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Harmonic Bergman Spaces on the Real Hyperbolic Ball: Atomic Decomposition, Interpolation and Inclusion Relations

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Abstract
For $\alpha > -1$ and $0 < p < \infty$, we study weighted Bergman spaces $\mathcal{B}^p_\alpha$ of harmonic functions on the real hyperbolic ball. We obtain an atomic decomposition of Bergman functions in terms of reproducing kernels. We show that an r-separated sequence $\{a_m\}$ with sufficiently large r is an interpolating sequence for $\mathcal{B}^p_\alpha$. Using these we determine precisely when a Bergman space $\mathcal{B}^p_\alpha$ is included in another Bergman space $\mathcal{B}^q_\beta$.

Keywords Real hyperbolic ball · Hyperbolic harmonic function · Bergman space · Atomic decomposition · Interpolation · Inclusion relations

Mathematics Subject Classification Primary 31C05; Secondary 46E22

1 Introduction

For $x, y \in \mathbb{R}^n$, let $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$ be the Euclidean inner product and $|x| = \sqrt{\langle x, x\rangle}$ be the corresponding norm. Let $\mathbb{B} = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball and $\mathbb{S} = \partial\mathbb{B}$ be the unit sphere. The hyperbolic ball is $\mathbb{B}$ endowed with the hyperbolic metric

$$ds^2 = \frac{4}{(1-|x|^2)^2} \sum_{i=1}^{n} dx_i^2.$$

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The Laplacian Δ_h and the gradient ∇^h with respect to the hyperbolic metric are given by (see [13, Chapter 3] for more details)

$$(\Delta_h f)(a) = \Delta(f \circ \varphi_a)(0) \qquad (f \in C^2(\mathbb{B})),$$

and

$$(\nabla^h f)(a) = -\nabla(f \circ \varphi_a)(0) \qquad (f \in C^1(\mathbb{B})),$$

where $\Delta = \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2$ and $\nabla = \big(\partial/\partial x_1, \ldots, \partial/\partial x_n\big)$ are the usual Euclidean Laplacian and gradient. Here φ_a is the canonical Möbius transformation mapping $\mathbb{B}$ to $\mathbb{B}$ and exchanging a and 0 given in (9). It is easy to show that

$$\Delta_h f(a) = (1-|a|^2)^2 \Delta f(a) + 2(n-2)(1-|a|^2)\langle a, \nabla f(a)\rangle,$$

and

$$\nabla^h f(a) = (1-|a|^2)\nabla f(a). \tag{1}$$

A twice continuously differentiable function $f : \mathbb{B} \to \mathbb{C}$ is called hyperbolic harmonic or $\mathcal{H}$-harmonic on $\mathbb{B}$ if $\Delta_h f(x) = 0$ for every $x \in \mathbb{B}$. We denote the set of all $\mathcal{H}$-harmonic functions by $\mathcal{H}(\mathbb{B})$.

Let ν be the Lebesgue measure on $\mathbb{R}^n$ normalized so that $\nu(\mathbb{B}) = 1$. For $\alpha > -1$, define the weighted measure $d\nu_\alpha(x)$ by

$$d\nu_\alpha(x) = (1-|x|^2)^\alpha \, d\nu(x),$$

and for $0 < p < \infty$, denote the Lebesgue space with respect to $d\nu_\alpha$ by $L^p_\alpha = L^p(d\nu_\alpha)$. The subspace of L^p_α consisting of $\mathcal{H}$-harmonic functions is called the weighted $\mathcal{H}$-harmonic Bergman space and is denoted by $\mathcal{B}^p_\alpha$,

$$\mathcal{B}^p_\alpha = \Big\{ f \in \mathcal{H}(\mathbb{B}) : \|f\|^p_{L^p_\alpha} = \int_{\mathbb{B}} |f(x)|^p \, d\nu_\alpha(x) < \infty \Big\}.$$

These are Banach spaces when $1 \le p < \infty$, and complete metric spaces with respect to the metric $d(f,g) = \|f-g\|^p_{L^p_\alpha}$ when $0 < p < 1$.

Point evaluation functionals are bounded on all $\mathcal{B}^p_\alpha$ and, in particular, $\mathcal{B}^2_\alpha$ is a reproducing kernel Hilbert space. Therefore, for every $x \in \mathbb{B}$, there exists $\mathcal{R}_\alpha(x,\cdot) \in \mathcal{B}^2_\alpha$ such that

$$f(x) = \int_{\mathbb{B}} f(y)\overline{\mathcal{R}_\alpha(x,y)}\, d\nu_\alpha(y) \qquad (f \in \mathcal{B}^2_\alpha). \tag{2}$$

The reproducing kernel $\mathcal{R}_\alpha(\cdot,\cdot)$ is symmetric in its variables, is real valued (so conjugation in (2) can be deleted) and is $\mathcal{H}$-harmonic with respect to each variable.

For $a, b \in \mathbb{B}$, let $\rho(a, b) = |\varphi_a(b)|$ be the pseudo-hyperbolic metric, and for $0 < r < 1$, let $E_r(a) = \{x \in \mathbb{B} : \rho(x, a) < r\}$ be the pseudo-hyperbolic ball of radius r centered at a. For $0 < r < 1$, a sequence $\{a_m\}$ of points of $\mathbb{B}$ is called r-separated if $\rho(a_k, a_m) \geq r$ when $k \neq m$. An r-separated sequence $\{a_m\}$ is called an r-lattice if $\bigcup_{m=1}^{\infty} E_r(a_m) = \mathbb{B}$, that is, if $\{a_m\}$ is maximal.

In [6, Theorem 2], it is shown by Coifman and Rochberg that if $\{a_m\}$ is an r-lattice with r sufficiently small, then every *holomorphic* Bergman function $f \in A^p$ on the unit ball of $\mathbb{C}^n$ (more generally on a symmetric Siegel domain of type two) can be represented in the form $f(z) = \sum_{m=1}^{\infty} \lambda_m \tilde{B}(z, a_m)$, where $\{\lambda_m\} \in \ell^p$ and $\tilde{B}(z, a_m)$ is determined by $B(\cdot, a_m)$, the reproducing kernel at the point a_m. This representation is called atomic decomposition, $B(\cdot, a_m)$ being the atoms. They further showed that a similar decomposition holds for (Euclidean) *harmonic* functions on the unit ball of $\mathbb{R}^n$. This last result is extended in [14, 15] to harmonic Bergman spaces on bounded symmetric domains of $\mathbb{R}^n$.

Our first aim in this work is to show that if $\{a_m\}$ is an r-lattice with small enough r, then an analogous series representation in terms of the reproducing kernels holds also for $\mathcal{H}$-harmonic Bergman spaces $\mathcal{B}^p_\alpha$. Atomic decomposition of $\mathcal{H}$-harmonic *Hardy* spaces on the real hyperbolic ball has been obtained in [7].

Theorem 1.1 *Let $\alpha > -1$ and $0 < p < \infty$. Pick s large enough to satisfy*

$$\begin{aligned} \alpha + 1 &< p(s+1), \quad \text{if } p \geq 1 \\ \alpha + n &< p(s+n), \quad \text{if } 0 < p < 1. \end{aligned} \tag{3}$$

There is an $r_0 < 1/8$ depending only on n, α, p, s such that if $\{a_m\}$ is an r-lattice with $r < r_0$, then for every $f \in \mathcal{B}^p_\alpha$, there exists $\{\lambda_m\} \in \ell^p$ such that

$$f(x) = \sum_{m=1}^{\infty} \lambda_m \frac{\mathcal{R}_s(x, a_m)}{\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha}} \qquad (x \in \mathbb{B}), \tag{4}$$

where the series converges absolutely and uniformly on compact subsets of $\mathbb{B}$ and in $\|\cdot\|_{\mathcal{B}^p_\alpha}$, and the norm $\|\{\lambda_m\}\|_{\ell^p}$ is equivalent to the norm $\|f\|_{\mathcal{B}^p_\alpha}$.

The decomposition above can be written in other forms. By Lemma 2.10 below, the estimate $\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha} \sim (1 - |a_m|^2)^{(\alpha+n)/p-(s+n)}$ holds and Theorem 1.1 remains true if (4) is replaced with (see Remark 3.8)

$$f(x) = \sum_{m=1}^{\infty} \lambda_m (1 - |a_m|^2)^{s+n-(\alpha+n)/p} \mathcal{R}_s(x, a_m) \qquad (x \in \mathbb{B}). \tag{5}$$

Also, $\mathcal{R}_s(a_m, a_m) \sim (1 - |a_m|^2)^{-(s+n)}$ by (17), and (4) can be replaced with

$$f(x) = \sum_{m=1}^{\infty} \lambda_m (1 - |a_m|^2)^{-(\alpha+n)/p} \frac{\mathcal{R}_s(x, a_m)}{\mathcal{R}_s(a_m, a_m)} \qquad (x \in \mathbb{B}).$$

We next consider the interpolation problem. If $\{a_m\}$ is r-separated and $f \in \mathcal{B}^p_\alpha$, then the sequence (see Proposition 3.3)

$$\left\{ f(a_m)(1-|a_m|^2)^{(\alpha+n)/p} \right\}$$

is in ℓ^p. If the converse holds, that is, if for every $\{\lambda_m\} \in \ell^p$, one can find an $f \in \mathcal{B}^p_\alpha$ such that $f(a_m)(1-|a_m|^2)^{(\alpha+n)/p} = \lambda_m$, then $\{a_m\}$ is called an interpolating sequence for $\mathcal{B}^p_\alpha$. We show that if the separation constant r is large enough, then $\{a_m\}$ is an interpolating sequence.

