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Generalized Convolution Operator Associated with the (k, a)-Generalized Fourier Transform on the Real Line and Applications

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Abstract

Recently in Amri (Product formula for one-dimensional (k, a)-generalized Fourier kernel. arXiv:2301.06587), the author proved the product formula for the one dimensional (k, a)-generalized Fourier kernel. Profiting for this result, the primary aim of the present paper, is to develop the harmonic analysis associated with (k, a) the generalized Fourier transform. Firstly we study the (k, a)-generalized translation operator associated with the (k, a)-generalized Fourier transform. By means of the generalized translation operator, we define and we investigate the generalized convolution product in the setting of the (k, a)-generalized Fourier transform. Nevertheless, significant attention is also devoted to the time-frequency analysis by examining some applications on the wavelet transform in the (k, a)-generalized Fourier transform setting.

Keywords (k, a)-Generalized Fourier transform · (k, a)-Generalized translation · Generalized convolution · (k, a)-Generalized wavelet transform

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1 Introduction

Harmonic analysis in $\mathbb{R}^d$ is governed by the following three operators

$$\triangle := \sum_{j=1}^{d} \partial_{x_j}^2, \quad ||x||^2 := \sum_{j=1}^{d} x_j^2, \quad \mathbb{E} := \sum_{j=1}^{d} x_j \partial_{x_j},$$

where $\triangle$ is the Laplace operator and $\mathbb{E}$ is the Euler operator. As observed in [25], the operators

$$E = \frac{||x||^2}{2}, \quad F = \frac{-\triangle}{2}, \quad \text{and} \quad H = E + \frac{d}{2}$$

are invariant under $O(d)$ and generate the Lie algebra $\mathfrak{sl}_2$:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H.$$

Recently, there has been a lot of interest in other differential or difference operator realizations of $\mathfrak{sl}_2$ or other Lie (super) algebras. The focus is in particular on the generalized Fourier transforms that subsequently arise. We mention the Dunkl transform [17], various discrete Fourier transforms [26], Fourier transforms in Clifford analysis [15], etc. For a more detailed review, we refer the reader to [16].

A hard problem in this context is to find explicit closed formulas for the integral kernel of the associated Fourier transforms.

The classical Fourier transform in $\mathbb{R}^d$ can be defined in many ways. In its most basic formulation, it is given by the integral transform

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\langle \lambda, x\rangle} f(x) dx.$$

Alternatively, one can rewrite the transform as

$$\mathcal{F}(f)(\lambda) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \mathcal{K}(\lambda, x) f(x) dx,$$

where $\mathcal{K}(\lambda, x)$ is the unique solution to the system of partial differential equations

$$\begin{cases} \partial_{x_j} \mathcal{K}(\lambda, x) = -i\lambda_j \mathcal{K}(\lambda, x), \quad j = 1, ..., d, \\ \mathcal{K}(\lambda, 0) = 1, \quad \lambda \in \mathbb{R}^d. \end{cases}$$

A third description was discovered by Howe [25],

$$\mathcal{F} = \exp\left(\frac{i\pi d}{4}\right) \exp\left(\frac{i\pi}{4}(\triangle - ||x||^2)\right).$$

Both of the previous representations have their uses, and it is explained in the overview paper [16] how to construct various extensions such as a fractional Fourier transform and Clifford algebra-valued analogues.

Recently, Ben Said and all in [4], have given a foundation of the deformation theory of the classical situation, by constructing a generalization $\mathcal{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathcal{I}_{k,a}(z)$ with infinitesimal generator

$$\mathcal{L}_{k,a} := ||x||^{2-a}\Delta_k - ||x||^a, \quad a > 0,$$

acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^d)$. Here Δ_k is the Dunkl Laplace operator (see [17]). The authors have analyzed these operators $\mathcal{F}_{k,a}$ and $\mathcal{I}_{k,a}(z)$ in the context of integral operators as well as representation theory. The deformation parameters consist of a real parameter a coming from the interpolation of the minimal unitary representations of two different reductive groups by keeping smaller symmetries (see DIAGRAM 1), and a parameter k coming from Dunkl's theory of differential-difference operators associated with a finite Coxeter group; also the dimension d and the complex variable z may be considered as a parameter of the theory. (See [4]).

A lot of attention has been given to various generalizations of the Fourier transform. This paper focuses on the (k, a)-generalized Fourier transform associated with the operator $\mathcal{L}_{k,a}$.

