Complex Analysis and Operator Theory (2024) 18:116
https://doi.org/10.1007/s11785-024-01561-x

Complex Analysis and Operator Theory

Representations of Some Classes of Quaternionic Hyperholomorphic Functions

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Received: 2 February 2024 / Accepted: 8 June 2024 / Published online: 18 June 2024

Abstract

In the algebra of complex quaternions $\mathbb{H}(\mathbb{C})$ we consider the left– and right–ψ–hyperholomorphic functions, and left–$\Lambda - \psi$–hyperholomorphic functions. We justify the transition in left– and right–ψ–hyperholomorphic functions to a simpler basis i.e., to the Cartan basis. Using Cartan's basis we find the solution of Cauchy–Fueter equation. By the same method we find representations of left– and right–ψ–hyperholomorphic functions, and representation of left–$\Lambda - \psi$–hyperholomorphic functions.

Keywords Complex quaternions · Cartan basis · Left– and right–ψ–hyperholomorphic function · Weighted Dirac operator · Cauchy–Fueter type equation · Left–$\Lambda - \psi$–hyperholomorphic function.

Mathematics Subject Classification Primary 30G35; Secondary 32A10

1 Introduction

Our main object of interest is the set which is usually called the set of complex quaternions and which is traditionally denoted as $\mathbb{H}(\mathbb{C})$. It turns out to be an associative, non–commutative complex algebra generated by the elements $1, I, J, K$ such that the

Communicated by Juan Bory Reyes.

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following multiplication rules hold:

$$I^2 = J^2 = K^2 = IJK = -1,$$
$$IJ = -JI = K, \quad JK = -KJ = I, \quad KI = -IK = J,$$

and the complex imaginary unit i commutes with I, J and K. For $\mathbb{H}(\mathbb{C})$ another name, the algebra of biquaternions, is used also.

Consider in $\mathbb{H}(\mathbb{C})$ another set $\{e_1, e_2, e_3, e_4\}$, which is Cartan's basis [1] such that

$$e_1 = \frac{1}{2}(1 + iI), \quad e_2 = \frac{1}{2}(1 - iI), \quad e_3 = \frac{1}{2}(iJ - K), \quad e_4 = \frac{1}{2}(iJ + K), \quad (1.1)$$

where i is the complex imaginary unit. It is direct to check that we got a new basis.

The multiplication table can be represented as

$\cdot$	e_1	e_2	e_3	e_4
e_1	e_1	0	e_3	0
e_2	0	e_2	0	e_4
e_3	0	e_3	0	e_1
e_4	e_4	0	e_2	0

. (1.2)

The unit 1 can be decomposed as $1 = e_1 + e_2$.

Note that the subalgebra with the basis $\{e_1, e_2\}$ is the algebra of bicomplex numbers $\mathbb{BC}$ or Segre's algebra of commutative quaternions (see, e.g., [2, 3]).

The following relations holds:

$$1 = e_1 + e_2\,, \quad I = -ie_1 + ie_2\,, \quad J = -ie_3 - ie_4\,, \quad K = e_4 - e_3\,. \quad (1.3)$$

Of course, formulas (1.1) and (1.3) give the transition from one basis to the other.

Together with the Hamilton and Cartan bases, we will also consider the Pauli basis. It is a well-known fact that complex quaternions admit the matrix representations via the famous Pauli spin matrices:

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the Pauli basis the multiplication table has the form:

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \sigma_0\,, \quad \sigma_1\sigma_2\sigma_3 = i\sigma_0\,,$$

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3\,, \quad \sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1\,, \quad \sigma_1\sigma_3 = -\sigma_3\sigma_1 = i\sigma_2\,.$$

Formulas for the transition from the Pauli basis to the Cartan basis have the form

$$e_1 = \frac{1}{2}(\sigma_0 - \sigma_3), \ e_2 = \frac{1}{2}(\sigma_0 + \sigma_3), \ e_3 = \frac{1}{2}(-\sigma_2 - i\sigma_1), \ e_4 = \frac{1}{2}(-\sigma_2 + i\sigma_1). \quad (1.4)$$

2 Classes of Hyperholomorphic Functions

Let $\psi_1, \psi_2, \psi_3, \psi_4$ be fixed elements in $\mathbb{H}(\mathbb{C})$ with the following representations in the Cartan's basis:

$$\psi_1 := \sum_{s=1}^{4} \alpha_s e_s, \quad \alpha_s \in \mathbb{C}, \qquad \psi_2 := \sum_{s=1}^{4} \beta_s e_s, \quad \beta_s \in \mathbb{C},$$
$$\psi_3 := \sum_{s=1}^{4} \gamma_s e_s, \quad \gamma_s \in \mathbb{C}, \qquad \psi_4 := \sum_{s=1}^{4} \delta_s e_s, \quad \delta_s \in \mathbb{C}. \tag{2.1}$$

Consider a variable $z = z_1e_1+z_2e_2+z_3e_3+z_4e_4,\ z_s \in \mathbb{C},\ s = 1, 2, 3, 4$ and consider a function

$$f(z) = \sum_{s=1}^{4} f_s(z_1, z_2, z_3, z_4)e_s, \quad f_s : \Omega \to \mathbb{H}(\mathbb{C}),$$

where Ω is a domain in $\mathbb{C}^4$. Let components $f_s,\ s = 1, 2, 3, 4$, be holomorphic functions of four complex variables z_1, z_2, z_3, z_4 in Ω.

Consider the operators

$${}^{\psi}D[f](z) := \psi_1 \frac{\partial f}{\partial z_1} + \psi_2 \frac{\partial f}{\partial z_2} + \psi_3 \frac{\partial f}{\partial z_3} + \psi_4 \frac{\partial f}{\partial z_4}, \tag{2.2}$$
$$D^{\psi}[f](z) := \frac{\partial f}{\partial z_1} \psi_1 + \frac{\partial f}{\partial z_2} \psi_2 + \frac{\partial f}{\partial z_3} \psi_3 + \frac{\partial f}{\partial z_4} \psi_4. \tag{2.3}$$

Definition 2.1 A function $f : \Omega \to \mathbb{H}(\mathbb{C}),\ \Omega \subset \mathbb{C}^4$, is called left–$\psi$–hyperholomorphic (or right–ψ–hyperholomorphic) if components f_s are holomorphic functions of four complex variables z_1, z_2, z_3, z_4 in Ω, and f satisfies the equation

$${}^{\psi}D[f](z) = 0. \tag{2.4}$$

(or $D^{\psi}[f](z) = 0$.)

The class of ψ–hyperholomorphic functions in the real quaternions algebra is introduced for the first time by Shapiro and Vasilevski in the papers [4, 5]. Since then, these functions have attracted the attention of many researchers. K. Gürlebeck and his student H. M.Nguyen pay a special attention to the applications of ψ–hyperholomorphic functions. See, for example, the papers [6–8] and dissertation of Nguyen [9]. We note also that operators (2.2) and (2.2) are also called the weighted Dirac operators. Analysis and application of such operators are studied in papers [10, 11].

There are different generalizations of ψ–hyperholomorphic functions, which are being actively researched. Recently, generalizations to the case of fractional derivatives have become interesting. We will mark the works [12, 13].

Also, operators of a more general form than (2.2) are considered. Namely, in the paper [14], an operator of the following form is investigated:

$$ {}^{\psi}_{\Lambda}D[f] := \Lambda f + \psi_1 \frac{\partial f}{\partial z_1} + \psi_2 \frac{\partial f}{\partial z_2} + \psi_3 \frac{\partial f}{\partial z_3} + \psi_4 \frac{\partial f}{\partial z_4}, \quad \Lambda \in \mathbb{H}(\mathbb{C}). \tag{2.5} $$

Definition 2.2 A function $f : \Omega \to \mathbb{H}(\mathbb{C})$, $\Omega \subset \mathbb{C}^4$, is called left–$\Lambda - \psi$–hyperholomorphic if components f_s are holomorphic functions of four complex variables z_1, z_2, z_3, z_4 in Ω, and f satisfies the equation

$$ {}^{\psi}_{\Lambda}D[f](z) = 0. \tag{2.6} $$

In the paper [15] it is develop the theory of so-called (ϕ, ψ)–hyperholomorphic functions. Following a matrix approach, for such functions a generalized Borel–Pompeiu formula and the corresponding Plemelj-Sokhotski formulae are established. Research from paper [15] was continued in the papers [16–19].

