Complex Analysis and Operator Theory (2024) 18:105
https://doi.org/10.1007/s11785-024-01543-z

Complex Analysis and Operator Theory

On Recovering Dirac Operators with Two Delays

Biljana Vojvodić[1] · Nebojša Djurić[2] · Vladimir Vladičić[3]

Received: 9 October 2023 / Accepted: 27 April 2024 / Published online: 29 May 2024
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2024

Abstract
We study the inverse spectral problems of recovering Dirac-type functional-differential operator with two constant delays a_1 and a_2 not less than one-third of the length the interval. It has been proved that the operator can be recovered uniquely from four spectra when $2a_1 + \frac{a_2}{2}$ is not less than the length of the interval, while it is not possible otherwise.

Keywords Dirac-type operator · Constant delay · Inverse spectral problem

Mathematics Subject Classification 34A55 · 34K29

1 Introduction

The theory of differential equations with delays is a a very significant area of the theory of ordinary differential equations (see [12, 13]). In last decades, there has been a growing interest in studying inverse spectral problems for different types of operators with one or more delays. It turned out that this type of operators is usually more adequate for modeling different real physical processes, frequently possessing

Communicated by Jussi Behrndt.

This article is part of the topical collection "Reproducing kernel spaces and applications" edited by Daniel Alpay.

✉ Vladimir Vladičić
vladimir.vladicic@ff.ues.rs.ba

Biljana Vojvodić
biljana.vojvodic@mf.unibl.org

Nebojša Djurić
nebojsa.djuric@etf.unibl.org

1 Faculty of Mechanical Engineering, University of Banja Luka, Banja Luka, Bosnia and Herzegovina

2 Faculty of Electrical Engineering, University of Banja Luka, Banja Luka, Bosnia and Herzegovina

3 Faculty of Philosophy, University of East Sarajevo, East Sarajevo, Bosnia and Herzegovina

a nonlocal nature. Inverse problems for Sturm–Liouville operators with one delay have been studied in most details (see [1, 5–9, 11, 15–18]). There is also considerable number of results related to the Sturm–Liouville operators with two constant delays (see [4, 14, 19–23]). In recent years, a significant number of results related to the inverse problems for Sturm–Liouville operators, have been extended to Dirac operators with one delay (see [3, 10, 25, 26]), as well as to Dirac operators with two delays (see [24]). The key issue in solving inverse problems for operators with delays is the question of inverse problem solution's uniqueness. Although inverse problem solution's uniqueness was for long thought to be indisputable, as in the case of inverse problems for classical operators (without delays), it turned out that the solution of inverse problems for operators with delays does not have to be unique. It has been shown in the papers [16, 18] that Sturm–Liouville operator with one delay can be recovered uniquely from two spectra if the delay belongs to $\left[\frac{2\pi}{5}, \pi\right)$, while in the papers [6, 7] has been shown that this is not possible for the delay from $\left[\frac{\pi}{3}, \frac{2\pi}{5}\right)$. There are the same results for Dirac operator with one delay (see [3, 10]). For operators with two delays, there are just results related to the inverse problem solution's uniqueness. So, in the papers [21, 22] it has been proven that Sturm–Liouville operator can be recovered uniquely from four spectra for the delays greater than $\frac{\pi}{2}$ under Robin and Dirichlet boundary conditions, respectively. In the paper [20] inverse problem solution's uniqueness has been proven for the Sturm–Liouville operator with two delays from $[\frac{2\pi}{5}, \frac{\pi}{2})$ under Robin boundary conditions. Papers [19, 23] deal with Sturm–Liouville operator with two delays such that first delay a_1 belongs to $\left[\frac{\pi}{3}, \frac{2\pi}{5}\right)$ and the second one to $[2a_1, \pi)$ under Robin and Dirichlet/Neumann boundary conditions, respectively and uniqueness of inverse problem solution has been proven. In the paper [24] it has been proven that Dirac operator with two delays from $\left[\frac{2\pi}{5}, \pi\right)$ can be recovered uniquely from four spectra. So far there are no results with non-unique solutions of inverse problems for operators with two delays, even for Sturm–Liouville operators.This paper will be the first result proving that the uniqueness of the inverse problem's solution does not have to be valid neither for operators with two delays.

In this paper we study Boundary value problems (BVPs) $D_j(P, Q, m), m \in \{0, 1\}, j \in \{1, 2\}$, for Dirac-type system of the form

$$\begin{aligned} &BY'(x) + (-1)^m P(x)Y(x - a_1) + Q(x)Y(x - a_2) = \lambda Y(x), x \in (0, \pi) \\ &y_1(0) = y_j(\pi) = 0 \end{aligned} \tag{1.1}$$

where

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Y(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \quad \frac{\pi}{3} \le a_1 < \frac{2\pi}{5}, \quad \frac{\pi}{3} \le a_2 < \pi, \ a_1 < a_2,$$

$$P(x) = \begin{bmatrix} p_1(x) & p_2(x) \\ p_2(x) & -p_1(x) \end{bmatrix}, \quad Q(x) = \begin{bmatrix} q_1(x) & q_2(x) \\ q_2(x) & -q_1(x) \end{bmatrix},$$

and $p_1(x), p_2(x), q_1(x), q_2(x) \in L^2[0, \pi]$ are complex-valued functions such that

$$P(x) = 0, \quad x \in (0, a_1), \quad Q(x) = 0, \quad x \in (0, a_2).$$

This paper shall answer the question whether the theorem of uniqueness holds or not in the case when the first delay belongs to $\left[\frac{\pi}{3}, \frac{2\pi}{5}\right)$ and the second one to $\left[\frac{\pi}{3}, \pi\right)$. In this way, the results from the paper [10] dealing with Dirac operator with one delay shall be generalized to Dirac operator with two delays. Besides research in inverse problems for delay(s) less than $\frac{\pi}{3}$, further research in this area should also answer the question weather the theorem of uniqueness holds or not for Sturm–Liouville operators with two delays greater than one-third of the interval.

Hereinafter we will assume that delays a_1 and a_2 are known. Also, in the following we will assume that $j \in \{1, 2\}$ and $m \in \{0, 1\}$.

Let $\{\lambda_{n,j}^m\}$ be the spectra of the BVPs $D_j(P, Q, m)$. The inverse problem of recovering matrix-functions $P(x), x \in (a_1, \pi)$ and $Q(x), x \in (a_2, \pi)$ from four spectra has been studied.

Inverse Problem 1. Given the spectra $\{\lambda_{n,j}^m\}$ of the BVPs $D_j(P, Q, m)$, find the matrix-functions $P(x)$ and $Q(x)$.

The paper is organized as follows: In Sect. 2 we construct characteristic functions and study asymptotic behavior of eigenvalues. Section 3 is devoted to the solving Inverse problem 1. We shall show that Theorem of uniqueness holds in the case when delays meet the condition $2a_1 + \frac{a_2}{2} \geq \pi$, as well as it does not in the case when $2a_1 + \frac{a_2}{2} < \pi$.

