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Strong *b*-Suprametric Spaces and Fixed Point Principles

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Abstract

In this paper, we introduce the strong b-suprametric spaces in which we prove the fixed point principles of Banach and Edelstein. Moreover, we prove a variational principle of Ekeland and deduce a Caristi fixed point theorem. Furthermore, we introduce the strong b-supranormed linear spaces in which we establish the fixed point principles of Brouwer and Schauder. As applications, we study the existence of solutions to an integral equation and to a third-order boundary value problem.

Keywords sb-Suprametric space · sb-Supranormed space · Fixed point theorem · Variational principle

Mathematics Subject Classification 54D35 · 54E99 · 47H10 · 47J20

Introduction

Let X be a nonempty set and $\mathbb{R}_+$ be the set of all nonnegative real numbers. A semimetric is a distance function $d\colon X \times X \to \mathbb{R}_+$ that satisfies two axioms: (d_1): $d(x, y) = 0$ if and only if $x = y$; (d_2): $d(x, y) = d(y, x)$ for all $x, y \in X$. It is well known that by adding the triangle inequality to the axioms of d it becomes continuous. In 1993, Czerwik [8] investigated a semimetric called b-metric, which satisfies the inequality: $d(x, y) \le b(d(x, z) + d(z, y))$, where b is a constant in $[1, +\infty)$ and $x, y, z \in X$. This notion has been studied previously by different authors, for the latest and rather complete bibliography, we refer the reader to the surveys of Berinde and Păcurar [2] and Karapınar [14]. Despite the b-metric is very useful in applications [7, 15, 27], it has a major drawback due to its lack of continuity [26]. In order to overcome this limitation, Kirk and Shahzad proposed a slight modification in the third axiom, see [16, 17].

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The suprametric, which was introduced by the author in [3], is a semimetric that fulfill $d(x, y) \leq d(x, z) + d(z, y) + \rho d(x, z)d(z, y)$, where ρ is a constant in $\mathbb{R}_+$ and $x, y, z \in X$. This distance function is very useful to construct projective metrics of Thompson's type [25], and to prove the existence of solutions to various classes of integral and matrix equations. Very recently, the suprametric has been utilized by Panda et al. [21, 22] to analyze complex valued fractional order neural networks and the existence of a solution of stochastic integral equations.

It is known that the space of p-integrable functions for $p \in (0, 1)$ is a b-metric space, but it is unclear whether it is a suprametric space. The author introduced the b-suprametric [4], which subsume such functional space. Note that the distance function of the b-suprametric is not necessarily continuous [4, Example 2.10], although the continuity is very useful. In order to overcome this drawback here we introduce the strong b-suprametric distance function, a subfamily of the b-suprametric, and shows its continuity.

The objectives of this work are fourfold: (1) To introduce the strong b-suprametric space in which we establish fixed point theorems of Banach and Edelstein types. (2) To prove a variational principle through the Cantor's intersection theorem, then to derive a Caristi fixed point result via this variational principle. (3) To introduce the strong b-supranormed linear space and to provide the fixed point principles of Brouwer and Schauder in such linear space. (4) To provide new sufficient conditions for the existence of a solution to an integral equation, via a Chebyshev type inequality, where the integral operator involved is not necessarily Lipschitzian with respect to a metric. Then, we show the existence of a unique solution to a third-order boundary value problem.

1 Strong *b*-Suprametric Spaces

Here and below, the symbols $\mathbb{R}$ and $\mathbb{N}$ will denote respectively the set of all real numbers and all nonnegative natural numbers. The symbol $\mathrm{cl}(A)$ stands for the closure of a set A. We first need to recall the b-suprametric spaces from [4].

Definition 1.1 Let (X, d) be a semimetric space and $b \geq 1$, $\rho \geq 0$ be two real constants. The function d is called b-suprametric if:

(d_3) $d(x, y) \leq b\,(d(x, z) + d(z, y)) + \rho d(x, z)d(z, y)$ for all $x, y, z \in X$.

A pair (X, d) is called b-suprametric space if X is a nonempty set and d is a b-suprametric.

In the previous definition, if $\rho = 0$ we obtain the b-metric [8] and if $b = 1$ we obtain the suprametric [3]. In the sequel, we focus on the following subclass of b-suprametric space.

Definition 1.2 Let (X, d) be a semimetric space and $b \geq 1$, $\rho \geq 0$ be two real constants. The function d is called strong b-suprametric (sb-suprametric space) if:

(d_3') $d(x, y) \leq b\, d(x, z) + d(z, y) + \rho d(x, z)d(z, y)$ for all $x, y, z \in X$.

A pair (X, d) is called sb-suprametric space if X is a nonempty set and d is an sb-suprametric.

Remark 1.3 From (d_2), it follows that we also have

d_3'' $d(x, y) \leq d(x, z) + b\, d(z, y) + \rho d(x, z) d(z, y)$, for all $x, y, z \in X$.

Examples 1.4
- All suprametric spaces of [3] are sb-suprametric spaces.
- Let $X = \{1, 2, 3\}$ and let $d\colon X \times X \to \mathbb{R}_+$ be a function defined by:

$$d(x, y) = \begin{cases} 0, & x = y, \\ 3, & (x, y) \in \{(1, 2), (2, 1)\}, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

  Then (X, d) is an sb-suprametric space with coefficient $b = \frac{3}{2}$ and $\rho = 8$.
- Let $X = C_+[0, 1]$ of continuous nonnegative functions endowed with

$$\delta(x, y) = \sup_{t \in [0,1]} |x(t) - y(t)|(|x(t) - y(t)| + \tfrac{1}{2}), \quad \text{for all } x, y \in X.$$

  Then (X, δ) is an sb-suprametric space for $b = \rho = 2$.

Proposition 1.5 *Let (X, d) be an sb-suprametric space, then for all $p, q, s, t \in X$*

$$\frac{\rho(d(p, q) - d(s, t))^2}{(b + \rho d(p, q) + \rho d(s, t))^2} \leq 2\big(d(p, t) + d(s, q) + \rho\, d(p, t)\, d(s, q)\big). \tag{1}$$

Proof Let (X, d) be an sb-suprametric space with $\rho > 0$ (the case $\rho = 0$ is trivial). Then,

$$\begin{aligned} d(p, q) &\leq d(p, s) + b\, d(s, q) + \rho\, d(p, s)\, d(s, q) \\ &\leq d(s, t) + b\, d(p, t) + \rho\, d(s, t)\, d(p, t) + b\, d(s, q) \\ &\quad + \rho\, \big(d(s, t) + b\, d(p, t) + \rho\, d(s, t)\, d(p, t)\big)\, d(s, q) \\ &\leq d(s, t) + \big(b + \rho\, d(s, t)\big)\big(d(p, t) + d(s, q) + \rho\, d(p, t)\, d(s, q)\big), \end{aligned}$$

which implies

$$\frac{d(p, q) - d(s, t)}{(b + \rho\, d(s, t))} \leq d(p, t) + d(s, q) + \rho\, d(p, t)\, d(s, q). \tag{2}$$

A similar argument shows that

$$\frac{d(s, t) - d(p, q)}{(b + \rho\, d(p, q))} \leq d(p, t) + d(s, q) + \rho\, d(p, t)\, d(s, q). \tag{3}$$

Adding (2) to (3), we obtain

$$\frac{\rho(d(p, q) - d(s, t))^2}{(b + \rho d(p, q))(b + \rho d(s, t))} \leq 2\big(d(p, t) + d(s, q) + \rho\, d(p, t)\, d(s, q)\big),$$

which implies (1). □

Remark 1.6 Let $u \geq 0$ and $\rho > 0$. Assume that a sequence $\{u_n\} \subset \mathbb{R}_+$ satisfies

$$\lim_{n\to\infty} \frac{\rho\,(u_n - u)^2}{(b + \rho\,u_n + \rho\,u)^2} = 0,$$

then u_n tends to u as $n \to \infty$. Otherwise, if u_n does not tends to u there exists $\varepsilon > 0$ such that for all integer $k > 0$, $n(k) > k$ and $|u_{n(k)} - u| > \varepsilon$. Then,

$$\frac{\sqrt{\rho}\,\varepsilon}{b + \rho\,u_{n(k)} + \rho\,u} < \frac{\sqrt{\rho}\,|u_{n(k)} - u|}{b + \rho\,u_{n(k)} + \rho\,u} \to 0 \text{ as } k \to \infty,$$

which implies that $u_{n(k)}$ tends to infinity as $k \to \infty$, and hence

$$\lim_{k\to\infty} \frac{\rho\,(u_{n(k)} - u)^2}{(b + \rho\,u_{n(k)} + \rho\,u)^2} = \frac{1}{\rho},$$

yields a contradiction.

Remark 1.7 Let $\{p_n\}$ and $\{q_n\}$ be sequences in X such that $\lim_{n\to\infty} d(p_n, t) = 0$ and $\lim_{n\to\infty} d(q_n, s) = 0$, then by (1) and Remark 1.6, $\lim\limits_{n\to\infty} d(p_n, q_n) = d(s, t)$, and this means that d is continuous.

Let (X, d) be an sb-suprametric space. An open ball and a closed ball centered at $a \in X$ and of radius $r > 0$, are respectively given by

$$B(a, r) := \{x \in X : d(a, x) < r\} \text{ and } B[a, r] := \{x \in X : d(a, x) \leq r\}.$$

Proposition 1.8 *Let (X, d) be an sb-suprametric space. Then*

(i) every open ball is an open set.
(ii) every closed ball is a closed set.

