the result of an abstraction process, while regions appear as more realistic entities rooted in the empirical experience. We have a better knowledge of a *small* region than a point and also a basic natural intuition of nearness between regions and adjacency between sets of regions. In Whitehedian geometry the primitives are the *regions* and the *connection relation* between regions, that is, the relation between two regions which either overlap, a mereological concept, or at least share a common boundary point, a topological concept.

What is the most natural world in which the regions can live? Hyperspaces of metric spaces are natural frameworks in which regions can be thought of as objects. This suggests to go back to possible distances between sets in metric spaces. And the Hausdorff excess, the prototype of non-symmetric distances called *quasi-metrics* appears as the most appropriate. The Hausdorff excess is a half of the celebrated Hausdorff metric which reveals itself as very useful in many branches of Mathematics and appears as the most natural one in measuring of natural world phenomena's changes and is also suitable to decribe reciprocal positions of regions in the space. A canonical model of space of regions is the hyperspace of the nonempty regular closed, or, equally well, regular open subsets of a metric space equipped with the Hausdorff excess. So, it seems natural to substitue the Hausdorff excess with a quasi-metric.

In [6], the authors proposed a point-free approach to geometry assuming as only primitives the concepts of region and quasi-metric. A quasi-metric is a non-symmetric distance and in real life non-symmetric distances are very common.

Quasi-metric space of regions is a framework including all known, recent and classical, models giving a new formalization of region-based geometric theory. We hope it can be a computationally efficient system.

Our target is a representation theorem for an abstract quasi-metric of regions as hyperspace of nonempty regular closed, or, equally well, regular open subsets of a metric space carrying the Hausdorff excess or, in other words, as canonical model. The final goal is to collect a system of conditions under which an abstract space of regions can be seen as a canonical model.

Essentially, we concentrate on geometric aspects. For that, we are interested in spaces of regions, when they *admit points* , *enough points* and *the regions, entities of same dimension, are fully determined by their own points.* Of course, we construct the notion of *point* by regions and state the relation *a point belongs to a region.* A quasi-metric of regions induces naturally a parthood relation between regions, that in turn yields a natural notion of diameter of a region. Nearness is in the core of our construction. It comes in consideration when defining point-representing sequences, that are special sequences whose elements become *smaller* and *smaller* and *nearer* and *nearer.* Here nearness is non-symmetric. In a first tentative, we might assume a point-representing sequence as a point, but this choice is not completely satisfactory. For instance, two vanishing decreasing sequences $\{x_n\}$ and $\{y_n\}$ of Euclidean spheres each done by internally tangent spheres at a same common point but with each $x_n$ externally tangent to each $y_m$ should codify two distinct *ideal points.* To exclude this kind of double visualization of a *same* point, inspired by the classical Cauchy construction of metric completion, we proceed to an identification by adjacency. There are in literature [9] point-free constructions having as primitives the notion of region, parthood relation between regions and diameter. But, different quasi-metrics, such as the Euclidean excess and the Manhattan excess, [11], give the same parthood, the same diameter function, and, being bi-Lipschitzian equivalent, the same nearness, consequentely the same point-space but very different geometries.

To achieve our representation theorem, we proceed gradually justifying by counterexamples at any step our constraints. To this end, we characterize the interior points of a point-region by introducing a strong parthood relation between regions that is finally recast as the strong inclusion, the dual of Efremovič proximity, associated with the metric of point-space.

When the point-space has been constructed, then a metric geometry in the sense of Blumenthal [3] can start. For instance, we can consider loci of lower dimension such as lines, planes and so on. Of course, geometric definitions and properties in a metric geometry so generated should be expressed in terms of properties of the generating quasi-metric. Blumenthal gave metric characterizations of the Euclidean geometry but also characterizations of the hyperbolic geometry and further of the elliptic geometry as well. The hyperspace of a metric space can be metrized, as well-known, by the Hausdorff metric that is the symmetrization of the Haudorff excess. But, unfortunately, only little work has been done on Hausdorff metric geometry of the hyperspace.

In the end, we quote as an example the Tarski Geometry of Solids in three dimensional Euclidean space. In our opinion, this model is not only interesting in itself but it is suggestive of other models, for example those deriving from Minkowski [11] and Chebyshev metric. Finally, by using groups of general similitudes preserving shape or other relevant features like colours or orientation, we introduce the notion of *featured-point.*

## 2 Preliminaries

- *Background on the Hausdorff excess*

What is the most natural world in which the regions can live? The answer is immediate and simple. Hyperspaces are natural frameworks in which regions can be thought of as objects. This suggests going back to possible distances between sets in metric spaces.