At the same time, the problem of representation (or description in the explicit form) of ψ–hyperholomorphic and left–$\Lambda - \psi$–hyperholomorphic functions is open. This paper is devoted to solving this problem.

2.1 Examples

At first, we consider examples of left– and right–ψ–hyperholomorphic functions.

Example 1 Consider a domain $\Omega \subset \mathbb{C}^2 \simeq \mathbb{BC}$ and consider a variable $\zeta = z_1e_1+z_2e_2$, and a function $f : \Omega \to \mathbb{H}(\mathbb{C})$ of the form

$$ f = \sum_{s=1}^{4} f_s(z_1, z_2)e_s, \quad f_s : \Omega \to \mathbb{C}. $$

This should be understood as follows. We identify $\mathbb{C}^2$ and $\mathbb{BC}$ after which the set Ω in $\mathbb{BC}$ becomes a subset in $\mathbb{H}(\mathbb{C})$, not in $\mathbb{C}^2$; next we consider some objects as being situated in $\mathbb{H}(\mathbb{C})$. In particular, the set Ω is situated in $\mathbb{H}(\mathbb{C})$. When saying that the domain of f is in $\mathbb{H}(\mathbb{C})$ we mean already the previous identifications. Hence we work with functions with both domains and ranges in $\mathbb{H}(\mathbb{C})$. Thus ζ is in a domain in $\mathbb{H}(\mathbb{C})$: we imbed everything in $\mathbb{H}(\mathbb{C})$.

With these agreements we introduce the following definitions.

A function $f : \Omega \to \mathbb{H}(\mathbb{C})$, $\Omega \subset \mathbb{BC}$, is called *right-$\mathbb{BC}$-hyperholomorphic* if there exists an element of the algebra $\mathbb{H}(\mathbb{C})$, $f_r'(\zeta)$ such that

$$ \lim_{\varepsilon \to 0} \frac{f(\zeta + \varepsilon h) - f(\zeta)}{\varepsilon} = h \cdot f_r'(\zeta) \qquad \forall h \in \mathbb{BC}. \tag{2.7} $$

A function $f : \Omega \to \mathbb{H}(\mathbb{C}),\ \Omega \subset \mathbb{BC}$, is called *left-$\mathbb{BC}$-hyperholomorphic* if there exists an element of the algebra $\mathbb{H}(\mathbb{C})$, $f_l'(\zeta)$ such that

$$\lim_{\varepsilon \to 0} \frac{f(\zeta + \varepsilon h) - f(\zeta)}{\varepsilon} = f_l'(\zeta) \cdot h \qquad \forall h \in \mathbb{BC}. \tag{2.8}$$

Condition (2.7) implies

$$\frac{\partial f}{\partial z_1} = e_1 f_r'(\zeta) \quad \text{for} \quad h = e_1 \tag{2.9}$$

and

$$\frac{\partial f}{\partial z_2} = e_2 f_r'(\zeta) \quad \text{for} \quad h = e_2. \tag{2.10}$$

From (2.9) and (2.10) follows the analog of the Cauchy–Riemann condition

$$e_2 \frac{\partial f}{\partial z_1} = e_1 \frac{\partial f}{\partial z_2}. \tag{2.11}$$

Analogously, from (2.8) follows

$$\frac{\partial f}{\partial z_1} e_2 = \frac{\partial f}{\partial z_2} e_1. \tag{2.12}$$

Thus, right– and left–$\mathbb{BC}$–hyperholomorphic function generalize the concepts of holomorphic functions in the algebra $\mathbb{BC}$ (see, e.g., [2, 3]).

It is easy to see that the set of right- and left–$\mathbb{BC}$–hyperholomorphic functions is a subset of left–ψ–hyperholomorphic and right–ψ–hyperholomorphic function, respectively. Indeed, for $\zeta = z_1 e_1 + z_2 e_2$ the equality (2.11) has the form of the equality (2.4) with $\psi_1 = e_2$, $\psi_2 = -e_1$, $\psi_3 = \psi_4 = 0$. Analogously, right–$\mathbb{BC}$–hyperholomorphic functions is a subset of a set of right–ψ–hyperholomorphic functions.

Another example of mappings from the domain in $\mathbb{R}^3$ into the algebra $\mathbb{H}(\mathbb{C})$, which are a particular case of left– and right–ψ–hyperholomorphic functions, is considered in [20, 21].

Example 2 In (2.4) we set $\psi_1 = 1,\ \psi_2 = I,\ \psi_3 = J,\ \psi_4 = K$. In this case

$$\alpha_1 = \alpha_2 = 1, \quad \alpha_3 = \alpha_4 = 0, \quad \beta_1 = -i, \quad \beta_2 = i, \quad \beta_3 = \beta_4 = 0,$$

$$\gamma_1 = \gamma_2 = 0, \quad \gamma_3 = -i, \quad \gamma_4 = -i, \quad \delta_1 = \delta_2 = 0, \quad \delta_3 = -1, \quad \delta_4 = 1.$$

Then (2.4) takes the form

$$\frac{\partial f}{\partial z_1} + I \frac{\partial f}{\partial z_2} + J \frac{\partial f}{\partial z_3} + K \frac{\partial f}{\partial z_4} = 0$$

that is well-known Cauchy–Fueter type equation (see, e.g., [22, 23]).

2.2 Main Property of Left– and Right–ψ–Hyperholomorphic functions

Theorem 2.3 *Let a function f be left–ψ–hyperholomorphic (or right–ψ–hyperholomorphic) in some basis of the algebra $\mathbb{H}(\mathbb{C})$. Then in another basis of $\mathbb{H}(\mathbb{C})$ there exist a vector $\Psi := (\Psi_1, \Psi_2, \Psi_3, \Psi_4)$, $\Psi_s \in \mathbb{H}(\mathbb{C})$, $s = 1, 2, 3, 4$, such that the function f is left–Ψ–hyperholomorphic (or right–Ψ–hyperholomorphic).*

Proof Let us prove the theorem for the case left–ψ–hyperholomorphic functions. Let $\{e_1, e_2, e_3, e_4\}$ be the Cartan basis in $\mathbb{H}(\mathbb{C})$ and let $\{i_1, i_2, i_3, i_4\}$ be another basis in $\mathbb{H}(\mathbb{C})$. It means that

$$\begin{aligned} e_1 &= k_1 i_1 + k_2 i_2 + k_3 i_3 + k_4 i_4, \\ e_2 &= m_1 i_1 + m_2 i_2 + m_3 i_3 + m_4 i_4, \\ e_3 &= n_1 i_1 + n_2 i_2 + n_3 i_3 + n_4 i_4, \\ e_4 &= r_1 i_1 + r_2 i_2 + r_3 i_3 + r_4 i_4, \end{aligned}$$

where k_i, m_i, n_i, r_i, $i = 1, 2, 3, 4$, are complex numbers.