As of now, the (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ has witnessed an ample amount of research in the realm of harmonic analysis, which include study of the kernel of the (k, a)-generalized Fourier transform [1, 12, 24], the generalized translation operator [7, 10, 39], the generalized maximal function [6, 7, 9], the Flett potentials [8], Pitt's inequalities [23], uncertainty principles [23, 27], the (k, a)-Fourier multipliers [29], the (k, a)-generalized wavelet multipliers [33], the (k, a)-generalized wavelet transform [34, 38], the (k, a)-generalized Gabor transform [36, 37], the (k, a)-generalized Stockwell transform [39], the (k, a)-generalized Wigner transform [44], Hardy inequalities for fractional (k, a)-generalized harmonic oscillator [47] and many more.

Yet there are still several gaps in our knowledge of the harmonic analysis associated with the (k, a)-generalized Fourier transform. One of the main reasons is the lack of tools related to the generalized translation operator. Unfortunately, the L^p-boundedness and the positivity of this generalised translation operator are not obtained in general. At the moment an explicit formula for the generalised translation operator is known only in the following cases:

- $a = 2,\ d \geq 1,\ \ k \equiv 0$. The generalized translation operator is the Euclidean translation operator. (See [45]).
- $a = 2,\ d = 1,\ \ k > 0$. Then we recover the Dunkl translation (see [42]).
- $a = 1,\ d \geq 1,\ \ k \equiv 0$. Then the generalized translation operator is the multivariable Bessel translation. (See [11, 32]).
- $a = 1,\ d = 1,\ \ k > 0$. The generalized translation operator is the k-Hankel translation studied in [5].
- $a = \frac{2}{n},\ d = 1,\ \ k > 0$. The generalized translation operator is the deformed Hankel translation studied in [10, 39].

We note that when $d > 1$, the generalized translation operators are defined for the suitable radial functions only in the spacial cases $a = 1$ and $a = 2$ (see [7, 46]).

This paper is a continuation of the papers [34, 39] on the study of the generalized translation operators and its applications. Indeed, we note that in [34], we have defined the generalized translation operators on the Lebesgue space $L^2_{k,a}(\mathbb{R}^d)$, next we defined and studied the generalized wavelet transform in the setting of the (k, a)-generalized Fourier transform, and we gave many applications on this transformation. Also, in [39], we have studied the positivity of the generalized translation operators in the special case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$, next we defined and study the generalized Stockwell transform in the setting of the (k, a)-generalized Fourier transform and we give many applications on this transformation.

In this paper, we consider the case $a > 0$ and $d = 1$. The purpose of this document is twofold. On one hand and profiting from the product formula proved in [1], we want to develop the harmonic analysis associated with the (k, a)-generalized Fourier transform. In particular, we introduce and we study the generalized translation operator on the (k, a)-generalized Fourier transform setting. Next, we introduce the generalized convolution operator and we prove its fundamental properties. The inversion theorem for the (k, a)-generalized Fourier transform is also proven. Profiting of the harmonic analysis associated with the (k, a)-generalized Fourier transform, the aim of the second part of this paper is to consider the generalized wavelet transform in the setting of the (k, a)-generalized Fourier transform, study its harmonic analysis and to give many applications for this transformation. The applications on the wavelet transforms have been studied by many authors for various Fourier transforms, for examples (cf. [3, 13, 14, 20–22, 43, 48, 49]) and others.

The main contributions of this article are as follows:

- To study the generalized translation operator on the Lebesgue spaces $L^p_{k,a}(\mathbb{R})$.
- To define and to study the generalized convolution operator on the Lebesgue spaces $L^p_{k,a}(\mathbb{R})$.
- To prove the inversion and Plancherel's formulas for the (k, a)-generalized Fourier transform.
- To study the harmonic analysis associated with the (k, a)-generalized wavelet transform.
- To introduce and to study the (k, a)-generalized Hardy operator.

The remainder of this paper is arranged as follows.

In Sect. 2, we recall the main results about the (k, a)-generalized Fourier transform. Section 3 is exclusively dedicated to study the generalized convolution operator. In Sect. 4, we give many applications for the generalized wavelet transform. Firstly, we prove the inversion, Plancherel's and Lieb's formulas. Next we develop the concept of the generalized wavelet transform in L^p-space and we derive the Parseval's and the inversion formulas. We define the composition of the (k, a)-generalized wavelet transforms and we obtain its Parseval's identity. Further, we discuss the generalized convolution operator and (k, a)-generalized wavelet transform as time-invariant filters. Finally, the (k, a)-generalized Hardy operator is investigated.

