Complex Analysis and Operator Theory (2024) 18:62
https://doi.org/10.1007/s11785-024-01511-7

**Complex Analysis and Operator Theory**

# Several Properties of a Class of Generalized Harmonic Mappings

**Bo-Yong Long[1] · Qi-Han Wang[1]**

Received: 17 November 2023 / Accepted: 16 February 2024 / Published online: 23 March 2024


**Abstract**

We call the solution of a kind of second order homogeneous partial differential equation as real kernel $\alpha$-harmonic mappings. In this paper, the representation theorem, the Lipschitz continuity, the univalency and the related problems of the real kernel $\alpha$-harmonic mappings are explored.

**Keywords** Weighted Laplacian operator · Univalency · Polyharmonic mappings · Lipschitz continuity · Gauss hypergeometric function

**Mathematics Subject Classification** Primary 30C45; Secondary 33C05 · 30B10

## 1 Introduction

Let $\mathbb{D}$ be the open unit disk and $\mathbb{T}$ the unit circle. For $\alpha \in \mathbb{R}$ and $z \in \mathbb{D}$, let

$$T_\alpha = -\frac{\alpha^2}{4}(1-|z|^2)^{-\alpha-1} + \frac{\alpha}{2}(1-|z|^2)^{-\alpha-1}\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) + (1-|z|^2)^{-\alpha}\triangle$$

be the second order elliptic partial differential operator, where $\triangle$ is the usual complex Laplacian operator

Communicated by Ronen Peretz.

Supported by NSFC (No. 12271001) and Natural Science Foundation of Anhui Province (2308085MA03), China.

✉ Bo-Yong Long
boyonglong@163.com

Qi-Han Wang
qihan@ahu.edu.cn

1 School of Mathematical Sciences, Anhui University, Hefei 230601, China

$$\triangle := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

The corresponding partial differential equation is

$$T_\alpha(u) = 0 \quad \text{in } \mathbb{D}. \tag{1.1}$$

The associated Dirichlet boundary value problem is

$$\begin{cases} T_\alpha(u) = 0 & \text{in } \mathbb{D}, \\ u = u^* & \text{on } \mathbb{T}. \end{cases} \tag{1.2}$$

Here, the boundary data $u^* \in \mathfrak{D}'(\mathbb{T})$ is a distribution on the boundary of $\mathbb{D}$, and the boundary condition in (1.2) is interpreted in the distributional sense that $u_r \to u^*$ in $\mathfrak{D}'(\mathbb{T})$ as $r \to 1^-$, where

$$u_r(e^{i\theta}) = u(re^{i\theta}), \quad e^{i\theta} \in \mathbb{T},$$

for $r \in [0, 1)$. In [24], Olofsson proved that, for the parameter $\alpha > -1$, if a function $u \in \mathcal{C}^2(\mathbb{D})$ satisfies (1.1) with $\lim_{r\to 1^-} u_r = u^* \in \mathfrak{D}'(\mathbb{T})$, then it has the form of Poisson type integral

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} K_\alpha(ze^{-i\tau})u^*(e^{i\tau})d\tau, \quad \text{for } z \in \mathbb{D}, \tag{1.3}$$

where

$$K_\alpha(z) = c_\alpha \frac{(1-|z|^2)^{\alpha+1}}{|1-z|^{\alpha+2}}, \tag{1.4}$$

$c_\alpha = \Gamma^2(\alpha/2+1)/\Gamma(1+\alpha)$ and $\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$ for $s > 0$ is the standard Gamma function. If $\alpha \le -1$, $u \in \mathcal{C}^2(\mathbb{D})$ satisfies (1.1), and the boundary limit $u^* = \lim_{r\to 1^-} u_r$ exists in $\mathfrak{D}'(\mathbb{T})$, then $u(z) = 0$ for all $z \in \mathbb{D}$. So, in the following of this paper, we always assume that $\alpha > -1$.

For $c \neq 0, -1, -2, \ldots$, the Gauss hypergeometric function is defined by the series

$$F(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

for $|x| < 1$, and has a continuation to the complex plane with branch points at 1 and $\infty$, where $(a)_0 = 1$ and $(a)_n = a(a+1)\ldots(a+n-1)$ for $n = 1, 2, \ldots$ are the Pochhammer symbols. Obviously, for $n = 0, 1, 2, \ldots$, $(a)_n = \Gamma(a+n)/\Gamma(a)$. It is easily to verified that

$$\frac{d}{dx}F(a, b; c; x) = \frac{ab}{c}F(a+1, b+1; c+1; x). \tag{1.5}$$

Furthermore, it holds that (cf. [3])

$$\lim_{x\to 1} F(a, b; c; x) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.6}$$

if $Re(c-a-b) > 0$.

The following Lemma 1.1 involves the determination of monotonicity of Gauss hypergeometric functions.

**Lemma 1.1** [24] *Let $c > 0$, $a \le c$, $b \le c$ and $ab \le 0$ ($ab \ge 0$). Then the function $F(a, b; c; x)$ is decreasing (increasing) on $x \in (0, 1)$.*

The following result of [24] is the homogeneous expansion of solutions of (1.1).

**Theorem 1.2** [24] *Let $\alpha \in \mathbb{R}$ and $u \in \mathcal{C}^2(\mathbb{D})$. Then $u$ satisfies (1.1) if and only if it has a series expansion of the form*

$$\begin{aligned} u(z) = &\sum_{k=0}^{\infty} c_k F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2\right) z^k \\ &+\sum_{k=1}^{\infty} c_{-k} F\left(-\frac{\alpha}{2}, k-\frac{\alpha}{2}; k+1; |z|^2\right) \bar{z}^k, \quad z \in \mathbb{D}, \end{aligned} \tag{1.7}$$

*for some sequence $\{c_k\}_{-\infty}^{\infty}$ of complex number satisfying*

$$\lim_{|k|\to\infty} \sup |c_k|^{\frac{1}{|k|}} \le 1. \tag{1.8}$$

*In particular, the expansion (1.7), subject to (1.8), converges in $\mathcal{C}^{\infty}(\mathbb{D})$, and every solution $u$ of (1.1) is $\mathcal{C}^{\infty}$-smooth in $\mathbb{D}$.*

Let

$$v(z) = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} \bar{z}^k, \quad z \in \mathbb{D}. \tag{1.9}$$

It is obvious that $v(z)$ is a harmonic mapping, i.e., $\Delta v = 0$. We observe that $u(z)$ of (1.7) and $v(z)$ have same coefficient sequence $\{c_k\}_{-\infty}^{\infty}$. Actually, if $\alpha = 0$, then $u(z) = v(z)$.