Consider the equation

$$^{\psi}D[f](t) := \psi_1 \frac{\partial f}{\partial t_1} + \psi_2 \frac{\partial f}{\partial t_2} + \psi_3 \frac{\partial f}{\partial t_3} + \psi_4 \frac{\partial f}{\partial t_4} = 0, \tag{2.13}$$

where $t := t_1 e_1 + t_2 e_2 + t_3 e_3 + t_4 e_4$, $t_1, t_2, t_3, t_4 \in \mathbb{C}$. In the variable t we passing to the basis $\{i_1, i_2, i_3, i_4\}$. Then

$$t = i_1(t_1 k_1 + t_2 m_1 + t_3 n_1 + t_4 r_1) + i_2(t_1 k_2 + t_2 m_2 + t_3 n_2 + t_4 r_2)$$

$$+ i_3(t_1 k_3 + t_2 m_3 + t_3 n_3 + t_4 r_3) + i_4(t_1 k_4 + t_2 m_4 + t_3 n_4 + t_4 r_4).$$

We set

$$\begin{aligned} z_1 &:= t_1 k_1 + t_2 m_1 + t_3 n_1 + t_4 r_1, \\ z_2 &:= t_1 k_2 + t_2 m_2 + t_3 n_2 + t_4 r_2, \\ z_3 &:= t_1 k_3 + t_2 m_3 + t_3 n_3 + t_4 r_3, \\ z_4 &:= t_1 k_4 + t_2 m_4 + t_3 n_4 + t_4 r_4. \end{aligned} \tag{2.14}$$

From equalities (2.14) we obtain

$$\begin{aligned} \frac{\partial f}{\partial t_1} &= k_1 \frac{\partial f}{\partial z_1} + k_2 \frac{\partial f}{\partial z_2} + k_3 \frac{\partial f}{\partial z_3} + k_4 \frac{\partial f}{\partial z_4}, \\ \frac{\partial f}{\partial t_2} &= m_1 \frac{\partial f}{\partial z_1} + m_2 \frac{\partial f}{\partial z_2} + m_3 \frac{\partial f}{\partial z_3} + m_4 \frac{\partial f}{\partial z_4}, \\ \frac{\partial f}{\partial t_3} &= n_1 \frac{\partial f}{\partial z_1} + n_2 \frac{\partial f}{\partial z_2} + n_3 \frac{\partial f}{\partial z_3} + n_4 \frac{\partial f}{\partial z_4}, \\ \frac{\partial f}{\partial t_4} &= r_1 \frac{\partial f}{\partial z_1} + r_2 \frac{\partial f}{\partial z_2} + r_3 \frac{\partial f}{\partial z_3} + r_4 \frac{\partial f}{\partial z_4}. \end{aligned}$$

Then Eq. (2.13) is equivalent to the following equation

$$
{}^{\psi}D[f](t) = (\psi_1 k_1 + \psi_2 m_1 + \psi_3 n_1 + \psi_4 r_1)\frac{\partial f}{\partial z_1} + (\psi_1 k_2 + \psi_2 m_2 + \psi_3 n_2 + \psi_4 r_2)\frac{\partial f}{\partial z_2}
$$

$$
+(\psi_1 k_3 + \psi_2 m_3 + \psi_3 n_3 + \psi_4 r_3)\frac{\partial f}{\partial z_3} + (\psi_1 k_4 + \psi_2 m_4 + \psi_3 n_4 + \psi_4 r_4)\frac{\partial f}{\partial z_4}. \tag{2.15}
$$

Using denotation (2.1), we have

$$
\psi_1 = \sum_{s=1}^{4} \alpha_s e_s = \sum_{s=1}^{4} i_s(\alpha_1 k_s + \alpha_2 m_s + \alpha_3 n_s + \alpha_4 r_s),
$$

$$
\psi_2 = \sum_{s=1}^{4} \beta_s e_s = \sum_{s=1}^{4} i_s(\beta_1 k_s + \beta_2 m_s + \beta_3 n_s + \beta_4 r_s),
$$

$$
\psi_3 = \sum_{s=1}^{4} \gamma_s e_s = \sum_{s=1}^{4} i_s(\gamma_1 k_s + \gamma_2 m_s + \gamma_3 n_s + \gamma_4 r_s),
$$

$$
\psi_4 = \sum_{s=1}^{4} \delta_s e_s = \sum_{s=1}^{4} i_s(\delta_1 k_s + \delta_2 m_s + \delta_3 n_s + \delta_4 r_s),
$$

From (2.15) we obtain

$$
{}^{\psi}D[f](t) = \sum_{s=1}^{4} i_s\Big[(\alpha_1 k_s + \alpha_2 m_s + \alpha_3 n_s + \alpha_4 r_s)k_1 + (\beta_1 k_s + \beta_2 m_s + \beta_3 n_s + \beta_4 r_s)m_1
$$

$$
+(\gamma_1 k_s + \gamma_2 m_s + \gamma_3 n_s + \gamma_4 r_s)n_1 + (\delta_1 k_s + \delta_2 m_s + \delta_3 n_s + \delta_4 r_s)r_1\Big]\frac{\partial f}{\partial z_1}
$$

$$
+\sum_{s=1}^{4} i_s\Big[(\alpha_1 k_s + \alpha_2 m_s + \alpha_3 n_s + \alpha_4 r_s)k_2 + (\beta_1 k_s + \beta_2 m_s + \beta_3 n_s + \beta_4 r_s)m_2
$$

$$
+(\gamma_1 k_s + \gamma_2 m_s + \gamma_3 n_s + \gamma_4 r_s)n_2 + (\delta_1 k_s + \delta_2 m_s + \delta_3 n_s + \delta_4 r_s)r_2\Big]\frac{\partial f}{\partial z_2}
$$

$$
+\sum_{s=1}^{4} i_s\Big[(\alpha_1 k_s + \alpha_2 m_s + \alpha_3 n_s + \alpha_4 r_s)k_3 + (\beta_1 k_s + \beta_2 m_s + \beta_3 n_s + \beta_4 r_s)m_3
$$

$$
+(\gamma_1 k_s + \gamma_2 m_s + \gamma_3 n_s + \gamma_4 r_s)n_3 + (\delta_1 k_s + \delta_2 m_s + \delta_3 n_s + \delta_4 r_s)r_3\Big]\frac{\partial f}{\partial z_3}
$$

$$+\sum_{s=1}^{4} i_s\Big[(\alpha_1 k_s+\alpha_2 m_s+\alpha_3 n_s+\alpha_4 r_s)k_4+(\beta_1 k_s+\beta_2 m_s+\beta_3 n_s+\beta_4 r_s)m_4$$

$$+(\gamma_1 k_s+\gamma_2 m_s+\gamma_3 n_s+\gamma_4 r_s)n_4+(\delta_1 k_s+\delta_2 m_s+\delta_3 n_s+\delta_4 r_s)r_4\Big]\frac{\partial f}{\partial z_4}$$

$$=: \Psi_1\frac{\partial f}{\partial z_1}+\Psi_2\frac{\partial f}{\partial z_2}+\Psi_3\frac{\partial f}{\partial z_3}+\Psi_4\frac{\partial f}{\partial z_4}=0.$$

□

Remark 2.4 In Clifford algebras, it is known that the equations

$$\frac{\partial f}{\partial t_0}+I\frac{\partial f}{\partial t_1}+J\frac{\partial f}{\partial t_2}+K\frac{\partial f}{\partial t_3}=0 \qquad (2.16)$$

and ${}^{\psi}D[f](t)=0$ coincide, up to an orthogonal transformation. Note that, in essence, Theorem 2.3 is a similar statement, but formulated in other terms.

Remark 2.5 It follows from Theorem 2.3 that in future investigation it is enough to consider constants ψ and function f in the simplest basis, i.e., in Cartan basis. The use of the Cartan basis is of principal importance, because in this basis the multiplication table has the simplest form. In addition, in the Cartan basis, Eqs. (2.4), (2.6) and (2.16) are reduced to systems of differential equations that we integrate in the explicit form. This is what we will do next.