Theorem 1.2 *Let $\alpha > -1$ and $0 < p < \infty$. There is an r_0 with $1/2 < r_0 < 1$ depending only on n, α, p such that if $\{a_m\}$ is an r-separated sequence with $r > r_0$, then for every $\{\lambda_m\} \in \ell^p$, there exists $f \in \mathcal{B}^p_\alpha$ such that*

$$f(a_m)(1-|a_m|^2)^{(\alpha+n)/p} = \lambda_m,$$

and the norm $\|f\|_{\mathcal{B}^p_\alpha}$ is equivalent to the norm $\|\{\lambda_m\}\|_{\ell^p}$.

The *holomorphic* analogue of the above theorem is proved in [2] for the unit ball and polydisc, and in [11] for more general domains of $\mathbb{C}^n$. For *harmonic* Bergman spaces on the upper half-space of $\mathbb{R}^n$, an analogous result is proved in [5].

Finally, we determine precisely when a Bergman space $\mathcal{B}^p_\alpha$ is contained in an another Bergman space $\mathcal{B}^q_\beta$.

Theorem 1.3 *Let $\alpha, \beta > -1$ and $0 < p, q < \infty$.*

(a) If $q \geq p$, then

$$\mathcal{B}^p_\alpha \subset \mathcal{B}^q_\beta \quad \text{if and only if} \quad \frac{\alpha+n}{p} \leq \frac{\beta+n}{q}$$

(b) If $q < p$, then

$$\mathcal{B}^p_\alpha \subset \mathcal{B}^q_\beta \quad \text{if and only if} \quad \frac{\alpha+1}{p} < \frac{\beta+1}{q}$$

In both cases the inclusion $i : \mathcal{B}^p_\alpha \to \mathcal{B}^q_\beta$ is continuous.

For holomorphic Bergman spaces on the unit ball of $\mathbb{C}^n$, the counterpart of this theorem has been proved in [8, Lemma 2.1]. However, this source uses gap series formed by using the so-called Ryll–Wojtaszczyk polynomials (see [12]). We do not know whether such type of $\mathcal{H}$-harmonic functions exist on the real hyperbolic ball. Our proof is based on the above atomic decomposition and interpolation theorems.

2 Preliminaries

In this section we collect some known facts about Möbius transformations and $\mathcal{H}$-harmonic Bergman spaces that will be used in the sequel.

2.1 Notation

We denote positive constants whose exact values are inessential with C. The value of C may differ from one occurrence to another. We write $X \lesssim Y$ if $X \le CY$, and $X \sim Y$ if both $X \le CY$ and $Y \le CX$.

For $x, y \in \mathbb{R}^n$, we write

$$[x, y] := \sqrt{1 - 2\langle x, y\rangle + |x|^2|y|^2},$$

which is symmetric in the variables x, y, and the following equality holds

$$[x, y]^2 = |x - y|^2 + (1 - |x|^2)(1 - |y|^2). \tag{6}$$

If either of the variables is 0, then $[x, 0] = [0, y] = 1$; otherwise

$$[x, y] = \left| |y|x - \frac{y}{|y|} \right| = \left| \frac{x}{|x|} - |x|y \right|, \tag{7}$$

and so

$$1 - |x||y| \le [x, y] \le 1 + |x||y| \qquad (x, y \in \mathbb{B}). \tag{8}$$

2.2 Möbius Transformations

For more details about the facts listed in this subsection we refer the reader to [1] or [13].

A Möbius transformation of $\hat{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is a finite composition of reflections (inversions) in spheres or planes. We denote the group of all Möbius transformations mapping $\mathbb{B}$ to $\mathbb{B}$ by $\mathcal{M}(\mathbb{B})$. For $a \in \mathbb{B}$, the mapping

$$\varphi_a(x) = \frac{a|x - a|^2 + (1 - |a|^2)(a - x)}{[x, a]^2} \qquad (x \in \mathbb{B}) \tag{9}$$

is in $\mathcal{M}(\mathbb{B})$, exchanges a and 0, and satisfies $\varphi_a \circ \varphi_a = \mathrm{Id}$. The group $\mathcal{M}(\mathbb{B})$ is generated by $\{\varphi_a : a \in \mathbb{B}\}$ and orthogonal transformations. A very useful identity involving φ_a is ([13, Eqn. 2.1.7])

$$1 - |\varphi_a(x)|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{[x, a]^2}. \tag{10}$$

The Jacobian J_{φ_a} of φ_a satisfies ([13, Theorem 3.3.1])

$$|J_{\varphi_a}(x)| = \frac{(1 - |\varphi_a(x)|^2)^n}{(1 - |x|^2)^n}. \tag{11}$$

The following lemma is a special case of [10, Theorem 1.1].

Lemma 2.1 *For $a, x \in \mathbb{B}$, the following equality holds*

$$[\varphi_a(x), a] = \frac{1-|a|^2}{[x, a]}.$$

Proof Replacing x in (10) with $\varphi_a(x)$ and noting that $\varphi_a \circ \varphi_a = \mathrm{Id}$ shows

$$[\varphi_a(x), a]^2 = \frac{(1-|a|^2)(1-|\varphi_a(x)|^2)}{1-|x|^2}.$$

Applying (10) again, we obtain the desired result. □

For $a, b \in \mathbb{B}$, the pseudo-hyperbolic metric $\rho(a, b) = |\varphi_a(b)|$ satisfies

$$\rho(a, b) = \frac{|a-b|}{[a, b]}, \tag{12}$$

by (10) and (6). It is Möbius invariant in the sense that $\rho(\psi(a), \psi(b)) = \rho(a, b)$ for every $\psi \in \mathcal{M}(\mathbb{B})$. It satisfies not only the triangle inequality, but the following strong triangle inequality (see [10, Theorem 1.2]).

Lemma 2.2 *For $a, b, x \in \mathbb{B}$, the following inequalities hold*

$$\frac{\big|\rho(a, x) - \rho(b, x)\big|}{1 - \rho(a, x)\rho(b, x)} \le \rho(a, b) \le \frac{\rho(a, x) + \rho(b, x)}{1 + \rho(a, x)\rho(b, x)}.$$

Lemma 2.3 *For $x, y \in \mathbb{B}$,*

$$1 - \rho(x, y) \le \frac{1-|x|^2}{[x, y]} \le 1 + \rho(x, y).$$

Proof The lemma clearly holds when $x = 0$. Otherwise, let $x^* := x/|x|^2$ be the inversion of x with respect to the unit sphere $\mathbb{S}$. Multiply the triangle inequality

$$|x^* - y| - |y - x| \le |x^* - x| \le |x^* - y| + |y - x|$$

by $|x|$. Noting that $|x||x^* - y| = [x, y]$ by (7), and $|x||x^* - x| = 1 - |x|^2$, we deduce

$$[x, y] - |x||y - x| \le 1 - |x|^2 \le [x, y] + |x||y - x|.$$

The lemma follows from the facts that $|y - x| = \rho(x, y)[x, y]$ by (12), and $|x| < 1$. □

The following lemma is a slight modification of [3, Lemma 2.1] and immediately follows from Lemma 2.3.

Lemma 2.4 *For* $x, y \in \mathbb{B}$,

$$\frac{1-\rho(x,y)}{1+\rho(x,y)} \le \frac{1-|x|^2}{1-|y|^2} \le \frac{1+\rho(x,y)}{1-\rho(x,y)}.$$

The next lemma is proved in [3, Lemma 2.2].

Lemma 2.5 *For* $a, x, y \in \mathbb{B}$,

$$\frac{1-\rho(x,y)}{1+\rho(x,y)} \le \frac{[x,a]}{[y,a]} \le \frac{1+\rho(x,y)}{1-\rho(x,y)}.$$

Let $\mathbb{B}_r = \{x \in \mathbb{R}^n : |x| < r\}$. The pseudo-hyperbolic ball $E_r(a) = \{x \in \mathbb{B} : \rho(x,a) < r\} = \varphi_a(\mathbb{B}_r)$ is also a Euclidean ball with (see [13, Theorem 2.2.2])

$$\text{center} = \frac{(1-r^2)a}{1-|a|^2r^2} \quad \text{and} \quad \text{radius} = \frac{(1-|a|^2)r}{1-|a|^2r^2}. \tag{13}$$

2.3 Separated Sequences and Lattices

There exists an r-lattice for every $0 < r < 1$ as explained in [6, p. 18], and every r-separated sequence can be completed to an r-lattice. The following lemma follows from an invariant volume argument.

Lemma 2.6 *Let* $0 < r, \delta < 1$ *and* $\{a_m\}$ *be* r*-separated. There exists* N *depending only on* n, r, δ *such that every* $x \in \mathbb{B}$ *belongs to at most* N *of the balls* $E_\delta(a_m)$.

Lemma 2.7 *Let* $\{a_m\}$ *be an* r*-lattice. There exists a sequence* $\{E_m\}$ *of disjoint sets such that* $\bigcup_{m=1}^\infty E_m = \mathbb{B}$ *and*

$$E_{r/2}(a_m) \subset E_m \subset E_r(a_m). \tag{14}$$

Proof Let $E_1 = E_r(a_1)\backslash \bigcup_{m=2}^\infty E_{r/2}(a_m)$ and given $E_1, \dots, E_{m-1}$, let

$$E_m = E_r(a_m)\backslash\left(\bigcup_{i=1}^{m-1} E_i \bigcup \bigcup_{i=m+1}^{\infty} E_{r/2}(a_i)\right).$$

□

Lemma 2.8 *Let* $\gamma \in \mathbb{R}$ *and* $0 < r < 1$.