Observe that the kernel $K_\alpha$ in (1.4) is real. We call $u$ of (1.3) or (1.7) as **real kernel $\alpha$-harmonic mappings**. Furthermore, suppose $u(z)$ and $v(z)$ have the expansions of (1.7) and (1.9), respectively. We call $v(z)$ as **the corresponding harmonic mapping** of $u(z)$. Conversely, we call $u(z)$ as **the corresponding real kernel $\alpha$-harmonic mapping** of $v(z)$.

If we take $\alpha = 2(p-1)$, then a real kernel $\alpha$-harmonic mapping $u$ is polyharmonic (or $p$-harmonic), where $p \in \{1, 2, \ldots\}$ (cf. [1, 2, 5, 6, 11, 13, 15, 27]). In particular, if $\alpha = 0$, then $u$ is harmonic (cf. [10, 18–20]). Thus, the real kernel $\alpha$-harmonic mapping is a kind of generalization of classical harmonic mapping. Furthermore, by Olofsson [25], we know that it is related to standard weighted harmonic mappings. For the related discussion on standard weighted harmonic mappings, see [8, 16, 17, 23].

For the real kernel $\alpha$-harmonic mappings, the Schwarz–Pick type estimates and coefficient estimates are obtained in [7]; the starlikeness, convexity and Landau type theorem are studied in [22]; the sharp Heinz type inequality is established and the extremal functions of Schwartz type lemma are explored in [21]; the Lipschitz continuity with respect to the distance ratio metric is proved in [14]. In [12], using the properties of the real kernel $\alpha$-harmonic mappings, the authors established some Schwarz type lemmas for mappings satisfying a class of inhomogeneous biharmonic Dirichlet problem.

In this paper, we continue to study the properties of the real kernel $\alpha$-harmonic mappings. The main idea of this paper is that by establishing the relationship between harmonic mapping and the corresponding real kernel $\alpha$-harmonic mapping, we use the harmonic mapping to characterize the corresponding real kernel $\alpha$-harmonic mapping. In Sect. 2, for a nonnegative even number $\alpha$, we get an explicit representation theorem which determines the relation between the real kernel $\alpha$-harmonic mapping and the corresponding harmonic mapping. As its application, in Sect. 3, we show that the Lipschitz continuity of a real kernel $\alpha$-harmonic mapping is determined by the corresponding harmonic mapping. In Sect. 4, for a subclass of the real kernel $\alpha$-harmonic mappings, we discuss its univalency and explore its Radó–Kneser–Choquet type theorem. In Sect. 5, we explore the influence of parameters $\alpha$ on the image area of the real kernel $\alpha$-harmonic mappings.

## 2 Representation Theorem

**Theorem 2.1** *Let* $v(z) = h(z) + \overline{g(z)} = \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} \bar{z}^k$ *be a harmonic mapping defined on the unit disk* $\mathbb{D}$. *If* $\frac{\alpha}{2} = p - 1$ *is a nonnegative integer, then the corresponding real kernel* $\alpha$*-harmonic mapping of* $v(z)$ *can be represented by*

$$u(z) = \sum_{n=0}^{p-1} |z|^{2n} \frac{(1-p)_n}{n!} \left(I_n + \overline{J_n}\right), \tag{2.1}$$

*where* $I_n$ *and* $J_n$ *satisfy the recurrence formulas*

$$I_n = I_{n-1} - p \frac{\int_0^z z^{n-1} I_{n-1} dz}{z^n}, \tag{2.2}$$

$$J_n = J_{n-1} - p \frac{\int_0^z z^{n-1} J_{n-1} dz}{z^n} \quad n = 1, 2, \ldots, p-1, \tag{2.3}$$

$I_0 = h(z)$, *and* $J_0 = g(z)$.

***Proof*** Let $H(z) = \sum_{k=0}^{\infty} c_k F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2) z^k$ and $G(z) = \sum_{k=1}^{\infty} \overline{c_{-k}}$ $F(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; |z|^2) z^k$. Then by the assumption and (1.7), we have

$$u(z) = H(z) + \overline{G(z)}. \tag{2.4}$$

When $\frac{\alpha}{2} = p - 1$, rewrite $H(z)$ as

$$\begin{aligned} H(z) &= \sum_{k=0}^{\infty} c_k F(1-p, k+1-p; k+1; |z|^2) z^k \\ &= \sum_{k=0}^{\infty} c_k z^k \left( \sum_{n=0}^{\infty} \frac{(1-p)_n (k+1-p)_n}{(k+1)_n} \frac{|z|^{2n}}{n!} \right) \\ &= \sum_{k=0}^{\infty} c_k z^k \left( \sum_{n=0}^{p-1} \frac{(1-p)_n (k+1-p)_n}{(k+1)_n} \frac{|z|^{2n}}{n!} \right) \\ &= \sum_{n=0}^{p-1} |z|^{2n} \frac{(1-p)_n}{n!} I_n, \end{aligned} \tag{2.5}$$

where

$$I_n = \sum_{k=0}^{\infty} \frac{(k+1-p)_n}{(k+1)_n} c_k z^k.$$

Because

$$\begin{aligned} \frac{(k+1-p)_n}{(k+1)_n} &= \frac{(k+1-p)_{n-1}(k+n-p)}{(k+1)_{n-1}(k+n)} \\ &= \frac{(k+1-p)_{n-1}}{(k+1)_{n-1}} - \frac{p}{k+n} \frac{(k+1-p)_{n-1}}{(k+1)_{n-1}}, \end{aligned}$$

we can get

$$\begin{aligned} I_n &= \sum_{k=0}^{\infty} \frac{(k+1-p)_{n-1}}{(k+1)_{n-1}} c_k z^k - \sum_{k=0}^{\infty} \frac{p}{k+n} \frac{(k+1-p)_{n-1}}{(k+1)_{n-1}} c_k z^k \\ &= I_{n-1} - p \frac{\int_0^z z^{n-1} I_{n-1} dz}{z^n}. \end{aligned}$$

This is (2.2).