2 Spectral Properties

Based on the results from the paper [24], we obtain that Eq. (1.1) is equivalent to the integral equation

$$Y(x,\lambda) = R(x,\lambda)C + (-1)^m \int_{a_1}^{x} P(t)S^{-1}(x-t,\lambda)Y(t-a_1,\lambda)dt$$
$$+ \int_{a_2}^{x} Q(t)S^{-1}(x-t,\lambda)Y(t-a_2,\lambda)dt \tag{2.1}$$

where

$$R(x,\lambda) = \begin{bmatrix} \cos\lambda x & -\sin\lambda x \\ \sin\lambda x & \cos\lambda x \end{bmatrix}, \quad S(x,\lambda) = \begin{bmatrix} \sin\lambda x & cos\lambda x \\ -\cos\lambda x & \sin\lambda x \end{bmatrix}$$

and

$$C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

is a constant vector. Let

$$Y^m(x,\lambda) = \begin{bmatrix} y_1^m(x,\lambda) \\ y_2^m(x,\lambda) \end{bmatrix}$$

be the the solution of Eq. (1.1) such that

$$Y^m(0,\lambda) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

From (2.1) we obtain

$$Y^m(x,\lambda) = Y_0(x,\lambda) + (-1)^m \int_{a_1}^{x} P(t)S^{-1}(x-t,\lambda)Y^m(t-a_1,\lambda)dt + \int_{a_2}^{x} Q(t)S^{-1}(x-t,\lambda)Y^m(t-a_2,\lambda)dt \tag{2.2}$$

where

$$Y_0(x,\lambda) = \begin{bmatrix} \sin \lambda x \\ -\cos \lambda x \end{bmatrix}.$$

We solve integral Eq. (2.2) by the method of successive approximation, where representation of the solution depends on the order of delays (see [24]).

Let us for $k, l \in \{1, 2\}$, introduce notations

$$\alpha^1_{p_k p_l}(x) = \begin{cases} \int_{x+a_1}^{\pi} p_k(t)p_l(t-x)dt, & x \in [a_1, \pi - a_1] \\ 0, & x \in [0, a_1) \cup (\pi - a_1, \pi] \end{cases}$$

$$\alpha^2_{q_k q_l}(x) = \begin{cases} \int_{x+a_2}^{\pi} q_k(t)q_l(t-x)dt, & x \in [a_2, \pi - a_2] \\ 0, & x \in [0, a_2) \cup (\pi - a_2, \pi] \end{cases}$$

$$\alpha^{12}_{p_k q_l}(x) = \begin{cases} \int_{x+\frac{a_1+a_2}{2}}^{\pi} p_k(t)q_l(t-x-\frac{a_1-a_2}{2})dt, & x \in [\frac{a_1+a_2}{2}, \pi - \frac{a_1+a_2}{2}] \\ 0, & x \in [0, \frac{a_1+a_2}{2}) \cup (\pi - \frac{a_1+a_2}{2}, \pi] \end{cases}$$

$$\alpha^{12}_{q_k p_l}(x) = \begin{cases} \int_{x+\frac{a_1+a_2}{2}}^{\pi} q_k(t)p_l(t-x-\frac{a_2-a_1}{2})dt, & x \in [\frac{a_1+a_2}{2}, \pi - \frac{a_1+a_2}{2}] \\ 0, & x \in [0, \frac{a_1+a_2}{2}) \cup (\pi - \frac{a_1+a_2}{2}, \pi] \end{cases}$$

Eigenvalues of the BVPs $D_j(P, Q, m)$ coincide with zeros of the entire function

$$\Delta^m_j(\lambda) = y^m_j(\pi,\lambda)$$

which is called *characteristic function* of BVPs $D_j(P, Q, m)$. It has been shown in [24] that characteristic functions of BVPs $D_j(P, Q, m)$ can be represented in the form

$$\Delta_1^m(\lambda) = \sin\lambda\pi + \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} K^m(x)\cos\lambda(\pi - 2x)dx - \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} G^m(x)\sin\lambda(\pi - 2x)dx, \tag{2.3}$$

$$\Delta_2^m(\lambda) = -\cos\lambda\pi + \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} K^m(x)\sin\lambda(\pi - 2x)dx + \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} G^m(x)\cos\lambda(\pi - 2x)dx \tag{2.4}$$

where

$$K^m(x) = K_1^m(x) + K_2(x) + K_{12}^m(x) + K_{21}^m(x) \tag{2.5}$$

$$G^m(x) = G_1^m(x) + G_2(x) + G_{12}^m(x) + G_{21}^m(x) \tag{2.6}$$

and for $x \in (\frac{a_1}{2}, \pi - \frac{a_1}{2})$

$$K_1^m(x) = (-1)^m p_1\left(x + \frac{a_1}{2}\right) - \alpha^1_{p_1p_2}(x) + \alpha^1_{p_2p_1}(x),$$

$$G_1^m(x) = (-1)^m p_2\left(x + \frac{a_1}{2}\right) - \alpha^1_{p_1p_1}(x) - \alpha^1_{p_2p_2}(x),$$

$$K_2(x) = q_1\left(x + \frac{a_2}{2}\right) - \alpha^2_{q_1q_2}(x) + \alpha^2_{q_2q_1}(x),$$

$$G_2(x) = q_2\left(x + \frac{a_2}{2}\right) - \alpha^1_{q_1q_1}(x) - \alpha^2_{q_2q_2}(x),$$

$$K_{12}^m(x) = (-1)^m\alpha^{12}_{p_2q_1}(x) - (-1)^m\alpha^{12}_{p_1q_2}(x),$$

$$K_{21}^m(x) = (-1)^m\alpha^{12}_{q_2p_1}(x) - (-1)^m\alpha^{12}_{q_1p_2}(x),$$

$$G_{12}^m(x) = -(-1)^m\alpha^{12}_{p_1q_1}(x) - (-1)^m\alpha^{12}_{p_2q_2}(x),$$

$$G_{21}^m(x) = -(-1)^m\alpha^{12}_{q_1p_1}(x) - (-1)^m\alpha^{12}_{q_2p_2}(x).$$

Now we consider the asymptotic behavior of eigenvalues of BVPs $D_j(P, Q, m)$. Using the standard approach involving Rouché's theorem or proof from [10], one can show that the next theorem holds.

Theorem 2.1 *The boundary value problems $D_j(P, Q, m)$ have infinitely many eigenvalues $\lambda_{n,j}^m$, $n \in \mathbb{Z}$, of the form*

$$\lambda_{n,j}^m = n + \frac{1-j}{2} + \varkappa_{n,j}^m$$

where for $\varkappa_{n,j}^m \neq 0$ and for $|n| \to \infty$

$$\begin{aligned}
\varkappa_{n,1}^m =& \frac{1}{\pi} \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} K^m(x) \sin 2nx \, dx + \frac{1}{\pi} \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} G^m(x) \cos 2nx \, dx + o(\varkappa_{n,1}^m), \\
\varkappa_{n,2}^m =& \frac{1}{\pi} \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} K^m(x) \sin(2n-1)x \, dx \\
&+ \frac{1}{\pi} \int_{\frac{a_1}{2}}^{\pi-\frac{a_1}{2}} G^m(x) \cos(2n-1)x \, dx + o(\varkappa_{n,2}^m).
\end{aligned}$$

3 Recovering of the Matrix-Functions

In order to recover the matrix-functions $P(x)$ and $Q(x)$ from the spectra $\{\lambda_{n,j}^m\}$, at the beginning we construct characteristic functions by Hadamard theorem of factorization (Fig. 1).