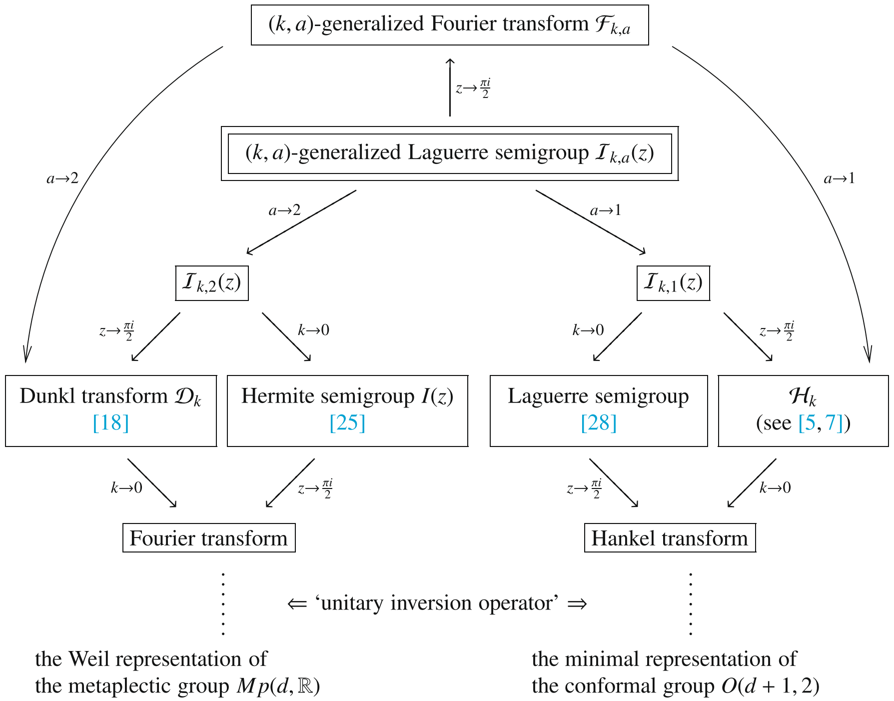

Fig. 1 Special values of holomorphic semigroup $\mathcal{I}_{k,a}(z)$

2 Preliminaries

We shall take a survey of the (k, a)-generalized Fourier transform together with the fundamental properties. Main references are [4, 24]. To facilitate the narrative, we set some notations as under:

- For $p \in [1, \infty]$, p' denotes as in all that follows, the conjugate exponent of p.
- $M_{k,a} := \dfrac{1}{2a^{\frac{2k-1}{a}}\Gamma(\frac{2k+a-1}{a})}$,
- $d\gamma_{k,a}(x) := M_{k,a}|x|^{2k+a-2}dx$, $k \geq \frac{2-a}{2}$.
- $L^p_{k,a}(\mathbb{R})$, $1 \leq p \leq \infty$, denotes the space of measurable functions f on $\mathbb{R}$ satisfying

$$||f||_{L^p_{k,a}(\mathbb{R})} := \left(\int_{\mathbb{R}} |f(x)|^p d\gamma_{k,a}(x)\right)^{1/p} < \infty, \quad \text{if} \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty_{k,a}(\mathbb{R})} := \text{ess} \sup_{x\in\mathbb{R}} |f(x)| < \infty.$$

In case $p = 2$, the inner product on the space $L^p_{k,a}(\mathbb{R})$ is given by

$$\langle f, g\rangle_{L^2_{k,a}(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}\,d\gamma_{k,a}(x).$$

- $C_{\mathbf{b}}(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$.
- $C_{\mathbf{b},e}(\mathbb{R})$ the space of even bounded continuous functions on $\mathbb{R}$.
- $C^p(\mathbb{R})$ the space of functions of class C^p on $\mathbb{R}$.
- $\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
- $\mathcal{S}_e(\mathbb{R})$ the Schwartz space of even rapidly decreasing functions on $\mathbb{R}$.
- Let E be a measurable subset of $\mathbb{R}$, χ_E denotes the characteristic function of the set E.