Proof To see (i), let $r > 0$ and $a \in X$. For $y \in B(a, r)$ let

$$r_1 := \frac{r - d(y, a)}{b + \rho d(y, a)},$$

then if $x \in B(y, r_1)$,

$$\begin{aligned} d(x, a) &\leq b\,d(x, y) + d(y, a) + \rho d(x, y)\,d(y, a) \\ &< b\,r_1 + d(y, a) + \rho r_1\,d(y, a) = r. \end{aligned}$$

Thus, $B(y, r_1) \subseteq B(a, r)$, and $B(a, r)$ is open.

Now, to see (ii), let $r > 0$ and $a \in X$ and take a sequence $\{x_n\}$ in $B[a, r]$ convergent to some x with respect to d. Then

$$\begin{aligned} d(a, x) &\leq d(a, x_n) + b\, d(x_n, x) + \rho d(a, x_n)\, d(x_n, x) \\ &\leq r + (b + \rho r)\, d(x_n, x), \end{aligned}$$

and as $n \to \infty$, we get $x \in B[a, r]$ which proves that $B[a, r]$ is closed. □

As a consequence, we obtain the following propositions.

Proposition 1.9 *Let (X, d) be an sb-suprametric space. The family of open balls form a base of a topology on X.*

Proof Let $u \in B(x, \varepsilon), B(y, \varepsilon')$ and choose $r > 0$ so that $(b + \rho r)d(x, u) + r < \varepsilon$ and $(b + \rho r)d(y, u) + r < \varepsilon'$. Then by taking a point $v \in B(u, r)$, we obtain

$$\begin{aligned} d(x, v) &\leq bd(x, u) + d(u, v) + \rho d(x, u)d(u, v) < (b + \rho r)d(x, u) + r < \varepsilon, \\ d(y, v) &\leq bd(y, u) + d(x, u) + \rho d(y, u)d(u, v) < (b + \rho r)d(y, u) + r < \varepsilon', \end{aligned}$$

which implies that $B(u, r) \subset B(x, \varepsilon) \cap B(y, \varepsilon')$. Finally, we conclude by using [10, Lemma I.4.7] and the fact that every $x \in X$ is also in $B(x, \tau)$ for some $\tau > 0$. □

Proposition 1.10 *An sb-suprametric space is normal.*

Proof Let (X, d) be an sb-suprametric space. If $x, y \in X$ such $x \neq y$, then $U{:=}B(x, \frac{d(x,y)}{2})$ and $V{:=}B(y, \frac{d(x,y)}{2b+\rho\, d(x,y)})$ are disjoint neighborhoods of x and y respectively. Otherwise, assume that $U \cap V \neq \emptyset$, so there exists $z \in U \cap V$. Thus, using that $d(x, z) < \frac{r}{2}$ and $d(y, z) < \frac{r}{2b+\rho r}$ where $r = d(x, y)$, we obtain

$$\begin{aligned} r = d(x, y) &\leq d(x, z) + d(z, y) + \rho\, d(x, z)d(z, y) \\ &< \frac{r}{2} + \frac{r}{2b + \rho r} + \rho \frac{r}{2} \frac{r}{2b + \rho r} = r, \end{aligned}$$

a contradiction, so our claim holds. We conclude therefore that X is Hausdorff.

Let now U and V be disjoint closed sets and let

$$d(x, U){:=} \inf_{u \in U} d(x, u) \text{ and } d(x, V){:=} \inf_{v \in V} d(x, v).$$

Define the sets

$$U'{:=}\big\{x \in X : d(x, U) < d(x, V)\big\} \text{ and } V'{:=}\big\{x \in X : d(x, V) < d(x, U)\big\}.$$

Then U' and V' are disjoint neighborhoods of U and V respectively. □

Proposition 1.11 *In an sb-suprametric space, if a sequence has a limit it is unique.*

Proposition 1.12 *The strong b-metric space [16] is normal.*

Definition 1.13 Let (X, d) be an sb-suprametric space.

(i) The sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ iff $\lim\limits_{n\to\infty} d(x_n, x) = 0$.
(ii) The sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy iff $\lim\limits_{n,m\to\infty} d(x_n, x_m) = 0$.
(iii) (X, d) is complete iff any Cauchy sequence in X is convergent.

Remark 1.14 Let $X = C_+[0, 1]$ be the set of continuous functions $x\colon [0, 1] \to \mathbb{R}_+$ endowed with δ the sb-suprametric of Examples 1.4. The completeness of (X, δ) follows from that of (X, d) with $d(x, y) = \sup\limits_{t\in[0,1]} |x(t) - y(t)|$ for $x, y \in X$.

The next lemma is a direct consequence of Definition 1.13.

Lemma 1.15 *In an sb-suprametric space, we have:*

(i) A convergent sequence is a Cauchy sequence.
(ii) A Cauchy sequence converges iff it has a convergent subsequence.
(iii) A point $u \in \mathrm{cl}(U)$ *iff there is a sequence* $\{u_n\} \subset U$ *converging to* u.

Remark 1.16 If a sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in a complete sb-suprametric (X, d), then there exists $x_* \in X$ such that $\lim\limits_{n\to\infty} d(x_n, x_*) = 0$. By d_3' follows that every subsequence $\{x_{n(k)}\}_{k\in\mathbb{N}}$ converges to x_*.

Remark 1.17 Let (X, d) be an sb-suprametric space. By d_3', we have:

$$d(x_0, x_n) \leq b \max\{1, \rho^n\} \sum_{i=1}^{n} \mathbf{e}_i(d_0, \ldots, d_{n-1}),$$

for all $n \in \mathbb{N}$, $x_0, \ldots, x_n \in X$, where $d_{i-1} := d(x_{i-1}, x_i)$ and $\mathbf{e}_i$ is the i^{th} elementary symmetric polynomial in n variables, that is,

$$\mathbf{e}_i(d_0, \ldots, d_{n-1}) = \sum_{0\leq j_1<j_2<\cdots<j_i\leq n-1} \prod_{k=1}^{i} d_{j_k}.$$

It is easy to see that these polynomials possess the following properties:

Proposition 1.18 *Let* $n \in \mathbb{N}$ *and* $\mathbf{e}_i(x_0, \ldots, x_{n-1})$ *be an elementary symmetric polynomial of index* $0 \leq i \leq n$. *Then,*

(i) $x_k \mapsto \mathbf{e}_i(x_0, \ldots, x_k, \ldots, x_{n-1})$ *is a nondecreasing function for* $0 \leq k \leq n$.
(ii) $\mathbf{e}_i(ax_0, \ldots, ax_{n-1}) = a^i \mathbf{e}_i(x_0, \ldots, x_{n-1})$ *for all* $a \in \mathbb{R}_+$.

A *covering* of a set U in X is a family of open sets whose union contains U. A set $U \subseteq X$ is called *compact* if and only if every covering of U by open sets in X contains a finite sub-covering. A subset U of a topological space X is sequentially compact, if every sequence of points in U has a subsequence converging to a point of X. A set $N \subset X$ is called an ε-*net* for a set $U \subseteq X$ ($\varepsilon > 0$), if there exists $x_\varepsilon \in N$ for every $x \in U$ such that $d(x, x_\varepsilon) < \varepsilon$. Next, we provide an extreme value theorem in sb-suprametric spaces.

Theorem 1.19 *Let (X, d) be an sb-suprametric space. Let U be a compact subset of X and $f: U \to \mathbb{R}$ be a continuous function. Then,*

(i) f is bounded on U,
(ii) f attains its supremum and its infimum.

Proof The proof is similar to that of [18, Chap. 5. Theorem 1] (see also [10, Lemma I.5.8]). □

The compactness is discussed in the rest of this section.

Theorem 1.20 *For a set U in an sb-suprametric space X to be compact, it is necessary, and in the case of completeness of X, sufficient that there is a finite ε-net for the set U for every $\varepsilon > 0$.*

Proof The proof is similar to that of [18, Chap. 5. Theorem 3], except for the value of the distance between any two points, which does not exceed $\varepsilon_n\,(1 + b + \rho\varepsilon_n)$. □

Corollary 1.21 *A subset U of an sb-suprametric space is compact if and only if it is closed and sequentially compact.*

Proof The proof is exactly similar to that of [10, Theorem I.6.13]. □

Corollary 1.22 *Let (X, d) be a sb-suprametric space and $U \subseteq X$. If U is compact, then it is bounded.*

Proof Let $S_n = \{x_1, \ldots, x_n\}$ be a 1-net for U. Let $a \in X$, $x \in X$ and $x_i \in S_n$ for $i = 1, \ldots, n$. Then,

$$\begin{aligned} d(x,a) &\leq bd(x,x_i) + d(x_i,a) + \rho d(x,x_i)\,d(x_i,a) \\ &\leq b\big(d(x,x_i) + d(x_i,a)\big) + \tfrac{\rho}{2}\big(d(x,x_i) + d(x_i,a)\big)^2 \\ &\leq (b + \tfrac{\rho}{2})\big(1 + \max_i d(x_i,a)\big)^2 < \infty. \end{aligned}$$

□

2 Banach and Edelstein Fixed Point Theorems

We start by presenting a Banach fixed point result in sb-suprametric spaces.

Theorem 2.1 *Let (X, d) be a complete sb-suprametric space and $f: X \to X$ be a given mapping. Assume that there exists $c \in [0, b^{-1})$ such that for all $x, y \in X$,*

$$d(fx, fy) \leq c\,d(x, y). \tag{4}$$

Then f has a unique fixed point and $\{f^n x\}_{n\in\mathbb{N}}$ converges to it for all $x \in X$.