Lemma 3.1 *The specification of the spectra $(\lambda_{n,j}^m)$ uniquely determines the characteristic functions Δ_1^m and Δ_2^m of BVPs $D_j(P, Q, m)$ by the formulas*

$$\Delta_1^m(\lambda) = \pi(\lambda_{0,1}^m - \lambda) \prod_{|n| \in N} \frac{\lambda_{n,1}^m - \lambda}{n} e^{\frac{\lambda}{n}},$$

$$\Delta_2^m(\lambda) = \prod_{n \in Z} \frac{\lambda_{n,2}^m - \lambda}{n - \frac{1}{2}} e^{\frac{\lambda}{n-\frac{1}{2}}}.$$

Proof See Theorem 5 in [2]. □

Using approach from the paper [10], we can recover functions $K^m(x)$ and $G^m(x)$ by formulas

$$K^m(x) = \sum_{n\in\mathbb{Z}} \left(\frac{(-1)^n}{\pi} \Theta_1^m(n) + i\frac{(-1)^n}{\pi} \Theta_2^m(n) \right) e^{2inx} \tag{3.1}$$

$$G^m(x) = \sum_{n\in\mathbb{Z}} \left(\frac{i(-1)^n}{\pi} \Theta_3^m(n) + \frac{(-1)^n}{\pi} \Theta_4^m(n) \right) e^{2inx} \tag{3.2}$$

where

$$\Theta_1^m(\lambda) = \frac{\Delta_1^m(\lambda) + \Delta_1^m(-\lambda)}{2},$$

$$\Theta_2^m(\lambda) = \frac{\Delta_2^m(\lambda) - \Delta_2^m(-\lambda)}{2},$$

$$\Theta_3^m(\lambda) = \frac{-\Delta_1^m(\lambda) + \Delta_1^m(-\lambda)}{2} + \sin\lambda\pi,$$

$$\Theta_4^m(\lambda) = \frac{\Delta_2^m(\lambda) + \Delta_2^m(-\lambda)}{2} + \cos\lambda\pi.$$

Now we come to our main result. Using functions $K^m(x)$ and $G^m(x)$ from (3.1) and (3.2) respectively, we shall answer the question whether the theorem of uniqueness for Inverse problem 1 holds on the set

$$R = \left\{ (a_1, a_2) : \ \frac{\pi}{3} < a_1 < \frac{2\pi}{5} \ \wedge \ \frac{\pi}{3} < a_2 < \pi \ \wedge \ a_1 < a_2 \right\}.$$

It will be shown that it is true on the subset

$$R_1 = \left\{ (a_1, a_2) \in R : \ \frac{\pi}{3} < a_1 < \frac{2\pi}{5} < a_2 < \pi \ \wedge \ 2a_1 + \frac{a_2}{2} \geq \pi \right\},$$

while on the subset

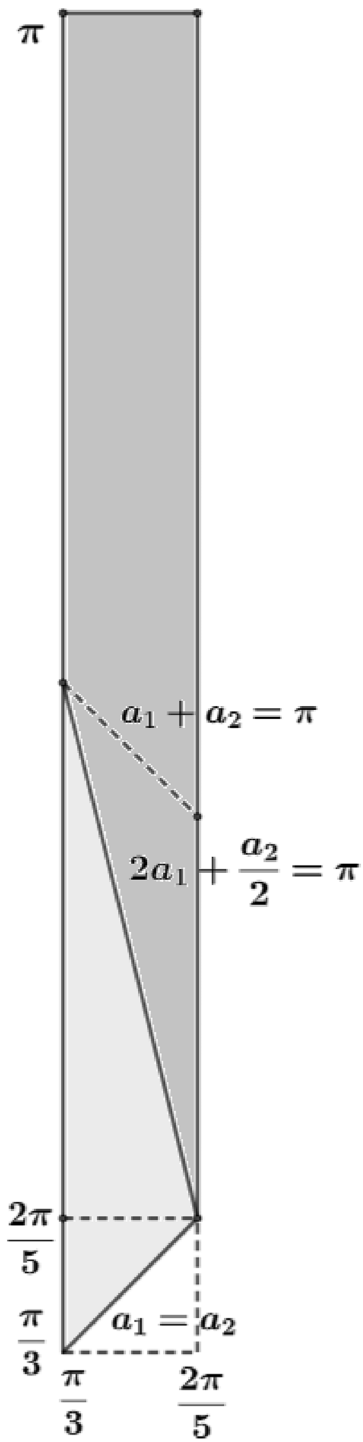
$$R_2 = R \setminus R_1 = \left\{ (a_1, a_2) \in R : \frac{\pi}{3} < a_1 < \frac{2\pi}{5} \ \wedge \ \frac{\pi}{3} < a_2 < \frac{2\pi}{3} \ \wedge \ 2a_1 + \frac{a_2}{2} < \pi \right\}$$

that is not true, Picture 3.1 (Fig. 1).

Firstly we will prove that the theorem of uniqueness holds on the subset R_1. We will recover functions $p_1(x)$, $p_2(x)$ by formulas

$$\begin{aligned} p_1(x) &= \frac{1}{2}\left(K^0\left(x - \frac{a_1}{2}\right) - K^1\left(x - \frac{a_1}{2}\right)\right) + \frac{1}{2}A_1\left(x - \frac{a_1}{2}\right), \\ p_2(x) &= \frac{1}{2}\left(G^0\left(x - \frac{a_1}{2}\right) - G^1\left(x - \frac{a_1}{2}\right)\right) + \frac{1}{2}A_2\left(x - \frac{a_1}{2}\right) \end{aligned} \tag{3.3}$$

Fig. 1 Picture 3.1



where for $x \in (\frac{a_1+a_2}{2}, \pi - \frac{a_1+a_2}{2})$

$$A_1(x) = -\alpha^{12}_{p_2 q_1}(x) + \alpha^{12}_{p_1 q_2}(x) - \alpha^{12}_{q_2 p_1}(x) + \alpha^{12}_{q_1 p_2}(x),$$

$$A_2(x) = \alpha^{12}_{p_1 q_1}(x) + \alpha^{12}_{p_2 q_2}(x) + \alpha^{12}_{q_1 p_1}(x) + \alpha^{12}_{q_2 p_2}(x)$$

and

$$A_1(x) = A_2(x) = 0, \quad x \notin (\frac{a_1 + a_2}{2}, \pi - \frac{a_1 + a_2}{2}).$$

Functions $q_1(x),\ q_2(x)$ will be recovered by formulas

$$\begin{aligned} q_1(x) &= \frac{1}{2}\left(K^0\left(x - \frac{a_2}{2}\right) + K^1\left(x - \frac{a_2}{2}\right)\right) + \frac{1}{2}B_1\left(x - \frac{a_2}{2}\right), \\ q_2(x) &= \frac{1}{2}\left(G^0\left(x - \frac{a_2}{2}\right) + G^1\left(x - \frac{a_2}{2}\right)\right) + \frac{1}{2}B_2\left(x - \frac{a_2}{2}\right) \end{aligned} \tag{3.4}$$

where for $x \in (a_1, a_2) \cup (\pi - a_2, \pi - a_1)$

$$B_1(x) = \alpha^1_{p_1 p_2}(x) - \alpha^1_{p_2 p_1}(x),$$

$$B_2(x) = \alpha^1_{p_1 p_1}(x) + \alpha^1_{p_2 p_2}(x),$$

for $x \in (a_2, \pi - a_2)$

$$B_1(x) = \alpha^1_{p_1 p_2}(x) - \alpha^1_{p_2 p_1}(x) + \alpha^2_{q_1 q_2}(x) - \alpha^2_{q_2 q_1}(x),$$

$$B_2(x) = \alpha^1_{p_1 p_1}(x) + \alpha^1_{p_2 p_1}(x) + \alpha^2_{q_1 q_1}(x) + \alpha^2_{q_2 q_2}(x),$$

and

$$B_1(x) = B_2(x) = 0, \quad x \notin (a_1, \pi - a_1).$$

Theorem 3.2 *Let $\frac{\pi}{3} \le a_1 < \frac{2\pi}{5} < a_2 < \pi$ and $2a_1 + \frac{a_2}{2} \ge \pi$. The spectra $(\lambda^m_{n,j})$ of BVPs $D_j(P, Q, m)$ uniquely determine matrix-functions $P(x),\ x \in (a_1, \pi)$ and $Q(x),\ x \in (a_2, \pi)$.*