Similarly, we can get

$$G(z) = \sum_{n=0}^{p-1} |z|^{2n} \frac{(1-p)_n}{n!} J_n, \tag{2.6}$$

where $J_n$ is defined as in (2.3). Therefore, Eq. (2.1) follows from Eqs. (2.4)–(2.6). □

***Example 2.1*** From the recurrence formula (2.1), we have the following:

(i) When $\alpha = 0$, i.e. $p = 1$,

$$u(z) = v(z);$$

(ii) When $\alpha = 2$, i.e. $p = 2$,

$$\begin{aligned} u(z) &= h + \bar{g} - |z|^2 \left( h - 2\frac{\int_0^z h(z)dz}{z} + \overline{g - 2\frac{\int_0^z g(z)dz}{z}} \right) \\ &= \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} \bar{z}^k - |z|^2 \left( \sum_{k=0}^{\infty} c_k \frac{k-1}{k+1} z^k + \sum_{k=1}^{\infty} c_{-k} \frac{k-1}{k+1} \bar{z}^k \right); \end{aligned} \tag{2.7}$$

(iii) When $\alpha = 4$, i.e. $p = 3$,

$$\begin{aligned} u(z) &= h + \bar{g} - 2|z|^2 \left( h - 3\frac{\int_0^z h(z)dz}{z} + \overline{g - 3\frac{\int_0^z g(z)dz}{z}} \right) \\ &\quad + |z|^4 \left( h - 3\frac{\int_0^z h(z)dz}{z} - 3\frac{\int_0^z zh(z)dz}{z^2} + 9\frac{\int_0^z \int_0^z h(z)dzdz}{z^2} \right. \\ &\quad \left. + \overline{g - 3\frac{\int_0^z g(z)dz}{z} - 3\frac{\int_0^z zg(z)dz}{z^2} + 9\frac{\int_0^z \int_0^z g(z)dzdz}{z^2}} \right) \\ &= \sum_{k=0}^{\infty} c_k z^k + \sum_{k=1}^{\infty} c_{-k} \bar{z}^k - 2|z|^2 \left( \sum_{k=0}^{\infty} c_k \frac{k-2}{k+1} z^k + \sum_{k=1}^{\infty} c_{-k} \frac{k-2}{k+1} \bar{z}^k \right) \\ &\quad + |z|^4 \left( \sum_{k=0}^{\infty} c_k \frac{(k-1)(k-2)}{(k+1)(k+2)} z^k + \sum_{k=1}^{\infty} c_{-k} \frac{(k-1)(k-2)}{(k+1)(k+2)} \bar{z}^k \right). \end{aligned}$$

## 3 Lipschitz Continuity

**Theorem 3.1** *Let $u(z)$ be the corresponding real kernel $\alpha$-harmonic mapping of $v(z) = h + \bar{g}$ on the unit disk $\mathbb{D}$. If $v(z)$ is Lipschitz continuous on the unit disk $\mathbb{D}$ and $\frac{\alpha}{2} = p - 1$ is a nonnegative integer, then $u$ is Lipschitz continuous on the unit disk $\mathbb{D}$ as well.*

***Proof*** By the assumption and (2.1), it is sufficient to prove that $I_n$ and $J_n$ are Lipschitz continuous on the unit disk $\mathbb{D}$ for $n = 0, 1, 2, \ldots, p-1$. In the following, we just prove the Lipschitz continuity of $I_n$. The case of $J_n$ is similar.

Observe that $I_0 = h(z)$ is holomorphic on $\mathbb{D}$. Then by the recurrence formula (2.2), it is easy to see that all $I_n$ are holomorphic on $\mathbb{D}$. It follows that all $I_n'$ are holomorphic on $\mathbb{D}$ too, where

$$I_n' = I_{n-1}' - p\frac{z^n I_{n-1} - n\int_0^z z^{n-1} I_{n-1} dz}{z^{n+1}}, \quad n = 1, 2, \ldots, p-1. \tag{3.1}$$

Taking account of the maximum modulus principle of holomorphic functions, from Eqs. (2.2) and (3.1), we get

$$\sup_{z\in\mathbb{D}} |I_n| \le \sup_{z\in\mathbb{D}} |I_{n-1}| + p \sup_{z\in\mathbb{D}} |I_{n-1}| = (p+1) \sup_{z\in\mathbb{D}} |I_{n-1}|$$

and

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} |I_n'| \le \sup_{z\in\mathbb{D}} |I_{n-1}'| + p \sup_{z\in\mathbb{D}} |I_{n-1}| + np \sup_{z\in\mathbb{D}} |I_{n-1}| \\
&= \sup_{z\in\mathbb{D}} |I_{n-1}'| + (n+1) p \sup_{z\in\mathbb{D}} |I_{n-1}|,
\end{aligned}$$

respectively. It follows that

$$\sup_{z\in\mathbb{D}} |I_n| \le (p+1)^n \sup_{z\in\mathbb{D}} |I_0|$$

and

$$\begin{aligned}
\sup_{z\in\mathbb{D}} |I_n'| &\le \sup_{z\in\mathbb{D}} |I_{n-1}'| + (n+1) p \sup_{z\in\mathbb{D}} |I_{n-1}| \\
&\le \sup_{z\in\mathbb{D}} |I_{n-2}'| + np \sup_{z\in\mathbb{D}} |I_{n-2}| + (n+1) p \sup_{z\in\mathbb{D}} |I_{n-1}| \\
&\le \cdots \\
&\le \sup_{z\in\mathbb{D}} |I_0'| + p \sum_{i=1}^{n} (i+1) \sup_{z\in\mathbb{D}} |I_{i-1}| \\
&\le \sup_{z\in\mathbb{D}} |I_0'| + p \sum_{i=1}^{n} (i+1)(p+1)^{i-1} \sup_{z\in\mathbb{D}} |I_0|.
\end{aligned} \tag{3.2}$$