3 Application to Solving Cauchy–Fueter Type Equation

Now, we will establish a connection between solutions of the equation

$$D[f](t):=\frac{\partial f}{\partial t_0}+I\frac{\partial f}{\partial t_1}+J\frac{\partial f}{\partial t_2}+K\frac{\partial f}{\partial t_3}=0, \qquad (3.1)$$

where $t:=t_0+t_1I+t_2J+t_3K,\ \ t_0,t_1,t_2,t_3\in\mathbb{C}$, and the solutions of Eq. (2.4). For this purpose, in t we passing to Cartan basis. We have

$$t=t_0(e_1+e_2)+t_1(-ie_1+ie_2)+t_2(-ie_3-ie_4)+t_3(e_4-e_3)$$

$$=(t_0-it_1)e_1+(t_0+it_1)e_2+(-it_2-t_3)e_3+(-it_2+t_3)e_4.$$

We set

$$z_1:=t_0-it_1,\quad z_2:=t_0+it_1,\quad z_3:=-it_2-t_3,\quad z_4:=-it_2+t_3. \qquad (3.2)$$

From equalities (3.2) we obtain

$$\frac{\partial f}{\partial t_0}=\frac{\partial f}{\partial z_1}+\frac{\partial f}{\partial z_2},\qquad \frac{\partial f}{\partial t_1}=-i\frac{\partial f}{\partial z_1}+i\frac{\partial f}{\partial z_2},$$

$$\frac{\partial f}{\partial t_2} = -i\frac{\partial f}{\partial z_3} - i\frac{\partial f}{\partial z_4}, \qquad \frac{\partial f}{\partial t_3} = -\frac{\partial f}{\partial z_3} + \frac{\partial f}{\partial z_4}.$$

Then Eq. (3.1) is equivalent to the following equation

$$\begin{aligned} D[f] &= \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_2} - iI\frac{\partial f}{\partial z_1} + iI\frac{\partial f}{\partial z_2} - iJ\frac{\partial f}{\partial z_3} - iJ\frac{\partial f}{\partial z_4} - K\frac{\partial f}{\partial z_3} + K\frac{\partial f}{\partial z_4} \\ &= (1 - iI)\frac{\partial f}{\partial z_1} + (1 + iI)\frac{\partial f}{\partial z_2} + (-iJ - K)\frac{\partial f}{\partial z_3} + (-iJ + K)\frac{\partial f}{\partial z_4} \\ &= 2\left(e_2\frac{\partial f}{\partial z_1} + e_1\frac{\partial f}{\partial z_2} - e_4\frac{\partial f}{\partial z_3} - e_3\frac{\partial f}{\partial z_4}\right) = 0. \end{aligned}$$

Thus, we proved the following theorem

Theorem 3.1 *A function f of the variable $t = t_0 + t_1 I + t_2 J + t_3 K$ satisfies Eq.* (3.1) *if and only if the function f of the variable $z = z_1 e_1 + z_2 e_2 + z_3 e_3 + z_4 e_4$ satisfies the equation*

$$e_2\frac{\partial f}{\partial z_1} + e_1\frac{\partial f}{\partial z_2} - e_4\frac{\partial f}{\partial z_3} - e_3\frac{\partial f}{\partial z_4} = 0, \tag{3.3}$$

where z and t are related by equalities (3.2).

Now, we solve Eq. (3.3).

$$\begin{aligned} e_2\frac{\partial f}{\partial z_1} &= e_2\left(\frac{\partial f_1}{\partial z_1}e_1 + \frac{\partial f_2}{\partial z_1}e_2 + \frac{\partial f_3}{\partial z_1}e_3 + \frac{\partial f_4}{\partial z_1}e_4\right) \\ &= \frac{\partial f_2}{\partial z_1}e_2 + \frac{\partial f_4}{\partial z_1}e_4, \\ e_1\frac{\partial f}{\partial z_2} &= \frac{\partial f_1}{\partial z_2}e_1 + \frac{\partial f_3}{\partial z_2}e_3, \\ e_4\frac{\partial f}{\partial z_3} &= \frac{\partial f_1}{\partial z_3}e_4 + \frac{\partial f_3}{\partial z_3}e_2, \\ e_3\frac{\partial f}{\partial z_4} &= \frac{\partial f_2}{\partial z_4}e_3 + \frac{\partial f_4}{\partial z_4}e_1. \end{aligned}$$

Then Eq. (3.3) is equivalent to the system

$$\frac{\partial f_1}{\partial z_2} = \frac{\partial f_4}{\partial z_4}, \qquad \frac{\partial f_2}{\partial z_1} = \frac{\partial f_3}{\partial z_3},$$

$$\frac{\partial f_3}{\partial z_2} = \frac{\partial f_2}{\partial z_4}, \qquad \frac{\partial f_4}{\partial z_1} = \frac{\partial f_1}{\partial z_3}.$$

We have pair of systems

$$\frac{\partial f_1}{\partial z_2} = \frac{\partial f_4}{\partial z_4}, \qquad \frac{\partial f_1}{\partial z_3} = \frac{\partial f_4}{\partial z_1} \tag{3.4}$$

and

$$\frac{\partial f_2}{\partial z_1} = \frac{\partial f_3}{\partial z_3}, \qquad \frac{\partial f_2}{\partial z_4} = \frac{\partial f_3}{\partial z_2}. \tag{3.5}$$

In a simple connected domain Ω, a solution of system (3.4) is an arbitrary holomorphic function

$$f_1 = f_1(z_2, z_3)$$

and

$$f_4 = z_4 \frac{\partial f_1}{\partial z_2} + z_1 \frac{\partial f_1}{\partial z_3}.$$

In a simple connected domain Ω, a solution of system (3.5) is an arbitrary holomorphic function

$$f_2 = f_2(z_1, z_4)$$

and

$$f_3 = z_3 \frac{\partial f_2}{\partial z_1} + z_2 \frac{\partial f_2}{\partial z_4}.$$

Thus, we have the following solution of Eq. (3.3):

$$\begin{aligned} f(z) &= f_1(z_2, z_3)e_1 + f_2(z_1, z_4)e_2 \\ &+ \left(z_3 \frac{\partial f_2}{\partial z_1} + z_2 \frac{\partial f_2}{\partial z_4} \right) e_3 + \left(z_4 \frac{\partial f_1}{\partial z_2} + z_1 \frac{\partial f_1}{\partial z_3} \right) e_4 . \end{aligned} \tag{3.6}$$

Thus, accordingly to Theorem 3.1 we obtain

Theorem 3.2 *In a simple connected domain, function* (3.6)*, where* z_1, z_2, z_3, z_4 *are given by relations* (3.2)*, satisfies* Eq. (3.1).

Proposition 3.3 *In a simple connected domain, function* (3.6) *satisfies the four-dimensional complex Laplace equation*

$$\Delta_{\mathbb{C}^4} f := \frac{\partial^2 f}{\partial t_1^2} + \frac{\partial^2 f}{\partial t_2^2} + \frac{\partial^2 f}{\partial t_3^2} + \frac{\partial^2 f}{\partial t_4^2} = 0. \tag{3.7}$$

About Eq. (3.7) and its relation to the Cauchy-Fueter equation see in [22].

4 Representation of Left–ψ–hyperholomorphic Functions in a Special Case

Now we will find a general solution of Eq. (2.4) for a special choice of parameters ψ_1, ψ_2, ψ_3 and ψ_4. For this purpose, we reduce Eq. (2.4) to a system of four PDEs. We have

$$\psi_1 \frac{\partial f}{\partial z_1} = (\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4) \left(\frac{\partial f_1}{\partial z_1} e_1 + \frac{\partial f_2}{\partial z_1} e_2 + \frac{\partial f_3}{\partial z_1} e_3 + \frac{\partial f_4}{\partial z_1} e_4 \right)$$