(a) If $\{a_m\}$ *is* r*-separated and* $\gamma > n-1$, *then* $\sum_{m=1}^\infty (1-|a_m|^2)^\gamma < \infty$.
(b) If $\{a_m\}$ *is an* r*-lattice, then* $\sum_{m=1}^\infty (1-|a_m|^2)^\gamma < \infty$ *if and only if* $\gamma > n-1$.

Proof To see part (a), note that

$$\int_{E_{r/2}(a_m)} (1-|y|^2)^{\gamma-n}\, d\nu(y) \sim (1-|a_m|^2)^\gamma, \tag{15}$$

where the implied constants depend only on the fixed parameters n, γ, r and are independent of a_m. This is true because for $y \in E_{r/2}(a_m)$, we have $(1-|y|^2) \sim (1-|a_m|^2)$ by Lemma 2.4 and $\nu(E_{r/2}(a_m)) \sim (1-|a_m|^2)^n$ by (13). Thus

$$\sum_{m=1}^{\infty}(1-|a_m|^2)^{\gamma} \lesssim \int_{\mathbb{B}}(1-|y|^2)^{\gamma-n}\,d\nu(y),$$

since the balls $E_{r/2}(a_m)$ are disjoint. If $\gamma > n-1$, then the above integral is finite.

For part (b), let E_m be as given in Lemma 2.7. By (14), we similarly have

$$\int_{E_m}(1-|y|^2)^{\gamma-n}\,d\nu(y) \sim (1-|a_m|^2)^{\gamma} \tag{16}$$

and therefore

$$\sum_{m=1}^{\infty}(1-|a_m|^2)^{\gamma} \sim \sum_{m=1}^{\infty}\int_{E_m}(1-|y|^2)^{\gamma-n}\,d\nu(y) = \int_{\mathbb{B}}(1-|y|^2)^{\gamma-n}\,d\nu(y).$$

The last integral is finite if and only if $\gamma > n-1$. □

2.4 Reproducing Kernels and Bergman Projection

The following upper estimates of the reproducing kernels $\mathcal{R}_\alpha$ of $\mathcal{H}$-harmonic Bergman spaces have been obtained in [16, Theorem 1.2].

Lemma 2.9 *For $\alpha > -1$, there exists a constant $C > 0$ such that for all $x, y \in \mathbb{B}$,*

(a) $|\mathcal{R}_\alpha(x,y)| \le \dfrac{C}{[x,y]^{\alpha+n}}$,

(b) $|\nabla_1\mathcal{R}_\alpha(x,y)| \le \dfrac{C}{[x,y]^{\alpha+n+1}}$.

Here ∇_1 means the gradient is taken with respect to the first variable.

More is true on the diagonal $y = x$ and the two-sided estimate ([16, Lemma 6.1])

$$\mathcal{R}_\alpha(x,x) \sim \frac{1}{(1-|x|^2)^{\alpha+n}} \tag{17}$$

holds. The following lemma is part of [16, Theorem 1.3].

Lemma 2.10 *If $\alpha, s > -1$, $0 < p < \infty$ and $p(s+n)-(\alpha+n) > 0$, then*

$$\int_{\mathbb{B}}\big|\mathcal{R}_s(x,y)\big|^p\,d\nu_\alpha(y) \sim \frac{1}{(1-|x|^2)^{p(s+n)-(\alpha+n)}}.$$

The implied constants depend only on n, α, s, p and are independent of x.

For $s > -1$ and suitable f, we define the projection operator P_s and the related operator Q_s by

$$
\begin{aligned}
P_s f(x) &= \int_{\mathbb{B}} f(y)\mathcal{R}_s(x, y)\, d\nu_s(y), \\
Q_s f(x) &= \int_{\mathbb{B}} \frac{f(y)}{[x, y]^{s+n}}\, d\nu_s(y).
\end{aligned} \tag{18}
$$

Lemma 2.11 *Let $1 \le p < \infty$ and $\alpha, s > -1$. The following are equivalent:*

(a) $P_s : L^p_\alpha \to \mathcal{B}^p_\alpha$ is bounded,
(b) $Q_s : L^p_\alpha \to L^p_\alpha$ is bounded,
(c) $\alpha + 1 < p(s + 1)$.

In case (c) holds, then $P_s f = f$ for every $f \in \mathcal{B}^p_\alpha$.

Proof (b) ⇒ (a) follows from Lemma 2.9 (a), (a) ⇒ (c) is proved in [16, Theorem 1.1], and (c) ⇒ (b) is well-known and included in the proof of [16, Theorem 1.1]. □

For a proof of the following estimates, see [9, Proposition 2.2].

Lemma 2.12 *Let $b > -1$ and $c \in \mathbb{R}$. For $x \in \mathbb{B}$, define*

$$
I_c(x) := \int_{\mathbb{S}} \frac{d\sigma(\zeta)}{|x - \zeta|^{n-1+c}} \quad \text{and} \quad J_{b,c}(x) := \int_{\mathbb{B}} \frac{(1 - |y|^2)^b}{[x, y]^{n+b+c}}\, d\nu(y),
$$

where σ is the normalized surface measure on $\mathbb{S}$. For all $x \in \mathbb{B}$,

$$
I_c(x) \sim J_{b,c}(x) \sim \begin{cases} \dfrac{1}{(1 - |x|^2)^c}, & \text{if } c > 0; \\ 1 + \log \dfrac{1}{1 - |x|^2}, & \text{if } c = 0; \\ 1, & \text{if } c < 0, \end{cases}
$$

where the implied constants depend only on n, b, c and are independent of x.

We record the following elementary facts about the sequence spaces ℓ^p for future reference.

Lemma 2.13 *(i) For $0 < p < q < \infty$, $\|\{\lambda_m\}\|_{\ell^q} \le \|\{\lambda_m\}\|_{\ell^p}$.*

(ii) Let $1 < p < \infty$ and p' be the conjugate exponent of p, $1/p + 1/p' = 1$. If $\sum_{m=1}^{\infty} |\lambda_m \kappa_m| < \infty$ for every $\{\kappa_m\} \in \ell^{p'}$, then $\{\lambda_m\} \in \ell^p$.

3 Atomic Decomposition

The purpose of this section is to prove Theorem 1.1. The main problem is to show that under the assumptions of the theorem, the operator $U : \ell^p \to \mathcal{B}^p_\alpha$ defined in (19) below is onto. We do this through a couple of propositions.

Proposition 3.1 *For $\alpha > -1$ and $0 < p < \infty$, choose s so that (3) holds. If $\{a_m\}$ is r-separated for some $0 < r < 1$, then the operator $U\colon \ell^p \to \mathcal{B}^p_\alpha$ mapping $\lambda = \{\lambda_m\}$ to*

$$U\lambda(x) = \sum_{m=1}^{\infty} \lambda_m \frac{\mathcal{R}_s(x, a_m)}{\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha}} \qquad (x \in \mathbb{B}) \tag{19}$$

is bounded. The above series converges absolutely and uniformly on compact subsets of $\mathbb{B}$, and also in $\|\cdot\|_{\mathcal{B}^p_\alpha}$.

Proof Throughout the proof we suppress the constants that depend on the fixed parameters n, α, p, s and r. Note that, by Lemma 2.10 and (3), for every $0 < p < \infty$,

$$\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha} \sim (1 - |a_m|^2)^{(\alpha+n)/p-(s+n)}, \tag{20}$$

since in (3), the inequality $p(s + n) > (\alpha + n)$ holds also in the case $p \geq 1$.

We begin with the case $0 < p \leq 1$. We first show that for $\lambda \in \ell^p$, the series in (19) converges absolutely and uniformly on compact subsets of $\mathbb{B}$ which implies that $U\lambda$ is $\mathcal{H}$-harmonic on $\mathbb{B}$ since so is each $\mathcal{R}_s(\cdot, a_m)$. If $|x| \leq R < 1$, then $|\mathcal{R}_s(x, a_m)| \lesssim 1$ by Lemma 2.9 (a), since $[x, a_m] \geq 1 - |x|$ by (8). Thus, using also (20), the fact that $(s + n) - (\alpha + n)/p > 0$ by (3), and Lemma 2.13 (i) we obtain

$$\sum_{m=1}^{\infty} |\lambda_m| \frac{|\mathcal{R}_s(x, a_m)|}{\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha}} \lesssim \sum_{m=1}^{\infty} |\lambda_m| (1 - |a_m|^2)^{(s+n)-(\alpha+n)/p} \leq \sum_{m=1}^{\infty} |\lambda_m| \leq \|\lambda\|_{\ell^p},$$

which proves the assertion. The inequality $\|U\lambda\|_{\mathcal{B}^p_\alpha} \leq \|\lambda\|_{\ell^p}$ immediately follows from Lemma 2.13 (i) and shows also that the series in (19) converges in $\|\cdot\|_{\mathcal{B}^p_\alpha}$.