Because $v = h + \bar{g}$ is Lipschitz, there exists a constant $M$ such that

$$|h'| = |I_0'| \le M \tag{3.3}$$

for $z \in \mathbb{D}$. It follows that

$$\sup_{z\in\mathbb{D}} |I_0| = \sup_{z\in\mathbb{D}} |h| \le M. \tag{3.4}$$

Therefore, by inequalities (3.2)–(3.4), we get that there exists a constant $C = C(M, p, n)$, such that

$$\sup_{z\in\mathbb{D}} |I_n'| \le \left(1 + p\sum_{i=1}^{n}(i+1)(p+1)^{i-1}\right) M =: C$$

for $n = 1, 2, \dots, p-1$. It means that $I_n$ is Lipschitz continuous on $\mathbb{D}$. □

## 4 Univalency of a Subclass of Real Kernel $\alpha$-Harmonic Mappings

In the rest of this paper, we use the following notations. Let $\alpha > -1$, $z = re^{i\theta}$, and

$$\begin{aligned} t &= |z|^2 = r^2, \\ F &= F_k = F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; t\right), \quad k = 1, 2, \dots, \\ F_t &= F_{k,t} = F_{k,t}\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; t\right) = \frac{dF_k}{dt} = \frac{dF}{dt}. \end{aligned}$$

Furthermore, let

$$F_k(1) = \lim_{t\to 1^-} F\left(-\frac{\alpha}{2}, k - \frac{\alpha}{2}; k+1; t\right).$$

Then by (1.6), we have

$$F_k(1) = \frac{\Gamma(k+1)\Gamma(1+\alpha)}{\Gamma\left(k+1+\frac{\alpha}{2}\right)\Gamma\left(1+\frac{\alpha}{2}\right)}. \tag{4.1}$$

**Lemma 4.1** *Let $r_n$ and $s_n$ $(n = 0, 1, 2, \dots)$ be real numbers, and let the power series*

$$R(x) = \sum_{n=0}^{\infty} r_n x^n \quad \textit{and} \quad S(x) = \sum_{n=0}^{\infty} s_n x^n$$

*be convergent for $|x| < r$, $(r > 0)$ with $s_n > 0$ for all $n$. If the non-constant sequence $\{r_n/s_n\}$ is increasing (decreasing) for all $n$, then the function $x \mapsto R(x)/S(x)$ is strictly increasing (resp. decreasing) on $(0, r)$.*

Lemma 4.1 is basically due to [4] (see also [28]) and in this form with a general setting was stated in [26] along with many applications which were later adopted by a number of researchers.

**Lemma 4.2** [22] *Let $\frac{\alpha}{2} \in (0, 1]$. Then it holds that*

(1) $\dfrac{F_k}{F_1} \le 1$ *for* $k = 1, 2, 3, \dots$ *and* $t \in [0, 1)$;

(2) $\dfrac{|F_{k,t}|}{F_1} < \dfrac{\left(k-\frac{\alpha}{2}\right)\Gamma(k+1)\Gamma\left(2+\frac{\alpha}{2}\right)}{2\Gamma\left(k+1+\frac{\alpha}{2}\right)}$ *for* $k = 1, 2, 3, \ldots$ *and* $t \in (0, 1)$.

**Theorem 4.3** *If* $\alpha \in (0, 2]$, $c_{-k} \in (-N, N)$, *where*

$$N = \frac{\alpha}{2\left(\frac{\left(k-\frac{\alpha}{2}\right)\Gamma(k+1)\Gamma\left(2+\frac{\alpha}{2}\right)}{\Gamma\left(k+1+\frac{\alpha}{2}\right)} + k\right)}, \tag{4.2}$$

*then the real kernel* $\alpha$*-harmonic mapping*

$$u(z) = F_1 z + c_{-k} F_k \bar{z}^k, \quad k = 1, 2, 3 \ldots, \tag{4.3}$$

*is sense-preserving univalent in* $\mathbb{D}$.

***Proof*** We divide the proof into two steps.

**First step**: Formula (4.3) implies that

$$u_z = F_1 + F_{1,t}t + c_{-k}F_{k,t}\bar{z}^{k+1}, \quad u_{\bar{z}} = F_{1,t}z^2 + c_{-k}\left(F_{k,t}z\bar{z}^k + kF_k\bar{z}^{k-1}\right).$$

It follows that

$$\begin{aligned}
|u_z| - |u_{\bar{z}}| &\geq F_1 - |F_{1,t}t| - |c_{-k}F_{k,t}\bar{z}^{k+1}| \\
&\quad - \left|F_{1,t}z^2\right| - \left|c_{-k}F_{k,t}z\bar{z}^k\right| - k\left|c_{-k}F_k\bar{z}^{k-1}\right| \\
&\quad > F_1 - |F_{1,t}| - |c_{-k}||F_{k,t}| - |F_{1,t}| - |c_{-k}||F_{k,t}| - k|c_{-k}||F_k| \\
&= F_1\left[1 - \frac{2|F_{1,t}|}{F_1} - |c_{-k}|\left(\frac{2|F_{k,t}|}{F_1} + k\frac{|F_k|}{F_1}\right)\right] \\
&\quad > F_1\left[1 - \left(1 - \frac{\alpha}{2}\right) - |c_{-k}|\left(\frac{\left(k-\frac{\alpha}{2}\right)\Gamma(k+1)\Gamma\left(2+\frac{\alpha}{2}\right)}{\Gamma\left(k+1+\frac{\alpha}{2}\right)} + k\right)\right] > 0
\end{aligned}$$

for $c_{-k} \in (-N, N)$. The third inequality of the above holds because of Lemma 4.2. Therefore, $u(z)$ is sense-preserving.

**Second step**: Let $c_{-k} = |c_{-k}|e^{i\beta}$. By assumption, we have $\beta = 0$ or $\pi$. Let $z = re^{i\theta}$ and $u(z) = Re^{i\varphi}$. Rewrite $u(z)$ of (4.3) as

$$\begin{aligned}
u(z) &= F_1 re^{i\theta} + |c_{-k}|F_k r^k e^{i(\beta - k\theta)} \\
&= F_1 r\cos\theta + |c_{-k}|F_k r^k\cos(\beta - k\theta) + i(F_1 r\sin\theta + |c_{-k}|F_k r^k\sin(\beta - k\theta)).
\end{aligned} \tag{4.4}$$