Let $(X, d)$ be a metric space. At a first glance, for any two nonempty subsets $a, b$ of $X$ the distance $d$ between $a$ and $b$ defined as:

$$d(a, b) := inf\ \{d(A, B) : A \in a, B \in b\}$$

and the *gap* $D$ defined as:

$$D(a, b) := sup\{d(A, B) : A \in a, B \in b\}$$

appear as natural but they both have serious liabilities. On one side, the $d-$distance between distinct sets can be zero and further does not satisfy the triangle inequality. On the other side, the gap of a set from itself is just the diameter of the same set. Because of that, at the same time in which region-based theory of Euclidean geometry appeared in literature [15,16], F. Hausdorff introduced a new distance between sets the *excess of a over b,* as just the same word suggests, as:

$$e_d(a, b) := sup\ \{d(A, b) : A \in a\},$$

where $d(A, b) := inf\ \{d(A, B) : B \in b\}$ is the usual distance between points and sets. The excess $e_d$ is located in between $d$ and $D$ :

$$d(a, b) \leq\ e_d(a, b) \leq\ D(a, b) \quad \text{for each } \ a, b \subseteq X.$$

Denote as $S(P, \epsilon) = \{Q : d(P, Q) < \epsilon\}$ the open sphere with center $P$ and radius $\epsilon$, and as $S(a, \epsilon) = \cup\{S(P, \epsilon) : P \in a\}$ the $\epsilon-$enlargement of $a$, union of all spheres having their center in $a$ and radius less than $\epsilon$, Fig. 1a.

A basic observation: if $e_d(a, b) \leq \epsilon$, then $a \subseteq S(b, \epsilon)$ and if $a \subseteq S(b, \epsilon)$, then $e_d(a, b) \leq \epsilon$, Figs. 1b, 1c.

Any metric structure $(X, d)$ yields a rich flow of many different and very interesting notions close each other. Associated with a metric $d$ there is the Efremovič proximity over the powerset of $X$ defined as:

$a$ is near $b$ if and only if $d(a, b) = 0$.

That can be dually recast as a strong inclusion, a binary relation on the hyperset of $X$:

$a \ll_d b$ if and only if $a$ is far from $X \backslash b$

or, equivalently, $a \ll_d b$, if and only if there is a positive real number $\epsilon$ such that the $\epsilon-$enlargement of $a$ is contained in $b$ [7,13], Fig. 1d. Accordingly, strong inclusions have been named in literature also as *non tangential inclusions* or *well inside inclusions.*

The Hausdorff excess $e_d$ is a non-symmetric distance satisfying the triangle inequality. Further, $e_d(a, b)$ is zero if and only if $a \subseteq Cl(b)$, where $Cl(b)$, as usual, denotes the closure of $b$. So, to get zero-self distance property, we are forced to limit the action of $e_d$ to pairs of closed subsets or equally well to pairs of regular open subsets. The Hausdorff excess is the prototype of non-symmetric distances which are called quasi-metrics.

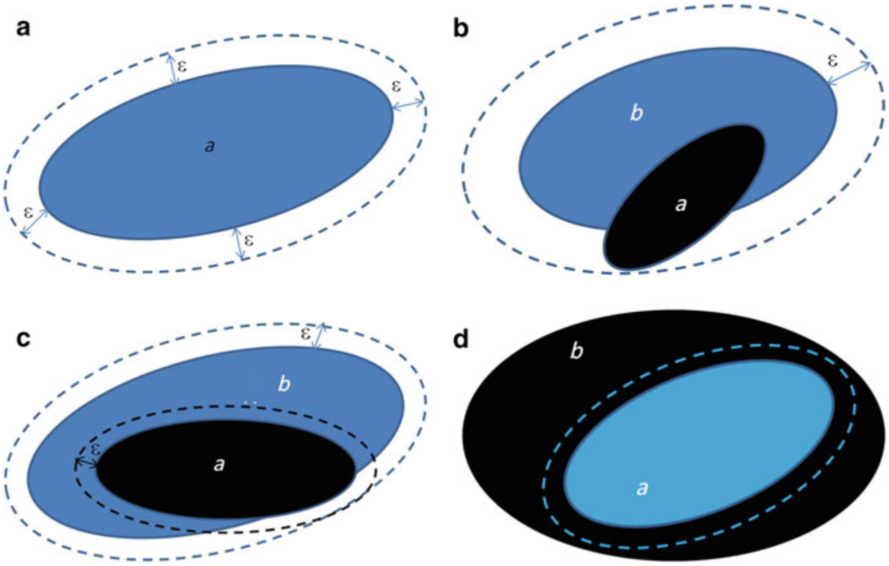


**Fig. 1** ε-enlargement of $a$ **(a)**, $e_d(a, b) < ε$ **(b)**, $e_d(a, b) < ε$, $e_d(b, a) < ε$ **(c)**, $a \ll_d b$ **(d)**

- *Interrelations*

We give now a list of interrelations among a metric $d$, the Hausdorff excess $e_d$ and the distance between points and sets. Whenever $P$, $Q$ are points and $x$, $y$ are nonempty subsets, then the following relations holds:

$e_d(\{P\}, \{Q\}) = d(P, Q)$; $e_d(\{P\}, x) = d(P, x)$; $e_d(\{x\}, P) = sup\{d(Q, P) : Q \in x\}$.

$e_d(x, y) \leq d(P, Q)$, $P \in x$, $Q \in y$; $\lim_{n \to +\infty} e_d\left(S_d\left(P, \frac{1}{n}\right), S_d\left(Q, \frac{1}{n}\right)\right) = d(P, Q)$.

Further, if $P$, $Q$ belong to $x$, $y$ respectively, then $max\ \{e_d(x, y), e_d(y, x)\} \leq d(P, Q)$.