We next consider the case $1 < p < \infty$. Let p' be the conjugate exponent of p. The series in (19) converges absolutely and uniformly on compact subsets of $\mathbb{B}$ because we again have $|\mathcal{R}_s(x, a_m)| \lesssim 1$, and by (20) and Hölder's inequality,

$$\begin{aligned}\sum_{m=1}^{\infty} |\lambda_m| \frac{|\mathcal{R}_s(x, a_m)|}{\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha}} &\lesssim \sum_{m=1}^{\infty} |\lambda_m| (1 - |a_m|^2)^{s+n-(\alpha+n)/p} \\ &\leq \|\lambda\|_{\ell^p} \Big(\sum_{m=1}^{\infty} (1 - |a_m|^2)^{p'(s+n-(\alpha+n)/p)} \Big)^{1/p'}.\end{aligned}$$

The last sum is finite by Lemma 2.8 (a) since the inequality $p'(s + n - (\alpha + n)/p) > n - 1$ is equivalent to (3). Thus $U\lambda$ is $\mathcal{H}$-harmonic on $\mathbb{B}$.

To show $\|U\lambda\|_{\mathcal{B}^p_\alpha} \lesssim \|\lambda\|_{\ell^p}$, following [4, 6, 15], we use the projection theorem. Denote by χ_A the characteristic function of a set A. For $\lambda \in \ell^p$, let

$$g(x) := \sum_{m=1}^{\infty} |\lambda_m| (1 - |a_m|^2)^{-(\alpha+n)/p} \chi_{E_{r/2}(a_m)}(x) \qquad (x \in \mathbb{B}).$$

We have $\|g\|_{L^p_\alpha} \sim \|\lambda\|_{\ell^p}$, since the balls $E_{r/2}(a_m)$ are disjoint and

$$\|g\|^p_{L^p_\alpha} = \sum_{m=1}^{\infty} |\lambda_m|^p (1-|a_m|^2)^{-(\alpha+n)} \nu_\alpha(E_{r/2}(a_m)) \sim \sum_{m=1}^{\infty} |\lambda_m|^p,$$

by (15). Next, with Q_s as in (18),

$$\begin{aligned} Q_s g(x) &= \sum_{m=1}^{\infty} |\lambda_m| (1-|a_m|^2)^{-(\alpha+n)/p} \int_{E_{r/2}(a_m)} \frac{(1-|y|^2)^s}{[x,y]^{s+n}} d\nu(y) \\ &\sim \sum_{m=1}^{\infty} \frac{|\lambda_m|(1-|a_m|^2)^{s+n-(\alpha+n)/p}}{[x,a_m]^{s+n}}, \end{aligned}$$

where in the last line we first use the fact that $[x,y] \sim [x,a_m]$ for $y \in E_{r/2}(a_m)$ by Lemma 2.5, and then use (15). This shows that $|U\lambda(x)| \lesssim Q_s g(x)$ by Lemma 2.9 (a) and (20). Since Q_s is bounded by Lemma 2.11 and (3), we conclude

$$\|U\lambda\|_{\mathcal{B}^p_\alpha} \lesssim \|Q_s g\|_{L^p_\alpha} \lesssim \|g\|_{L^p_\alpha} \sim \|\lambda\|_{\ell^p}.$$

□

To verify that the above operator $U : \ell^p \to \mathcal{B}^p_\alpha$ is onto under the additional assumption that $\{a_m\}$ is an r-lattice with r small enough, we need to consider a second operator. We first recall the following sub-mean value inequality for $\mathcal{H}$-harmonic functions. For a proof see [13, Section 4.7]. Here, $d\tau(x) = (1-|x|^2)^{-n} d\nu(x)$ is the invariant measure on $\mathbb{B}$.

Lemma 3.2 *Let $f \in \mathcal{H}(\mathbb{B})$ and $0 < p < \infty$. For all $a \in \mathbb{B}$ and all $0 < \delta < 1$,*

$$|f(a)|^p \le \frac{C}{\delta^n} \int_{E_\delta(a)} |f(y)|^p \, d\tau(y),$$

where $C = 1$ if $p \ge 1$ and $C = 2^{n/p}$ if $0 < p < 1$.

Proposition 3.3 *Let $\alpha > -1$, $0 < p < \infty$ and $\{a_m\}$ be r-separated for some $0 < r < 1$. Then the operator $T : \mathcal{B}^p_\alpha \to \ell^p$ defined by*

$$Tf = \left\{ f(a_m)(1-|a_m|^2)^{(\alpha+n)/p} \right\} \tag{21}$$

is bounded.

Proof Applying Lemma 3.2 with $\delta = r/2$ and noting that $(1-|y|^2) \sim (1-|a_m|^2)$ for $y \in E_{r/2}(a_m)$ by Lemma 2.4, we obtain

$$|f(a_m)|^p (1-|a_m|^2)^{\alpha+n} \lesssim \int_{E_{r/2}(a_m)} |f(y)|^p \, d\nu_\alpha(y).$$

Since the balls $E_{r/2}(a_m)$ are disjoint, we deduce

$$\|Tf\|_{\ell^p}^p = \sum_{m=1}^{\infty} |f(a_m)|^p (1-|a_m|^2)^{\alpha+n} \lesssim \sum_{m=1}^{\infty} \int_{E_{r/2}(a_m)} |f(y)|^p \, d\nu_\alpha(y) \le \|f\|_{\mathcal{B}_\alpha^p}^p.$$

□

We need a slightly modified version of the above operator T.

Proposition 3.4 *For $\alpha > -1$ and $0 < p < \infty$, choose s so that (3) holds. If $\{a_m\}$ is an r-lattice for some $0 < r < 1$ and $\{E_m\}$ is the associated sequence as given in Lemma 2.7, then the operator $\hat{T} : \mathcal{B}_\alpha^p \to \ell^p$ defined by*

$$\hat{T} f = \left\{ f(a_m) \, \|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}_\alpha^p} \, \nu_s(E_m) \right\} \tag{22}$$

is bounded.

Proof Since $\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}_\alpha^p} \nu_s(E_m) \sim (1-|a_m|^2)^{(\alpha+n)/p}$ by (16) and (20), the result follows from Proposition 3.3. □

Proposition 3.5 *For $\alpha > -1$ and $0 < p < \infty$, choose s so that (3) holds. There exists a constant $C > 0$ depending only on n, α, p, s such that if $\{a_m\}$ is an r-lattice with $r < 1/8$, then $\|I - U\hat{T}\|_{\mathcal{B}_\alpha^p \to \mathcal{B}_\alpha^p} \le Cr$.*

In the proofs of the previous propositions we allowed the constants to depend on the separation constant r. This time we need to be careful that the suppressed constants are independent of r. We prove the cases $p \ge 1$ and $0 < p < 1$ separately.

Proof of Proposition 3.5 when $p \ge 1$ By (19) and (22),

$$U\hat{T} f(x) = \sum_{m=1}^{\infty} \int_{E_m} f(a_m) \mathcal{R}_s(x, a_m) \, d\nu_s(y),$$

and by (3) and Lemma 2.11 we have $P_s f = f$ and so

$$f(x) = \sum_{m=1}^{\infty} \int_{E_m} f(y) \mathcal{R}_s(x, y) \, d\nu_s(y).$$

Therefore

$$\begin{aligned}(I - U\hat{T}) f(x) &= \sum_{m=1}^{\infty} \int_{E_m} \big(\mathcal{R}_s(x, y) - \mathcal{R}_s(x, a_m)\big) f(y) \, d\nu_s(y) \\ &\quad + \sum_{m=1}^{\infty} \int_{E_m} \mathcal{R}_s(x, a_m) \big(f(y) - f(a_m)\big) \, d\nu_s(y) \\ &=: h_1(x) + h_2(x).\end{aligned} \tag{23}$$

We first estimate h_1. Let $y \in E_m$. By the mean value theorem of calculus, there exists $\tilde{y}$ lying on the line segment joining a_m and y such that

$$\mathcal{R}_s(x, y) - \mathcal{R}_s(x, a_m) = \big\langle y - a_m, \nabla_2 \mathcal{R}_s(x, \tilde{y}) \big\rangle,$$

where ∇_2 means the gradient is taken with respect to the second variable. Observe that because r is bounded above by $1/8$, there are constants independent of r such that for $y \in E_m \subset E_r(a_m)$, we have $[y, a_m] \sim [y, y] = 1 - |y|^2$ by Lemma 2.5. Thus, by (12),

$$|y - a_m| = \rho(y, a_m)[y, a_m] < r[y, a_m] \lesssim r(1 - |y|^2).$$

Next, since a_m and y are both in the ball $E_r(a_m)$, so is $\tilde{y}$. Hence $\rho(y, \tilde{y}) < 1/4$ and by Lemma 2.5, $[x, y] \sim [x, \tilde{y}]$ for every $x \in \mathbb{B}$ with the constants again not depending on r. Therefore, by Lemma 2.9 (b) and the symmetry of $\mathcal{R}_s(\cdot, \cdot)$,

$$\big|\nabla_2 \mathcal{R}_s(x, \tilde{y})\big| \lesssim \frac{1}{[x, \tilde{y}]^{s+n+1}} \sim \frac{1}{[x, y]^{s+n+1}}.$$

Combining these we see that for $y \in E_m$ and $x \in \mathbb{B}$,

$$|\mathcal{R}_s(x, y) - \mathcal{R}_s(x, a_m)| \lesssim \frac{r(1 - |y|^2)}{[x, y]^{s+n+1}} \lesssim \frac{r}{[x, y]^{s+n}}, \tag{24}$$

where in the last inequality we use $[x, y] \geq 1 - |y|$ by (8). Thus

$$|h_1(x)| \lesssim r \sum_{m=1}^{\infty} \int_{E_m} \frac{|f(y)|}{[x, y]^{s+n}}\, d\nu_s(y) = r \int_{\mathbb{B}} \frac{|f(y)|}{[x, y]^{s+n}}\, d\nu_s(y) = r Q_s(|f|)(x),$$

and since Q_s is bounded on L^p_α by Lemma 2.11, we obtain $\|h_1\|_{L^p_\alpha} \lesssim r \|f\|_{B^p_\alpha}$.