$$
\begin{aligned}
&= \frac{\partial f_1}{\partial z_1}\alpha_1 e_1 + \frac{\partial f_3}{\partial z_1}\alpha_1 e_3 + \frac{\partial f_2}{\partial z_1}\alpha_2 e_2 + \frac{\partial f_4}{\partial z_1}\alpha_2 e_4 \\
&\quad + \frac{\partial f_2}{\partial z_1}\alpha_3 e_3 + \frac{\partial f_4}{\partial z_1}\alpha_3 e_1 + \frac{\partial f_1}{\partial z_1}\alpha_4 e_4 + \frac{\partial f_3}{\partial z_1}\alpha_4 e_2 \\
&= \frac{\partial}{\partial z_1}\left(\alpha_1 f_1 + \alpha_3 f_4\right) e_1 + \frac{\partial}{\partial z_1}\left(\alpha_2 f_2 + \alpha_4 f_3\right) e_2 \\
&\quad + \frac{\partial}{\partial z_1}\left(\alpha_1 f_3 + \alpha_3 f_2\right) e_3 + \frac{\partial}{\partial z_1}\left(\alpha_2 f_4 + \alpha_4 f_1\right) e_4.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\psi_2 \frac{\partial f}{\partial z_2} &= \frac{\partial}{\partial z_2}\left(\beta_1 f_1 + \beta_3 f_4\right) e_1 + \frac{\partial}{\partial z_2}\left(\beta_2 f_2 + \beta_4 f_3\right) e_2 \\
&\quad + \frac{\partial}{\partial z_2}\left(\beta_1 f_3 + \beta_3 f_2\right) e_3 + \frac{\partial}{\partial z_2}\left(\beta_2 f_4 + \beta_4 f_1\right) e_4, \\
\psi_3 \frac{\partial f}{\partial z_3} &= \frac{\partial}{\partial z_3}\left(\gamma_1 f_1 + \gamma_3 f_4\right) e_1 + \frac{\partial}{\partial z_3}\left(\gamma_2 f_2 + \gamma_4 f_3\right) e_2 \\
&\quad + \frac{\partial}{\partial z_3}\left(\gamma_1 f_3 + \gamma_3 f_2\right) e_3 + \frac{\partial}{\partial z_3}\left(\gamma_2 f_4 + \gamma_4 f_1\right) e_4, \\
\psi_4 \frac{\partial f}{\partial z_4} &= \frac{\partial}{\partial z_4}\left(\delta_1 f_1 + \delta_3 f_4\right) e_1 + \frac{\partial}{\partial z_4}\left(\delta_2 f_2 + \delta_4 f_3\right) e_2 \\
&\quad + \frac{\partial}{\partial z_4}\left(\delta_1 f_3 + \delta_3 f_2\right) e_3 + \frac{\partial}{\partial z_4}\left(\delta_2 f_4 + \delta_4 f_1\right) e_4.
\end{aligned}
$$

Then Eq. (2.4) is equivalent to the following system

$$
\begin{aligned}
&\frac{\partial}{\partial z_1}(\alpha_1 f_1 + \alpha_3 f_4) + \frac{\partial}{\partial z_2}(\beta_1 f_1 + \beta_3 f_4) + \frac{\partial}{\partial z_3}(\gamma_1 f_1 + \gamma_3 f_4) + \frac{\partial}{\partial z_4}(\delta_1 f_1 + \delta_3 f_4) = 0, \\
&\frac{\partial}{\partial z_1}(\alpha_2 f_2 + \alpha_4 f_3) + \frac{\partial}{\partial z_2}(\beta_2 f_2 + \beta_4 f_3) + \frac{\partial}{\partial z_3}(\gamma_2 f_2 + \gamma_4 f_3) + \frac{\partial}{\partial z_4}(\delta_2 f_2 + \delta_4 f_3) = 0, \\
&\frac{\partial}{\partial z_1}(\alpha_1 f_3 + \alpha_3 f_2) + \frac{\partial}{\partial z_2}(\beta_1 f_3 + \beta_3 f_2) + \frac{\partial}{\partial z_3}(\gamma_1 f_3 + \gamma_3 f_2) + \frac{\partial}{\partial z_4}(\delta_1 f_3 + \delta_3 f_2) = 0, \\
&\frac{\partial}{\partial z_1}(\alpha_2 f_4 + \alpha_4 f_1) + \frac{\partial}{\partial z_2}(\beta_2 f_4 + \beta_4 f_1) + \frac{\partial}{\partial z_3}(\gamma_2 f_4 + \gamma_4 f_1) + \frac{\partial}{\partial z_4}(\delta_2 f_4 + \delta_4 f_1) = 0.
\end{aligned}
\tag{4.1}
$$

Theorem 4.1 *Let*

$$
\begin{aligned}
\psi_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4, \qquad \alpha_1\alpha_2 \neq \alpha_3\alpha_4\,, \\
\psi_2 &= \lambda\alpha_1 e_1 + \mu\alpha_2 e_2 + \mu\alpha_3 e_3 + \lambda\alpha_4 e_4, \\
\psi_3 &= \theta\alpha_1 e_1 + \vartheta\alpha_2 e_2 + \vartheta\alpha_3 e_3 + \theta\alpha_4 e_4, \\
\psi_4 &= \nu\alpha_1 e_1 + \eta\alpha_2 e_2 + \eta\alpha_3 e_3 + \nu\alpha_4 e_4,
\end{aligned}
\tag{4.2}
$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \lambda, \mu, \theta, \vartheta, \nu, \eta$ *are arbitrary complex numbers. Then every left–ψ–hyperholomorphic function is of the form*

$$f(z) = f_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)e_1 + f_2(\zeta_2, \zeta_3, \zeta_4)e_2 + f_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)e_3 + f_4(\zeta_2, \zeta_3, \zeta_4)e_4, \quad (4.3)$$

where

$$\begin{aligned} &\widetilde{\zeta}_2 := \lambda z_1 - z_2, \quad \widetilde{\zeta}_3 := \theta z_1 - z_3, \quad \widetilde{\zeta}_4 := \nu z_1 - z_4, \\ &\zeta_2 := \mu z_1 - z_2, \quad \zeta_3 := \vartheta z_1 - z_3, \quad \zeta_4 := \eta z_1 - z_4, \end{aligned} \quad (4.4)$$

and f_1, f_2, f_3, f_4 *are arbitrary holomorphic functions of three their arguments.*

Proof For given parameters (4.2) the first equation of system (4.1) takes the form

$$\begin{aligned} &\frac{\partial}{\partial z_1}(\alpha_1 f_1 + \alpha_3 f_4) + \frac{\partial}{\partial z_2}(\lambda\alpha_1 f_1 + \mu\alpha_3 f_4) \\ &\quad + \frac{\partial}{\partial z_3}(\theta\alpha_1 f_1 + \vartheta\alpha_3 f_4) + \frac{\partial}{\partial z_4}(\nu\alpha_1 f_1 + \eta\alpha_3 f_4) = 0. \end{aligned} \quad (4.5)$$

Similarly, for given parameters (4.2) the fourth equation of system (4.1) takes the form

$$\begin{aligned} &\frac{\partial}{\partial z_1}(\alpha_4 f_1 + \alpha_2 f_4) + \frac{\partial}{\partial z_2}(\lambda\alpha_4 f_1 + \mu\alpha_2 f_4) \\ &\quad + \frac{\partial}{\partial z_3}(\theta\alpha_4 f_1 + \vartheta\alpha_2 f_4) + \frac{\partial}{\partial z_4}(\nu\alpha_4 f_1 + \eta\alpha_2 f_4) = 0. \end{aligned} \quad (4.6)$$

Consider the difference between Eq. (4.5) multiplied by α_2 and Eq. (4.6) multiplied by α_3. Then we obtain the following equation

$$\begin{aligned} &\frac{\partial}{\partial z_1}\Big(f_1(\alpha_1\alpha_2 - \alpha_3\alpha_4) + f_4(\alpha_2\alpha_3 - \alpha_2\alpha_3)\Big) \\ &\quad + \frac{\partial}{\partial z_2}\Big(f_1(\lambda\alpha_1\alpha_2 - \lambda\alpha_3\alpha_4) + f_4(\mu\alpha_2\alpha_3 - \mu\alpha_2\alpha_3)\Big) \\ &\quad + \frac{\partial}{\partial z_3}\Big(f_1(\theta\alpha_1\alpha_2 - \theta\alpha_3\alpha_4) + f_4(\vartheta\alpha_2\alpha_3 - \vartheta\alpha_2\alpha_3)\Big) \\ &\quad + \frac{\partial}{\partial z_4}\Big(f_1(\nu\alpha_1\alpha_2 - \nu\alpha_3\alpha_4) + f_4(\eta\alpha_2\alpha_3 - \eta\alpha_2\alpha_3)\Big) = 0. \end{aligned}$$

Thus, we obtain the equation

$$\frac{\partial f_1}{\partial z_1} + \lambda\frac{\partial f_1}{\partial z_2} + \theta\frac{\partial f_1}{\partial z_3} + \nu\frac{\partial f_1}{\partial z_4} = 0. \quad (4.7)$$

For Eq. (4.7) consider the characteristic equation

$$\frac{dz_1}{1} = \frac{dz_2}{\lambda} = \frac{dz_3}{\theta} = \frac{dz_4}{\nu}. \quad (4.8)$$

The solutions of system (4.8) are the following integrals

$$c_2 = \lambda z_1 - z_2, \quad c_3 = \theta z_1 - z_3, \quad c_4 = \nu z_1 - z_4.$$

Therefore, the general solution of Eq. (4.7) has the form

$$f_1 = f_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4),$$

where $\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4$ are defined by equalities (4.4).