Proof We distinguish cases when the sum of delays is greater or less than π.
1. Let $a_1 + a_2 \ge \pi$. We differ two cases

$$\frac{\pi}{3} \le a_1 < \frac{2\pi}{5} < a_2 < 2a_1 < \pi$$

and

$$\frac{\pi}{3} \le a_1 < \frac{2\pi}{5} < 2a_1 < a_2 < \pi.$$

We will prove the theorem for the first case, i.e. assuming that $a_2 < 2a_1$, while the proof for the second case differs only in the order of the intervals on which functions will be determined. From (2.5) and (2.6), for $x \in (\frac{a_1}{2}, \frac{a_2}{2}) \cup (\pi - \frac{a_2}{2}, \pi - \frac{a_1}{2})$, we have

$$p_1\left(x + \frac{a_1}{2}\right) = K^0(x), \qquad p_2\left(x + \frac{a_1}{2}\right) = G^0(x)$$

i.e. we determine functions

$$p_1(x), \quad p_2(x), \quad x \in \left(a_1, \frac{a_1 + a_2}{2}\right) \cup \left(\pi - \frac{a_2}{2} + \frac{a_1}{2}, \pi\right).$$

If $x \in (\frac{a_2}{2}, a_1) \cup (\pi - a_1, \pi - \frac{a_2}{2})$, functions $K^m(x)$ and $G^m(x)$ have the form

$$K^m(x) = (-1)^m p_1\left(x + \frac{a_1}{2}\right) + q_1(x + \frac{a_2}{2})$$

and

$$G^m(x) = (-1)^m p_2\left(x + \frac{a_1}{2}\right) + q_2\left(x + \frac{a_2}{2}\right).$$

Then we recover functions

$$p_1(x), \quad p_2(x), \quad x \in \left(\frac{a_1 + a_2}{2}, \frac{3a_1}{2}\right) \cup \left(\pi - \frac{a_1}{2}, \pi - \frac{a_2}{2} + \frac{a_1}{2}\right)$$

by formulas (3.3) for $A_1(x) = A_2(x) = 0$, as well as functions

$$q_1(x), \quad q_2(x), \quad x \in \left(a_2, a_1 + \frac{a_2}{2}\right) \cup \left(\pi - a_1 + \frac{a_2}{2}, \pi\right)$$

by formulas (3.4) for $B_1(x) = B_2(x) = 0$. Finally, from (2.5) and (2.6) we obtain that for $x \in (a_1, \pi - a_1)$ functions $K^m(x)$ and $G^m(x)$ have the form

$$K^m(x) = (-1)^m p_1\left(x + \frac{a_1}{2}\right) + q_1\left(x + \frac{a_2}{2}\right) - \alpha^1_{p_1 p_2}(x) + \alpha^1_{p_2 p_1}(x) \quad (3.5)$$

$$G^m(x) = (-1)^m p_2\left(x + \frac{a_1}{2}\right) + q_2\left(x + \frac{a_2}{2}\right) - \alpha^1_{p_1 p_1}(x) - \alpha^1_{p_2 p_2}(x). \quad (3.6)$$

Then we recover functions

$$p_1(x), \quad p_2(x), \quad x \in \left(\frac{3a_1}{2}, \pi - \frac{a_1}{2}\right)$$

by formulas (3.3) for $A_1(x) = A_2(x) = 0$. In this way functions $p_1(x)$ and $p_2(x)$ are recovered on (a_1, π). Then integrals $\alpha^1_{p_k p_l}(x)$, $k, l \in \{1, 2\}$ are known, too. Now, using formulas (3.4) for $B_1(x) = \alpha^1_{p_1p_2}(x) - \alpha^1_{p_2p_1}(x)$ and $B_2(x) = \alpha^1_{p_1p_1}(x) + \alpha^1_{p_2p_2}(x)$, we determine functions

$$q_1(x), \quad q_2(x), \quad x \in \left(a_1 + \frac{a_2}{2}, \pi - a_1 + \frac{a_2}{2}\right),$$

so they are also completely recovered on (a_2, π).

2. Let us now consider the case $a_1 + a_2 < \pi$. Taking the definition of the subset R_1 into account, we have[1]

$$\frac{2\pi}{5} < a_2 < \frac{2\pi}{3} < 2a_1.$$

At the beginning, in the same way as in the proof of previous case, for $x \in (\frac{a_1}{2}, \frac{a_2}{2}) \cup (\pi - \frac{a_2}{2}, \pi - \frac{a_1}{2})$ and $x \in (\frac{a_2}{2}, a_1) \cup (\pi - a_1, \pi - \frac{a_2}{2})$, using formulas (3.3) and (3.4), we determine functions

$$p_1(x), \quad p_2(x), \quad x \in \left(a_1, \frac{3a_1}{2}\right) \cup \left(\pi - \frac{a_1}{2}, \pi\right)$$

and functions

$$q_1(x), \quad q_2(x), \quad x \in (a_2, a_1 + \frac{a_2}{2}) \cup (\pi - a_1 + \frac{a_2}{2}, \pi).$$

For $x \in (a_1, \frac{a_1+a_2}{2}) \cup (\pi - \frac{a_1+a_2}{2}, \pi - a_1)$ functions $K^m(x)$ and $G^m(x)$ have the form (3.5) and (3.6) and we determine functions

$$p_1(x), \quad p_2(x), \quad x \in \left(\frac{3a_1}{2}, a_1 + \frac{a_2}{2}\right) \cup \left(\pi - \frac{a_2}{2}, \pi - \frac{a_1}{2}\right)$$

by formulas (3.3) for $A_1(x) = A_2(x) = 0$. In order to determine functions $q_1(x), q_2(x)$, it is needed to show that integrals $\alpha^1_{p_kp_l}(x)$, $k, l \in \{1, 2\}$, are known. For arguments of subintegral functions $p_1(x), p_2(x)$ is valid

$$t > x + a_1 > 2a_1 > \pi - \frac{a_2}{2},$$
$$t - x < \pi - a_1 < a_1 + \frac{a_2}{2}$$

due to the assumption

$$2a_1 + \frac{a_2}{2} \geq \pi.$$

Therefore, arguments of subintegral functions $p_1(x), p_2(x)$ belong to the interval $(a_1, a_1 + \frac{a_2}{2}) \cup (\pi - \frac{a_2}{2}, \pi)$, hence integrals $\alpha^1_{p_kp_l}(x)$ are known.Then, from (3.4) for

[1] The order of delays $2a_1 < a_2 < a_1 + a_2 < \pi$ is not possible since $2a_1 < a_2 \implies \pi < 3a_1 < a_1 + a_2$

$B_1(x) = \alpha^1_{p_1p_2}(x) - \alpha^1_{p_2p_1}(x)$ and $B_2(x) = \alpha^1_{p_1p_1}(x) + \alpha^1_{p_2p_2}(x)$, we can determine functions

$$q_1(x), \quad q_2(x), \quad x \in \left(a_1 + \frac{a_2}{2}, a_2 + \frac{a_1}{2}\right) \cup \left(\pi - \frac{a_1}{2}, \pi - a_1 + \frac{a_2}{2}\right).$$

In the following we have in mind that the functions $p_1(x)$ and $p_2(x)$ are recovered on the interval $(a_1, a_1 + \frac{a_2}{2}) \cup (\pi - \frac{a_2}{2}, \pi)$ and functions $q_1(x)$ and $q_2(x)$ on the interval $(a_2, a_2 + \frac{a_1}{2}) \cup (\pi - \frac{a_1}{2}, \pi)$. We differ two cases, depending on the condition whether the second delay is grater or less than $\frac{\pi}{2}$.