We now estimate h_2. Let $y \in E_m$. As above, we have $P_s f = f$, and so

$$f(y) - f(a_m) = \int_{\mathbb{B}} \big(\mathcal{R}_s(y, z) - \mathcal{R}_s(a_m, z)\big) f(z)\, d\nu_s(z).$$

Since $\mathcal{R}_s(\cdot, \cdot)$ is symmetric, by (24),

$$|\mathcal{R}_s(y, z) - \mathcal{R}_s(a_m, z)| \lesssim \frac{r}{[y, z]^{s+n}},$$

for all $z \in \mathbb{B}$ with the constants not depending on r. Thus

$$|f(y) - f(a_m)| \lesssim r \int_{\mathbb{B}} \frac{|f(z)|}{[y, z]^{s+n}}\, d\nu_s(z) = r Q_s(|f|)(y),$$

and so

$$|h_2(x)| \lesssim r \sum_{m=1}^{\infty} \int_{E_m} |\mathcal{R}_s(x, a_m)|\, Q_s(|f|)(y)\, d\nu_s(y) \lesssim r \sum_{m=1}^{\infty} \int_{E_m} \frac{Q_s(|f|)(y)}{[x, a_m]^{s+n}}\, d\nu_s(y),$$

where in the last inequality we use Lemma 2.9 (a). By Lemma 2.4 again, we have $[x, a_m] \sim [x, y]$ for $y \in E_m \subset E_r(a_m)$ since $r < 1/8$. Hence

$$|h_2(x)| \lesssim r \sum_{m=1}^{\infty} \int_{E_m} \frac{Q_s(|f|)(y)}{[x, y]^{s+n}} d\nu_s(y) = r \int_{\mathbb{B}} \frac{Q_s(|f|)(y)}{[x, y]^{s+n}} d\nu_s(y)$$
$$= r\, Q_s\big(Q_s(|f|)\big)(x),$$

and since Q_s is bounded on L^p_α we obtain that $\|h_2\|_{L^p_\alpha} \lesssim r \|f\|_{\mathcal{B}^p_\alpha}$.

We conclude that $\|(I - U\hat{T})f\|_{\mathcal{B}^p_\alpha} \leq Cr\|f\|_{\mathcal{B}^p_\alpha}$, with C depending only on n, α, p, s. This finishes the proof when $p \geq 1$. □

In order to prove the case $0 < p < 1$, we need to do some preparation. The following inequality is proved in [13, Theorem 4.7.4 part (b)].

Lemma 3.6 *Let $0 < p < \infty$ and $0 < \delta < 1/2$. There exists a constant $C > 0$ depending only on n, p, δ such that for all $a \in \mathbb{B}$ and $f \in \mathcal{H}(\mathbb{B})$,*

$$|\nabla^h f(a)|^p \leq \frac{C}{\delta^n} \int_{E_\delta(a)} |f(y)|^p\, d\tau(y).$$

The next lemma is a special case of Theorem 1.3 part (a).

Lemma 3.7 *Let $0 < p < 1$ and $\alpha > -1$. Then $\mathcal{B}^p_\alpha \subset \mathcal{B}^1_{(\alpha+n)/p-n}$ and the inclusion is continuous.*

Proof By [13, Eqn. (10.1.5)], there exists a constant $C > 0$ depending only on n, α, p such that

$$|f(x)| \leq \frac{C}{(1 - |x|^2)^{(\alpha+n)/p}} \|f\|_{\mathcal{B}^p_\alpha}, \tag{25}$$

for all $x \in \mathbb{B}$ and $f \in \mathcal{B}^p_\alpha$. In the integral below writing $|f(x)| = |f(x)|^p |f(x)|^{1-p}$ and applying (25) to the factor $|f(x)|^{1-p}$, we deduce

$$\int_{\mathbb{B}} |f(x)|(1 - |x|^2)^{(\alpha+n)/p-n}\, d\nu(x) \leq C^{1-p} \|f\|^{1-p}_{\mathcal{B}^p_\alpha} \int_{\mathbb{B}} |f(x)|^p (1 - |x|^2)^\alpha\, d\nu(x)$$
$$= C^{1-p} \|f\|_{\mathcal{B}^p_\alpha}.$$

□

Proof of Proposition 3.5 when $0 < p < 1$ In this part of the proof we can not use the projection theorem which requires $p \geq 1$. Instead, we follow [6, p. 19] and use a suitable rearrangement of the sequence $\{a_m\}$ as described below.

Pick a $1/2$-lattice $\{b_m\}$ and fix it throughout the proof. Denote the sequence of sets associated to the lattice $\{b_m\}$ as described in Lemma 2.7 by $\{D_m\}$. That is, the sets D_m are disjoint with $\bigcup_{m=1}^{\infty} D_m = \mathbb{B}$ and

$$E_{1/4}(b_m) \subset D_m \subset E_{1/2}(b_m) \qquad (m = 1, 2, \dots).$$

Given an r-lattice $\{a_m\}$ with $r < 1/8$, renumber $\{a_m\}$ in the following way. Call the elements of $\{a_m\}$ that are in D_1 as $a_{11}, a_{12}, \dots, a_{1\kappa_1}$ and in general call the points of $\{a_m\}$ that are in D_m as $a_{m1}, a_{m2}, \dots, a_{m\kappa_m}$. Denote the sets given in Lemma 2.7 corresponding to this renumbering by E_{mk}. Thus, the sets E_{mk} are disjoint, $\bigcup_{m=1}^{\infty} \bigcup_{k=1}^{\kappa_m} E_{mk} = \mathbb{B}$ and

$$E_{r/2}(a_{mk}) \subset E_{mk} \subset E_r(a_{mk}) \qquad (m = 1, 2, \dots,\ k = 1, 2, \dots, \kappa_m).$$

By the above construction, since $a_{mk} \in D_m \subset E_{1/2}(b_m)$, we have

$$\rho(a_{mk}, b_m) < 1/2 \qquad (m = 1, 2, \dots,\ k = 1, 2, \dots, \kappa_m), \tag{26}$$

and by the triangle inequality and the fact that $r < 1/8$,

$$E_{mk} \subset E_{5/8}(b_m). \tag{27}$$

Suppose now $f \in \mathcal{B}^p_\alpha$. We claim that $P_s f = f$. This is true because by Lemma 3.7, f is in $\mathcal{B}^1_{(\alpha+n)/p-n}$ and for this space the required condition in Lemma 2.11 (c) is $s > (\alpha + n)/p - n$ which holds by (3). Therefore

$$f(x) = \int_{\mathbb{B}} f(y)\mathcal{R}_s(x, y)\, d\nu_s(y) = \sum_{m=1}^{\infty} \sum_{k=1}^{\kappa_m} \int_{E_{mk}} f(y)\mathcal{R}_s(x, y)\, d\nu_s(y).$$

Next, with the above rearrangement, by (19) and (22),

$$U\hat{T} f(x) = \sum_{m=1}^{\infty} \sum_{k=1}^{\kappa_m} f(a_{mk})\nu_s(E_{mk})\mathcal{R}_s(x, a_{mk})$$

and so, similar to (23), we have

$$\begin{aligned}(I - U\hat{T})f(x) &= \sum_{m=1}^{\infty} \sum_{k=1}^{\kappa_m} \int_{E_{mk}} \big(\mathcal{R}_s(x, y) - \mathcal{R}_s(x, a_{mk})\big) f(y)\, d\nu_s(y) \\ &\quad + \sum_{m=1}^{\infty} \sum_{k=1}^{\kappa_m} \int_{E_{mk}} \mathcal{R}_s(x, a_{mk})\big(f(y) - f(a_{mk})\big)\, d\nu_s(y) \\ &=: h_1(x) + h_2(x).\end{aligned}$$

We first estimate h_1. We will again be careful that in the estimates below the suppressed constants are independent of the separation constant r. Let $y \in E_{mk}$. First, as is shown in (24), for all $x \in \mathbb{B}$,

$$|\mathcal{R}_s(x, y) - \mathcal{R}_s(x, a_{mk})| \lesssim \frac{r}{[x, y]^{s+n}} \lesssim \frac{r}{[x, b_m]^{s+n}}, \tag{28}$$

where in the last inequality we use Lemma 2.5 with (27). Next, applying Lemma 3.2 with $\delta = 1/8$ and noting that $E_{1/8}(y) \subset E_{3/4}(b_m)$, we obtain

$$\begin{aligned} |f(y)|^p \lesssim \int_{E_{1/8}(y)} |f(z)|^p \, d\tau(z) &\leq \int_{E_{3/4}(b_m)} |f(z)|^p \, d\tau(z) \\ &\lesssim (1 - |b_m|^2)^{-(\alpha+n)} \int_{E_{3/4}(b_m)} |f(z)|^p \, d\nu_\alpha(z), \end{aligned} \tag{29}$$

where in the last inequality we use $(1 - |z|^2) \sim (1 - |b_m|^2)$ for $z \in E_{3/4}(b_m)$ by Lemma 2.4. Combining (28) and (29) we deduce

$$|h_1(x)| \lesssim r \sum_{m=1}^{\infty} \frac{(1 - |b_m|^2)^{-(\alpha+n)/p}}{[x, b_m]^{s+n}} \left(\int_{E_{3/4}(b_m)} |f(z)|^p d\nu_\alpha(z) \right)^{\frac{1}{p}} \sum_{k=1}^{\kappa_m} \nu_s(E_{mk}). \tag{30}$$