Proof Assume that $\rho > 0$, since the case $\rho = 0$ is treated in [8] (see also [13]). Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$, where f^n is n^{th} iterates of f. For simplification let us introduce the notation: $d_{i,j} := d(x_i, x_j)$, where $i, j \in \mathbb{N}$. Now, from (4), we get

$$d_{n,n+1} \leq c d_{n-1,n} < d_{n-1,n}.$$

Hence, $\{d_{n,n+1}\}$ is decreasing sequence and for all $k \in \mathbb{N}$, we have

$$d_{n,n+1} \leq c^{n-k}\, d_{k,k+1}, \quad \text{for all } n > k. \tag{5}$$

So $\lim_{n \to \infty} d_{n,n+1} = 0$, and therefore there exits $k \in \mathbb{N}$ such that for all $n \geq k$,

$$d_{n,n+1} \leq 1. \tag{6}$$

Next, we shall prove that the sequence $\{x_n\}$ is Cauchy. Using d_3' and (6), and for sufficiently large integers p, q such that $q > p > k$ it follow that

$$\begin{aligned} d_{p,q} &\leq d_{p,p+1} + b\, d_{p+1,q} + \rho\, d_{p,p+1}\, d_{p+1,q} \\ &\leq c^{p-k} d_{k,k+1} + b\, d_{p+1,q} + \rho\, c^{p-k} d_{k+1,k}\, d_{p+1,q} \\ &\leq c^{p-k} + (b + \rho\, c^{p-k}) d_{p+1,q}, \end{aligned}$$

where

$$\begin{aligned} d_{p+1,q} &\leq d_{p+1,p+2} + b\, d_{p+2,q} + \rho\, d_{p+1,p+2}\, d_{p+2,q} \\ &\leq c^{p-k+1} d_{k+1,k} + b\, d_{p+2,q} + \rho\, c^{p-k+1} d_{k+1,k}\, d_{p+2,q} \\ &\leq c^{p-k+1} + (b + \rho\, c^{p-k+1}) d_{p+2,q}. \end{aligned}$$

By combining the previous inequalities, we obtain

$$d_{p,q} \leq c^{p-k} + c^{p-k+1}(b + \rho\, c^{p-k}) + (b + \rho\, c^{p-k})(b + \rho\, c^{p-k+1}) d_{p+2,q}.$$

Using (6) in all terms of the sum, we obtain by induction

$$d_{p,q} \leq c^{p-k} \sum_{i=0}^{q-p-1} c^i \prod_{j=0}^{i-1} (b + \rho\, c^{p-k+j}).$$

Now, since $c \in [0, b^{-1})$, then

$$d_{p,q} \leq c^{p-k} \sum_{i=0}^{q-p-1} c^i \prod_{j=0}^{i-1} (b + \rho\, c^j).$$

Using d'Alembert's criterion of convergence of real series, we deduce that $\sum_{i=0}^{\infty} u_i$ converges, where

$$u_i := c^i \prod_{j=0}^{i-1} (b + \rho\, c^j).$$

We conclude $d_{p,q} \to 0$ as $p, q \to \infty$, so the sequence $\{x_n\}$ is Cauchy. Thus, it follows that $\{x_n\}$ converges to some $x_* \in X$, say, since X is sb-complete, which proves that $\omega_f(x_0)$ is nonempty. We now shall show that x_* is a fixed point of f. By using (4), we get

$$d(f x_{n(k)}, f x_*) \le c\, d(x_{n(k)}, x_*).$$

By letting $k \to \infty$, we obtain by Proposition 1.11 and Remark 1.16 that $x_* = f x_*$. Finally, the uniqueness of the fixed point follows immediately from (4). □

Remark 2.2 Theorem 2.1 generalizes [3, Theorem 2.1]. Note also that for the extended suprametric, introduced by Panda et al. [22], an additional continuity assumption was added to obtain the main fixed point theorem.

Proposition 2.3 *Let (X, d) be an sb-suprametric space and let $f : X \to X$ be a Lipschitz mapping, that is, there is a constant $\lambda \in [0, \infty)$ such that for all $x, y \in X$,*

$$d(f x, f y) \le \lambda\, d(x, y).$$

Then f is continuous.

Proof The proof is exactly the same as that of [3, Proposition 1.8]. □

Let X be a topological space and $f : X \to X$ be a mapping. For $x_0 \in X$ the ω-limit set is given by

$$\omega_f(x_0) := \bigcap_{n \in \mathbb{N}} \mathrm{cl}\left(\left\{f^k x_0 : k \ge n\right\}\right).$$

The next result follows immediately from Remark 1.7, Proposition 2.3 and [19, Theorem 1].

Theorem 2.4 *Let (X, d) be an sb-suprametric space and let $f : X \to X$ be a contractive mapping, that is, for all $x, y \in X$ with $x \ne y$,*

$$d(f x, f y) < d(x, y).$$

If there exists $x_0 \in X$ such that $\omega_f(x_0)$ is nonempty, then f has a unique fixed point and $\left\{f^n x\right\}_{n \in \mathbb{N}}$ converges to this fixed point for all $x \in X$.

Remark 2.5 Theorem 2.4 generalize [11, Theorem 1] and [3, Theorem 2.3]. In this connection, see also [12, Chapter 1, Theorem 1.2] and [5, Section 6].

3 Ekeland Variational Principle and Caristi Fixed Point Theorem

We first present a Cantor's intersection theorem in sb-suprametric spaces.

Theorem 3.1 *Let (X, d) be a complete sb-suprametric space, and let $\{C_n\}_{n\in\mathbb{N}}$ be a decreasing nested sequence of nonempty closed sets of X with*

$$\operatorname{diam}(C_n):=\sup\{d(x, y) : x, y \in C_n\} \to 0 \text{ as } n \to \infty.$$

Then $\bigcap_{n\in\mathbb{N}} C_n = \{z\}$ for some $z \in X$.

Proof Since C_n is nonempty for all n, we take z_n in every C_n. We then construct a Cauchy sequence $\{z_n\}$ because $d(z_m, z_n) \le \operatorname{diam}(C_N)$ for all m, n greater than some integer N and $\operatorname{diam}(C_N) \to 0$ as $N \to \infty$. Now, by completeness of (X, d), we deduce that $\{z_n\}$ converges to some z. Next, since $z_n \in C_N$ for all $n \ge N$ and C_N is closed, $z \in \bigcap_{n\ge N} C_n$, which implies by the nestedness property that $z \in \bigcap_{n\in\mathbb{N}} C_n$. Assume now that there exists $z' \in \bigcap_{n\in\mathbb{N}} C_n$ such that $z' \ne z$, then $d(z, z') > 0$, which implies that there exists $m \in \mathbb{N}$ such that $\operatorname{diam}(C_n) < d(z, z')$ for all $n \ge m$. Consequently, $z' \notin C_n$ for all $n \ge m$ and therefore $z' \notin \bigcap_{n\in\mathbb{N}} C_n$. This proves that $\bigcap_{n\in\mathbb{N}} C_n = \{z\}$. □

We next present an Ekeland's variational principle in the new spaces.

Theorem 3.2 *Let (X, d) be a complete sb-suprametric space $(b > 1)$ and let $\phi : X \to \mathbb{R} \cup \{\pm\infty\}$ be a lower semicontinuous function which is proper and lower bounded. Then, for every $x_0 \in X$ and $\varepsilon > 0$ with*

$$\phi(x_0) \le \inf_{x\in X} \phi(x) + \varepsilon,$$

there exist $x_\varepsilon \in X$ and a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that:

(i) $\lim_{i\to\infty} d(x_i, x_\varepsilon) = 0$.
(ii) $d(x_i, x_\varepsilon) \le 2^{-n}\varepsilon$, *for all* $i \in \mathbb{N}$.
(iii) $\phi(x_\varepsilon) + \sum_{i=0}^{\infty} b^{-i} d(x_\varepsilon, x_i) \le \phi(x_0)$.
(iv) $\phi(x_\varepsilon) + \sum_{i=0}^{\infty} b^{-i} d(x_\varepsilon, x_i) < \phi(x) + \sum_{i=0}^{\infty} b^{-i} d(x, x_i)$ *for all* $x \ne x_\varepsilon$.

Proof Let $x_0 \in X$ and $\varepsilon > 0$ and define the set

$$C_0:=\big\{x \in X : \phi(x) + d(x, x_0) \le \phi(x_0)\big\}.$$

Clearly, C_0 is nonempty and closed since it contains x_0, d is continuous and ϕ is lower semi-continuous. Now, for all $y \in C_0$, we have

$$d(x, x_0) \le \phi(x_0) - \phi(y) \le \phi(x_0) - \inf_{x\in X} \phi(x) \le \varepsilon. \tag{7}$$

Choose $x_1 \in C_0$ such that

$$\phi(x_1) + d(x_1, x_0) \leq \inf_{x \in C_0} \{\phi(x) + d(x, x_0)\} + (2b)^{-1}\varepsilon,$$

and consider the set

$$C_1 := \left\{x \in C_0 : \phi(x) + d(x, x_0) + b^{-1}d(x, x_1) \leq \phi(x_1) + d(x_1, x_0)\right\}.$$

By induction, we choose $x_{n-1} \in C_{n-2}$ $(n \geq 2)$ and consider

$$C_{n-1} := \left\{x \in C_{n-2} : \phi(x) + \sum_{i=0}^{n-1} b^{-i} d(x, x_i) \leq \phi(x_{n-1}) + \sum_{i=0}^{n-2} b^{-i} d(x_{n-1}, x_i)\right\}.$$

Then, we choose $x_n \in C_{n-1}$ such that

$$\phi(x_n) + \sum_{i=0}^{n-1} b^{-i} d(x_n, x_i) \leq \inf_{x \in C_{n-1}} \left\{\phi(x) + \sum_{i=0}^{n-1} b^{-i} d(x_{n-1}, x_i)\right\} + (2b)^{-n}\varepsilon. \quad (8)$$