## 3 Quasi-Metrics

A *quasi*-metric $q$ on a nonempty set $X$ is a function operating on the pairs of elements of $X$ towards $R^+$, the non negative real numbers, satisfying the following conditions:

$q_1)$ $q(x, y) = q(y, x) = 0$ if and only if $x = y$, *zero-self distance* plus *antisymmetry*
$q_2)$ $q(x, y) \leq q(x, z) + q(z, y)$, for each $x, y, z \in X$, *triangle inequality*.

If $q$ is a quasi-metric on $X$, then the function $q^{-1}$ on $X \times X$ defined by $q^{-1}(x, y) = q(y, x)$ for all $x, y \in X$ is a quasi-metric as well, called the *conjugate of* $q$, and the function $q^s$ defined on $X \times X$ as $q^s(x, y) = max\{q(x, y), q(y, x)\}$ for all $x, y \in X$ is a metric on $X$ called the *symmetrization of* $d$. The symmetrization of a metric excess is the Hausdorff metric.

Each quasi-metric $q$ on $X$ induces a $T_1$ topology $\tau(q)$ on $X$ whose a base is the family of $q$*−spheres* : $S_q(x, r) = \{y \in X : q(x, y) < r\}$, where $x \in X$ and $r$ is a positive real number.

- *Quasi-proximity*

Any quasi-metric determines a non-symmetric nearness.

Let $(R, q)$ be a quasi-metric space. Jointly with $q$, a quasi-proximity, a binary relation over the powerset $exp\ R$, can be given as follows, [12].

Let $R_1, R_2 \subseteq R$. Then, $R_1$ is *near* to $R_2$ provided that, for each positive real number $\epsilon$, there is $x \in R_1$ and $y \in R_2$ such that $q(x, y) < \epsilon$. It happens that:

$p_1$) $R$ is far from $\emptyset$ and $\emptyset$ is far from $R$.
$p_2$) $R_1$ is near to $R_2 \cup R_3$ iff $R_1$ is near to $R_2$ or $R_3$ and $R_1 \cup R_2$ is near to $R_3$ iff $R_1$ is near to $R_3$ or $R_2$ is near to $R_3$.
$p_3$) If $R_1$ is far from $R_2$, then there is $R_3$ such that $R_1$ is far from $R_3$ and $R_2$ is far from $R \backslash R_3$.

- *Parthood*

Any quasi-metric generates naturally a parthood.

Let $X$ be a nonempty set. A *parthood* $P$ on $X$ is a binary relation over "some" hyperset of $X$ satisfying the following axioms:

$P_1$) $\forall x[P(x,x)]$ *reflexivity*
$P_2$) $\forall x, y[(P(x,y) \wedge P(y,x))] \rightarrow x = y$ *antisymmetry*
$P_3$) $\forall x, y[(P(x,y) \wedge P(y,z)) \rightarrow P(x,z)]$ *transitivity*
$P_4$) $PP(x,y) \equiv_{def} P(x,y) \wedge \neg P(y,x)$ *proper parthood.*

Any parthood relation yields as subsequent binary relation the *overlap relation* $O$ defined as: $O(x,y) \equiv_{def} \exists z(P(z,x) \wedge P(z,y))$.

In mereology, [10], the following properties are required to a parthood:

$\forall x, y[\neg P(y,x) \rightarrow \exists z(P(z,y) \wedge \neg O(z,x))]\, strong\, supplementation$

The strong supplementation axiom makes the parthood extensional:

$\forall x, y[\forall z(O(x,z) \longleftrightarrow (O(z,y)) \rightarrow x = y]\, extensionality.$

Furthermore, it is required the existence of both the mereological sum and intersection conditional on overlap:

$\forall x, y[\exists z \forall u((O(u,z) \longleftrightarrow O(u,x) \vee O(u,y))]$ (sum $z = x + y$)
$\forall x, y[O(x,y) \rightarrow \exists z \forall u(P(u,z) \longleftrightarrow P(u,x) \wedge P(u,y))]$ $(intersection\ z = x \cdot y)$.

Any quasi-metric $q$ gets a parthood relation $P_q$, that we denote $\leq_q$ as follows:

$x \leq_q y$ if and only if $q(x,y) = 0$

and in turn the overlap relation:

$x$ and $y$ overlap if and only if there exists a region $z$ such that $z \leq_q x$ and $z \leq_q y$.

Any quasi-metric is monotone increasing in its first component and monotone decreasing in the second one :

$\forall x, x', y \in X,\ x \leq x' \Rightarrow q(x,y) \leq q(x',y)$ and $x \leq x' \Rightarrow q(y,x) \geq q(y,x')$.

The parthood naturally associated with the Hausdorff excess on the hyperspace of all nonempty closed subsets and also on the hyperspace of regular open subsets of a given metric space is the usual inclusion. But, when limited to bounded regularly closed subsets, the inclusion satisfies some more interesting properties as the strong supplementation property and the mereological sum and mereological intersection of two regions both make sense. The former as their usual union and the latter, when they overlap, as the closure of the interior of their usual intersection.

- *Diameter*

Let $(R, q)$ a quasi-metric space. A quasi-metric, plus the natural parthood associated with it, gives the notion of diameter in the following natural way:

$q(x) := sup\ \{q(x', x'') :\ x' \leq_q x,\ x'' \leq_q x\}$

or equivalently:

$q(x) := \{\, q(x, x') :\ x' \leq_q x\}$.