Since the sets E_{mk} are disjoint and $E_{mk} \subset E_{5/8}(b_m)$ for every $k = 1, \ldots, \kappa_m$ by (27), we have $\sum_{k=1}^{\kappa_m} \nu_s(E_{mk}) \leq \nu_s(E_{5/8}(b_m))$. Also $\nu_s(E_{5/8}(b_m)) \sim (1 - |b_m|^2)^{s+n}$ by Lemma 2.4 and (13). Using this and then Lemma 2.13 (i) yields

$$|h_1(x)|^p \lesssim r^p \sum_{m=1}^{\infty} \frac{(1 - |b_m|^2)^{p(s+n)-(\alpha+n)}}{[x, b_m]^{p(s+n)}} \int_{E_{3/4}(b_m)} |f(z)|^p \, d\nu_\alpha(z).$$

Integrating over $\mathbb{B}$ with respect to $d\nu_\alpha$, applying Fubini's theorem, and noting that

$$(1 - |b_m|^2)^{p(s+n)-(\alpha+n)} \int_{\mathbb{B}} \frac{d\nu_\alpha(x)}{[x, b_m]^{p(s+n)}} \lesssim 1,$$

by Lemma 2.12 and (3), we obtain

$$\|h_1\|_{L^p_\alpha}^p \lesssim r^p \sum_{m=1}^{\infty} \int_{E_{3/4}(b_m)} |f(z)|^p \, d\nu_\alpha(z). \tag{31}$$

Finally, by Lemma 2.6, there exists N such that every $z \in \mathbb{B}$ belongs at most N of the balls $E_{3/4}(b_m)$, and so $\sum_{m=1}^{\infty} \int_{E_{3/4}(b_m)} |f(z)|^p \, d\nu_\alpha(z) \leq N \int_{\mathbb{B}} |f(z)|^p \, d\nu_\alpha(z)$. We conclude that

$$\|h_1\|_{L^p_\alpha}^p \lesssim r^p \|f\|_{L^p_\alpha}^p. \tag{32}$$

We next estimate h_2. Let $y \in E_{mk}$. By the mean-value theorem of calculus, there exists $\tilde{y}$ lying on the line segment joining a_{mk} and y such that

$$\begin{aligned} |f(y) - f(a_{mk})| \leq |y - a_{mk}|\, |\nabla f(\tilde{y})| &= \rho(y, a_{mk})[y, a_{mk}]\, \frac{|\nabla^h f(\tilde{y})|}{1 - |\tilde{y}|^2} \\ &< r\, \frac{[y, a_{mk}]}{1 - |\tilde{y}|^2}\, |\nabla^h f(\tilde{y})|, \end{aligned}$$

where we also use (1), (12) and the fact that $\rho(y, a_{mk}) < r$ because $E_{mk} \subset E_r(a_{mk})$. Since the point $\tilde{y}$ is also in the ball $E_r(a_{mk})$ and $r < 1/8$, we have

$$\rho(\tilde{y}, a_{mk}) < 1/8, \tag{33}$$

and therefore $(1 - |\tilde{y}|^2) \sim (1 - |a_{mk}|^2)$ by Lemma 2.4. Similarly, since $\rho(y, a_{mk}) < 1/8$, we have $[y, a_{mk}] \sim [a_{mk}, a_{mk}] = (1 - |a_{mk}|^2)$ by Lemma 2.5 and we conclude

$$|f(y) - f(a_{mk})| \lesssim r|\nabla^h f(\tilde{y})|.$$

Next, applying Lemma 3.6 with $\delta = 1/8$ and then using $E_{1/8}(\tilde{y}) \subset E_{3/4}(b_m)$ which follows from (33) and (26), we obtain

$$\begin{aligned} |\nabla^h f(\tilde{y})|^p \lesssim \int_{E_{1/8}(\tilde{y})} |f(z)|^p\, d\tau(z) &\leq \int_{E_{3/4}(b_m)} |f(z)|^p\, d\tau(z) \\ &\lesssim (1 - |b_m|^2)^{-(\alpha+n)} \int_{E_{3/4}(b_m)} |f(z)|^p\, d\nu_\alpha(z), \end{aligned}$$

similar to (29). Using also that

$$|\mathcal{R}_s(x, a_{mk})| \lesssim \frac{1}{[x, a_{mk}]^{s+n}} \sim \frac{1}{[x, b_m]^{s+n}},$$

which follows from Lemma 2.9 (a) and Lemma 2.5 with (26), we conclude that

$$|h_2(x)| \lesssim r \sum_{m=1}^{\infty} \frac{(1 - |b_m|^2)^{-(\alpha+n)/p}}{[x, b_m]^{s+n}} \left(\int_{E_{3/4}(b_m)} |f(z)|^p d\nu_\alpha(z) \right)^{\frac{1}{p}} \sum_{k=1}^{\kappa_m} \nu_s(E_{mk}).$$

This estimate is same as (30). Thus we again have $\|h_2\|_{L^p_\alpha}^p \lesssim r^p \|f\|_{\mathcal{B}^p_\alpha}^p$ and hence $\|(I - U\hat{T})f\|_{\mathcal{B}^p_\alpha}^p \leq \|h_1\|_{L^p_\alpha}^p + \|h_2\|_{L^p_\alpha}^p \lesssim r^p \|f\|_{\mathcal{B}^p_\alpha}^p$. We conclude that $\|(I - U\hat{T})\| \leq Cr$, where C depends only on n, p, α, s. □

Proposition 3.5 immediately implies Theorem 1.1.

Proof of Theorem 1.1 By Proposition 3.5, if r is small enough, then $\|I - U\hat{T}\| < 1$, and so $U\hat{T}$ has bounded inverse. Given $f \in \mathcal{B}^p_\alpha$, let $\lambda = \hat{T}(U\hat{T})^{-1} f$. Then $\lambda \in \ell^p$, $U\lambda = f$, and $\|\lambda\|_{\ell^p} \sim \|f\|_{\mathcal{B}^p_\alpha}$. We note that in the equivalence $\|\lambda\|_{\ell^p} \sim \|f\|_{\mathcal{B}^p_\alpha}$, the suppressed constants depend also on r. □

Remark 3.8 One can replace (4) in Theorem 1.1 with (5) because of Lemma 2.10. The only change needed in the above proof is to replace $U\lambda$ in (19) with

$$U\lambda(x) = \sum_{m=1}^{\infty} \lambda_m (1 - |a_m|^2)^{s+n-(\alpha+n)/p} \, \mathcal{R}_s(x, a_m),$$

and $\hat{T} f$ in (22) with

$$\hat{T} f = \{f(a_m)(1 - |a_m|^2)^{(\alpha+n)/p-(s+n)} \, \nu_s(E_m)\}.$$

Then $U\hat{T}$ remains the same and so does Proposition 3.5. In the proofs of Propositions 3.1 and 3.4 we omit the references to (20).

4 Interpolation

To prove Theorem 1.2 we again consider two operators. One is $\hat{U} : \ell^p \to \mathcal{B}^p_\alpha$, a slightly modified version of U given in (19) and the other is $T : \mathcal{B}^p_\alpha \to \ell^p$,

$$Tf = \left\{ f(a_m)(1 - |a_m|^2)^{(\alpha+n)/p} \right\},$$

given in (21). Our main purpose is to show that the composition $T\hat{U} : \ell^p \to \ell^p$ is invertible when the separation constant is large enough.

Proposition 4.1 *For $\alpha > -1$ and $0 < p < \infty$, choose s so that (3) holds. If $\{a_m\}$ is r-separated for some $0 < r < 1$, then the operator $\hat{U} : \ell^p \to \mathcal{B}^p_\alpha$ mapping $\lambda = \{\lambda_m\}$ to*

$$\hat{U}\lambda(x) = \sum_{m=1}^{\infty} \lambda_m (1 - |a_m|^2)^{-(\alpha+n)/p} \frac{\mathcal{R}_s(x, a_m)}{\mathcal{R}_s(a_m, a_m)} \qquad (x \in \mathbb{B}) \tag{34}$$

is bounded. The above series converges absolutely and uniformly on compact subsets of $\mathbb{B}$, and also in $\|\cdot\|_{\mathcal{B}^p_\alpha}$.

We have $\mathcal{R}_s(a_m, a_m)(1 - |a_m|^2)^{(\alpha+n)/p} \sim \|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha}$ by (17) and Lemma 2.10, and this proposition can be proved in the same way as Proposition 3.1. The minor changes required are omitted.

Proposition 4.2 *For $\alpha > -1$ and $0 < p < \infty$, choose s so that (3) holds. There exists $1/2 < r_0 < 1$ depending only on n, α, p, s such that if $\{a_m\}$ is r-separated with $r > r_0$, then $\|T\hat{U} - I\|_{\ell^p \to \ell^p} < 1$.*

This proposition immediately implies Theorem 1.2, similar to the proof of Theorem 1.1 above.