Define again a set

$$C_n := \left\{x \in C_{n-1} : \phi(x) + \sum_{i=0}^{n} b^{-i} d(x, x_i) \leq \phi(x_n) + \sum_{i=0}^{n-1} b^{-i} d(x_n, x_i)\right\}. \quad (9)$$

Clearly, C_n is nonempty and closed since it contains x_n, d is continuous and ϕ is lower semicontinuous. Next, for all $y \in C_n$, we deduce from (8) and (9) that

$$\begin{aligned} b^{-n} d(y, x_n) &\leq \left[\phi(x_n) + \sum_{i=0}^{n-1} b^{-i} d(x_n, x_i)\right] - \left[\phi(y) + \sum_{i=0}^{n-1} b^{-i} d(y, x_i)\right] \\ &\leq \left[\phi(x_n) + \sum_{i=0}^{n-1} b^{-i} d(x_n, x_i)\right] - \inf_{x \in C_{n-1}} \left[\phi(x) + \sum_{i=0}^{n-1} b^{-i} d(x_n, x_i)\right] \\ &\leq (2b)^{-n}\varepsilon. \end{aligned}$$

Hence, for all $y \in C_n$, we have

$$d(y, x_n) \leq 2^{-n}\varepsilon. \quad (10)$$

this implies that (i) holds. Consequently, $\lim_{n\to\infty} \operatorname{diam}(C_n) = 0$ and the sequence $\{C_n\}_{n\in\mathbb{N}}$ is decreasing nested sequence of nonempty closed sets of X. So, by Theorem 3.1 it follows that $\bigcap_{n\in\mathbb{N}} C_n = \{x_\varepsilon\}$ for some $x_\varepsilon \in X$. Note that (ii) follows from (7) and (10). Now, since for all $x \neq x_\varepsilon$, $x \notin \bigcap_{n\in\mathbb{N}} C_n$, thus there exists $m \in \mathbb{N}$ such that $x \notin C_m$, then

$$\phi(x_m) + \sum_{i=0}^{m-1} b^{-i} d(x_m, x_i) < \phi(x) + \sum_{i=0}^{m} b^{-i} d(x, x_i).$$

But $x \notin C_m$ means that $x \notin C_k$ for all $k \geq m$, so from the previous inequalities we conclude that for all $k \geq m$, we have

$$\begin{aligned}\phi(x_\varepsilon) + \sum_{i=0}^{k} b^{-i} d(x_\varepsilon, x_i) &\leq \phi(x_k) + \sum_{i=0}^{k-1} b^{-i} d(x_k, x_i) \\ &\leq \phi(x_m) + \sum_{i=0}^{m-1} b^{-i} d(x_m, x_i) \leq \phi(x_0).\end{aligned}$$

Consequently (iii) and (iii) hold. □

Remark 3.3 Theorem 3.2 generalizes [6, Theorem 2.2].

Corollary 3.4 *Let (X, d) be a complete sb-suprametric space $(b > 1)$ and let $\phi : X \to \mathbb{R} \cup \{\pm\infty\}$ be a lower semi-continuous function which is proper and lower bounded. Then, for every $\varepsilon > 0$ there exist $x_\varepsilon \in X$ and a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that:*

(i) $\lim_{i\to\infty} d(x_i, x_\varepsilon) = 0$.

(ii) $\phi(x_\varepsilon) + \sum_{i=0}^{\infty} b^{-i} d(x_\varepsilon, x_i) \leq \inf_{x\in X} \phi(x) + \varepsilon$.

(iii) $\phi(x_\varepsilon) + \sum_{i=0}^{\infty} b^{-i} d(x_\varepsilon, x_i) \leq \phi(x) + \sum_{i=0}^{\infty} b^{-i} d(x, x_i)$ *for all* $x \in X$.

We next present a fixed point theorem in sb-suprametric spaces.

Theorem 3.5 *Let (X, d) be a complete sb-suprametric space $(b > 1)$. Let $f : X \to X$ be a mapping for which there exists a proper, lower semicontinuous and lower bounded function $\phi : X \to \mathbb{R} \cup \{\pm\infty\}$ such that for all $x \in X$,*

$$\frac{b^2 + \rho}{b - 1} d(x, f(x)) \leq \phi(x) - \phi(f(x)). \tag{11}$$

Then f has a fixed point.

Proof Assume that for all $x \in X$, $f(x) \neq x$. By applying Corollary 3.4, we deduce that for every $\varepsilon > 0$ there exist $x_\varepsilon \in X$ and a sequence $\{x_i\}_{i\in\mathbb{N}}$ in X such that:

$$\phi(x_\varepsilon) + \sum_{i=0}^{\infty} b^{-i} d(x_\varepsilon, x_i) \leq \phi(x) + \sum_{i=0}^{\infty} b^{-i} d(x, x_i),$$

for all $x \in X$. By taking $x = f(x_\varepsilon)$, where here $x \neq x_\varepsilon$, we get

$$\phi(x_\varepsilon) - \phi(f(x_\varepsilon)) < \sum_{i=0}^{\infty} b^{-i} d(f(x_\varepsilon), x_i) - \sum_{i=0}^{\infty} b^{-i} d(x_\varepsilon, x_i).$$

Now, by d_3' it follows that

$$\phi(x_\varepsilon) - \phi(f(x_\varepsilon)) < \sum_{i=0}^{\infty} b^{1-i} d(f(x_\varepsilon), x_\varepsilon) + \rho \sum_{i=0}^{\infty} b^{-i} d(f(x_\varepsilon), x_\varepsilon) d(x_\varepsilon, x_i).$$

Since $\lim_{i\to\infty} d(x_\varepsilon, x_i) = 0$, then there exists an integer $N > 0$ such that for all $i \geq N$, we have $d(x_\varepsilon, x_i) \leq 1$. Hence,

$$\phi(x_\varepsilon) - \phi(f(x_\varepsilon)) < \left(\frac{b^2+\rho}{b-1} + \max_{0\leq i\leq N}\left\{b^{-i} d(x_\varepsilon, x_i)\right\}\right) d(f(x_\varepsilon), x_\varepsilon).$$

Next, we take $x = x_\varepsilon$ in (11), we obtain

$$\frac{b^2+\rho}{b-1} d(x_\varepsilon, f(x_\varepsilon)) \leq \phi(x_\varepsilon) - \phi(f(x_\varepsilon)),$$

and this inequality combined with the previous one yield a contradiction. We conclude that f has a fixed point. □

The Caristi's fixed point theorem in sb-suprametric spaces follows immediately by taking $\psi = \frac{b-1}{b^2+\rho}\phi$ in the previous theorem.

Corollary 3.6 *Let (X, d) be a complete sb-suprametric space $(b > 1)$. Let $f : X \to X$ be a mapping for which there exists a proper, lower semicontinuous and lower bounded function $\psi : X \to \mathbb{R} \cup \{\pm\infty\}$ such that for all $x \in X$,*

$$d(x, f(x)) \leq \psi(x) - \psi(f(x)).$$

Then f has a fixed point.

Proposition 3.7 *Corollary 3.6 generalizes [17, Theorem 2.14].*

4 Strong *b*-Supranormed Spaces

In this section, we introduce the concept of strong b-supranormed spaces and derive some of its properties.

Definition 4.1 Let X be a nonempty linear space and $b \geq 1$, $\rho \geq 0$ are two real constants. A function $\|\cdot\| : X \to \mathbb{R}_+$ is called b-supranorm if the following conditions hold:

(n_1) $\|x\| = 0$ if and only if $x = 0$,
(n_2) $\|\lambda x\| = |\lambda|\|x\|$, for all $x \in X$ and $\lambda \in \mathbb{R}$
(n_3) $\|x + y\| \leq b(\|x\| + \|y\|) + \rho \|x\| \|y\|$ for all $x, y \in X$.

A pair $(X, \|\cdot\|)$ is called a b-supranorm space if X is a nonempty set and $\|\cdot\|$ is a b-supranorm. The pair $(X, \|\cdot\|)$ is called a supranorm space if $b = 1$.

Definition 4.2 Let X be a nonempty linear space and $b \geq 1$, $\rho \geq 0$ are two real constants. A function $\|\cdot\| : X \to \mathbb{R}_+$ is called strong b-supranorm (sb-supranorm) if it satisfies (n_1), (n_2) and

(n_3') $\|x + y\| \leq b\|x\| + \|y\| + \rho\|x\|\|y\|$ for all $x, y \in X$.

A pair $(X, \|\cdot\|)$ is called a strong b-supranormed (sb-supranormed) linear space if X is a nonempty set and $\|\cdot\|$ is a string b-supranorm. The pair $(X, \|\cdot\|)$ is called a strong supranormed linear space if $b = 1$.

Remark 4.3 Using (n_2), it follows that

(n_3'') n_3'' $\|x + y\| \leq \|x\| + b\,\|y\| + \rho\|x\|\|y\|$ for all $x, y \in X$.

Examples 4.4
- Clearly, strong b-normed spaces of [16] are sb-supranormed spaces.
- If $\|\cdot\|$ is an sb-supranorm linear space X, then the function $d\colon X \times X \to \mathbb{R}_+$ given by $d(x, y) = \|x - y\|$ is an sb-suprametric.
- Consider the set $X = \mathbb{R}^2$ endowed with a function $\|\cdot\|\colon X \to \mathbb{R}$ defined by

$$\|(x, y)\| = |x - y| + \min(|x|, |y|).$$

  It is not difficult to see that $(X, \|\cdot\|)$ is an sb-supranormed space for $b = \rho = 2$.