Then

$$\tan\varphi = \frac{F_1 r\sin\theta + |c_{-k}|F_k r^k\sin(\beta - k\theta)}{F_1 r\cos\theta + |c_{-k}|F_k r^k\cos(\beta - k\theta)}, \tag{4.5}$$

where $\varphi$ is the argument of $u(z)$. It follows that

$$\begin{aligned}\frac{d}{d\theta}(\tan\varphi) &= \frac{d}{d\theta}\left(\frac{F_1 r\sin\theta + |c_{-k}|F_k r^k\sin(\beta-k\theta)}{F_1 r\cos\theta + |c_{-k}|F_k r^k\cos(\beta-k\theta)}\right)\\ &= \frac{F_1^2 - |c_{-k}|^2F_k^2 r^{2(k-1)}k - (k-1)|c_{-k}|F_1F_k r^{k-1}\cos(\beta-(k+1)\theta)}{\left[F_1\cos\theta + |c_{-k}|F_k r^{k-1}\cos(\beta-k\theta)\right]^2}\\ &\geq \frac{F_1^2 - |c_{-k}|^2F_k^2 r^{2(k-1)}k - (k-1)|c_{-k}|F_1F_k r^{k-1}}{\left[F_1\cos\theta + |c_{-k}|F_k r^{k-1}\cos(\beta-k\theta)\right]^2}\\ &= \frac{\left(F_1 + |c_{-k}|F_k r^{k-1}\right)\left(F_1 - |c_{-k}|kF_k r^{k-1}\right)}{\left[F_1\cos\theta + |c_{-k}|F_k r^{k-1}\cos(\beta-k\theta)\right]^2} > 0 \qquad (4.6)\end{aligned}$$

for $|c_{-k}| < \frac{1}{k}$. The last inequality of the above holds because of Lemma 4.2(1). That is to say, $\tan\varphi$ is strictly increasing with respect to $\theta$. So is $\varphi$, too.

In the following we divide into two cases to discuss.

**Case 1** $\beta = 0$. It follows from (4.5) that

$$\cot\varphi = \frac{\cos\theta + |c_{-k}|\frac{F_k}{F_1}r^{k-1}\cos k\theta}{\sin\theta - |c_{-k}|\frac{F_k}{F_1}r^{k-1}\sin k\theta}. \qquad (4.7)$$

Let's take a close look at the changes in the value of the function $\cot\varphi$. Firstly, as is well-known, it is easy to verify by mathematical induction that

$$\left|\frac{\sin k\theta}{\sin\theta}\right| \leq k \qquad (4.8)$$

for $k = 1, 2, \dots$ and $\theta \in [0, 2\pi)$. If $|c_{-k}| < \frac{1}{k}$, $\alpha \in (0, 2]$ and $\sin\theta \neq 0$, then Lemma 4.2(1) and inequality (4.8) imply that $|\sin\theta| > |c_{-k}|\frac{F_k}{F_1}r^{k-1}|\sin k\theta|$. So, $\sin\theta \neq 0$ implies $\sin\theta - |c_{-k}|\frac{F_k}{F_1}r^{k-1}\sin k\theta \neq 0$. In another words, the zero of the denominator of the right side of equation (4.7) comes only from the zero of $\sin\theta$. Secondly, $\sin\theta$ only have two zeros in the intervals $[0, 2\pi)$. That is $\theta = 0$ and $\pi$. By (4.7), we have that if $\theta = 0^+$, then $\cot\varphi = +\infty$; if $\theta = \pi^-$, then $\cot\varphi = -\infty$; if $\theta = \pi^+$, then $\cot\varphi = +\infty$; if $\theta = 2\pi^-$, then $\cot\varphi = -\infty$. Therefore, considering the continuity and monotonicity of $\cot\varphi$, we can get that the $u(re^{i\theta})$ maps every circle $|z| = r < 1$ in a one-to-one manner onto a closed Jordan curve.

**Case 2** $\beta = \pi$. Considering (4.5), we have

$$\cot\varphi = \frac{\cos\theta - |c_{-k}|\frac{F_k}{F_1}r^{k-1}\cos k\theta}{\sin\theta + |c_{-k}|\frac{F_k}{F_1}r^{k-1}\sin k\theta}.$$

Follow the discussion of Case 1. We omit the further details.

It is easy to see that $N < \frac{1}{k}$, where $N$ defined by (4.2). Therefore, considering the above two steps of the proof, by degree principle [9], we can get that $u(z)$ is univalent in $\mathbb{D}$. □

The following is the well known Radó–Kneser–Choquet theorem, which can be seen in the page 29 of [10].

**Theorem 4.4** *If $\Omega \in \mathbb{C}$ is a bounded convex domain whose boundary is a Jordan curve $\gamma$ and $f$ is a homeomorphism of the unit circle $\mathbb{T}$ onto $\gamma$, then its harmonic extension*

$$u(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{it}-z|^2} f(e^{it})dt$$

*is univalent in $\mathbb{D}$ and defines a harmonic mapping of $\mathbb{D}$ onto $\Omega$.*

Next, we want to explore the Radó–Kneser–Choquet type theorem for real kernel $\alpha$-harmonic mappings. We need the following Proposition at first.

**Proposition 4.5** *Suppose $\alpha > -1$ and $c_{-k} \in \mathbb{R}$. Let*

$$f(e^{i\theta}) = F_1(1)e^{i\theta} + c_{-k}F_k(1)e^{-ik\theta}, \quad k = 1, 2, 3, \ldots. \tag{4.9}$$

*Then $f$ maps the unit circle $\mathbb{T}$ onto a convex Jordan curve if and only if $c_{-k} \in (-M, M)$, where*

$$M = \frac{\Gamma\left(k+1+\frac{\alpha}{2}\right)}{k^2\Gamma(k+1)\Gamma\left(2+\frac{\alpha}{2}\right)}. \tag{4.10}$$

***Proof*** Direct computation leads to

$$\frac{d}{d\theta}\left(f(e^{i\theta})\right) = -F_1(1)\sin\theta - kc_{-k}F_k(1)\sin k\theta + i(F_1(1)\cos\theta - kc_{-k}F_k(1)\cos k\theta).$$