The diameter function is, as expected, a monotone increasing function. "How bad diameter can be" is illustrated in [14].

In the metric space two diameter functions emerge : the usual one and that deriving from the Haudorff excess. It happens that:

$e_d(x) \leq diam(x).$

Moreover, the two diameter functions match when the hyperspace is $BCL(X)$, the hyperspace of all nonempty closed and bounded subsets, or $RBCL(X)$, the hyperspace of all nonempty regular closed and bounded subsets or, equally well, $RO(X)$, the hyperspace of all regular open and bounded subsets.

## 4 Points

Our target is to construct a *space of regions* admitting *points, enough points* and with the *regions fully determined by their own points.* For that, we introduce the notion of *point* and the relation *a point belongs to a region.*

A point is the achievement of an inductive abstraction process. In an approximation play, which sequences might be the natural candidates? Of course, those whose elements become *smaller* and *smaller* and also *nearer* and *nearer*. This basic observation suggests to introduce the natural notion of a point-representing sequence.

Let $(R, q)$ be a quasi-metric space. A sequence $\{x_n\}$ in $R$ is a *point-representing sequence* when:

1) *For each positive real number $\epsilon$, there exists a positive integer $\nu$ such that $q(x_n, x_m) < \epsilon$ for each $n, m > \nu$ or, equivalently, $\lim_{n,m\to+\infty} q(x_n, x_m) = 0$ ($q^s-$Cauchy).*
2) *The diameter sequence $\{q(x_n)\}$ tends to zero ( vanishing ).*

In a first attempt, we might assume a point-representing sequence as a point, but this choice should reveal not completely satisfactory. For instance, two vanishing decreasing sequences $\{x_n\}$ and $\{y_n\}$ of Euclidean spheres each done by internally tangent spheres at a same common point but with each $x_n$ externally tangent to each $y_m$ should determine two distinct *ideal points*. The case forces us to proceed to an identification by using the natural nearness associated with a quasi-metric.

We say that a point-representing sequence $\{x_n\}$ is *adjacent* to a point-representing sequence $\{y_n\}$ if and only if the underlying set of $\{x_n\}$ is $q-$near to the underlying set of $\{y_n\}$ or, in other words:

$\{x_n\}$ is adjacent to $\{y_n\}$ if and only if $\lim_{n\to+\infty} q(x_n, y_n) = 0.$

Adjacency relation is reflexive and transitive but not symmetrical, [6]. It appears natural to think that two point-representing sequences only when are adjacent each other then they can represent a same point.

We will say that a quasi-metric space $(R, q)$ is a *quasi-metric space of regions*, or, briefly, a *space of regions* if the quasi-metric $q$ satisfies the following:

**Symmetry gap axiom:**

$|q(x, y) - q(y, x)| \leq \max\{q(x), q(y)\}, \quad \text{for all } x, y.$

The symmetry gap axiom guarantees that $q$ is approximately symmetric when evaluated on regions of *small size*. In some sense the size, when small, forces symmetry. Accordingly, for a space of regions the adjacency between point-representing sequences is symmetrical, [6]. This takes naturally to the notion of point. *A* **point** *$P$ in a space of regions $(R, q)$ is a class of adjacent point-representing sequences of $R$.*

Unfortunately, it can happen that a quasi-metric has no bounded regions as, for instance, the standard quasi-metric on the reals or has no vanishing sequences as, for instance, the discrete quasi-metric on the reals, see next examples. Thus, it appears unavoidable to require the following:

**Point-existence axiom:** *There exist point-representing sequences.*

We denote as $P_t(R)$, or simply $P_t$, the set of all points of $(R, q)$ and refer to it as the *full space of points* of $(R, q)$.

- *Point-distance*

Let $(R, q)$ be a space of regions and $\{x_n\}$, $\{y_n\}$ point-representing sequences. Since they are $q^s$−Cauchy, then $\lim_{n+\infty} q(x_n, y_n)$ and $\lim_{n+\infty} q(y_n, x_n)$ both do exist. But, they are vanishing too. So, thanks to the symmetry gap axiom $\lim_{n\to\infty} q(x_n, y_n)$ does coincide with $\lim_{n\to\infty} q(y_n, x_n)$. The previous observations yield a natural definition of *point-distance* between points.

Let $(R, q)$ be a space of regions with points. Taken any two points $P = [\{x_n\}],\ Q = [\{x_n\}]$ in $P_t(R)$ we define their *point-distance* by the following formula, which is independent of codification:

$$d_q(P, Q) = \lim_{n\to+\infty} q(x_n, y_n).$$

**Theorem 4.1** *Let $(R, q)$ be a space of regions with points. Then full point-space $P_t(R)$ equipped with the point-distance $d_q$ is a metric space.*

*Proof* Being defined as a limit and thanks to the symmetry gap axiom the point-distance is a metric.