Remark 4.5 Let $(X, \|\cdot\|)$ be an sb-supranormed linear space. If a sequence $\{x_n\}$ converges simultaneously to x and y, that is, $\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \|x_n - y\| = 0$, then $x = y$, and this follows from (n_1) and (n_3'), since we have

$$\|x - y\| \leq \|x - x_n\| + b\|x_n - y\| + \rho\|x - x_n\|\|x_n - y\|.$$

Moreover, we have the following inequality:

$$\left\|\sum_{i=0}^{n} x_i\right\| \leq b \max\{1, \rho^n\} \sum_{i=1}^{n} \mathbf{e}_i(\|x_0\|, \ldots, \|x_n\|),$$

for all $n \in \mathbb{N}$ and $x_0, \ldots, x_n \in X$.

Lemma 4.6 *Let $(X, \|\cdot\|)$ be an sb-supranormed linear space. Then, $\|\cdot\|$ is a continuous function.*

Proof Assume that $\rho > 0$ and let $x, y \in X$, then

$$\|x\| = \|y + (x - y)\| \leq \|y\| + b\,\|x - y\| + \rho\|y\|\,\|x - y\|,$$

and consequently,

$$\frac{\|x\| - \|y\|}{b + \rho\|y\|} \leq \|x - y\|. \tag{12}$$

Similarly,

$$\frac{\|y\| - \|x\|}{b + \rho\|x\|} \leq \|x - y\|. \tag{13}$$

Hence, from (12) and (13), one gets

$$\frac{\rho(\|x\| - \|y\|)^2}{(b + \rho\|x\| + \rho\|y\|)^2} \leq 2\|x - y\|.$$

Therefore, if $\|x_n - x\| \to 0$ as $n \to \infty$, then

$$\frac{\rho(\|x_n\| - \|x\|)^2}{(b + \rho\|x_n\| + \rho\|x\|)^2} \leq 2\|x_n - x\| \to 0 \text{ as } n \to \infty,$$

and using Remark 1.6 it follows that $\|x_n\| \to \|x\|$ as $n \to \infty$, which implies that $\|\cdot\|$ is a continuous function. □

Let (X_n, d) be an n-dimensional sb-supranormed linear space and let $\{u_1, \ldots, u_n\}$ be a base of X_n. For every $x \in X$ there exist unique coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{n} \alpha_i u_i,$$

and define $\|\cdot\|_0 \colon X \to \mathbb{R}_+$ by

$$\|x\|_0 = \sum_{i=1}^{n} \mathbf{e}_i(|\alpha_1|, \ldots, |\alpha_n|).$$

Theorem 4.7 *Let $(X_n, \|\cdot\|)$ be an n-dimensional sb-supranormed linear space. Then, there exists $\beta > 0$ such that*

$$\|x\| \leq \beta\|x\|_0, \quad \textit{for all } x \in X. \tag{14}$$

Proof Let $x \in X_n$ and $\{u_1, \ldots, u_n\}$ be a base of X_n. Then, there exist unique coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^{n} \alpha_i u_i$. Hence,

$$\begin{aligned}
\|x\| &= \left\| \sum_{i=1}^{n} \alpha_i u_i \right\| \\
&\leq b \max\{1, \rho^n\} \sum_{i=1}^{n} \mathbf{e}_i(|\alpha_1|\, \|u_1\|, \ldots, |\alpha_n|\, \|u_n\|) \\
&\leq b \max\{1, \rho^n\} \Big(\max_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} \prod_{k=1}^{i} \|u_{j_k}\| \Big) \sum_{i=1}^{n} \mathbf{e}_i(|\alpha_1|, \ldots, |\alpha_n|) \\
&= \beta\, \|x\|_0,
\end{aligned}$$

where

$$\beta := b \max\{1, \rho^n\} \Big(\max_{1 \leq j_1 < j_2 < \cdots < j_i \leq n} \prod_{k=1}^{i} \|u_{j_k}\| \Big). \tag{15}$$

□

Theorem 4.8 *Let $(X_n, \|\cdot\|)$ be an n-dimensional sb-supranormed linear space. Then, $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent, that is, there exists $\alpha, \beta > 0$ such that for all $x \in X$*

$$\alpha\|x\|_0 \leq \|x\| \leq \beta\|x\|_0.$$

Proof Let $x \in X_n$ and $\{u_1, \ldots, u_n\}$ be a base of X_n. Then, there exist unique coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n \alpha_i u_i$. Define a set U by

$$U := \{x \in X : \|x\|_0 = 1\}.$$

We first show that U is bounded. Let $\alpha_1^j, \ldots, \alpha_n^j \in \mathbb{R}$ for $j = 1, 2$ such that $x_j = \sum_{i=1}^n \alpha_i^j u_i \in U$. Then

$$\begin{aligned}
\|x_1 - x_2\| &= \left\| \sum_{i=1}^n (\alpha_i^1 - \alpha_i^2) e_i \right\| \\
&\leq b \max\{1, \rho^n\} \sum_{i=1}^n \mathbf{e}_i(|\alpha_1^1 - \alpha_1^2|\, \|u_1\|, \ldots, |\alpha_n^1 - \alpha_n^2|\, \|u_n\|) \\
&\leq \beta \sum_{i=1}^n \mathbf{e}_i(|\alpha_1^1 - \alpha_1^2|, \ldots, |\alpha_n^1 - \alpha_n^2|) \\
&\leq \beta \sum_{i=1}^n \mathbf{e}_i(|\alpha_1^1| + |\alpha_1^2|, \ldots, |\alpha_n^1| + |\alpha_n^2|) \\
&\leq \beta K,
\end{aligned}$$

where β is given by (15) and $K := \max\left\{2^k n \binom{n}{k} : k = 1, \ldots, n\right\}$, which proves that U is bounded.

Define now a function $\phi\colon X \to \mathbb{R}_+$ by $\phi(x) = \|x\|$. It follows by Lemma 4.6 that f is continuous. Note that U is strongly compact, since it is bounded and closed subset of $\mathbb{R}^n$. Using Theorem 1.19, we deduce that ϕ has an infimum α in U, which is different from zero because $\|x\|_0 = 1$ for every vector $x \in U$. Hence,

$$\alpha = \inf\{\phi(x) : x \in U\} = \inf\{\|x\| : x \in U\} > 0.$$

Thus, from the fact $\frac{x}{\|x\|_0} \in U$ for all $x \in X$, it follows that

$$\left\| \frac{x}{\|x\|_0} \right\| \geq \alpha > 0, \quad \text{for all } x \in X,$$

which implies that

$$\alpha\|x\|_0 \leq \|x\|, \quad \text{for all } x \in X. \tag{16}$$

Finally, combine (14) and (16), we obtain the result. □

Remark 4.9 As an immediate consequence of Theorem 4.8, any two sb-supranorms on a finite-dimensional space are equivalent.

Lemma 4.10 *Let $(X_n, \|\cdot\|)$ be an n-dimensional sb-supranormed linear space, let U be a bounded set of X. Then, U is compact.*

Proof Let $x \in X_n$ and $\{u_1, \ldots, u_n\}$ be a base of X_n. Then, there exist unique coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n \alpha_i u_i$. Let $\overline{x} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Define the function $\phi\colon U \to \mathbb{R}^n$ by $\phi(x) = \overline{x}$ for all $x \in U$ and let $V = \phi(U)$. Since the function $\|\cdot\|_1\colon X \to \mathbb{R}_+$ by $\|x\|_1 := \|\phi(x)\|_n$ is an sb-supranorm on X_n, where $\|\cdot\|_n$ is an sb-supranorm on $\mathbb{R}^n$. Hence, according to Remark 4.9, $\|\cdot\|_1$ is equivalent to the sb-supranorm $\|\cdot\|$. We conclude that there exist $\alpha, \beta > 0$ such that

$$\alpha\|\overline{x}\|_n \leq \|x\| \leq \beta\|\overline{x}\|_n. \tag{17}$$

As consequences of (17), U bounded in X if and only if U bounded in $\mathbb{R}^n$, and a sequence $\{x_n\}$ is convergent in $(X, \|\cdot\|)$ if and only if the corresponding sequence $\{\overline{x}_n\}$ is convergent in $\mathbb{R}^n$. Consequently, the compactness of U bounded in X follows from the compactness of U bounded in $\mathbb{R}^n$. □

5 Brouwer and Schauder Fixed Point Principles

We first recall the Brouwer fixed point principle in $\mathbb{R}^n$.

Theorem 5.1 (Brouwer) *Let U be a bounded closed convex set of $\mathbb{R}^n$. If a mapping $f\colon U \to U$ is continuous, then it has a fixed point.*

The Brouwer fixed point principle in sb-supranormed space is given next.

Theorem 5.2 *Let $(X_n, \|\cdot\|)$ be an n-dimensional sb-supranormed linear space and let U be a bounded closed convex set of X_n. If a mapping $f\colon U \to U$ is continuous, then it has a fixed point.*

Proof Let $x \in X_n$ and $\{u_1, \ldots, u_n\}$ be a base of X_n. Then, there exist unique coefficients $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that $x = \sum_{i=1}^n \alpha_i u_i$. Let $\overline{x} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$. Define the function $\phi\colon U \to \mathbb{R}^n$ by $\phi(x) = \overline{x}$ for all $x \in U$ and let $V = \phi(U)$. The function $\phi\colon U \to V$ is bijective, where the mapping $\phi^{-1}\colon V \to U$ is given by $\phi^{-1}(\overline{x}) = x$ for all $\overline{x} \in V$. Next, we will prove several claims: □

Claim 1 $\phi\colon U \to V$ is an homeomorphism. Indeed ϕ is continuous in U, since by (16), we have

$$\|\phi(x) - \phi(x_0)\|_0 = \|\overline{x} - \overline{x}_0\|_0 \leq \alpha^{-1}\|x - x_0\|, \quad \text{for all } x, x_0 \in U,$$

Similarly, ϕ^{-1} is continuous in V, because by (14), we have

$$\|\phi^{-1}(\overline{x}) - \phi^{-1}(\overline{x}_0)\| = \|x - x_0\| \leq \beta\|\overline{x} - \overline{x}_0\|_0, \quad \text{for all } \overline{x}, \overline{x}_0 \in V.$$

Hence, Claim 1 holds.