Note that polynomials (4.4) are similarly to the well-known Fueter's polynomials [24].

Similarly, we obtain the representations for the components f_2, f_3, f_4. □

Thus, formula (4.3) gives a representation of left–ψ–hyperholomorphic function for a special choice of ψ.

Remark 4.2 Using formulas (1.4) we can rewrite representation (4.3) in the Pauli basis:

$$f(z) = \big(f_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) + f_2(\zeta_2, \zeta_3, \zeta_4)\big)\,\sigma_0 + \big(i f_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - i f_4(\zeta_2, \zeta_3, \zeta_4)\big)\,\sigma_1$$
$$+ \big(-f_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - f_4(\zeta_2, \zeta_3, \zeta_4)\big)\,\sigma_2 + \big(f_2(\zeta_2, \zeta_3, \zeta_4) - f_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)\big)\,\sigma_3 .$$

5 Representation of Right–ψ–hyperholomorphic Functions in a Special Case

In this section we will find a general solution of the equation

$$D^{\psi}[f](z) = \frac{\partial f}{\partial z_1}\,\psi_1 + \frac{\partial f}{\partial z_2}\,\psi_2 + \frac{\partial f}{\partial z_3}\,\psi_3 + \frac{\partial f}{\partial z_4}\,\psi_4 = 0 \tag{5.1}$$

for a special choice of parameters ψ_1, ψ_2, ψ_3 and ψ_4. For this purpose, we reduce Eq. (5.1) to a system of four PDEs. We have

$$\begin{aligned}
\frac{\partial f}{\partial z_1}\,\psi_1 &= \left(\frac{\partial f_1}{\partial z_1} e_1 + \frac{\partial f_2}{\partial z_1} e_2 + \frac{\partial f_3}{\partial z_1} e_3 + \frac{\partial f_4}{\partial z_1} e_4\right)(\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4) \\
&= \frac{\partial f_1}{\partial z_1}\alpha_1 e_1 + \frac{\partial f_1}{\partial z_1}\alpha_3 e_3 + \frac{\partial f_2}{\partial z_1}\alpha_2 e_2 + \frac{\partial f_2}{\partial z_1}\alpha_4 e_4 \\
&\quad + \frac{\partial f_3}{\partial z_1}\alpha_2 e_3 + \frac{\partial f_3}{\partial z_1}\alpha_4 e_1 + \frac{\partial f_4}{\partial z_1}\alpha_1 e_4 + \frac{\partial f_4}{\partial z_1}\alpha_3 e_2 \\
&= \frac{\partial}{\partial z_1}(\alpha_1 f_1 + \alpha_4 f_3)\, e_1 + \frac{\partial}{\partial z_1}(\alpha_2 f_2 + \alpha_3 f_4)\, e_2 \\
&\quad + \frac{\partial}{\partial z_1}(\alpha_3 f_1 + \alpha_2 f_3)\, e_3 + \frac{\partial}{\partial z_1}(\alpha_4 f_2 + \alpha_1 f_4)\, e_4.
\end{aligned}$$

Similarly,

$$\frac{\partial f}{\partial z_2}\,\psi_2 = \frac{\partial}{\partial z_2}\,(\beta_1 f_1 + \beta_4 f_3)\,e_1 + \frac{\partial}{\partial z_2}\,(\beta_2 f_2 + \beta_3 f_4)\,e_2 + \frac{\partial}{\partial z_2}\,(\beta_3 f_1 + \beta_2 f_3)\,e_3 + \frac{\partial}{\partial z_2}\,(\beta_4 f_2 + \beta_1 f_4)\,e_4,$$

$$\frac{\partial f}{\partial z_3}\,\psi_3 = \frac{\partial}{\partial z_3}\,(\gamma_1 f_1 + \gamma_4 f_3)\,e_1 + \frac{\partial}{\partial z_3}\,(\gamma_2 f_2 + \gamma_3 f_4)\,e_2 + \frac{\partial}{\partial z_3}\,(\gamma_3 f_1 + \gamma_2 f_3)\,e_3 + \frac{\partial}{\partial z_3}\,(\gamma_4 f_2 + \gamma_1 f_4)\,e_4,$$

$$\frac{\partial f}{\partial z_4}\,\psi_4 = \frac{\partial}{\partial z_4}\,(\delta_1 f_1 + \delta_4 f_3)\,e_1 + \frac{\partial}{\partial z_4}\,(\delta_2 f_2 + \delta_3 f_4)\,e_2 + \frac{\partial}{\partial z_4}\,(\delta_3 f_1 + \delta_2 f_3)\,e_3 + \frac{\partial}{\partial z_4}\,(\delta_4 f_2 + \delta_1 f_4)\,e_4.$$

Then Eq. (5.1) is equivalent to the following system

$$\begin{aligned}
&\frac{\partial}{\partial z_1}(\alpha_1 f_1 + \alpha_4 f_3) + \frac{\partial}{\partial z_2}(\beta_1 f_1 + \beta_4 f_3) + \frac{\partial}{\partial z_3}(\gamma_1 f_1 + \gamma_4 f_3) + \frac{\partial}{\partial z_4}(\delta_1 f_1 + \delta_4 f_3) = 0,\\
&\frac{\partial}{\partial z_1}(\alpha_2 f_2 + \alpha_3 f_4) + \frac{\partial}{\partial z_2}(\beta_2 f_2 + \beta_3 f_4) + \frac{\partial}{\partial z_3}(\gamma_2 f_2 + \gamma_3 f_4) + \frac{\partial}{\partial z_4}(\delta_2 f_2 + \delta_3 f_4) = 0,\\
&\frac{\partial}{\partial z_1}(\alpha_3 f_1 + \alpha_2 f_3) + \frac{\partial}{\partial z_2}(\beta_3 f_1 + \beta_2 f_3) + \frac{\partial}{\partial z_3}(\gamma_3 f_1 + \gamma_2 f_3) + \frac{\partial}{\partial z_4}(\delta_3 f_1 + \delta_2 f_3) = 0,\\
&\frac{\partial}{\partial z_1}(\alpha_4 f_2 + \alpha_1 f_4) + \frac{\partial}{\partial z_2}(\beta_4 f_2 + \beta_1 f_4) + \frac{\partial}{\partial z_3}(\gamma_4 f_2 + \gamma_1 f_4) + \frac{\partial}{\partial z_4}(\delta_4 f_2 + \delta_1 f_4) = 0.
\end{aligned} \tag{5.2}$$

Theorem 5.1 *Let*

$$\begin{aligned}
\psi_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4, \qquad \alpha_1\alpha_2 \neq \alpha_3\alpha_4\,,\\
\psi_2 &= \mu\alpha_1 e_1 + \lambda\alpha_2 e_2 + \mu\alpha_3 e_3 + \lambda\alpha_4 e_4,\\
\psi_3 &= \vartheta\alpha_1 e_1 + \theta\alpha_2 e_2 + \vartheta\alpha_3 e_3 + \theta\alpha_4 e_4,\\
\psi_4 &= \eta\alpha_1 e_1 + \nu\alpha_2 e_2 + \eta\alpha_3 e_3 + \nu\alpha_4 e_4,
\end{aligned} \tag{5.3}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \lambda, \mu, \theta, \vartheta, \nu, \eta$ *are arbitrary complex numbers. Then every right–*ψ*–hyperholomorphic function is of the form*

$$f(z) = f_1(\zeta_2, \zeta_3, \zeta_4)e_1 + f_2(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)e_2 + f_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)e_3 + f_4(\zeta_2, \zeta_3, \zeta_4)e_4, \tag{5.4}$$

where $\zeta_2, \zeta_3, \zeta_4, \widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4$ *are defined by relations* (4.4) *and* $f_1,\ f_2,\ f_3,\ f_4$ *are arbitrary holomorphic functions of their three arguments.*