2.1. Let $\frac{\pi}{2} \le a_2 < \frac{2\pi}{3}$. Then it remains to recover matrix-functions $P(x)$ and $Q(x)$ on the interval $(\frac{a_1+a_2}{2}, \pi - \frac{a_1+a_2}{2})$. We have

$$\begin{aligned} K^m(x) = (-1)^m p_1\left(x + \frac{a_1}{2}\right) + q_1\left(x + \frac{a_2}{2}\right) - \alpha^1_{p_1p_2}(x) + \alpha^1_{p_2p_1}(x) \\ + (-1)^m \alpha^{12}_{p_2q_1}(x) - (-1)^m \alpha^{12}_{p_1q_2}(x) + (-1)^m \alpha^{12}_{q_2p_1}(x) - (-1)^m \alpha^{12}_{q_1p_2}(x) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} G^m(x) = (-1)^m p_2\left(x + \frac{a_1}{2}\right) + q_2\left(x + \frac{a_2}{2}\right) - \alpha^1_{p_1p_1}(x) - \alpha^1_{p_2p_2}(x) \\ - (-1)^m \alpha^{12}_{p_1q_1}(x) - (-1)^m \alpha^{12}_{p_2q_2}(x) - (-1)^m \alpha^{12}_{q_1p_1}(x) - (-1)^m \alpha^{12}_{q_2p_2}(x). \end{aligned} \tag{3.8}$$

One can easily obtain that integrals $\alpha^1_{p_kp_l}(x)$ are known. We have

$$t > x + a_1 > \frac{a_1 + a_2}{2} + a_1 > \pi - \frac{a_2}{2},$$
$$t - x < \pi - \frac{a_1 + a_2}{2} < a_1 + \frac{a_2}{2}$$

since

$$\frac{3a_1}{2} + a_2 > 2a_1 + \frac{a_2}{2} > \pi.$$

Then, from (3.4) we can determine functions

$$q_1(x), \quad q_2(x), \quad x \in \left(a_2 + \frac{a_1}{2}, \pi - \frac{a_1}{2}\right).$$

In that way functions $q_1(x)$, $q_2(x)$ are completely recovered on (a_2, π). Now it is not difficult to show that "mixed" integrals $\alpha^{12}_{p_kq_l}(x)$ and $\alpha^{12}_{q_kp_l}(x)$ are also known. Indeed, for arguments of subintegral functions $p_1(x)$, $p_2(x)$ in "mixed" integrals is valid

$$t > x + \frac{a_1 + a_2}{2} > a_1 + a_2 > \pi - \frac{a_2}{2},$$

$$t-x-\frac{a_2-a_1}{2}<\pi-\frac{a_1+a_2}{2}-\frac{a_2-a_1}{2}=\pi-a_2<a_1+\frac{a_2}{2}$$

since

$$a_1+\frac{3a_2}{2}>2a_1+\frac{a_2}{2}>\pi.$$

Therefore, arguments of subintegral functions $p_1(x)$, $p_2(x)$ belong to the interval $(a_1, a_1+\frac{a_2}{2})\cup(\pi-\frac{a_2}{2},\pi)$ and "mixed" integrals are known. Then, using formulas (3.3), we can determine functions

$$p_1(x),\quad p_2(x),\quad x\in\left(a_1+\frac{a_2}{2},\pi-\frac{a_2}{2}\right)$$

so they are also completely recovered on (a_1,π).

2.2. Let $\frac{2\pi}{5}\le a_2<\frac{\pi}{2}$. It remains to show that theorem of uniqueness is valid on the intervals $(\frac{a_1+a_2}{2},a_2)\cup(\pi-a_2,\pi-\frac{a_1+a_2}{2})$ and $(a_2,\pi-a_2)$. On the interval $(\frac{a_1+a_2}{2},a_2)\cup(\pi-a_2,\pi-\frac{a_1+a_2}{2})$ functions $K^m(x)$ and $G^m(x)$ have the form (3.7) and (3.8) respectively. In the same way as in the proof for the case 2.1, one can show that integrals $\alpha^1_{p_kp_l}(x)$ are known, so we can determine functions

$$q_1(x),\quad q_2(x),\quad x\in\left(a_2+\frac{a_1}{2},\frac{3a_2}{2}\right)\cup\left(\pi-\frac{a_2}{2},\pi-\frac{a_1}{2}\right)$$

by formulas (3.4). Now we will show that “mixed” integrals $\alpha^{12}_{p_kq_l}(x)$ and $\alpha^{12}_{q_kp_l}(x)$ are also known. In the same way as in previous case we show that arguments of subintegral functions $p_1(x)$, $p_2(x)$ belong to the interval $(a_1,a_1+\frac{a_2}{2})\cup(\pi-\frac{a_2}{2},\pi)$. For arguments of subintegral functions $q_1(x)$, $q_2(x)$ in "mixed" integrals we have

$$\begin{aligned}&t>x+\frac{a_1+a_2}{2}>a_1+a_2>\pi-\frac{a_2}{2},\\&t-x-\frac{a_1-a_2}{2}<\pi-\frac{a_1+a_2}{2}-\frac{a_1-a_2}{2}=\pi-a_1<\frac{3a_2}{2}\end{aligned}$$

since

$$a_1+\frac{3a_2}{2}>2a_1+\frac{a_2}{2}>\pi.$$

Then, using formulas (3.3), we can determine functions

$$p_1(x),\quad p_2(x),\quad x\in\left(a_1+\frac{a_2}{2},a_2+\frac{a_1}{2}\right)\cup\left(\pi-a_2+\frac{a_1}{2},\pi-\frac{a_2}{2}\right).$$

Let us finally consider the interval $(a_2, \pi - a_2)$. Then functions $K^m(x)$, $G^m(x)$ have the form

$$\begin{aligned}K^m(x) = (-1)^m p_1\left(x + \frac{a_1}{2}\right) + q_1\left(x + \frac{a_2}{2}\right) - \alpha^1_{p_1p_2}(x) \\ + \alpha^1_{p_2p_1}(x) - \alpha^2_{q_1q_2}(x) + \alpha^2_{q_2q_1}(x) + (-1)^m\alpha^{12}_{p_2q_1}(x) \\ - (-1)^m\alpha^{12}_{p_1q_2}(x) + (-1)^m\alpha^{12}_{q_2p_1}(x) - (-1)^m\alpha^{12}_{q_1p_2}(x)\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}G^m(x) = (-1)^m p_2\left(x + \frac{a_1}{2}\right) + q_2\left(x + \frac{a_2}{2}\right) - \alpha^1_{p_1p_1}(x) \\ - \alpha^1_{p_2p_2}(x) - \alpha^2_{q_1q_1}(x) - \alpha^2_{q_2q_2}(x) - (-1)^m\alpha^{12}_{p_1q_1}(x) \\ - (-1)^m\alpha^{12}_{p_2q_2}(x) - (-1)^m\alpha^{12}_{q_1p_1}(x) - (-1)^m\alpha^{12}_{q_2p_2}(x).\end{aligned} \tag{3.10}$$