Claim 2 V is convex. Let $\overline{x} = (\alpha_1, \ldots, \alpha_n)$, $\overline{y} = (\beta_1, \ldots, \beta_n) \in V$, where $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \ldots, n$. For all $\lambda \in [0, 1]$, we obtain by convexity of U that

$$\begin{aligned}\lambda\overline{x} + (1-\lambda)\overline{y} &= (\lambda\alpha_1 + (1-\lambda)\beta_1, \ldots, \lambda\alpha_n + (1-\lambda)\beta_n) \\ &= \phi(\lambda x + (1-\lambda)y) \in V,\end{aligned}$$

which implies that Claim 2 holds.

Claim 3 V is bounded. Let $\overline{x}, \overline{y} \in V$, then by the boundedness of U and Theorem 4.7 it follows that

$$\|\overline{x} - \overline{y}\|_0 \leq \alpha^{-1}\|x - y\| \leq \alpha^{-1}d(U),$$

where $d(U) := \max\{\|x - y\| : x, y \in U\}$, which proves Claim 3.

Claim 4 V is closed. Let $x = \sum_{i=1}^{n} \alpha_i u_i$, $x_0 = \sum_{i=1}^{n} \beta_i u_i$, $\overline{x} = (\alpha_1, \ldots, \alpha_n) \in V$, $\overline{x}_0 = (\beta_1, \ldots, \beta_n)$, where $\alpha_i, \beta_i \in \mathbb{R}$ for $i = 1, \ldots, n$. Assume that $\|\overline{x} - \overline{x}_0\|_0$ tends to zero. Now, by (16), we have

$$\|x - x_0\| \leq \beta\|\overline{x} - \overline{x}_0\|_0,$$

so it follows that by closedness of U that $x_0 \in U$, which implies that $\overline{x}_0 \in V$ and this prove the claim.

Claim 5 f has a fixed point. To show this, define the function $F : V \to V$ by $F = \phi f \phi^{-1}$. By Theorem 5.1 and the previous claims, we deduce that there exists $\overline{x} \in V$ such that $F(x) = x$, that is,

$$\phi f \phi^{-1}(x) = x,$$

which is equivalent to $f(\phi^{-1}(x)) = \phi^{-1}(x)$, and since $\phi^{-1}(x) \in U$, then f has a fixed point in U. □

Before establishing the fixed point principle of Schauder type in sb-supranormed spaces, we need to develop some auxiliary results. Let $(E, \|\cdot\|)$ be an sb-supranormed linear space and $N := \{c_1, \ldots, c_n\}$ be a finite subset of E. For any fixed $\varepsilon > 0$, define the set

$$(N, \varepsilon) := \bigcup_{i=1}^{n} B(c_i, \varepsilon),$$

where

$$B(c_i, \varepsilon) := \{x \in E : \|x - c_i\| < \varepsilon\}, \quad i = 1, \ldots, n.$$

Define a mapping $\mu_i : (N, \varepsilon) \to \mathbb{R}$ by

$$\mu_i(x) := \max\left[0, \varepsilon - \|x - c_i\|\right], \quad i = 1, \ldots, n.$$

Consider the Schauder projection $p_\varepsilon\colon (N,\varepsilon) \to \operatorname{conv}(N)$ given by

$$p_\varepsilon(x) = \left[\sum_{i=1}^{n} \mu_i(x)\right]^{-1} \sum_{i=1}^{n} \mu_i(x) c_i,$$

Note that $p_\varepsilon((N,\varepsilon)) \subset \operatorname{conv}(N)$ as a convex combination of $\{c_1, \dots, c_n\}$. Moreover, if $x \in (N,\varepsilon)$, then there exists i such that $x \in B(C_i, \varepsilon)$, so $\sum_{i=1}^{n} \mu_i(x) \neq 0$, which means that p_ε is well defined.

Lemma 5.3 *Let $(E, \|\cdot\|)$ be an sb-supranormed linear space, U be a convex subset of E and $N = \{c_1, \dots, c_n\} \subset U$. Then for a sufficiently small $\varepsilon > 0$, we have*

(i) $\|x - p_\varepsilon(x)\| \leq n\, b\, \varepsilon \max\{1, \rho^n\}$ *for all $x \in (N,\varepsilon)$,*
(ii) $p_\varepsilon\colon (N,\varepsilon) \to \operatorname{conv}(N) \subset U$ *is a continuous compact mapping.*

Proof Let $\varepsilon \in (0,1]$ be sufficiently small such that for every $1 \leq i \leq n$ and any x in (N,ε), $P_i(x) \leq P_1(x)$, where $P_i(x) := \mathbf{e}_i(\mu_1(x), \dots, \mu_n(x))$. Then,

$$\begin{aligned}
\|x - p_\varepsilon(x)\| &= \left[\sum_{i=1}^{n} \mu_i(x)\right]^{-1} \left\|\sum_{i=1}^{n} \mu_i(x)(x - c_i)\right\| \\
&\leq b \max\{1, \rho^n\}(P_1(x))^{-1} \sum_{i=1}^{n} \mathbf{e}_i(\mu_1(x)\|x - c_1\|, \dots, \mu_n(x)\|x - c_n\|) \\
&\leq b \max\{1, \rho^n\}(P_1(x))^{-1} \sum_{i=1}^{n} \mathbf{e}_i(\mu_1(x)\varepsilon, \dots, \mu_n(x)\varepsilon) \\
&\leq b\, \varepsilon \max\{1, \rho^n\}(P_1(x))^{-1} \sum_{i=1}^{n} P_i(x) \\
&\leq n\, b\, \varepsilon \max\{1, \rho^n\}.
\end{aligned}$$

Now, since p_ε is a finite sum of continuous functions and $\|\cdot\|$ is continuous according to Lemma 4.6, then p_ε is continuous. The compactness of p_ε follows from Lemma 4.10, since its codomain is with finite-dimension. □

Lemma 5.4 *Let X be a topological space and E be an sb-supranormed linear space. Let U be a convex set of E and $f\colon X \to U$ be a compact mapping. Then for a sufficiently small $\varepsilon > 0$, there exists a finite set*

$$N = \{c_1, \dots, c_n\} \subset f(X) \subset U,$$

and a finite-dimensional mapping $f_\varepsilon\colon X \to U$ such that:

(i) $\|f_\varepsilon(x) - f(x)\| \leq n\, b\, \varepsilon \max\{1, \rho^n\}$ *for all $x \in X$,*
(ii) $f_\varepsilon(X) \subset \operatorname{conv}(N) \subset U$.

Proof **(i):** By Theorem 1.20 and for sufficiently small $\varepsilon \in (0,1)$ there exists a finite ε-net $\{c_1, \dots, c_n\} \subset f(X)$ because $f(X)$ is compact in E. Now, if $y \in f(X)$, then $d(y, c_i) < \varepsilon$ for some $i \in \{1, \ldots, n\}$, thus $y \in B(c_i, \varepsilon)$, so $y \in (N, \varepsilon)$ and this proves that $f(X) \subset (N, \varepsilon)$. Let $f_\varepsilon = p_\varepsilon f$. We deduce by Lemma 5.3 that

$$\|f_\varepsilon(x) - f(x)\| = \|p_\varepsilon y - y\| \leq n\, b\, \varepsilon \max\{1, \rho^n\},$$

where $y = f(x) \in (N, \varepsilon)$, for all $x \in X$.

(ii): Let $y \in f_\varepsilon(X)$. Thus, there is $z = f(x) \in (N, \varepsilon)$ for some $x \in X$ such that $y = p_\varepsilon(z)$. Consider

$$y = p_\varepsilon(z) = \sum_{i=1}^{n} \lambda_i c_i, \quad \sum_{i=1}^{n} \lambda_i = 1, \quad \lambda_i \in \mathbb{R}, \quad i = 1, \ldots, n.$$

Thus, $y \in \mathrm{conv}(N) \subset U$, and by convexity of U it follows that $f_\varepsilon \subset \mathrm{conv}(N) \subset U$. □

Let (X, d) be an sb-suprametric space, U be a nonempty set of X and $f : U \to X$ be a given mapping. If for a given $\varepsilon > 0$, there exists a point $x \in U$ such that $d(x, f(x)) < \varepsilon$, then we say that x is an *ε-fixed point* for f.