To verify Proposition 4.2, let $\lambda = \{\lambda_m\} \in \ell^p$. Then the m-th term of the sequence $(T\hat{U} - I)\lambda$ is given by

$$\{(T\hat{U} - I)\lambda\}_m = (1 - |a_m|^2)^{(\alpha+n)/p} \sum_{\substack{k=1 \\ k\neq m}}^{\infty} \lambda_k (1 - |a_k|^2)^{-(\alpha+n)/p} \frac{\mathcal{R}_s(a_m, a_k)}{\mathcal{R}_s(a_k, a_k)},$$

and by Lemma 2.9 (a) and (17), we have

$$\left|\{(T\hat{U} - I)\lambda\}_m\right| \leq C(1 - |a_m|^2)^{(\alpha+n)/p} \sum_{\substack{k=1 \\ k\neq m}}^{\infty} |\lambda_k| \frac{(1 - |a_k|^2)^{s+n-(\alpha+n)/p}}{[a_m, a_k]^{s+n}}, \quad (35)$$

where the constant C depends only on n, α, p and s.

To estimate the norm $\|(T\hat{U} - I)\lambda\|_{\ell^p}$, we need an estimate of the series on the right of (35) (without the $|\lambda_k|$ term) as given in Lemma 4.4 below. We first prove this lemma and complete the proof of Proposition 4.2 at the end of the section.

Observe that by Lemma 2.12, for $b > -1$ and $c > 0$, there exists $C > 0$ (depending only on n, b, c) such that

$$(1 - |a|^2)^c \int_{\mathbb{B}} \frac{(1 - |y|^2)^b}{[a, y]^{n+b+c}} \, d\nu(y) \leq C,$$

uniformly for all $a \in \mathbb{B}$. The next result will be needed in the proof of Lemma 4.4.

Lemma 4.3 *Let $b > -1$ and $c > 0$. For $\varepsilon > 0$, there exists $0 < r_\varepsilon < 1$ such that if $r_\varepsilon < r < 1$, then for all $a \in \mathbb{B}$,*

$$(1 - |a|^2)^c \int_{\mathbb{B}\setminus E_r(a)} \frac{(1 - |y|^2)^b}{[a, y]^{n+b+c}} \, d\nu(y) < \varepsilon.$$

Proof Let

$$F(a, r) := (1 - |a|^2)^c \int_{\mathbb{B}\setminus E_r(a)} \frac{(1 - |y|^2)^b d\nu(y)}{[a, y]^{n+b+c}},$$

and in the integral make the change of variable $y = \varphi_a(z)$. Since $\varphi_a(\mathbb{B}_r) = E_r(a)$ and $|J_{\varphi_a}|$ is as given in (11), we obtain

$$F(a, r) = (1 - |a|^2)^c \int_{\mathbb{B}\setminus\mathbb{B}_r} \frac{(1 - |\varphi_a(z)|^2)^{b+n} \, d\nu(z)}{[a, \varphi_a(z)]^{n+b+c} (1 - |z|^2)^n}.$$

Applying Lemma 2.1 and (10), and simplifying shows

$$F(a,r)=\int_{\mathbb{B}\setminus\mathbb{B}_r}\frac{(1-|z|^2)^b}{[a,z]^{n+b-c}}\,d\nu(z)=n\int_r^1 t^{n-1}(1-t^2)^b\int_{\mathbb{S}}\frac{d\sigma(\zeta)}{|ta-\zeta|^{n+b-c}}\,dt,$$

where in the second equality we integrate in polar coordinates and use the fact that $[a,t\zeta]=|ta-\zeta|$ by (7). By Lemma 2.12 and the inequality $1-|a|^2t^2\geq 1-t^2$,

$$\int_{\mathbb{S}}\frac{d\sigma(\zeta)}{|ta-\zeta|^{n+b-c}}\leq Cg(t):=\begin{cases}\dfrac{1}{(1-t^2)^{1+b-c}}, & \text{if } 1+b-c>0;\\ 1+\log\dfrac{1}{1-t^2}, & \text{if } 1+b-c=0;\\ 1, & \text{if } 1+b-c<0,\end{cases}$$

where the constant C depends only on n,b,c and do not depend on a. Thus

$$F(a,r)\leq Cn\int_r^1 t^{n-1}(1-t^2)^b\,g(t)\,dt.$$

In all the three cases the integral $\int_0^1 t^{n-1}(1-t^2)^b\,g(t)\,dt$ is finite because $b>-1$ and $c>0$ and hence, one can make $F(a,r)<\varepsilon$ by choosing r close to 1. □

The next lemma is an analogue of Lemma 3.1 of [11].

Lemma 4.4 *Let $b>n-1$ and $c>0$. For $1/2<r<1$, there exists $C(r)>0$ (depending also on n, b and c) such that for every r-separated sequence $\{a_m\}$ and for every $m=1,2,\dots,$*

$$(1-|a_m|^2)^c\sum_{\substack{k=1\\k\neq m}}^{\infty}\frac{(1-|a_k|^2)^b}{[a_m,a_k]^{b+c}}\leq C(r).$$

Moreover, one can choose $C(r)$ to be arbitrarily small by making r sufficiently close to 1.

Proof By the Lemmas 2.4, 2.5 and (13), there exists $C>0$ depending only on n,b,c such that

$$\frac{(1-|a|^2)^b}{[x,a]^{b+c}}\leq C\int_{E_{1/4}(a)}\frac{(1-|y|^2)^{b-n}}{[x,y]^{b+c}}\,d\nu(y),$$

for all $a,x\in\mathbb{B}$. If $\{a_m\}$ is r-separated with $r>1/2$, then the balls $E_{1/4}(a_m)$ are disjoint and therefore

$$(1-|a_m|^2)^c\sum_{\substack{k=1\\k\neq m}}^{\infty}\frac{(1-|a_k|^2)^b}{[a_m,a_k]^{b+c}}\leq C(1-|a_m|^2)^c\int_{\bigcup_{\substack{k=1\\k\neq m}}^{\infty}E_{1/4}(a_k)}\frac{(1-|y|^2)^{b-n}}{[a_m,y]^{b+c}}\,d\nu(y).$$

Set

$$R := \frac{r - 1/4}{1 - r/4}.$$

Clearly, $0 < R < 1$. We claim that $\bigcup_{\substack{k=1\\k\neq m}}^{\infty} E_{1/4}(a_k) \subset \mathbb{B}\backslash E_R(a_m)$. To see this, let $z \in E_{1/4}(a_k)$ with $k \neq m$. Then, by the strong triangle inequality in Lemma 2.2,

$$\rho(z, a_m) \geq \frac{\rho(a_m, a_k) - \rho(z, a_k)}{1 - \rho(a_m, a_k)\rho(z, a_k)} \geq \frac{r - \rho(z, a_k)}{1 - r\rho(z, a_k)} \geq \frac{r - 1/4}{1 - r/4},$$

where in the second and third inequalities we use $\rho(a_m, a_k) \geq r$ and $\rho(z, a_k) < 1/4$, and the elementary fact that for $0 \leq t_0 < 1$, the function $f(t) = (t - t_0)/(1 - tt_0)$ is increasing on the interval $0 \leq t < 1$ and $-f$ is decreasing. Thus

$$(1 - |a_m|^2)^c \sum_{\substack{k=1\\k\neq m}}^{\infty} \frac{(1 - |a_k|^2)^b}{[a_m, a_k]^{b+c}} \leq C(1 - |a_m|^2)^c \int_{\mathbb{B}\backslash E_R(a_m)} \frac{(1 - |y|^2)^{b-n}}{[a_m, y]^{b+c}}\, d\nu(y),$$

and since $R \to 1^-$ as $r \to 1^-$, the desired result follows from Lemma 4.3. □

We now complete the proof of Proposition 4.2. We consider the cases $0 < p \leq 1$ and $p > 1$ separately.

Proof of Proposition 4.2 when $0 < p \leq 1$ For $\lambda = \{\lambda_m\} \in \ell^p$, by (35), Lemma 2.13 (i) and Fubini's theorem,

$$\begin{aligned}
\|(T\hat{U} - I)\lambda\|_{\ell^p}^p &\leq C^p \sum_{m=1}^{\infty} (1 - |a_m|^2)^{\alpha+n} \left(\sum_{\substack{k=1\\k\neq m}}^{\infty} |\lambda_k| \frac{(1 - |a_k|^2)^{s+n-(\alpha+n)/p}}{[a_m, a_k]^{s+n}} \right)^p \\
&\leq C^p \sum_{m=1}^{\infty} (1 - |a_m|^2)^{\alpha+n} \sum_{\substack{k=1\\k\neq m}}^{\infty} |\lambda_k|^p \frac{(1 - |a_k|^2)^{p(s+n)-(\alpha+n)}}{[a_m, a_k]^{p(s+n)}} \\
&= C^p \sum_{k=1}^{\infty} |\lambda_k|^p (1 - |a_k|^2)^{p(s+n)-(\alpha+n)} \sum_{\substack{m=1\\m\neq k}}^{\infty} \frac{(1 - |a_m|^2)^{\alpha+n}}{[a_m, a_k]^{p(s+n)}}.
\end{aligned}$$

By Lemma 4.4, there exists $C(r)$ such that (note that $\alpha + n > n - 1$ since $\alpha > -1$, and $p(s + n) - (\alpha + n) > 0$ by (3))

$$\|(T\hat{U} - I)\lambda\|_{\ell^p}^p \leq C^p\, C(r) \|\lambda\|_{\ell^p}^p.$$

Since $C(r)$ can be made arbitrarily small by making r close enough to 1, the proposition follows. □

We next deal with the case $1 < p < \infty$. Let p' be the conjugate exponent of p. We employ Schur's test which, for the sequence space ℓ^p, has the following form (see [11, Lemma 3.2]): Let $A = (A_{mk})_{1 \le m,k < \infty}$ be an infinite matrix with nonnegative entries and $L_A : \ell^p \to \ell^p$ be the corresponding operator taking $\lambda = \{\lambda_m\}$ to

$$\{L_A \lambda\}_m = \sum_{k=1}^{\infty} A_{mk} \lambda_k, \qquad m = 1, 2, \dots .$$

If there exists a constant $C > 0$ and a positive sequence $\{\gamma_m\}$ such that

$$\sum_{k=1}^{\infty} A_{mk} \gamma_k^{p'} \le C \gamma_m^{p'}, \qquad m = 1, 2, \dots ,$$

and

$$\sum_{m=1}^{\infty} A_{mk} \gamma_m^{p} \le C \gamma_k^{p}, \qquad k = 1, 2, \dots ,$$

then the operator $L_A : \ell^p \to \ell^p$ is bounded and $\|L_A\| \le C$.