Proof For given parameters (5.3) the first equation of system (5.2) takes the form

$$\frac{\partial}{\partial z_1}\,(\alpha_1 f_1 + \alpha_4 f_3) + \frac{\partial}{\partial z_2}\,(\mu\alpha_1 f_1 + \lambda\alpha_4 f_3)$$

$$+ \frac{\partial}{\partial z_3}(\vartheta\alpha_1 f_1 + \theta\alpha_4 f_3) + \frac{\partial}{\partial z_4}(\eta\alpha_1 f_1 + \nu\alpha_4 f_3) = 0. \tag{5.5}$$

Similarly, for given parameters (5.3) the third equation of system (5.2) takes the form

$$\frac{\partial}{\partial z_1}(\alpha_3 f_1 + \alpha_2 f_3) + \frac{\partial}{\partial z_2}(\mu\alpha_3 f_1 + \lambda\alpha_2 f_3)$$
$$+ \frac{\partial}{\partial z_3}(\vartheta\alpha_3 f_1 + \theta\alpha_2 f_3) + \frac{\partial}{\partial z_4}(\eta\alpha_3 f_1 + \nu\alpha_2 f_3) = 0. \tag{5.6}$$

Consider the difference between Eq. (5.5) multiplied by α_2 and Eq. (5.6) multiplied by α_4. Then we obtain the following equation

$$\frac{\partial}{\partial z_1}\Big(f_1(\alpha_1\alpha_2 - \alpha_3\alpha_4) + f_3(\alpha_2\alpha_4 - \alpha_2\alpha_4)\Big)$$
$$+ \frac{\partial}{\partial z_2}\Big(\mu f_1(\alpha_1\alpha_2 - \alpha_3\alpha_4) + \lambda f_3(\alpha_2\alpha_4 - \alpha_2\alpha_4)\Big)$$
$$+ \frac{\partial}{\partial z_3}\Big(\vartheta f_1(\alpha_1\alpha_2 - \alpha_3\alpha_4) + \theta f_3(\alpha_2\alpha_4 - \alpha_2\alpha_4)\Big)$$
$$+ \frac{\partial}{\partial z_4}\Big(\eta f_1(\alpha_1\alpha_2 - \alpha_3\alpha_4) + \nu f_3(\alpha_2\alpha_3 - \alpha_2\alpha_3)\Big) = 0.$$

Thus, we obtain the equation

$$\frac{\partial f_1}{\partial z_1} + \mu\frac{\partial f_1}{\partial z_2} + \vartheta\frac{\partial f_1}{\partial z_3} + \eta\frac{\partial f_1}{\partial z_4} = 0. \tag{5.7}$$

For Eq. (5.7) consider the characteristic equation

$$\frac{dz_1}{1} = \frac{dz_2}{\mu} = \frac{dz_3}{\vartheta} = \frac{dz_4}{\eta}. \tag{5.8}$$

The solutions of system (5.8) are the following integrals

$$c_2 = \mu z_1 - z_2, \quad c_3 = \vartheta z_1 - z_3, \quad c_4 = \eta z_1 - z_4.$$

be of the for Therefore, the general solution of Eq. (5.7) has the form

$$f_1 = f_1(\zeta_2, \zeta_3, \zeta_4),$$

where $\zeta_2, \zeta_3, \zeta_4$ are defined by equalities (4.4).

Similarly, we obtain the representations for the components f_2, f_3, f_4. □

Thus, formula (5.4) gives a representation of right–ψ–hyperholomorphic function for a special choice of ψ.

Comparing representations (4.3) and (5.4), we obtain the following statement.

Proposition 5.2 *Let the conditions of Theorem 4.1 be satisfied. Then the function f is simultaneously left– and right–ψ–hyperholomorphic if it takes values on the set* $\{h_3e_3 + h_4e_4 : h_3, h_4 \in \mathbb{C}\}$.

Remark 5.3 Using formulas (1.4), we can rewrite representation (5.4) in the Pauli basis:

$$f(z) = \big(f_1(\zeta_2, \zeta_3, \zeta_4) + f_2(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)\big)\,\sigma_0 + \big(if_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - if_4(\zeta_2, \zeta_3, \zeta_4)\big)\,\sigma_1$$
$$+ \big(-f_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - f_4(\zeta_2, \zeta_3, \zeta_4)\big)\,\sigma_2 + \big(f_2(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - f_1(\zeta_2, \zeta_3, \zeta_4)\big)\,\sigma_3\,.$$

6 Representation of Left–$\Lambda - \psi$–hyperholomorphic Functions in a Special Case

In this section we consider operator (2.5) and Eq. (2.6). Now we will find a representation of left–$\Lambda - \psi$–hyperholomorphic functions for a special choice of parameters $\psi_1, \psi_2, \psi_3, \psi_4$ and Λ. For this purpose, we reduce Eq. (2.6) to a system of four PDEs. Denote

$$\Lambda := \Lambda_1 e_1 + \Lambda_2 e_2 + \Lambda_3 e_3 + \Lambda_4 e_4. \tag{6.1}$$

Using result of Sect. 4, we obtain that Eq. (2.6) is equivalent to the non-homogeneous system of PDEs:

$$\begin{aligned}
&\tfrac{\partial}{\partial z_1}(\alpha_1 f_1 + \alpha_3 f_4) + \tfrac{\partial}{\partial z_2}(\beta_1 f_1 + \beta_3 f_4) + \tfrac{\partial}{\partial z_3}(\gamma_1 f_1 + \gamma_3 f_4) + \tfrac{\partial}{\partial z_4}(\delta_1 f_1 + \delta_3 f_4) \\
&\qquad = -\Lambda_1 f_1 - \Lambda_3 f_4, \\
&\tfrac{\partial}{\partial z_1}(\alpha_2 f_2 + \alpha_4 f_3) + \tfrac{\partial}{\partial z_2}(\beta_2 f_2 + \beta_4 f_3) + \tfrac{\partial}{\partial z_3}(\gamma_2 f_2 + \gamma_4 f_3) + \tfrac{\partial}{\partial z_4}(\delta_2 f_2 + \delta_4 f_3) \\
&\qquad = -\Lambda_2 f_2 - \Lambda_4 f_3, \\
&\tfrac{\partial}{\partial z_1}(\alpha_1 f_3 + \alpha_3 f_2) + \tfrac{\partial}{\partial z_2}(\beta_1 f_3 + \beta_3 f_2) + \tfrac{\partial}{\partial z_3}(\gamma_1 f_3 + \gamma_3 f_2) + \tfrac{\partial}{\partial z_4}(\delta_1 f_3 + \delta_3 f_2) \\
&\qquad = -\Lambda_1 f_3 - \Lambda_3 f_2, \\
&\tfrac{\partial}{\partial z_1}(\alpha_2 f_4 + \alpha_4 f_1) + \tfrac{\partial}{\partial z_2}(\beta_2 f_4 + \beta_4 f_1) + \tfrac{\partial}{\partial z_3}(\gamma_2 f_4 + \gamma_4 f_1) + \tfrac{\partial}{\partial z_4}(\delta_2 f_4 + \delta_4 f_1) \\
&\qquad = -\Lambda_4 f_1 - \Lambda_2 f_4.
\end{aligned} \tag{6.2}$$

Theorem 6.1 *Let*

$$\begin{aligned}
\psi_1 &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4, \qquad \alpha_1\alpha_2 \neq \alpha_3\alpha_4\,, \\
\psi_2 &= \lambda\alpha_1 e_1 + \mu\alpha_2 e_2 + \mu\alpha_3 e_3 + \lambda\alpha_4 e_4, \\
\psi_3 &= \theta\alpha_1 e_1 + \vartheta\alpha_2 e_2 + \vartheta\alpha_3 e_3 + \theta\alpha_4 e_4, \\
\psi_4 &= \nu\alpha_1 e_1 + \eta\alpha_2 e_2 + \eta\alpha_3 e_3 + \nu\alpha_4 e_4,
\end{aligned} \tag{6.3}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \lambda, \mu, \theta, \vartheta, \nu, \eta$ are arbitrary complex numbers. In additional, let Λ be of the form (6.1) such that

$$\alpha_2\Lambda_3 = \alpha_3\Lambda_2\,, \qquad \alpha_1\Lambda_4 = \alpha_4\Lambda_1\,. \tag{6.4}$$