Theorem 5.5 *Let (X, d) be an sb-suprametric space and U be a closed set of X. If a mapping $f : U \to X$ is compact, then f has a fixed point if and only if for each $\varepsilon > 0$ it has an ε-fixed point.*

Proof The necessary condition is trivial, so we only show the sufficient condition. Let $\varepsilon_n = \frac{1}{n}$, $n \in \mathbb{N}$. Assume there exists $u_n \in U$ for all $n \in \mathbb{N}$ such that u_n are ε_n-fixed point, that is,

$$d(u_n, f(u_n)) < \tfrac{1}{n}, \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

The mapping f is compact, so there exists a compact K such that $f(X) \subseteq K$. Thus, there exists a subsequence $\{u_{n_k}\}$ such that $f(u_{n_k})$ converges to some $u \in X$ as k tends to infinity. Now, using (18), it follows that for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for $k \geq k_0$, we have

$$\begin{aligned} d(u_{n_k}, u) &\leq bd(u_{n_k}, f(u_{n_k})) + d(f(u_{n_k}), u) + \rho\, d(u_{n_k}, f(u_{n_k}))d(f(u_{n_k}), u) \\ &\leq \tfrac{b}{n_k} + \varepsilon + \tfrac{\rho\,\varepsilon}{n_k} < \varepsilon(b + 1 + \varepsilon), \end{aligned}$$

which implies that $\{u_{n_k}\}$ converges to u in U because U is closed. Observe that $\{f(u_{n_k})\}$ converges to u and by continuity of f it converges also to $f(u)$, which means by Proposition 1.11 that $u = f(u)$. □

Remark 5.6 In Theorem 5.5, if $f : U \to U$ is compact, the assumption of closeness of U may be dropped, since the sequence $f(u_{n_k})$ converges to some $u \in \mathrm{cl}\,(f(U))$ which is a subset of U.

Finally, we present a Schauder fixed point principle.

Theorem 5.7 *Let $(X, \|\cdot\|)$ be an sb-supranormed linear space and U be a convex set (not necessarily closed) of X. If a mapping $f: U \to U$ is compact, then it has a fixed point.*

Proof It suffice to show that f has an ε-fixed point. By Lemma 5.4 it follows that for a sufficiently small $\varepsilon > 0$ there exists $f_\varepsilon : U \to U$ such that

(i) $\|f_\varepsilon(x) - f(x)\| \leq n\, b\, \varepsilon\, \max\{1, \rho^n\}$ for all $x \in U$,
(ii) $f_\varepsilon(U) \subset \mathrm{conv}(N) \subset U$.

Since $\mathrm{conv}(N) \subset U$, we get $f_\varepsilon(\mathrm{conv}(N)) \subset f_\varepsilon(U) \subset \mathrm{conv}(N)$, which implies that $f_\varepsilon : \mathrm{conv}(N) \to \mathrm{conv}(N)$ is well defined. Since $\mathrm{conv}(N)$ is bounded closed convex (see also [24, Propositions C.2 and C5]), we deduce by Theorem 5.2 that there exists $x_\varepsilon \in \mathrm{conv}(N) \subset U$ such that $f_\varepsilon x_\varepsilon = x_\varepsilon$, so by (i), we obtain

$$\|f(x_\varepsilon) - x_\varepsilon\| = \|f(x_\varepsilon) - f_\varepsilon(x_\varepsilon)\| \leq n\, b\, \varepsilon\, \max\{1, \rho^n\},$$

and by letting ε tends to zero together with the continuity of f, we obtain the result by Theorem 5.5 and Remark 5.6. □

Remark 5.8 Observe that U is not necessary closed, since Theorem 5.2 is applied to the selfmap f_ε defined on the closed set $\mathrm{conv}(N)$. Moreover and according to Remark 5.6, Theorem 5.5 can be applied without requiring the closeness of U. This answer the question in [9, Remark 13].

6 Applications

In this section, we study the existence of a unique solution to an integral equation as well as to a boundary value problem, as applications to the fixed point theorem proved in Section 2. We consider the integral equation:

$$x(t) = \lambda(t) + \int_0^1 G(t,s)h(s,x(s))\mathrm{d}s, \quad t \in [0,1]. \tag{19}$$

The problem of existence of a solution for the integral equation (19) will be discussed under the following assumptions:

(a_1) $\lambda\colon [0,1] \to \mathbb{R}_+$ is a continuous function.
(a_2) $h\colon [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and there exists a continuous function $u\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ such that for all $(s,p,q) \in [0,1] \times \mathbb{R}_+^2$,

$$u(p,p) = 0, \tag{20a}$$
$$|h(s,p) - h(s,q)| \leq u(p,q), \tag{20b}$$
$$u(p,q)^2 + \tfrac{1}{2}u(p,q) \leq |p-q|^2 + \tfrac{1}{2}|p-q|. \tag{20c}$$

(a_3) $G\colon [0,1]^2 \to \mathbb{R}_+$ is a continuous function such that

$$c := \max_{s,t \in [0,1]} G(s,t) < 1.$$

Before presenting the main result of this section, we derive an inequality of Chebyshev type. For more details on Chebyshev inequalities, we refer to Chapter IX of [20] and for more recent references, see for instance [1, 23].

Lemma 6.1 *Let a and b be real numbers such that $a < b$, and let $w(t, \cdot)$ be a nonnegative measurable function for every $t \in [a, b]$. Let $x(s) = (x_1(s), x_2(s), \ldots, x_n(s))$ such that $\{x_i\}_{1 \le i \le n}$ are nonnegative functions defined on $[a, b]$, and let u be a nonnegative function defined on $[a, b] \times \mathbb{R}_+^n$ such that $s \mapsto u(s, x(s))$ is integrable with respect to $w(t, s)$ for every $t \in [a, b]$. Then*

$$\left(\int_a^b u(s, x(s))w(t, s)\mathrm{d}s\right)^2 \le \int_a^b u(s, x(s))^2 w(t, s)\mathrm{d}s \int_a^b w(t, s)\mathrm{d}s, \ \ t \in [a, b].$$

Proof We have

$$\begin{aligned}
0 &\le \int_a^b \int_a^b \big(u(r, x(r)) - u(s, x(s))\big)^2 w(t, s)w(t, r)\mathrm{d}s\mathrm{d}r \\
&= \int_a^b \int_a^b \big(u(r, x(r))^2 - 2u(r, x(r))u(s, x(s)) + u(s, x(s))^2\big) w(t, s)w(t, r)\mathrm{d}s\mathrm{d}r \\
&= \int_a^b \Big(u(r, x(r))^2 \int_a^b w(t, s)\mathrm{d}s - 2u(r, x(r)) \int_a^b u(s, x(s))w(t, s)\mathrm{d}s \\
&\quad + \int_a^b u(s, x(s))^2 w(t, s)\mathrm{d}s\Big) w(t, r)\mathrm{d}r \\
&= 2\int_a^b u(s, x(s))^2 w(t, s)\mathrm{d}s \int_a^b w(t, s)\mathrm{d}s - 2\left(\int_a^b u(s, x(s))w(t, s)\mathrm{d}s\right)^2.
\end{aligned}$$

□

Theorem 6.2 *Under assumptions (a_1)–(a_3), the integral equation (19) has a unique solution in $C_+([0, 1])$.*

Proof Let $X = C_+([0, 1])$ be the set of continuous functions $x : [0, 1] \to \mathbb{R}_+$, endowed with the suprametric δ of Examples 1.4. First, by Remark 1.14, (X, δ) is a complete. Consider the operator $T : X \to X$ defined by

$$Tx(t) = \lambda(t) + \int_0^1 G(t, s)h(s, x(s))\mathrm{d}s, \quad t \in [0, 1].$$

Observe first that T is well defined. Let $x, y \in X$, then by using the assumptions (a_1)–(a_3) and Lemma 6.1, we get

$$\begin{aligned}
&|Tx(t) - Ty(t)|(|Tx(t) - Ty(t)| + \tfrac{1}{2}) \\
&= \left|\int_0^1 G(t, s)\big(h(s, x(s)) - h(s, y(s))\big)\mathrm{d}s\right| \left(\left|\int_0^1 G(t, s)\big(h(s, x(s)) - h(s, y(s))\big)\mathrm{d}s\right| + \tfrac{1}{2}\right) \\
&\le \int_0^1 G(t, s)\big|h(s, x(s)) - h(s, y(s))\big|\mathrm{d}s \left(\int_0^1 G(t, s)\big|h(s, x(s)) - h(s, y(s))\big|\mathrm{d}s + \tfrac{1}{2}\right) \\
&\le \int_0^1 G(t, s)u(x(s), y(s))\mathrm{d}s \left(\int_0^1 G(t, s)u(x(s), y(s))\mathrm{d}s + \tfrac{1}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_0^1 G(t,s)\mathrm{d}s \int_0^1 G(t,s)u(x(s),y(s))^2\mathrm{d}s + \tfrac{1}{2}\int_0^1 G(t,s)u(x(s),y(s))\mathrm{d}s \\
&\leq \int_0^1 G(t,s)\big(u(x(s),y(s))^2 + \tfrac{1}{2}u(x(s),y(s))\big)\mathrm{d}s \\
&\leq \int_0^1 G(t,s)|x(s)-y(s)|\big(|x(s)-y(s)| + \tfrac{1}{2}\big)\mathrm{d}s \\
&\leq c\delta(x,y),
\end{aligned}
$$

and this implies

$$\delta(Tx, Ty) \leq c\delta(x, y).$$

By Theorem 2.1, we conclude that the integral equation (19) has a unique solution in X. ☐

Next by Theorem 6.2, we show the existence of a unique solution in $C_+[0,1]$ to the following nonlinear third-order boundary value problem:

$$x'''(t) + \sqrt{t\,x(t)+1}(1 - e^{-t\,x(t)}) = 0, \quad t \in [0,1], \tag{22a}$$

$$x(0) = x'(1) = 0 \text{ and } x(1) = 1. \tag{22b}$$

Proposition 6.3 *The boundary value problem (22) has a unique solution in $C_+[0,1]$.*