The Dunkl operator T_k, on $\mathbb{R}$ is given for f in $C^1(\mathbb{R})$ by

$$T_k f(x) := f'(x) + 2k\frac{f(x) - f(-x)}{x}. \tag{2.1}$$

We define the Dunkl Laplace operator $\triangle_k$ on $\mathbb{R}$ for f in $C^2(\mathbb{R})$, by

$$\triangle_k f(x) := T_k^2 f(x) = f''(x) + 2k\Big(\frac{f'(x)}{x} - \frac{f(x) - f(-x)}{x^2}\Big).$$

Consider the operator

$$\Delta_{k,a} := |x|^{2-a}\triangle_k - |x|^a. \tag{2.2}$$

In the following we recall some spectral properties of the operator $\Delta_{k,a}$.

Proposition 2.1 *Let a and k be as above.*

(1) The differential-difference operator $\Delta_{k,a}$ is an essentially self-adjoint operator on $L^2_{k,a}(\mathbb{R})$.

(2) There is no continuous spectrum of $\Delta_{k,a}$.

(3) The discrete spectrum of $-\Delta_{k,a}$ is given by

$$\{2ma + 2k + a \pm 1 : m \in \mathbb{N}\}.$$

For $k \geq \max(\frac{2-a}{2}, 0)$, the (k, a)-generalized kernel $B_{k,a}(\lambda, x)$ is given by

$$\begin{aligned} B_{k,a}(x, y) &= j_{\frac{2k-1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}) \\ &+ \frac{\Gamma(\frac{2k+a-1}{a})}{\Gamma(\frac{2k+a+1}{a})}\frac{xy}{(i\,a)^{\frac{2}{a}}} j_{\frac{2k+1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}). \end{aligned} \tag{2.3}$$

Here j_ν is the normalized Bessel function given by

$$j_\nu(t) = \Gamma(\nu + 1)\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^{2n} n!\Gamma(n + \nu + 1)}. \tag{2.4}$$

Let $n \in \mathbb{Z}$, we denote by $X_{n,a}$ the set defined by

$$X_{n,a} := e^{\frac{in\pi}{a}}\mathbb{R}.$$

It is clear that $X_{0,a}=\mathbb{R}$, $X_{n,1}=\mathbb{R}$ and $X_{n,2}=i\mathbb{R}$.

We extend the definition of the (k,a)-generalized kernel $B_{k,a}$ on $X_{n,a}$ as follow:

$$\forall x,y\in\mathbb{R},\ B_{k,a}(e^{\frac{in\pi}{a}}x,y)=j_{\frac{2k-1}{a}}\left(e^{\frac{in\pi}{2}}\frac{2}{a}|xy|^{\frac{a}{2}}\right)$$
$$+\frac{\Gamma(\frac{2k+a-1}{a})}{\Gamma(\frac{2k+a+1}{a})}\frac{e^{\frac{in\pi}{a}}xy}{(i\,a)^{\frac{2}{a}}}j_{\frac{2k+1}{a}}\left(e^{\frac{in\pi}{2}}\frac{2}{a}|xy|^{\frac{a}{2}}\right). \tag{2.5}$$

In the following result, we present some important properties of the (k,a)-generalized kernel $B_{k,a}$.

Proposition 2.2 [4, 24] (i) *For $x,y\in\mathbb{R}$, we have*

$$B_{k,a}(x,y)=B_{k,a}(y,x),\quad B_{k,a}(x,0)=1,$$

and $B_{k,a}(\lambda x,y)=B_{k,a}(x,\lambda y)$ for all $\lambda\in\mathbb{R}$.

ii*) If $a=\frac{2}{n}$, $n\in\mathbb{N}$ and $k\geq\frac{1}{2}$, then for all $x,y\in\mathbb{R}$, we have*

$$|B_{k,a}(x,y)|\leq 1. \tag{2.6}$$

iii*) The conditions*

$$0<a\leq 2,\quad k\geq\frac{1}{2}-\frac{a}{4},\qquad \text{or}\quad a\geq 2,\quad k\geq 0, \tag{2.7}$$

are necessary and sufficient for boundedness of the kernel $B_{k,a}(x,y)$.

iv*) If k and a satisfying (2.7), there exists a finite positive constant C only depends on a and k, such that*

$$\forall x,y\in\mathbb{R},\quad |B_{k,a}(x,y)|\leq C. \tag{2.8}$$

v*) The distribution $B_{k,a}(.,.)$ solves the following differential-difference equations on $\mathbb{R}\times\mathbb{R}$*

$$\begin{cases}|\lambda|^{2-a}\,\triangle_k^{\lambda}B_{k,a}(\lambda,x)=-|x|^a\,B_{k,a}(\lambda,x),\\ |x|^{2-a}\,\triangle_k^{x}B_{k,a}(\lambda,x)=-|\lambda|^a\,B_{k,a}(\lambda,x).\end{cases} \tag{2.9}$$

Here, the superscript in $\triangle_k^x$, etc indicates the relevant variable.