Let us show that integrals $\alpha^1_{p_kp_l}(x)$, as well as integrals $\alpha^2_{q_kq_l}(x)$, are known. For arguments of subintegral functions $p_1(x)$, $p_2(x)$ we have

$$\begin{aligned}t > x + a_1 > a_1 + a_2 > \pi - a_2 + \frac{a_1}{2}, \\ t - x < \pi - a_2 < a_2 + \frac{a_1}{2}\end{aligned}$$

since

$$2a_2 + \frac{a_1}{2} > 2a_1 + \frac{a_2}{2} > \pi.$$

For arguments of subintegral functions $q_1(x)$, $q_2(x)$ the following is valid

$$\begin{aligned}t > x + a_2 > 2a_2 > \pi - \frac{a_2}{2}, \\ t - x < \pi - a_2 < \frac{3a_2}{2}\end{aligned}$$

since

$$\frac{5a_2}{2} > \pi.$$

In that way, using formulas (3.4), we can recover functions

$$q_1(x), \quad q_2(x), \quad x \in \left(\frac{3a_2}{2}, \pi - \frac{a_2}{2}\right)$$

so they are completely recovered on (a_2, π). It remains to show that "mixed" integrals $\alpha^1_{p_kq_l}(x)$ and $\alpha^1_{q_kp_l}(x)$ are also known. Due to the condition $\frac{5a_2}{2} > \pi$, for arguments of subintegral functions $p_1(x)$, $p_2(x)$ is valid

$$t > x + \frac{a_1 + a_2}{2} > a_2 + \frac{a_1 + a_2}{2} > \pi - a_2 + \frac{a_1}{2},$$
$$t - x - \frac{a_2 - a_1}{2} < \pi - a_2 - \frac{a_2 - a_1}{2} < a_2 + \frac{a_1}{2}.$$

Then we can determine functions

$$p_1(x), \quad p_2(x), \quad x \in \left(a_2 + \frac{a_1}{2}, \pi - a_2 + \frac{a_1}{2}\right)$$

by formulas (3.3), and they are also completely recovered on (a_1, π). Theorem is proved. □

Now we will show that theorem of uniqueness does not hold on the subset R_2. For that purpose let us for fixed a_1, a_2 and define the integral operator $M : L^2\left(a_1 + \frac{a_2}{2}, \pi - a_1\right) \to L^2\left(a_1 + \frac{a_2}{2}, \pi - a_1\right)$

$$M(f(x)) = \int_{a_1+\frac{a_2}{2}}^{\pi - x + \frac{a_2}{2}} f(t) h\left(t + x - \frac{a_2}{2}\right) dt \tag{3.11}$$

for some non-zero real function $h(x) \in L^2(2a_1 + \frac{a_2}{2}, \pi)$. Operator $M(f(x))$ is self-adjoint since

$$\int_{a_1+\frac{a_2}{2}}^{\pi - a_1} M(f(x)) g(x) dx = \int_{a_1+\frac{a_2}{2}}^{\pi - a_1} \left(\int_{a_1+\frac{a_2}{2}}^{\pi - x + \frac{a_2}{2}} f(t) h\left(t + x - \frac{a_2}{2}\right) dt\right) g(x) dx$$
$$= \int_{a_1+\frac{a_2}{2}}^{\pi - a_1} f(t) \left(\int_{a_1+\frac{a_2}{2}}^{\pi - t - \frac{a_2}{2}} h\left(t + x - \frac{a_2}{2}\right) g(x) dx\right) dt = \int_{a_1+\frac{a_2}{2}}^{\pi - a_1} f(t) M(g(t)) dt.$$

We can choose function $h(x)$ such that the operator $M(f(x))$ has eigenvalue $\eta_1 = 1$ with corresponding eigenfunction $e_1(x)$ (see [6]), i.e.

$$M(e_1(x)) = e_1(x), \quad x \in \left(a_1 + \frac{a_2}{2}, \pi - a_1\right). \tag{3.12}$$

We construct the family of functions

$$D_\beta = \{p_1(x), p_2^\beta(x), q_1(x), q_2^\beta(x) : \beta \in C\}$$

where

$$p_1(x) = 0, \; x \in (0, \pi),$$
$$p_2^\beta(x) = \begin{cases} 0, \quad x \in \left(0, a_1 + \frac{a_2}{2}\right) \cup \left(\pi - a_1, 2a_1 + \frac{a_2}{2}\right) \\ \beta e_1(x), \quad x \in \left(a_1 + \frac{a_2}{2}, \pi - a_1\right) \\ h(x), \quad x \in \left(2a_1 + \frac{a_2}{2}, \pi\right) \end{cases} \tag{3.13}$$

$$q_1(x) = 0, \quad x \in (0, \pi),$$
$$q_2^\beta(x) = \begin{cases} 0, & x \in \left(0, a_1 + \frac{a_2}{2}\right) \cup (\pi - a_1, \pi) \\ \beta e_1(x), & x \in \left(a_1 + \frac{a_2}{2}, \pi - a_1\right). \end{cases} \tag{3.14}$$

Using this family of functions, we will prove that the solution of Inverse problem 1 is not unique if $2a_1 + \frac{a_2}{2} < \pi$. Let us for this purpose denote

$$P_\beta(x) = \begin{bmatrix} 0 & p_2^\beta(x) \\ p_2^\beta(x) & 0 \end{bmatrix}, \quad Q_\beta(x) = \begin{bmatrix} 0 & q_2^\beta(x) \\ q_2^\beta(x) & 0 \end{bmatrix}.$$

Theorem 3.3 *Let* $\frac{\pi}{3} \le a_1 < \frac{2\pi}{5}$, $\frac{\pi}{3} \le a_2 < \frac{2\pi}{3}$, $a_1 < a_2$ *and* $2a_1 + \frac{a_2}{2} < \pi$. *The spectra* $(\lambda_{n,j}^m)$ *of BVPs* $D_j(P_\beta, Q_\beta, m)$ *is independent of* β.

Proof Notice that in this case obviously $a_1 + a_2 < \pi$ and functions $K^m(x)$ and $G^m(x)$ have the form (3.7) and (3.8) or (3.9) and (3.10) respectively, depending on the condition whether the second delay is less or not than $\frac{\pi}{2}$. Taking into account that $p_1(x) = q_1(x) = 0$, as well as that function $q_2^\beta(x)$ is vanishing for $x \in (\pi - a_1, \pi)$, we obtain that

$$K^m(x) = 0, \quad x \in (0, \pi), \tag{3.15}$$

while in functions $G^m(x)$, besides functions $p_2(x)$ and $q_2(x)$, only integrals $\alpha^1_{p_2p_2}(x)$ and $\alpha^{12}_{p_2q_2}(x)$ remain. Since

$$p_2\left(x + \frac{a_1}{2}\right) = 0, \; x \in \left(0, \frac{a_1 + a_2}{2}\right) \cup \left(\pi - \frac{3a_1}{2}, \frac{3a_1}{2} + \frac{a_2}{2}\right)$$

and

$$q_2\left(x + \frac{a_2}{2}\right) = 0, \; x \in (0, a_1) \cup \left(\pi - a_1 - \frac{a_2}{2}, \pi - \frac{a_2}{2}\right)$$

we obtain

$$\alpha^1_{p_2p_2}(x) = 0, \quad x \in \left(\pi - a_1 - \frac{a_2}{2}, \pi - a_1\right)$$

and

$$\alpha^{12}_{p_2q_2}(x) = 0, \; x \in \left(\pi - \frac{3a_1}{2}, \pi - \frac{a_1 + a_2}{2}\right).$$