Proof The boundary value problem (22) has a solution $x \in C_+[0,1]$ if and only if the operator $T\colon C_+[0,1] \to C_+[0,1]$ defined by

$$Tx(t) = \int_0^1 G(t,s)\sqrt{s\,x(s)+1}(1 - e^{-s\,x(s)})\mathrm{d}s, \quad t \in [0,1],$$

has a fixed point in $C_+[0,1]$, where the Green's function associated to the homogeneous problem $x'''(t) = 0$ that satisfies the boundary condition (22b) is given by

$$G(t,s) = \begin{cases} \frac{1}{2}s^2(t-1)^2, & 0 \leq s \leq t \leq 1, \\ \frac{1}{2}t(s-1)(s(t-2)+t)), & 0 \leq t \leq s \leq 1. \end{cases}$$

Firstly, observe that T is well defined and (a_1) holds, where $\lambda = 0$. Moreover, it is easy to see that G is continuous and satisfies (a_3), since we have

$$0 \leq G(t,s) \leq \tfrac{1}{2}, \text{ for all } t, s \in [0,1].$$

Consider now the functions $h\colon [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ and $u\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ given by

$$h(s,p) = \sqrt{s\,p+1}(1 - e^{-s\,p}) \text{ and } u(p,q) = \sqrt{|p-q|+1}(1 - e^{-|p-q|}).$$

In order to use Theorem 6.2 and conclude that T has a unique solution in $C_+[0,1]$, we have to check (a_2). Note that h and u are continuous and it is not difficult to see that (20b) and (20c) follow from the next lemma. □

Lemma 6.4 *For all* $(p,q,s) \in \mathbb{R}_+^2 \times [0,1]$, *we have* $A \geq 0$ *and* $B \geq 0$, *where*

$$A = \sqrt{|p-q|+1}(1-e^{-|p-q|}) - |\sqrt{p+1}(1-e^{-p}) - \sqrt{q+1}(1-e^{-q})|,$$
$$B = p^2 + \tfrac{1}{2}p - (sp+1)(1-e^{-sp})^2 - \tfrac{1}{2}\sqrt{sp+1}(1-e^{-sp}).$$

Proof Suppose, without loss of generality, that $p > q$. Then,

$$A = \sqrt{p-q+1}(1-e^{q-p}) - \sqrt{p+1}(1-e^{-p}) + \sqrt{q+1}(1-e^{-q})$$

Using the mean value theorem twice, it follows that there exists $c \in (q,p)$ such that

$$A = \sqrt{p-q+1}(1-e^{q-p}) - \tfrac{1}{2}(p-q)\tfrac{1+e^{-c}+2ce^{-c}}{\sqrt{c+1}},$$

and also there exists $c' \in (p-q,p)$ such that

$$A = \sqrt{q+1}(1-e^{-q}) - \tfrac{1}{2}q\tfrac{1+e^{-c'}+2c'e^{-c'}}{\sqrt{c'+1}}.$$

Now, since the function $h_1 \colon \mathbb{R}_+ \to \mathbb{R}$ given by $h_1(t) = \frac{1+e^{-t}+2te^{-t}}{\sqrt{t+1}}$ is decreasing on $\mathbb{R}_+$, we obtain

$$A \geq \sqrt{p-q+1}(1-e^{q-p}) - \tfrac{1}{2}(p-q)\tfrac{1+e^{-q}+2qe^{-q}}{\sqrt{q+1}},$$

and

$$A \geq \sqrt{q+1}(1-e^{-q}) - \tfrac{1}{2}q\tfrac{1+e^{-(p-q)}+2(p-q)e^{-(p-q)}}{\sqrt{p-q+1}}.$$

Hence, it suffice to know the sign of $h_2(p-q) - h_1(q)$ and $h_2(q) - h_1(p-q)$, where the function $h_2 \colon \mathbb{R}_+ \to \mathbb{R}$ is given by $h_2(t) = 2t^{-1}\sqrt{t+1}(1-e^{-t})$. It is not difficult to see that h_2 is decreasing, so if $p-q \leq q$, $h_2(p-q)-h_1(q) \geq h_2(p-q)-h_1(p-q)$ and if $p-q > q$, $h_2(q) - h_1(p-q) \geq h_2(q) - h_1(q)$. We conclude from the fact that $t \mapsto (h_2 - h_1)(t)$ is positive that $A \geq 0$. Finally, we have

$$B \geq p^2 + \tfrac{1}{2}p - (sp+1)(1-e^{-sp})^2 - \sqrt{sp+1}(1-e^{-sp})$$
$$\geq (sp)^2 + \tfrac{1}{2}sp - (sp+1-\sqrt{sp+1})(1-e^{-sp}) \geq 0.$$

□

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References

1. Bakherad, M., Dragomir, S.S.: Noncommutative Chebyshev inequality involving the Hadamard product. Azerb. J. Math. **9**, 46–58 (2019)
2. Berinde, V., Păcurar, M.: The early developments in fixed point theory on b-metric spaces. Carpathian J. Math. **38**, 523–538 (2022)
3. Berzig, M.: First results in suprametric spaces with applications. Mediterr. J. Math. **19**, 1–18 (2022). https://doi.org/10.1007/s00009-022-02148-6
4. Berzig, M.: Nonlinear contraction in b-suprametric spaces. J. Anal. (2024). https://doi.org/10.1007/s41478-024-00732-5
5. Berzig, M., Kedim, I.: Eilenberg–Jachymski collection and its first consequences for the fixed point theory. J. Fixed Point Theory Appl. **23**, 1–13 (2021). https://doi.org/10.1007/s11784-021-00854-4
6. Bota, M., Molnar, A., Varga, C.: On Ekeland's variational principle in b-metric spaces. Fixed Point Theory **12**, 21–28 (2011)
7. Chifu, C., Petruşel, G.: Fixed points for multivalued contractions in b-metric spaces with applications to fractals. Taiwan. J. Math. **18**, 1365–1375 (2014). https://doi.org/10.11650/tjm.18.2014.4137 https://doi.org/10.11650/tjm.18.2014.4137 https://doi.org/10.11650/tjm.18.2014.4137
8. Czerwik, S.: Contraction mappings in b-metric spaces. Acta Math. Univ. Ostrav. **1**, 5–11 (1993)
9. Czerwik, S.: On b-metric spaces and Brower and Schauder fixed point principles. In: Approximation Theory and Analytic Inequalities, pp. 71–86. Springer (2021). https://doi.org/10.1007/978-3-030-60622-0_6
10. Dunford, N., Schwartz, J.: Linear Operators, Part 1: General Theory, vol. 10. Wiley, New York (1988)
11. Edelstein, M.: On fixed and periodic points under contractive mappings. J. Lond. Math. Soc. **1**, 74–79 (1962). https://doi.org/10.1112/jlms/s1-37.1.74
12. Goebel, K., Reich, S.: Uniform Convexity, Hyperbolic Geometry, and Non-expansive Mappings. Marcel Dekker, New York (1984)
13. Kajántó, S., Lukács, A.: A note on the paper "contraction mappings in b-metric spaces" by Czerwik. Acta Math. Univ. Ostrav. **10**, 85–89 (2018). https://doi.org/10.2478/ausm-2018-0007
14. Karapınar, E.: A short survey on the recent fixed point results on b-metric spaces. Constr. Math. Anal. **1**, 15–44 (2018). https://doi.org/10.33205/cma.453034
15. Karapınar, E., Noorwali, M.: Dragomir and Gosa type inequalities on b-metric spaces. J. Inequal. Appl. **2019**, 1–7 (2019). https://doi.org/10.1186/s13660-019-1979-9
16. Kirk, W., Shahzad, N.: Fixed Point Theory in Distance Spaces. Springer, Cham (2014)
17. Kirk, W., Shahzad, N.: Fixed points and Cauchy sequences in semimetric spaces. J. Fixed Point Theory Appl. **17**, 541–555 (2015). https://doi.org/10.1007/s11784-015-0233-4
18. Liusternik, K., Sobolev, V.: Elements of Functional Analysis. Hindustan Publishing Co, New Delhi (1974)
19. Liepiņš, A.: Edelstein's fixed point theorem in topological spaces. Numer. Funct. Anal. Optim. **2**, 387–396 (1980). https://doi.org/10.1080/01630568008816066
20. Mitrinović, D.S., Pečarić, J.E., Fink, A.M.: Classical and New Inequalities in Analysis, vol. 61. Springer, Berlin (2013)
21. Panda, S.K., Kalla, K., Nagy, A., Priyanka, L.: Numerical simulations and complex valued fractional order neural networks via (ε-μ)-uniformly contractive mappings. Chaos Solitons Fractals **173**, 113738 (2023). https://doi.org/10.1016/j.chaos.2023.113738
22. Panda, S.K., Agarwal, R.P., Karapınar, E.: Extended suprametric spaces and Stone-type theorem. AIMS Math. **8**, 23183–23199 (2023). https://doi.org/10.3934/math.20231179
23. Rahman, G., Nisar, K.S., Ghanbari, B., Abdeljawad, T.: On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals. Adv. Differ. Equ. **2020**(1), 1–19 (2020). https://doi.org/10.1186/s13662-020-02830-7

24. Shapiro, J.: A Fixed-Point Farrago. Springer, Cham (2016)
25. Thompson, A.C.: On certain contraction mappings in a partially ordered vector space. Proc. Am. Math. Soc. **14**, 438–443 (1963). https://doi.org/10.2307/2033816
26. Van An, T., Tuyen, L., Van Dung, N.: Stone-type theorem on b-metric spaces and applications. Topol. Appl. **185**, 50–64 (2015). https://doi.org/10.1016/j.topol.2015.02.005
27. Younis, M., Singh, D., Abdou, A.: A fixed point approach for tuning circuit problem in dislocated b-metric spaces. Math. Methods Appl. Sci. **45**, 2234–2253 (2022). https://doi.org/10.1002/mma.7922