Let $\psi = \psi(\theta) = \arg\{\frac{d}{d\theta}f(e^{i\theta})\}$. Then we have

$$\begin{aligned}\frac{d}{d\theta}(\tan\psi(\theta)) &= \frac{(F_1(1))^2 - k^3(c_{-k}F_k(1))^2 + k(k-1)c_{-k}F_1(1)F_k(1)\ \cos((k+1)\theta)}{(F_1(1)\sin\theta + kc_{-k}F_k(1)\sin k\theta)^2}\\ &\geq \frac{(F_1(1))^2 - k^3(c_{-k}F_k(1))^2 - k(k-1)|c_{-k}|F_1(1)F_k(1))}{(F_1(1)\sin\theta + kc_{-k}F_k(1)\sin k\theta)^2}\\ &= \frac{(F_1(1)+k|c_{-k}|F_k(1))(F_1(1)-k^2|c_{-k}|F_k(1))}{(F_1(1)\sin\theta + kc_{-k}F_k(1)\sin k\theta)^2}\end{aligned}$$

Hence, $\frac{d}{d\theta}(\tan\psi(\theta)) \geq 0$ if and only if $|c_{-k}| \leq \frac{F_1(1)}{k^2F_k(1)} = \frac{\Gamma(k+1+\frac{\alpha}{2})}{k^2\Gamma(k+1)\Gamma(2+\frac{\alpha}{2})}$. □

Now let $f(e^{i\theta})$ be defined as in (4.9) with $\alpha \in (0, 2]$, $c_{-k} \in (-L, L)$, where $L = \min\{M, N\}$. Observe that $\lim_{r\to 1} u(z) := u^*(e^{i\theta}) = f(e^{i\theta})$, where $u(z)$ are defined by (4.3). Similar to the second step of the proof of Theorem 4.3, we can verify that $f(e^{i\theta})$ maps unit circle $\mathbb{T}$ onto a closed Jordan curve in a one-to-one manner, too. Therefore, considering Theorem 3.3 of [24] and Theorem 4.3 of the above, we actually get a Radó–Kneser–Choquet type theorem as follows:

**Proposition 4.6** *Let $u^*(e^{i\theta}) = f(e^{i\theta})$ be defined by (4.9) with $k = 1, 2, 3, \ldots$, $\alpha \in (0, 2]$, $c_{-k} \in (-L, L)$, where $L = \min\{M, N\}$, $N$ and $M$ are defined by (4.2) and (4.10), respectively. Then $u^*(e^{i\theta})$ is a homeomorphism of the unit circle $\mathbb{T}$ onto a convex Jordan curve $\gamma$ which is a boundary of a bounded convex domain $\Omega \subset \mathbb{C}$. Furthermore, $u(z)$ defined by (1.3) defines a univalent real kernel $\alpha$-harmonic mapping of $\mathbb{D}$ onto $\Omega$.*

Let us have a look at some special cases of Theorem 4.3 or Proposition 4.6.

***Example 4.1*** Let $\alpha = 2$. Then $M = \frac{k+1}{2k^2}$ and $N = \frac{k+1}{k^2+3k-2}$. Formula (4.9) deduces to

$$f(e^{i\theta}) = e^{i\theta} + \frac{2}{k+1} c_{-k} e^{-ik\theta}.$$

Furthermore, let $u^*(e^{i\theta}) = f(e^{i\theta})$. Then (1.3), or (2.7), implies that the corresponding real kernel $\alpha$-harmonic mapping is

$$u(z) = F_1 z + c_{-k} F_k \bar{z}^k = z + c_{-k} \left(1 - \frac{k-1}{k+1}|z|^2\right) \bar{z}^k. \tag{4.11}$$

Actually, it is biharmonic.

(1) If $k = 1$ or $k = 2$, then $L = M = N$. If $c_{-k} \in (-L, L)$, then Proposition 4.6 says that the $u(z)$ given by (4.11) is univalent, and $u(\mathbb{D}) = \Omega$ is a convex domain.
(2) If $k = 3, 4, 5, \ldots$, then a direct computation leads to $N > M$. Taking $c_{-k} \in (M, N)$, Theorem 4.3 and Proposition 4.5 imply that the above $u(z)$ is still univalent, but $u(\mathbb{D}) = \Omega$ is not a convex domain.

## 5 Area $S_u$

Let $S_u(\alpha)$ denote the area of the Riemann surface of real kernel $\alpha$-harmonic mapping $u$. Then we have the following results.

**Theorem 5.1** *Let $u$ be a sense-preserving real kernel $\alpha$-harmonic mapping that has the series expansion of the form (1.7) with $c_0 = 0$, continuous on $\overline{\mathbb{D}}$. Let $v$ be the corresponding sense-preserving harmonic mapping that has the series expansion of the form (1.9), continuous on $\overline{\mathbb{D}}$. If $|c_k| \geq |c_{-k}|$ for $k = 1, 2, \ldots$, then*

(1) *$S_u(\alpha) < S_u(0)$ for $\alpha \in (0, 2)$ and $S_u > S_u(0)$ for $\alpha \in (-1, 0)$;*
(2) *$S_u(\alpha)$ is strictly decreasing with respect to $\alpha \in (-1, \alpha_0)$, where $\alpha_0$ is the unique solution of equation*

$$\psi(1+\alpha) - \psi\left(1 + \frac{\alpha}{2}\right) - \frac{1}{2+\alpha} = 0$$

*in the interval* $(0.8, 1)$, $\psi$ *is the digamma function. Furthermore, the constant* $\alpha_0$ *is sharp.*

***Proof*** By (1.7), direct computation leads to

$$\begin{aligned} u_z &= \sum_{k=1}^{\infty} c_k \left[ F_t \bar{z} z^k + k F z^{k-1} \right] + \sum_{k=1}^{\infty} c_{-k} F_t \bar{z}^{k+1} \\ &= \sum_{k=1}^{\infty} c_k \left[ F_t r^{k+1} + k F r^{k-1} \right] e^{i(k-1)\theta} + \sum_{k=1}^{\infty} c_{-k} F_t r^{k+1} e^{-i(k+1)\theta} \end{aligned}$$

and

$$\begin{aligned} u_{\bar{z}} &= \sum_{k=1}^{\infty} c_k F_t z^{k+1} + \sum_{k=1}^{\infty} c_{-k} \left[ F_t z \bar{z}^k + k F \bar{z}^{k-1} \right] \\ &= \sum_{k=1}^{\infty} c_{-k} \left[ F_t r^{k+1} + k F r^{k-1} \right] e^{-i(k-1)\theta} + \sum_{k=1}^{\infty} c_k F_t r^{k+1} e^{i(k+1)\theta}. \end{aligned}$$