Indeed, for $p_2(x)$ subintegral function's argument in integral $\alpha^1_{p_2p_2}(x)$ on the interval $(\pi - a_1 - \frac{a_2}{2}, \pi - a_1)$ the following is valid

$$t - x < \pi - \pi + a_1 + \frac{a_2}{2} = a_1 + \frac{a_2}{2},$$

while $q_2(x)$ function's argument in $\alpha^{12}_{p_2q_2}(x)$ on the interval $(\pi - \frac{3a_1}{2}, \pi - \frac{a_1+a_2}{2})$ satisfies condition

$$t - x - \frac{a_1 - a_2}{2} < \pi - \pi + \frac{3a_1}{2} - \frac{a_1 - a_2}{2} = a_1 + \frac{a_2}{2}.$$

Then we have

$$\begin{aligned}
&G^m(x) = 0, \quad x \in \left(\frac{a_1}{2}, a_1\right) \cup \left(\pi - \frac{3a_1}{2}, \frac{3a_1}{2} + \frac{a_2}{2}\right), \\
&G^m(x) = (-1)^m \left(p_2(x + \frac{a_1}{2}) - \alpha^{12}_{p_2q_2}(x)\right) \mathbb{1}_{\left(\frac{a_1+a_2}{2}, \pi - \frac{3a_1}{2}\right)}(x) \\
&\quad + \left(q_2\left(x + \frac{a_2}{2}\right) - \alpha^{1}_{p_2p_2}(x)\right) \mathbb{1}_{(a_1, \pi - a_1 - \frac{a_2}{2})}(x), \quad x \in \left(a_1, \pi - \frac{3a_1}{2}\right), \\
&G^m(x) = (-1)^m p_2\left(x + \frac{a_1}{2}\right), \quad x \in \left(\frac{3a_1}{2} + \frac{a_2}{2}, \pi - \frac{a_1}{2}\right).
\end{aligned} \tag{3.16}$$

Then from (2.3), (3.15) and (3.16), we obtain that characteristic functions $\Delta^m_1(\lambda)$ for family of functions D_β have the form

$$\begin{aligned}
\Delta^m_1(\lambda) &= \sin\lambda\pi + \int_{\frac{a_1}{2}}^{\pi - \frac{a_1}{2}} K^m(x) \cos\lambda(\pi - 2x) dx - \int_{\frac{a_1}{2}}^{\pi - \frac{a_1}{2}} G^m(x) \sin\lambda(\pi - 2x) dx \\
&= \sin\lambda\pi - (-1)^m \int_{\frac{a_1+a_2}{2}}^{\pi - \frac{3a_1}{2}} \left(p^\beta_2\left(x + \frac{a_1}{2}\right) - \alpha^{12}_{p^\beta_2 q^\beta_2}(x)\right) \sin\lambda(\pi - 2x) dx \\
&\quad - \int_{a_1}^{\pi - a_1 - \frac{a_2}{2}} \left(q^\beta_2\left(x + \frac{a_2}{2}\right) \alpha^{1}_{p^\beta_2 p^\beta_2}(x)\right) \sin\lambda(\pi - 2x) dx \\
&\quad - (-1)^m \int_{\frac{3a_1}{2} + \frac{a_2}{2}}^{\pi - \frac{a_1}{2}} p^\beta_2\left(x + \frac{a_1}{2}\right) \sin\lambda(\pi - 2x) dx \\
&= \sin\lambda\pi - (-1)^m \int_{a_1 + \frac{a_2}{2}}^{\pi - a_1} \left(p^\beta_2(x) - \alpha^{12}_{p^\beta_2 q^\beta_2}\left(x - \frac{a_1}{2}\right)\right) \sin\lambda(\pi - 2x + a_1) dx \\
&\quad - \int_{a_1 + \frac{a_2}{2}}^{\pi - a_1} \left(q^\beta_2(x) - \alpha^{1}_{p^\beta_2 p^\beta_2}\left(x - \frac{a_2}{2}\right)\right) \sin\lambda(\pi - 2x + a_2) dx
\end{aligned}$$

$$-(-1)^m \int\limits_{2a_1+\frac{a_2}{2}}^{\pi} p_2^\beta(x) \sin\lambda(\pi - 2x + a_1)dx.$$

Let us show that

$$U(x) = p_2^\beta(x) - \alpha^{12}_{p_2^\beta q_2^\beta}\left(x - \frac{a_1}{2}\right) = 0, \quad x \in \left(a_1 + \frac{a_2}{2}, \pi - a_1\right).$$

Using the definition and properties of the integral operator $M(f)$ from (3.11) and (3.12), as well as the form of functions $p_1^\beta(x)$ and $q_2^\beta(x)$ from (3.13) and (3.14), we obtain

$$\begin{aligned} U(x) &= p_2^\beta(x) - \int\limits_{x+\frac{a_2}{2}}^{\pi} p_2^\beta(t) q_2^\beta\left(t - x + \frac{a_2}{2}\right)dt \\ &= p_2^\beta(x) - \int\limits_{a_2}^{\pi - x + \frac{a_2}{2}} p_2^\beta\left(s + x - \frac{a_2}{2}\right) q_2^\beta(s)ds \\ &= p_2^\beta(x) - \int\limits_{a_1+\frac{a_2}{2}}^{\pi - x + \frac{a_2}{2}} q_2^\beta(s) p_2^\beta\left(s + x - \frac{a_2}{2}\right)ds. \end{aligned}$$

Since

$$\pi - x + \frac{a_2}{2} < \pi - a_1 \ \wedge \ s + x - \frac{a_2}{2} > 2a_1 + \frac{a_2}{2}$$

we obtain

$$\begin{aligned} U(x) &= \beta e_1(x) - \int\limits_{a_1+\frac{a_2}{2}}^{\pi - x + \frac{a_2}{2}} \beta e_1(s) h\left(s + x - \frac{a_2}{2}\right)ds \\ &= \beta e_1(x) - \beta M(e_1(x)) = \beta e_1(x) - \beta e_1(x) = 0. \end{aligned}$$

In the same way one can show that

$$V(x) = q_2^\beta(x) - \alpha^1_{p_2^\beta p_2^\beta}\left(x - \frac{a_2}{2}\right) = 0, \ x \in \left(a_1 + \frac{a_2}{2}, \pi - a_1\right).$$

Indeed, we have

$$
\begin{aligned}
V(x) &= q_2^\beta(x) - \int\limits_{x-\frac{a_2}{2}+a_1}^{\pi} p_2^\beta(t) p_2^\beta\left(t - x + \frac{a_2}{2}\right) dt \\
&= q_2^\beta(x) - \int\limits_{a_1}^{\pi-x+\frac{a_2}{2}} p_2^\beta\left(s + x - \frac{a_2}{2}\right) p_2^\beta(s) ds \\
&= q_2^\beta(x) - \int\limits_{a_1+\frac{a_2}{2}}^{\pi-x+\frac{a_2}{2}} p_2^\beta(s) p_2^\beta\left(s + x - \frac{a_2}{2}\right) ds \\
&= \beta e_1(x) - \int\limits_{a_1+\frac{a_2}{2}}^{\pi-x+\frac{a_2}{2}} \beta e_1(s) h\left(s + x - \frac{a_2}{2}\right) ds \\
&= \beta e_1(x) - \beta M(e_1(x)) = \beta e_1(x) - \beta e_1(x) = 0.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
\Delta_1^m(\lambda) &= \sin \lambda\pi - (-1)^m \int\limits_{2a_1+\frac{a_2}{2}}^{\pi} p_2^\beta(x) \sin \lambda(\pi - 2x + a_1) dx \\
&= \sin \lambda\pi - (-1)^m \int\limits_{2a_1+\frac{a_2}{2}}^{\pi} h(x) \sin \lambda(\pi - 2x + a_1) dx.
\end{aligned}
$$