Remark 2.1 (i) Gorbachev and all in [24], proved that in either of the following cases

- $0<a\leq 1$, and $k=\frac{1}{2}-\frac{a}{4}$, or
- $a\in(1,2)\cup(2,\infty)$, and $k\geq 0$,

we have

$$||B_{k,a}||_\infty:=\sup_{x,y\in\mathbb{R}}|B_{k,a}(x,y)|>1.$$

(ii) For $x, y \in \mathbb{R}$, we have

$$\begin{aligned}\overline{B_{k,a}(x,y)} &= j_{\frac{2k-1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}) + \frac{\Gamma(\frac{2k+a-1}{a})}{\Gamma(\frac{2k+a+1}{a})} e^{\frac{i\pi}{a}} \frac{xy}{(a)^{\frac{2}{a}}} j_{\frac{2k+1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}) \\ &= j_{\frac{2k-1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}) + \frac{\Gamma(\frac{2k+a-1}{a})}{\Gamma(\frac{2k+a+1}{a})} \frac{e^{\frac{2i\pi}{a}} xy}{(ia)^{\frac{2}{a}}} j_{\frac{2k+1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}).\end{aligned}$$

Thus involving (2.5), we derive that

$$\overline{B_{k,a}(x,y)} = B_{k,a}(e^{\frac{2i\pi}{a}}x, y). \tag{2.10}$$

(iii) For $x, y \in \mathbb{R}$, we have

$$\overline{B_{k,a}(x,y)} = \begin{cases} B_{k,a}(\xi, x), & \text{if } a = \frac{1}{r},\ r \in \mathbb{N}, \\ B_{k,a}(-\xi, x), & \text{if } a = \frac{2}{2r+1},\ r \in \mathbb{N}_0. \end{cases} \tag{2.11}$$

Convention: When k and a satisfy (2.7), we shall replace $d\gamma_{k,a}$ by the rescaled version $d\gamma_{k,a}/C$ but continue to use the same symbol $d\gamma_{k,a}$.

For $k \geq \max(\frac{2-a}{2}, 0)$, and $f \in L^1_{k,a}(\mathbb{R})$, the (k,a)-generalized Fourier transform is defined by

$$\mathcal{F}_{k,a}(f)(\lambda) = \int_{\mathbb{R}} f(x) B_{k,a}(\lambda, x) d\gamma_{k,a}(x), \quad \text{for all } \lambda \in \mathbb{R}. \tag{2.12}$$

Remark 2.2 (i) We note that the previous Proposition implies that the (k,a)-generalized Fourier transform is bounded on the space $L^1_{k,a}(\mathbb{R})$, and we have

$$||\mathcal{F}_{k,a}(f)||_{L^\infty_{k,a}(\mathbb{R})} \leq ||f||_{L^1_{k,a}(\mathbb{R})}, \tag{2.13}$$

for all f in $L^1_{k,a}(\mathbb{R})$.

(ii) The (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ provides a natural generalization of the Hankel transform. Indeed, if we set

$$\begin{aligned}B^{even}_{k,a}(x,y) &= \frac{1}{2}(B_{k,a}(x,y) + B_{k,a}(x,-y)) \\ &= j_{\frac{2k-1}{a}}(\frac{2}{a}|xy|^{\frac{a}{2}}).\end{aligned}$$

Then, the transform $\mathcal{F}_{k,a}$ of an even function f on the real line specializes to a Hankel type transform on $\mathbb{R}_+$.

The authors in [4] have proved the following.