So,

$$\begin{aligned} S_u(\alpha) &= \int_0^{2\pi} \int_0^1 J_u(z) r dr d\theta \\ &= 2\pi \int_0^1 \sum_{k=1}^{\infty} \left[ \left| c_k \left( F_t r^{k+1} + k F r^{k-1} \right) \right|^2 + \left| c_{-k} F_t r^{k+1} \right|^2 \right. \\ &\quad \left. - \left| c_{-k} \left( F_t r^{k+1} + k F r^{k-1} \right) \right|^2 - \left| c_k F_t r^{k+1} \right|^2 \right] r dr \\ &= 2\pi \int_0^1 \left[ \sum_{k=1}^{\infty} \left( |c_k|^2 - |c_{-k}|^2 \right) \left( k^2 F^2 r^{2k-1} + 2k F F_t r^{2k+1} \right) \right] dr \\ &= 2\pi \sum_{k=1}^{\infty} \left[ \left( |c_k|^2 - |c_{-k}|^2 \right) k \int_0^1 \left( k F^2 r^{2k-1} + 2 F F_t r^{2k+1} \right) dr \right] \\ &= \pi \sum_{k=1}^{\infty} \left[ \left( |c_k|^2 - |c_{-k}|^2 \right) k \int_0^1 d(F^2 r^{2k}) \right] \\ &= \pi \frac{\Gamma^2(1+\alpha)}{\Gamma^2\left(1+\frac{\alpha}{2}\right)} \sum_{k=1}^{\infty} \left[ k \left( |c_k|^2 - |c_{-k}|^2 \right) \frac{\Gamma^2(k+1)}{\Gamma^2\left(k+1+\frac{\alpha}{2}\right)} \right]. \end{aligned} \tag{5.1}$$

The last equality holds because of (1.6).

Particularly, we have

$$S_u(0) = \pi \sum_{k=1}^{\infty} k \left( |c_k|^2 - |c_{-k}|^2 \right). \tag{5.2}$$

(1) Recall that the digamma function is defined as $\psi(x) = \Gamma'(x)/\Gamma(x)$. It is well known that (cf. [3]) $\psi(x)$ is strictly increasing on $(0, +\infty)$.

Let

$$f(x) = \frac{\Gamma(1+\alpha)\Gamma(x+1)}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(x+1+\frac{\alpha}{2}\right)}.$$

Then we have

$$(\log f(x))' = \psi(x+1) - \psi\left(x+1+\frac{\alpha}{2}\right).$$

It follows that $(\log f(x))' < 0$ provided $\alpha > 0$, and $(\log f(x))' > 0$ provided $\alpha < 0$. Observe that

$$f(\alpha/2) = 1.$$

Therefore, for $k = 1, 2, \ldots$, we have $f(k) < 1$ if $\alpha \in (0, 2)$ as well as $f(k) > 1$ if $\alpha \in (-1, 0)$. Taking account of (5.1) and (5.2), we can get Theorem 5.1(1).

(2) As to digamma function $\psi(x)$, we have (cf. [3])

$$\psi(1+x) = \frac{1}{x} + \psi(x), \tag{5.3}$$

$$\psi(x) = -\gamma + \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+x} \right), \tag{5.4}$$

and

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2} \tag{5.5}$$

for any $x \in (0, +\infty)$, where $\gamma$ is the Euler–Mascheroni constant.

Let

$$h(\alpha) = \psi(1+\alpha) - \psi\left(1+\frac{\alpha}{2}\right) - \frac{1}{2+\alpha}.$$

Using (5.4), direct computation or numerical computation lead to

$$h(1) = \psi(2) - \psi(\frac{3}{2}) - \frac{1}{3}$$

$$
\begin{aligned}
&= -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) - \left[-\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+3/2}\right)\right] - \frac{1}{3} \\
&= 2\sum_{N=1}^{\infty}\left(\frac{1}{2N+1} - \frac{1}{2N+2}\right) - \frac{1}{3} \\
&= 2\left(\log 2 - \frac{1}{2}\right) - \frac{1}{3} > 0
\end{aligned}
$$

and

$$
h(0.8) = -0.0108 < 0.
$$

Furthermore, (5.5) implies that

$$
\begin{aligned}
h'(\alpha) &= \psi'(1+\alpha) - \frac{1}{2}\psi'(1+\frac{\alpha}{2}) + \frac{1}{(2+\alpha)^2} \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{(1+\alpha+n)^2} - \frac{1}{2\left(1+\frac{\alpha}{2}+n\right)^2}\right) + \frac{1}{(2+\alpha)^2} \\
&= \sum_{n=0}^{\infty}\frac{(n+1)^2 - \frac{\alpha^2}{2}}{2\left(1+\frac{\alpha}{2}+n\right)^2(1+\alpha+n)^2} + \frac{1}{(2+\alpha)^2} > 0
\end{aligned}
$$

for $\alpha \in (-1, 1]$. Thus, there exists a unique $\alpha_0 \in (0.8, 1)$, such that $h(\alpha_0) = 0$ and $h(\alpha) < 0$ for $\alpha \in (-1, \alpha_0)$. Let

$$
g(\alpha) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\frac{\alpha}{2})\Gamma(k+1+\frac{\alpha}{2})}, \quad k = 1, 2, \ldots .
$$

Then it follows that

$$
\begin{aligned}
\frac{d\log g(\alpha)}{d\alpha} &= \psi(1+\alpha) - \frac{1}{2}\psi\left(1+\frac{\alpha}{2}\right) - \frac{1}{2}\psi\left(k+1+\frac{\alpha}{2}\right) \\
&< \psi(1+\alpha) - \frac{1}{2}\psi\left(1+\frac{\alpha}{2}\right) - \frac{1}{2}\psi\left(2+\frac{\alpha}{2}\right) \\
&= h(\alpha).
\end{aligned}
$$

That is to say $g(\alpha)$ is strictly decreasing on $(-1, \alpha_0)$. Therefore, (5.1) implies that $S_u(\alpha)$ is strictly decreasing with respect to $\alpha \in (-1, \alpha_0)$.