**Theorem 4.2** *The full point-space $P_t(R)$ of a quasi-metric space of regions with points, when carrying the point-distance, embeds as isometric subspace in the Cauchy metric completion of $(R, q^s)$, where $q^s$ is the metric symmetrization of $q$. Furthermore, it is a complete metric space.*

*Proof* If $\{P_h\}$, where $P_h = [\{x_n^h\}]$, is a Cauchy sequence in the point-distance $d_q$, then $\{P_h\}$ converges to the point determined by the point-representing sequence $\{x_n^h\}$, both $h, n$ running, as can be proved easily by a classical diagonal procedure.

**Metric case** Of course, any metric $d$ is a quasi-metric. The natural parthood associated with $d$ is the identity relation: *any region is just an atom, thus with zero diameter.* Consequently, any point-representing sequence of regions is a $d$−Cauchy sequence and the identification process gives the metric completion.

**Examples of quasi-metric spaces**

- Example 1 : *Any metric.*
- Example 2: The *discrete quasi-metric* on a set $P$ carrying a partial order $\leq$ defined as: $x, y \in P, q(x, y) = 0$ when $x \leq y$ while $q(x, y) = 1$, otherwise. When $P$ is the real number space with the usual order, the symmetrization is the dicrete metric. Since any sequence has diameter equal to one, no sequence can be vanishing. Thus, no point.
- Example 3: The *standard quasi-metric* $\div$ on $\mathbb{R}$ defined as: $x \div y = x - y$, when $x \geq y$ and $x \div y = 0$, otherwise. The metric symmetrization is in this case the Euclidean metric. Since any region is unbounded, again no point.
- Example 4: *Any Haudorff excess.* Let $(X, d)$ stand for a metric space. The hyperspace $BRCL(X)$ of nonempty regular closed and bounded sets of $X$ and the hyperspace $RO(X)$ of nonempty regular open subsets of $X$, when both carrying the Hausdorff excess $e_d$, are spaces of regions with points.
- Example 5: *A combination*: Let $e_d$ be the Hausdorff excess on the hyperspace of all nonempty bounded subsets of a metric space. Then, $q(x, y) = e_d(x, y) + |e_d(x) - e_d(y)|$ is a quasi-metric but not a quasi-metric of regions.

**Point-regions**

Our target now is to express *points in term of regions in which they in turn must be located naturally.*

Let $(R, q)$ be a space of regions with points. To any region $x$ comes naturally associated the *point-region $P_t x$* by saying that:

$P = [\{x_n\}]$ belongs to $P_t x$ if and only if $q(P, x) = \lim_{n\to+\infty} q(x_n, x) = 0.$

Now, we wonder: *Can a point not belong to any region? Can a region have no point? Can a point-region have no inside points?*

Answer to the first question: yes, it can. Let $D$ be the unit closed disc of the Euclidean plane. Choose as space of regions the collection of all closed discs in $\mathbb{R}^2 \setminus D$ equipped with the Euclidean excess. Then, any point on the boundary of $D$, which, as easily seen, can be codified by point-representing sequences, but, having a positive distance from any region, is just for that outside of any region.

Answer to the second question: yes, it can. Any region in any discrete quasi-metric space.

Answer to the third question: it can. Let $D$ the unit disc in the Euclidean plane. Suppose the space of regions done by $D$ and all closed balls contained in $\mathbb{R}^2 \setminus D$ and equipped with the Euclidean excess. Then $D$ is an atom. Thus, it contains a unique but not inside point.

**Proposition 4.1** *Let $(R, q)$ be a space of regions with points and $x$ a bounded region, i.e. $q(x) < +\infty$, then the point-region $P_t x$ is a bounded subset of the full point-space $P_t(R)$.*

*Proof* Let $P, Q \in P_t x$. By the generalized triangle inequality:

$d_q(P, Q) \leq q(P, x) + q(x, Q) \leq q(x, Q),$

and thanks to the relation: $|q(Q, x) - q(x, Q)| \leq q(x)$, it follows that $diam(P_t x) \leq q(x)$. So, the result is acquired.

**Theorem 4.3** *Any point-region is a closed subset of the full point-space.*

*Proof* By the triangle inequality if $\{P_n\}$ is a sequence points in $x$ convergent to a point $P$, then $P$ is in $x$.

**Theorem 4.4** *If a region $x$ is a part of a region $y$ $(x \leq_q y)$, then $P_t x$ is contained in $P_t y$, but the vice versa is not true.*

*Proof* By the triangle inequality if $x \leq_q y$, then $P_t x \subseteq P_t y$. Vice versa. Let $BRCL(R^2)$ be the hyperspace of all bounded regular closed nonempty subsets of the Euclidean plane $R^2$. Consider the quasi-metric space whose underlying set is $R = \{(h, k) : h, k \in BRCL(R^2), \ h \subseteq k\}$ and $q[(h_1, k_1), (h_2, k_2)] = e_d(h_1, k_1) \oplus e_d(h_2, k_2)$, where $\oplus$ is the average and $d$ the Euclidean metric. It easily proved that $(R, q)$ is a quasi-metric space of regions. The associated parthood is:

$(h_1, k_1) \leq_q (h_2, k_2)$ if and only if $h_1 \subseteq h_2$ and $k_1 \subseteq k_2$.