Proposition 2.3 (i) *(Plancherel's theorem for $\mathcal{F}_{k,a}$). The (k,a)-generalized Fourier transform $f \mapsto \mathcal{F}_{k,a}(f)$ is an isometric isomorphism on $L^2_{k,a}(\mathbb{R})$ and we have*

$$\int_{\mathbb{R}} |f(x)|^2 d\gamma_{k,a}(x) = \int_{\mathbb{R}} |\mathcal{F}_{k,a}(f)(\lambda)|^2 d\gamma_{k,a}(\lambda). \tag{2.14}$$

(ii) *(Parseval's formula for $\mathcal{F}_{k,a}$). For all f, g in $L^2_{k,a}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} d\gamma_{k,a}(x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\overline{\mathcal{F}_{k,a}(g)(\lambda)} d\gamma_{k,a}(\lambda). \tag{2.15}$$

(iii) *(Inversion formula). The (k,a)-generalized Fourier transform $\mathcal{F}_{k,a}$ is of finite order if and only if $a \in \mathbb{Q}$. If $a \in \mathbb{Q}$ is of the form $a = \frac{s}{t}$, with s, t positive, then $\mathcal{F}^{2s}_{k,a} = Id$. In particular*

$$\mathcal{F}^{-1}_{k,a} = \mathcal{F}^{2s-1}_{k,a}. \tag{2.16}$$

$L^1_{k,a}(\mathbb{R})$, we

Proposition 2.4 *Let f be in $L^p_{k,a}(\mathbb{R})$, $p \in [1, 2]$. Then $\mathcal{F}_{k,a}(f)$ belongs to $L^{p'}_{k,a}(\mathbb{R})$ and we have*

$$\left\| \mathcal{F}_{k,a}(f) \right\|_{L^{p'}_{k,a}(\mathbb{R})} \leq \| f \|_{L^p_{k,a}(\mathbb{R})}. \tag{2.17}$$

3 Generalized Convolution Operator

On the follow we recall the definition and the properties of the (k,a)-generalized translation operator.

Definition 3.1 ([34]) Let $x \in \mathbb{R}$. The (k,a)-generalized translation operator $f \mapsto \tau^{k,a}_x f$ is defined on $L^2_{k,a}(\mathbb{R})$ by

$$\mathcal{F}_{k,a}(\tau^{k,a}_x f)(\xi) = \overline{B_{k,a}(\xi, x)} \mathcal{F}_{k,a}(f)(\xi). \tag{3.1}$$

It is useful to have a class of functions in which (3.1) holds pointwise. One such class is given by the generalized Wigner space $\mathcal{W}_{k,a}(\mathbb{R})$ given by

$$\mathcal{W}_{k,a}(\mathbb{R}) := \left\{ f \in L^1_{k,a}(\mathbb{R}) : \mathcal{F}_{k,a}(f) \in L^1_{k,a}(\mathbb{R}) \right\}.$$

On the follow we give several properties of the generalized translation operator.

Proposition 3.1 [34]

(i) Let f be in $L^2_{k,a}(\mathbb{R})$, we have

$$\|\tau^{k,a}_x f\|_{L^2_{k,a}(\mathbb{R})} \leq \|f\|_{L^2_{k,a}(\mathbb{R})}, \quad \forall x \in \mathbb{R}. \tag{3.2}$$

(ii) For all f in $\mathcal{W}_{k,a}(\mathbb{R})$ or for all f in $L^2_{k,a}(\mathbb{R})$ such that $\mathcal{F}_{k,a}(f)$ belongs to $L^1_{k,a}(\mathbb{R})$ and $x \in \mathbb{R}$, we have for almost every $y \in \mathbb{R}$

$$\tau^{k,a}_x f(y) = \int_{\mathbb{R}} \overline{B_{k,a}(\xi, x) B_{k,a}(\xi, y)} \mathcal{F}_{k,a}(f)(\xi) d\gamma_{k,a}(\xi). \tag{3.3}$$

(iii) For all f in $\mathcal{W}_{k,a}(\mathbb{R})$ and for all $x, y \in \mathbb{R}$, we have

$$\tau^{k,a}_x f(y) = \tau^{k,a}_y f(x). \tag{3.4}$$

On the follow we give the (k, a)-generalized translation of the generalized Gaussian (the generalized heat kernel associated with $\Delta_{k,a}$):

Proposition 3.2 *For every $\delta > 0$ and for every $x \in \mathbb{R}$, we have*

$$\tau^{k,a}_x [e^{-\frac{|s|^a}{\delta a^2}}](y) = e^{-\frac{|x|^a + |y|^a}{\delta a^2}} B_{k,a}\left(\frac{x}{(\delta a)^{\frac{2}{a}}}, e^{\frac{i\pi}{a}} y\right).$$