Let

$$
u(z) = F\left(-\frac{\alpha}{2}, 1-\frac{\alpha}{2}; 2; |z|^2\right) z.
$$

Obviously, it satisfies the conditions of Theorem 5.1. Thus

$$S_u(\alpha) = \pi \left( \frac{\Gamma(1+\alpha)}{\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(2+\frac{\alpha}{2}\right)} \right)^2.$$

It follows that $\frac{d \log S_u(\alpha)}{d\alpha} = h(\alpha)$. Considering the positivity and negativity of function $h(\alpha)$, we have that $S_u(\alpha)$ is strictly decreasing with respect to $\alpha$ in $(-1, \alpha_0)$ and strictly increasing in $(\alpha_0, 1)$. Therefore, the constant $\alpha_0$ is sharp. □

**Acknowledgements** The authors heartily thank the anonymous reviewers for their careful review and for their effective suggestions.

**Data Availability Statement** This manuscript has no associated date.

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

## References

1. Abdulhadi, Z., Abu Muhanna, Y.: Landau's theorem for biharmonic mappings. J. Math. Anal. Appl. **338**(1), 705–709 (2008)
2. Amozova, K.F., Ganenkova, E.G., Ponnusamy, S.: Criteria of univalence and fully $\alpha$-accessibility for $p$-harmonic and $p$-analytic functions. Complex Var. Elliptic Equ. **62**(8), 1165–1183 (2017)
3. Andrews, G.E., Askey, R., Roy, R.: Special functions. In: Rota, G.-C. (ed.) Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
4. Biernacki, M., Krzyż, J.: On the monotonicity of certain functionals in the theory of analytic functions. Ann. Univ. M. Curie-Skłodowska **9**, 134–145 (1955)
5. Chen, J.-L., Rasila, A., Wang, X.-T.: Coefficient estimates and radii problems for certain classes of polyharmonic mappings. Complex Var. Elliptic Equ. **60**(3), 354–371 (2015)
6. Chen, S.-L., Ponnusamy, S., Wang, X.-T.: Bloch constant and Landau's theorem for planar $p$-harmonic mappings. J. Math. Anal. Appl. **373**(1), 102–110 (2011)
7. Chen, S.-L., Vuorinen, M.: Some properties of a class of elliptic partial differential operators. J. Math. Anal. Appl. **431**(2), 1124–1137 (2015)
8. Chen, X.-D., Kalaj, D.: A representation theorem for standard weighted harmonic mappings with an integer exponent and its applications. J. Math. Anal. Appl. **444**(2), 1233–1241 (2016)
9. Cristea, M.: A generalization of the argument principle. Complex Var. Theory Appl. **42**(4), 333–345 (2000)
10. Duren, P.: Harmonic mappings in the plane. In: Bollobas, B., Fulton, W., katok, A., Kirwan, F., Sarnak, P. (eds.) Cambridge Tracts in Mathematics, vol. 156. Cambridge University Press, Cambridge (2004)
11. El Hajj, L.: On the univalence of polyharmonic mappings. J. Math. Anal. Appl. **452**(2), 871–882 (2017)
12. Khalfallah, A., Haggui, F., Mhamdi, M.: Generalized harmonic functions and Schwarz lemma for biharmonic mappings. Mon. Math. **196**(4), 823–849 (2021)
13. Li, P.-J., Khuri, S.A., Wang, X.-T.: On certain geometric properties of polyharmonic mappings. J. Math. Anal. Appl. **434**(2), 1462–1473 (2016)
14. Li, P.-J., Ponnusamy, S.: Lipschitz continuity of quasiconformal mappings and of the solutions to second order elliptic PDE with respect to the distance ratio metric. Complex Anal. Oper. Theory **12**(8), 1991–2001 (2018)
15. Li, P.-J., Ponnusamy, S., Wang, X.-T.: Some properties of planar $p$-harmonic and log-$p$-harmonic mappings. Bull. Malays. Math. Sci. Soc. **36**(3), 595–609 (2013)
16. Li, P.-J., Wang, X.-T.: Lipschitz continuity of $\alpha$-harmonic functions. Hokkaido Math. J. **48**(1), 85–97 (2019)

17. Li, P.-J., Wang, X.-T., Xiao, Q.-H.: Several properties of $\alpha$-harmonic functions in the unit disk. Mon. Math. **184**(4), 627–640 (2017)
18. Liu, M.-S., Chen, H.-H.: The Landau–Bloch type theorems for planar harmonic mappings with bounded dilation. J. Math. Anal. Appl. **468**(2), 1066–1081 (2018)
19. Liu, Z.-H., Jiang, Y.-P., Sun, Y.: Convolutions of harmonic half-plane mappings with harmonic vertical strip mappings. Filomat **31**(7), 1843–1856 (2017)
20. Long, B.-Y., Sugawa, T., Wang, Q.-H.: Completely monotone sequences and harmonic mappings. Ann. Fenn. Math. **47**(1), 237–250 (2022)
21. Long, B.-Y., Wang, Q.-H.: Some coefficient estimates on real kernel $\alpha$-harmonic mappings. Proc. Am. Math. Soc. **150**(4), 1529–1540 (2022)
22. Long, B.-Y., Wang, Q.-H.: Starlikeness, convexity and Landau type theorem of the real kernel $\alpha$-harmonic mappings. Filomat **35**(8), 2629–2644 (2021)
23. Long, B.-Y., Wang, Q.-H.: Some geometric properties of complex-valued kernel $\alpha$-harmonic mappings. Bull. Malays. Math. Sci. Soc. **44**(4), 2381–2399 (2021)
24. Olofsson, A.: Differential operators for a scale of Poisson type kernels in the unit disc. J. Anal. Math. **123**, 227–249 (2014)
25. Olofsson, A., Wittsten, J.: Poisson integrals for standard weighted Laplacians in the unit disc. J. Math. Soc. Jpn. **65**(2), 447–486 (2013)
26. Ponnusamy, S., Vuorinen, M.: Asymptotic expansions and inequalities for hypergeometric functions. Mathematika **44**(2), 278–301 (1997)
27. Qiao, J.-J.: Univalent harmonic and biharmonic mappings with integer coefficients in complex quadratic fields. Bull. Malays. Math. Sci. Soc. **39**(4), 1637–1646 (2016)
28. Yang, Z.-H., Chu, Y.-M., Wang, M.-K.: Monotonicity criterion for the quotient of power series with applications. J. Math. Anal. Appl. **428**(1), 587–604 (2015)