Let $s, s'$ be two disjoint closed discs. Then, $(s, s \cup s') \not\leq_q (s, s')$ but $P_t(s, s \cup s') = P_t(s, s')$. A point $(P, Q) = [\{(a_n, b_n)\}]$ is in $P_t(s, s \cup s')$ if and only if the two points $[\{a_n\}], [\{b_n\}]$ in the point-space associated with the quasi-metric space $BRCL(R^2)$ with the Euclidean excess are in $P_t s$ and in $P_t(s \cup s')$ respectively. But $P = Q$ since $a_n \subseteq b_n$ for each $n$ and $\{b_n\}$ is vanishing. Moreover, observe that the region $(s, s \cup s')$ is not a part of $(s, s)$ but any part of $(s, s \cup s')$ coincides with $(s, s)$. So, the parthood is not extensional. □

How to avoid this inconvenience? By requiring a stronger form of extensionality of the parthood associated with the quasi-metric, that we call:

**Geometrical extensionality** *If a region $x$ is not a part of a region $y$, then there is a point in $x$ not in $y$*. By geometrical extensionality it follows that:

$x \ll_q y$ if and only if $P_t x = P_t y$.

Now, when a point $P = [\{x_n\}]$ and a region $x$ are given, we can consider on one side $q(P, x) := \lim_{n+\infty} q(x_n, x)$ and on the other $d_q(P, P_t x)$, this second being the usual distance between a point and a subset in the full point-space equipped with the point-distance. In general, $q(P, x)$ is different from $d_q(P, P_t x)$, but always $q(P, x) \leq d_q(P, P_t x)$. A positive gap indicates no enough points are in $x$. The quasi-metric space examined in Theorem 4.4 has a positive gap.

If there is no gap, it happens that:

**Theorem 4.5** *Suppose that $q(P, x) = d_q(P, P_t x)$ for any point $P$ and any region $x$. Then, for any two regions $x, y$, it happens that $q(x, y) \geq e_{d_q}(P_t x, P_t y)$, [6].*

As example with positive gap we can take the hyperspace of nonempty regular closed subsets of a metric space $BRCL(X)$ of a metric space $(X, d)$ endowed with the quasi-metric defined as: $e_d(x, y) + e_d(x) \div e_d(y)$.

In the case of no coincidence it will mean again that there are too few points in a region. But then, we might try to remove the obstacle enforcing the point-existence axiom in the following one:

**Nested point-existence axiom** *Any region $x$ contains as a part a region $y$ with diameter $q(y)$ less than* $\frac{q(x)}{2}$ and requiring:

**No gap axiom**

$q(P, x) = d_p(P, P_t x)$ and $d(x, y) = sup\{e_d(P, P_t y) : P \in P_t x\}$, for all $P \in P_t$, $x, y \in R$.

Of course, the nested point-existence implies the existence of points codified by nested vanishing sequence. And, moreover the full point-space has no isolated points, [6].

**Theorem 4.6** *Let $(R, q)$ be a space of regions with points. The set of points codified by nested vanishing sequences is a, generally proper, subspace of the full point space $P_t(R)$.*

*Proof* If $(X, d)$ is a metric space, then the full point-space agrees with its metric completion, while the nested point-space is just $X$.

A point $[\{x_n\}]$ represented by a nested sequence such that any region $x_n$ is a part of the region $x$ (the same $x_n \leq_q x$) is an inside point of $P_t x$? Generally, no. It is enough to observe that a boundary point of an Euclidean sphere can be codified with a vanishing decreasing sequence of spheres all internally tangent to the starting one.

How to recognize the interior points of a point-region? To exhibit a characterization of the interior of a point-region in the full point-space we need to introduce the notion of strong inclusion.

- *Strong inclusion associated with a quasi-metric*

Let $x, y$ be two regions. We say that $x$ is *strongly contained* in $y$, or is a *non tangential part of* $y$, and write $x \ll_q y$, if and only if $x \leq_q y$ and there exists a positive real number $\epsilon$ such that for any region $z$ with distance $q(z, x) < \epsilon$, it happens that $q(z, y) = 0$ or equivalently $z \leq_q y$.

- *Relations between strong inclusions*

It happens that:

**Theorem 4.7** *Whenever $x \ll_q y$, then $P_t x \ll_d P_t y$, or, in other words, strong inclusion of regions yields the strong inclusion between their associated point-regions w.r.t. the point-distance.*

*Proof* First, suppose $x \ll_q y$. Next, choose $\epsilon$ so that any region $z$ with $q(z, x) < \epsilon$ is part of $y$. Then, we show that the $\frac{\epsilon}{2}$-enlargement of $P_t x$ is contained in $P_t y$. Of course, $x \leq_q y$. Let $P = [\{x_n\}]$, $Q = [\{y_n\}]$ two points such that $q(P, x) = 0$ and $d_q(P, Q) \leq \frac{\epsilon}{2}$. Then, all distances $q(x_n, y_n)$, $q(x_n, x)$ and all diameters $q(x_n), q(y_n)$, are less than $\frac{\epsilon}{2}$ residually. Thus, $q(y_n, x)$ is less than $\epsilon$ residually. Consequently, $q(y_n, y) = 0$ residually. That, finally, implies $q(Q, y) = 0 = d_q(Q, P_t y)$.

Vice versa holds true only partially.