Then every left–$\Lambda - \psi$–hyperholomorphic function is of the form

$$\begin{aligned} f(z) = e_1\, \Phi_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) \cdot \exp\left(\frac{\alpha_3 \Lambda_4 - \alpha_2 \Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4}\, z_1\right) \\ + e_2\, \Phi_2(\zeta_2, \zeta_3, \zeta_4) \cdot \exp\left(\frac{\alpha_4 \Lambda_3 - \alpha_1 \Lambda_2}{\alpha_1\alpha_2 - \alpha_3\alpha_4}\, z_1\right) \\ + e_3\, \Phi_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) \cdot \exp\left(\frac{\alpha_3 \Lambda_4 - \alpha_2 \Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4}\, z_1\right) \\ + e_4\, \Phi_4(\zeta_2, \zeta_3, \zeta_4) \cdot \exp\left(\frac{\alpha_4 \Lambda_3 - \alpha_1 \Lambda_2}{\alpha_1\alpha_2 - \alpha_3\alpha_4}\, z_1\right), \end{aligned} \tag{6.5}$$

where $\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4, \zeta_2, \zeta_3, \zeta_4$ *are defined by relations* (4.4)*, and* $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ *are arbitrary holomorphic functions of three complex variables.*

Proof For given parameters (6.3) the first equation of system (6.2) takes the form

$$\begin{aligned} \frac{\partial}{\partial z_1}(\alpha_1 f_1 + \alpha_3 f_4) + \frac{\partial}{\partial z_2}(\lambda\alpha_1 f_1 + \mu\alpha_3 f_4) \\ + \frac{\partial}{\partial z_3}(\theta\alpha_1 f_1 + \vartheta\alpha_3 f_4) + \frac{\partial}{\partial z_4}(\nu\alpha_1 f_1 + \eta\alpha_3 f_4) = -\Lambda_1 f_1 - \Lambda_3 f_4. \end{aligned} \tag{6.6}$$

Similarly, for given parameters (6.3) the fourth equation of system (6.2) takes the form

$$\begin{aligned} \frac{\partial}{\partial z_1}(\alpha_4 f_1 + \alpha_2 f_4) + \frac{\partial}{\partial z_2}(\lambda\alpha_4 f_1 + \mu\alpha_2 f_4) \\ + \frac{\partial}{\partial z_3}(\theta\alpha_4 f_1 + \vartheta\alpha_2 f_4) + \frac{\partial}{\partial z_4}(\nu\alpha_4 f_1 + \eta\alpha_2 f_4) = -\Lambda_4 f_1 - \Lambda_2 f_4. \end{aligned} \tag{6.7}$$

Consider the difference between Eq. (6.6) multiplied by α_2 and Eq. (6.7) multiplied by α_3. Then, taking into account (6.4), we obtain the following equation

$$\begin{aligned} &\frac{\partial}{\partial z_1}\Big(f_1(\alpha_1\alpha_2 - \alpha_3\alpha_4) + f_4(\alpha_2\alpha_3 - \alpha_2\alpha_3)\Big) \\ &+ \frac{\partial}{\partial z_2}\Big(f_1(\lambda\alpha_1\alpha_2 - \lambda\alpha_3\alpha_4) + f_4(\mu\alpha_2\alpha_3 - \mu\alpha_2\alpha_3)\Big) \\ &+ \frac{\partial}{\partial z_3}\Big(f_1(\theta\alpha_1\alpha_2 - \theta\alpha_3\alpha_4) + f_4(\vartheta\alpha_2\alpha_3 - \vartheta\alpha_2\alpha_3)\Big) \\ &+ \frac{\partial}{\partial z_4}\Big(f_1(\nu\alpha_1\alpha_2 - \nu\alpha_3\alpha_4) + f_4(\eta\alpha_2\alpha_3 - \eta\alpha_2\alpha_3)\Big) = (\Lambda_4\alpha_3 - \Lambda_1\alpha_2) f_1. \end{aligned}$$

Thus, we obtain the equation

$$\frac{\partial f_1}{\partial z_1} + \lambda\frac{\partial f_1}{\partial z_2} + \theta\frac{\partial f_1}{\partial z_3} + \nu\frac{\partial f_1}{\partial z_4} = \frac{\Lambda_4\alpha_3 - \Lambda_1\alpha_2}{\alpha_1\alpha_2 - \alpha_3\alpha_4}\, f_1. \tag{6.8}$$

For Eq. (6.8) consider the characteristic equation

$$\frac{dz_1}{1} = \frac{dz_2}{\lambda} = \frac{dz_3}{\theta} = \frac{dz_4}{\nu} = \frac{(\alpha_1\alpha_2 - \alpha_3\alpha_4)\, d f_1}{(\Lambda_4\alpha_3 - \Lambda_1\alpha_2) f_1}. \tag{6.9}$$

The solutions of system (6.9) are the following integrals

$$c_2 = \lambda z_1 - z_2, \quad c_3 = \theta z_1 - z_3, \quad c_4 = \nu z_1 - z_4,$$

$$c_5 = \ln f_1 + \frac{\alpha_2\Lambda_1 - \alpha_3\Lambda_4}{\alpha_1\alpha_2 - \alpha_3\alpha_4} z_1.$$

Therefore, the general solution of Eq. (6.8) has the form

$$f_1 = \Phi_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) \cdot \exp\left(\frac{\alpha_3\Lambda_4 - \alpha_2\Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4} z_1\right),$$

where $\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4$ are defined by equalities (4.4), and Φ_1 is an arbitrary holomorphic function of three complex variables.

Similarly, we obtain the representations for the components f_2, f_3, f_4. □

Thus, formula (6.5) gives a representation of left–$\Lambda - \psi$–hyperholomorphic function for a special choice of Λ and ψ.

Remark 6.2 It is clear from the Eqs. (2.2) and (2.5) that for $\Lambda = 0$ a set of left–$\Lambda - \psi$–hyperholomorphic functions coincide with a set of left–ψ–hyperholomorphic functions. This fact is also confirmed by Theorems 4.1 and 6.1, because representation (6.5) coincides with representation (4.3) for $\Lambda = 0$.

Remark 6.3 Using formulas (1.4), we can rewrite representation (6.5) in the Pauli basis:

$$\begin{aligned} f(z) = {} & \sigma_0 \, \big(\Phi_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) + \Phi_2(\zeta_2, \zeta_3, \zeta_4)\big) \exp\left(\frac{\alpha_3\Lambda_4 - \alpha_2\Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4} z_1\right) \\ & + \sigma_1 \, \big(i\Phi_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - i\Phi_4(\zeta_2, \zeta_3, \zeta_4)\big) \exp\left(\frac{\alpha_3\Lambda_4 - \alpha_2\Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4} z_1\right) \\ & + \sigma_2 \, \big(-\Phi_3(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4) - \Phi_4(\zeta_2, \zeta_3, \zeta_4)\big) \exp\left(\frac{\alpha_3\Lambda_4 - \alpha_2\Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4} z_1\right) \\ & + \sigma_3 \, \big(\Phi_2(\zeta_2, \zeta_3, \zeta_4) - \Phi_1(\widetilde{\zeta}_2, \widetilde{\zeta}_3, \widetilde{\zeta}_4)\big) \exp\left(\frac{\alpha_3\Lambda_4 - \alpha_2\Lambda_1}{\alpha_1\alpha_2 - \alpha_3\alpha_4} z_1\right). \end{aligned}$$

Acknowledgements The authors express their gratitude to Professors M. E. Luna-Elizarraras and M. Shapiro for the discussion of the results and valuable advice. This work was supported by a grant from the Simons Foundation (1030291,V.S.Sh.). The main ideas of the study belong to V. Sh. The authors received all the results of the article together. The authors declare no competing interest.

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