Proof of Proposition 4.2 when $1 < p < \infty$ Without loss of generality we can assume that the r-separated sequence $\{a_m\}$ is maximal, that is $\{a_m\}$ is an r-lattice and so is an infinite sequence.

For $m, k = 1, 2, \dots$, let $A_{mk} = 0$ if $k = m$; and if $k \ne m$, let

$$A_{mk} = (1 - |a_m|^2)^{(\alpha+n)/p} \frac{(1 - |a_k|^2)^{s+n-(\alpha+n)/p}}{[a_m, a_k]^{s+n}}.$$

Let $A = (A_{mk})$ and $L_A : \ell^p \to \ell^p$ be the corresponding operator. Then by (35),

$$\left|\{(T\hat{U} - I)\lambda\}_m\right| \le C \{L_A \lambda\}_m .$$

To estimate $\|L_A\|$ with the Schur's test, we take $\{\gamma_m\} = \{(1 - |a_m|^2)^{(n-1)/pp'}\}$. Then

$$\sum_{k=1}^{\infty} A_{mk} \gamma_k^{p'} = (1 - |a_m|^2)^{(\alpha+n)/p} \sum_{\substack{k=1 \\ k \ne m}}^{\infty} \frac{(1 - |a_k|^2)^{s+n-(\alpha+1)/p}}{[a_m, a_k]^{s+n}},$$

and by Lemma 4.4, there exists $C_1(r)$ such that (we check that $s+n-(\alpha+1)/p > n-1$ by (3), and $(\alpha+1)/p > 0$)

$$\sum_{k=1}^{\infty} A_{mk} \gamma_k^{p'} \le (1 - |a_m|^2)^{(\alpha+n)/p} \frac{C_1(r)}{(1 - |a_m|^2)^{(\alpha+1)/p}} = C_1(r) \gamma_m^{p'} .$$

Observe next that

$$\sum_{m=1}^{\infty} A_{mk}\gamma_m^p = (1-|a_k|^2)^{s+n-(\alpha+n)/p} \sum_{\substack{m=1\\ m\neq k}}^{\infty} \frac{(1-|a_m|^2)^{(\alpha+n)/p+(n-1)/p'}}{[a_m, a_k]^{s+n}}.$$

To apply Lemma 4.4 we check that $(\alpha+n)/p+(n-1)/p' = (\alpha+1)/p+n-1 > n-1$, and $s+n-\big((\alpha+n)/p+(n-1)/p'\big) = s+1-(\alpha+1)/p > 0$ by (3). Thus there exists $C_2(r)$ such that

$$\sum_{m=1}^{\infty} A_{mk}\gamma_m^p \leq (1-|a_k|^2)^{s+n-(\alpha+n)/p} \frac{C_2(r)}{(1-|a_k|^2)^{s+n-(\alpha+n)/p-(n-1)/p'}} = C_2(r)\gamma_k^p.$$

We conclude that L_A is bounded and $\|L_A\| \leq \max\{C_1(r), C_2(r)\}$. Therefore $\|T\hat{U}-I\| \leq C\max\{C_1(r), C_2(r)\}$ and since both $C_1(r)$ and $C_2(r)$ can be made arbitrarily small by making r close enough to 1, we conclude that $\|T\hat{U}-I\|$ can be made small. This finishes the proof of Proposition 4.2. □

5 Inclusion Relations

In this section we prove Theorem 1.3.

Proof of Theorem 1.3 We first prove part (a). Suppose $\mathcal{B}_\alpha^p \subset \mathcal{B}_\beta^q$. Since point evaluations are bounded on $\mathcal{H}$-harmonic Bergman spaces, the inclusion $i\colon \mathcal{B}_\alpha^p \to \mathcal{B}_\beta^q$ is continuous by the closed graph theorem. For every $s > -1$ and $a \in \mathbb{B}$, the reproducing kernel $\mathcal{R}_s(a,\cdot)$ is bounded on $\mathbb{B}$ by Lemma 2.9 (a) and (8), so belongs to every Bergman space. By Lemma 2.10, for large enough s, we have

$$\frac{\|\mathcal{R}_s(a,\cdot)\|_{\mathcal{B}_\beta^q}}{\|\mathcal{R}_s(a,\cdot)\|_{\mathcal{B}_\alpha^p}} \sim (1-|a|^2)^{(\beta+n)/q-(\alpha+n)/p}, \tag{36}$$

and the right-hand side is bounded as $|a| \to 1^-$ only if $(\beta+n)/q \geq (\alpha+n)/p$.

Suppose now that

$$\frac{\alpha+n}{p} \leq \frac{\beta+n}{q}. \tag{37}$$

Pick s large enough so that (3) holds both for α, p and β, q. Let r_0 be as asserted in the atomic decomposition theorem for $\mathcal{B}_\alpha^p$ and let $\{a_m\}$ be an r-lattice with $r < r_0$. Then for every $f \in \mathcal{B}_\alpha^p$, there exists $\{\lambda_m\} \in \ell^p$ with $\|\{\lambda_m\}\|_{\ell^p} \sim \|f\|_{\mathcal{B}_\alpha^p}$ such that

$$f(x) = \sum_{m=1}^{\infty} \lambda_m \frac{\mathcal{R}_s(x,a_m)}{\|\mathcal{R}_s(\cdot,a_m)\|_{\mathcal{B}_\alpha^p}} = \sum_{m=1}^{\infty} \kappa_m \frac{\mathcal{R}_s(x,a_m)}{\|\mathcal{R}_s(\cdot,a_m)\|_{\mathcal{B}_\beta^q}},$$

where

$$\kappa_m = \lambda_m \frac{\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^q_\beta}}{\|\mathcal{R}_s(\cdot, a_m)\|_{\mathcal{B}^p_\alpha}}.$$

By (36) and (37), $|\kappa_m| \sim |\lambda_m|(1-|a_m|^2)^{(\beta+n)/q-(\alpha+n)/p} \le |\lambda_m|$, and so the sequence $\{\kappa_m\}$ is in ℓ^p. Thus, $\{\kappa_m\} \in \ell^q$ by Lemma 2.13 (i), and it follows from Proposition 3.1 that $f \in \mathcal{B}^q_\beta$ with $\|f\|_{\mathcal{B}^q_\beta} \lesssim \|\{\kappa_m\}\|_{\ell^q} \le \|\{\kappa_m\}\|_{\ell^p} \lesssim \|\{\lambda_m\}\|_{\ell^p} \lesssim \|f\|_{\mathcal{B}^p_\alpha}$.

We next prove part (b). Note first that in this case $p/q > 1$ and the conjugate exponent of p/q is $p/(p-q)$. To see the if part, suppose

$$\frac{\alpha+1}{p} < \frac{\beta+1}{q}. \tag{38}$$

By Hölder's inequality,

$$\int_{\mathbb{B}} |f(x)|^q \, d\nu_\beta(x) \le \left(\int_{\mathbb{B}} |f(x)|^p \, d\nu_\alpha(x)\right)^{\frac{q}{p}} \left(\int_{\mathbb{B}} (1-|x|^2)^{(\beta-\alpha\frac{q}{p})\frac{p}{p-q}} \, d\nu(x)\right)^{\frac{p-q}{p}},$$

and since the exponent $(\beta - \alpha\frac{q}{p})\frac{p}{p-q} > -1$ by (38), we obtain $\|f\|_{\mathcal{B}^q_\beta} \lesssim \|f\|_{\mathcal{B}^p_\alpha}$.

Suppose now that $\mathcal{B}^p_\alpha \subset \mathcal{B}^q_\beta$. Let r_0 be as asserted in the interpolation theorem for $\mathcal{B}^p_\alpha$ and let $\{a_m\}$ be an r-lattice with $r > r_0$. Given $\{\lambda_m\} \in \ell^{p/q}$, we have $\{|\lambda_m|^{1/q}\} \in \ell^p$ and there exists a function $f \in \mathcal{B}^p_\alpha$ such that

$$f(a_m) = |\lambda_m|^{1/q}(1-|a_m|^2)^{-(\alpha+n)/p}.$$

Since f is also in $\mathcal{B}^q_\beta$, the sequence $\{f(a_m)(1-|a_m|^2)^{(\beta+n)/q}\}$ is in ℓ^q by Proposition 3.3, and so

$$\sum_{m=1}^{\infty} |\lambda_m|(1-|a_m|^2)^{(\beta+n)-(\alpha+n)q/p} < \infty.$$

By Lemma 2.13 (ii), this implies that the sequence $\{(1-|a_m|^2)^{(\beta+n)-(\alpha+n)q/p}\}$ is in $\ell^{p/(p-q)}$ and by Lemma 2.8 (b) this holds only if

$$\left((\beta+n) - (\alpha+n)\frac{q}{p}\right)\frac{p}{p-q} > n-1,$$

which is equivalent to (38). □

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