**Theorem 4.8** *If $S(P_t x, \epsilon) \subseteq P_t y$ and $q(x) < \frac{\epsilon}{2}$, then $x \ll_q y$.*

*Proof* Suppose in contrast that $S(P_t x, \epsilon) \subseteq P_t y$, but $x$ non strongly contained in $y$. Since, from geometrical extensionality, $P_t x \subseteq P_t y$ it follows that $x \leq_q y$. Consequently, there exists a region $z$ with $q(z, x) < \frac{\epsilon}{2}$ but $q(z, y) \neq 0$. Again from geometrical extensionality there is a point $P$ which is in $z$ but not in $y$. Now, let $Q$ a point in $x$. Then:

$$d_q(P, Q) \leq q(P, x) + q(x, Q) \leq q(z, x) + q(x) < \epsilon.$$

In conclusion, $P$ being in a sphere having as center a point in $x$ and radius less than $\epsilon$ belongs to $P_t y$. A contradiction.

When a point $P$ is strongly contained in a region $x$? It appears as natural to say that *a point $P$ is strongly contained in a region $x$*, $P \ll_q x$, when $P$ belongs to a region $y$ which in turn is strongly contained in the region $x$.

When a point is strongly contained in a region, is it an inside point of the associated point-region? What about the vice versa?

**Theorem 4.9** *A point $P$ is strongly contained in a region $x$ if and only if it is an interior point of the point-region $P_t x$.*

*Proof* If $P \ll_q x$ then $P$ is inside $P_t x$. We can apply Theorem 4.7. Vice versa. If $P$ is an inside point of a point-region $P_t x$ there is a sphere $S(P, 2\epsilon)$ contained in $P_t x$. Let $P$ codified by $\{x_n\}$. Choose $x_h$ so that $q(x_h)$ and $q(P, x_h)$ both are less than $\frac{\epsilon}{2}$. Then, by the generalized triangle inequality, $d_q(P, Q) \leq q(P, x_h) + q(x_h, Q) \leq \frac{\epsilon}{2} + q(x_h) < \epsilon$, when $Q$ is in $x_h$. This gets $S(P_t x_h, \frac{\epsilon}{2}) \subseteq P_t x$. And the result follows from Theorem 4.8.

For having point-regions with inside points we enforce the point-existence axiom in the following way:

**Inside point-existence axiom** *Any region $x$ contains as strong part a region $y$ with diameter $q(y) < \frac{q(x)}{2}$.*

The inside point-existence axiom guarantees the existence of points codified by strongly nested vanishing sequences. This condition excludes that regions of different dimension can coexist.

Now, let $(R, q)$ be a quasi-metric space of regions verifying the inside point-existence axiom. Then, we can identify any region $x$ in $R$ with $Cl(int(P_t x))$, the closure of the interior of $P_t x$, which is a nonempty regular closed subset of the full point-space $P_t R$.

To summarize:

**Theorem 4.10** (Representation theorem) *A quasi-metric space $(R, q)$ is isometric via the injection $x \to Cl(int(P_t x))$ to a subspace of the hyperspace of nonempty regular closed subsets of the full point-space $P_t R$ carrying the Hausdorff excess associated with the point-distance if and only if it satisfies to the symmetry gap axiom, no gap axiom, inside point-existence axiom, geometrical extensionality.*

## 5 Metric Geometry and Shape

When the point-space has been constructed, then a metric geometry in the sense of Blumenthal [3] can start. For instance we can consider loci of lower dimension such as lines, planes and so on. Of course, geometric definitions and properties in a metric geometry so generated should be expressed in terms of properties of the generating quasi-metric. Blumenthal gave metric characterizations of the Euclidean geometry but also characterizations of the hyperbolic geometry and further of the elliptic geometry as well. The hyperspace of a metric space can be metrized, as we have seen, by the Hausdorff metric that is the symmetrization of the Haudorff excess. But, unfortunately only little work has been done on Hausdorff metric geometry of the hyperspace.

We quote as an example the Tarski Geometry of Solids in three dimensional Euclidean space. In our opinion, this model is not only interesting in itself but it is suggestive of other models, for example ones deriving from Minkowki and Chebyshev metric, [11]. In Tarski Geometry, in addition to the usual inclusion, an extensional Lesniewski mereology, the notion of sphere is the only "geometrical" primitive notion. Here too a point is just the "class" of all spheres which are concentric with a given sphere. We list the previous definitions needed for giving the notion of two concentric spheres and show that they all are expressable in terms of the Euclidean excess. Remind that the diameter of a sphere is the same as the diameter of the Euclidean excess being regular open when open and regular closed when closed; and, further, that two open spheres are disjoint if and only if they don't overlap and this happens if and only if their Hausdorff distance, the symmetrization of the Euclidean excess, is greater than or equal to the sum of their diameters.

- *External tangency* ET: A sphere $a$ is externally tangent to a sphere $b$ if (1) $a$ is disjoint from $b$ and (2) given two spheres $x$, $y$ containing $a$ and disjoint from $b$, then at least one of them has to be contained in the other one, Fig. 2a.