In the same way from (2.4) we obtain

$$
\Delta_2^m(\lambda) = -\cos \lambda\pi + (-1)^m \int\limits_{2a_1+\frac{a_2}{2}}^{\pi} h(x) \cos \lambda(\pi - 2x + a_1) dx,
$$

i.e. characteristic functions are independent of β. Theorem is proved. □

Remark 3.4 If we consider the case when the first delay is greater than the second one, then both delays are less then $\frac{2\pi}{5}$ and Theorem of uniqueness does not hold on the triangle which completes the set R up to the rectangle (Picture 3.1). But if we do not limit the first delay to the interval $[\frac{\pi}{3}, \frac{2\pi}{5})$ and consider the set

$$
S = \{(a_1, a_2) : \ \frac{\pi}{3} \le a_2 < a_1 < \pi\}
$$

it is clear that the theorem of uniqueness holds on the subset

$$S_1 = \{(a_1, a_2) \in S : \frac{2\pi}{5} \leq a_2 < a_1\}.$$

On the subset

$$S_2 = S \setminus S_1 = \left\{(a_1, a_2) \in S : \frac{\pi}{3} \leq a_1 < \pi \wedge \frac{\pi}{3} < a_2 < \frac{2\pi}{5} \wedge a_2 < a_1\right\}$$

the theorem of uniqueness does not hold which follows from the results for Dirac operator with one delay. Indeed, if we take $P(x) = 0$ in (1.1), then we obtain two BVPs with one delay a_2. It is known that inverse problem's solution is not unique in the case $a_2 < \frac{2\pi}{5}$ (see [10]) and then we conclude that the theorem of uniqueness does not hold on the subset S_2. Taking into account this result, as well as the result from the Theorem 3.2, we conclude that position of $(-1)^m$ in the BVPs setting is very important and significantly affects the size of the set on which the theorem of uniqueness is valid.

Author Contributions The authors have an equal contributions to the manuscript.

Declarations

Conflict of interest The authors declare no Conflict of interest.

References

1. Bondarenko, N., Yurko, V.: An inverse problem for Sturm–Liouville differential operators with deviating argument. Appl. Math. Lett. **83**, 140–144 (2018)
2. Buterin, S.A.: On the uniform stability on recovering sine-type functions with asimptotically separated zeros Math. Notes **111**(3), 343–355 (2022)
3. Buterin, S.A., Djurić, N.: Inverse problems for Dirac operators with constant delay: uniqueness, characterization, uniform stability. Lobachevskii J. Math. **43**(6), 1492–1501 (2022)
4. Buterin, S.A., Malyugina, M.A., Shieh, C.-T.: An inverse spectral problem for second-order functional-differential pencils with two delays. Appl. Math. Comput. **411**, 126475 (2021)
5. Buterin, S.A., Yurko, V.A.: An inverse spectral problem for Sturm–Liouville operators with a large constant delay. Anal. Math. Phys. **9**(1), 17–27 (2019)
6. Djurić, N., Buterin, S.: On an open question in recovering Sturm–Liouville-type operators with delay. Appl. Math. Lett. **113**, 106862 (2021)
7. Djurić, N., Buterin, S.: On non-uniqueness of recovering Sturm–Liouville operators with delay. Commun. Nonlinear Sci. Numer. Simulat. **102**, 105900 (2021)
8. Djurić, N., Buterin, S.: Iso-bispectral potentials for Sturm–Liouville-type operators with small delay. Nonlin. Analysis: Real World Appl. **63**, 103390 (2022)
9. Djurić, N., Vladičić, V.: Incomplete inverse problem for Sturm–Liouville type differential equation with constant delay. Res. Math. **74**, 161 (2019)
10. Djuric, N., Vojvodic, B.: Inverse problem for Dirac operators with a constant delay less than half the length of the interval. Appl. Anal. Discret. Math. **17**(1), 249–261 (2023)
11. Freiling, G., Yurko, V.A.: Inverse problems for Sturm–Liouville differential operators with a constant delay. Appl. Math. Lett. **25**(11), 1999–2004 (2012)

12. Myshkis, A.D.: Linear Differential Equations with a Delay Argument, "Nauka", Moscow (1951)
13. Norkin, S.B.: Second Order Differential Equations with a Delay Argument, "Nauka", Moscow (1965)
14. Pavlović, N., Pikula, M., Vojvodić, B.: First regularized trace of the limit assignment of Sturm–Liouville type with two constant delays. Filomat **29.1**(1), 51–62 (2015)
15. Pikula, M.: Determination of a Sturm–Liouville-type differential operator with delay argument from two spectra. Mat. Vesnik **43**(3–4), 159–171 (1991)
16. Pikula, M., Vladičić, V., Vojvodić, B.: Inverse spectral problems for Sturm–Liouville operators with a constant delay less than half the length of the interval and Robin boundary conditions. Res. Math. **74**(1), 45 (2019)
17. Vladičić, V., Bošković, M., Vojvodić, B.: Inverse problems for Sturm–Liouville-type differential equation with a constant delay under Dirichlet/plynomial boundary conditions. Bull. Iran. Math. Soc. **48**(4), 1829–1843 (2022)
18. Vladicic, V., Pikula, M.: An inverse problems for Sturm–Liouville-type differential equation with a constant delay. Sarajevo J. Math **12**(1), 83–88 (2016)
19. Vojvodić, B., Pavlović Komazec, N.: Inverse problems for Sturm–Liouville operator with potential functions from $L_2[0, \pi]$. Math. Montisnigri **49**, 28–38 (2020)
20. Vojvodić, B., Pavlović Komazec, N., Çetinkaya, F.A.: Recovering differential operators with two retarded arguments. Boletín de la Sociedad Matemática Mexicana **28**(3), 68 (2022)
21. Vojvodić, B., Pikula, M., Vladičić, V.: Inverse problems for Sturm–Liouville differential operators with two constant delays under Robin boundary conditions. Res. Appl. Math. **5**, 100082 (2020)
22. Vojvodić, B., Pikula, M., Vladičić, V., Çetinkaya, F.A.: Inverse problems for differential operators with two delays larger than half the length of the interval and Dirichlet conditions. Turk. J. Math. **44.3**(3), 900–905 (2020)
23. Vojvodic, B., Vladicic, V.: Recovering differential operators with two constant delays under Dirichlet/Neumann boundary conditions. J. Inverse Ill-Posed Probl. **28**(2), 237–241 (2020)
24. Vojvodić, B., Vladičić, V., Djurić, N.: Inverse problem for Dirac operators with two constant delays. J. Inverse Ill-Posed Probl. (forthcoming). https://doi.org/10.1515/jiip-2023-0047
25. Wang, F., Yang, C.F.: Inverse problems for Dirac operators with a constant delay less than half of the interval, preprint (2023). arXiv:2305.12631
26. Wang, F., Yang, C.F.: Incomplete inverse problem for Dirac operator with constant delay, preprint (2023). arXiv:2305.10752

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.