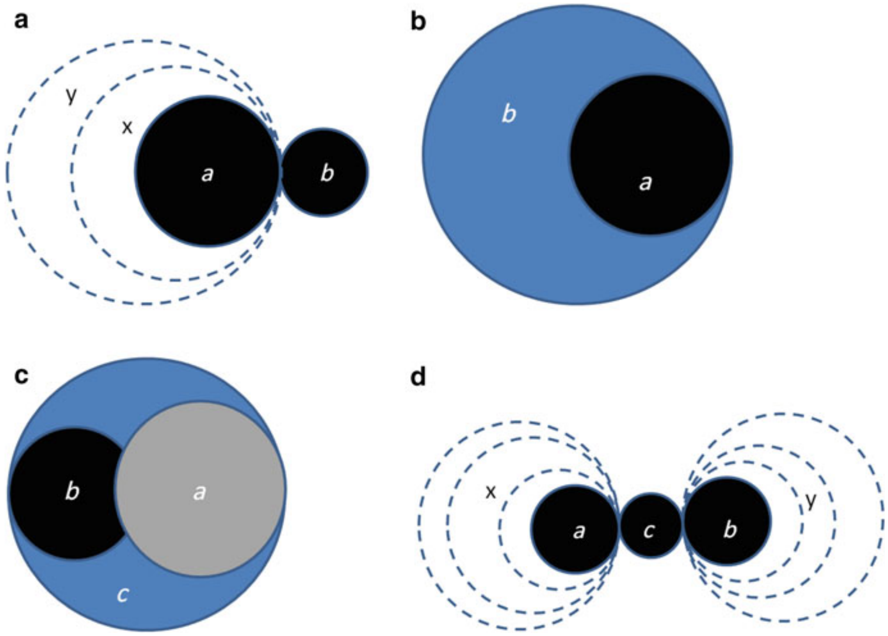


**Fig. 2** $a, b$ externally tangent **(a)**, $a$ internally tangent to $b$ **(b)**, $a, b$ internally diametrically tangent to $c$ **(c)**, $a, b$ externally diametrically tangent to $c$ **(d)**

- *Internal tangency* IT: A sphere $a$ is internally tangent to a sphere $b$ if (1) $a$ is a proper part of $b$ and (2) given two spheres $x, y$ contained in $b$ and containing $a$, then at least one of them is contained in the other one, Fig. 2b.
- *Internally diametrical* ID: Spheres $a, b$ are internally diametrical tangent to a sphere $c$ if (1) $a, b$ are both internally tangent to $c$ and (2) given two spheres $x, y$ both disjoint from $c$, and such that $a$ is externally tangent to $x$ and $b$ to $y$ then $x$ is disjoint from $y$, Fig. 2c.
- *Externally diametrical* ED: Spheres $a, b$ are externally diametrical tangent to a sphere $c$ if (1) $a, b$ are both externally tangent to $c$ and (2) given two spheres $x, y$ containing $a$ and $b$ respectively both disjoint from $c$, then $x$ is disjoint from $y$, Fig. 2d.
- *Concentric* $\circledcirc$: A sphere $a$ is concentric with a sphere $b$ if one of the following conditions holds: (1) $a$ and $b$ coincide, (2) $a$ is contained in $b$ and, given two spheres $x, y$ externally diametrical to $a$ and internally tangent to $b$,then $x, y$ are both internally diametrical to $b$, (3) $b$ is a proper part of $a$, and, given two spheres $x, y$ externally diametrical to $b$ and internally tangent to $a$, then are both internally diametrical to $a$.
- *Equidistant points:* Points $A$ and $B$ are *equidistant* from a point $C$ if there exists a sphere $a$ which belongs as element to the point $C$ and is such that no sphere $b$ belonging as element to the point $A$ or to the point $B$ is a part of $a$ or is disjoint from $a$.

It is enough to show that two spheres $a, b$ are externally tangent if and only if $e_d(a, b) = diam(a)$ and $e_d(b, a) = diam(b)$; and $a$ is internally tangent to $b$ if and only if $e_d(a, b) = 0$ and $e_d(b, a) = diam(b) - diam(a)$.

- *Shape*

In the general construction of point-free geometries the shape seems irrelevant, but it can play a discriminating role. The plane $\mathbb{R}^2$ can be metrized by the the Minkowski metric defined as: $d_M((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$ known also as the Manhattan or taxi-cab metric [11] considered a more appropriate measure of distance in an urban environment as Manhattan. It is very well known that Euclidean and Minkowski metrics are bi-Lipschitzian equivalent. For that the Euclidean and Minkowski excess determine both as parthood the set-inclusion and also the same diameter function and consequently the same space of points and the same point-regions.

But, if we construct points by using only spheres, then points are approximated by regions of very different shape. In the Euclidean case we visualize points by circles and in the Minkowski case by squares.

Let $(R, q)$ a space of regions. A *similitude* $S$ of $(R, q)$ is a bijection of $R$ on itself for which there exists a positive real number $\rho$ such that $q(x, y) = \rho\, q(S(x), S(y)),\ x, y \in R$. Of course, similitudes form a group $S(R)$.

If a family $\mathcal{S}$ of bounded regions is invariant under $S(R)$ and is a base for the underlying topology of the full point-space, then codifications by sequences extracted from $\mathcal{S}$ give a natural notion of *point with shape*. Of course, we can consider subgroups of the group of similitudes preserving relevant features other than shape. We can think to preservations of colours or also orientation and so on. This way we can get *featured points*.

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