Before proving this proposition we need the following lemma:

Lemma 3.1 *Let $\delta > 0$ and $x, y \in \mathbb{R}$. Then*

$$\int_{\mathbb{R}} e^{-\delta|\xi|^a} B_{k,a}(x, \xi) B_{k,a}(y, \xi) d\gamma_{k,a}(\xi)$$
$$= \frac{e^{-(1/\delta a^2)(|x|^a + |y|^a)}}{(\delta a)^{\frac{2k+a-1}{a}}} B_{k,a}(x/(\delta a)^{\frac{2}{a}}, e^{\frac{-i\pi}{a}} y). \tag{3.5}$$

Proof Involving the following formula

$$\int_0^\infty e^{-\delta\xi^2} j_\nu(x\xi) j_\nu(y\xi) d\gamma_{k,a}(\xi) = \frac{\Gamma(\nu+1)}{2\delta^{\nu+1}} e^{-(1/4\delta)(|x|^2+|y|^2)} j_\nu(ixy/2\delta) \tag{3.6}$$

and the fact that

$$B_{k,a}(x, y) = j_{\frac{2k-1}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right) + \frac{\Gamma(\frac{2k+a-1}{a})}{\Gamma(\frac{2k+a+1}{a})} \frac{xy}{(i a)^{\frac{2}{a}}} j_{\frac{2k+1}{a}}\left(\frac{2}{a}|xy|^{\frac{a}{2}}\right),$$

we derive that the left term of the equation (3.5) takes when $\delta > 0$ the following form:

$$\begin{aligned}&\int_{\mathbb{R}} e^{-\delta|\xi|^a} B_{k,a}(x,\xi)B_{k,a}(y,\xi)d\gamma_{k,a}(\xi)\\&=\int_{\mathbb{R}} e^{-\delta|\xi|^a} J_{\frac{2k-1}{a}}\Big(\frac{2}{a}|x\xi|^{\frac{a}{2}}\Big)J_{\frac{2k-1}{a}}\Big(\frac{2}{a}|y\xi|^{\frac{a}{2}}\Big)d\gamma_{k,a}(\xi)\\&\quad+\frac{xy}{(i\,a)^{\frac{4}{a}}}\Big(\frac{\Gamma(\frac{2k+a-1}{a})}{\Gamma(\frac{2k+a+1}{a})}\Big)^2\int_{\mathbb{R}}\xi^2 e^{-\delta|\xi|^a} J_{\frac{2k+1}{a}}\Big(\frac{2}{a}|x\xi|^{\frac{a}{2}}\Big)J_{\frac{2k+1}{a}}\Big(\frac{2}{a}|y\xi|^{\frac{a}{2}}\Big)d\gamma_{k,a}(\xi).\end{aligned}\tag{3.7}$$

After the change of variables $\xi = t^{\frac{2}{a}}$ the relation (3.6), implies that

$$\begin{aligned}&\int_{\mathbb{R}} e^{-\delta|\xi|^a} J_{\frac{2k-1}{a}}\Big(\frac{2}{a}|x\xi|^{\frac{a}{2}}\Big)J_{\frac{2k-1}{a}}\Big(\frac{2}{a}|y\xi|^{\frac{a}{2}}\Big)d\gamma_{k,a}(\xi)\\&=\frac{1}{(\delta\,a)^{\frac{2k-1+a}{a}}}\,e^{-(1/\delta a^2)(|x|^a+|y|^a)}\,j_{\frac{2k-1}{a}}(2i|xy|^{\frac{a}{2}}/a^2\delta)\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} \xi^2 e^{-\delta|\xi|^a} J_{\frac{2k+1}{a}}\Big(\frac{2}{a}|x\xi|^{\frac{a}{2}}\Big)J_{\frac{2k+1}{a}}\Big(\frac{2}{a}|y\xi|^{\frac{a}{2}}\Big)d\gamma_{k,a}(\xi)\\&=\frac{\Gamma(\frac{2k+1+a}{a})}{\delta^{\frac{2}{a}}\Gamma(\frac{2k-1+a}{a})(\delta\,a)^{\frac{2k-1+a}{a}}}\,e^{-(1/\delta a^2)(|x|^a+|y|^a)}\,j_{\frac{2k+1}{a}}(2i|xy|^{\frac{a}{2}}/a^2\delta).\end{aligned}$$

Thus involving equation (3.7) and (2.5), we derive the result. □

Proof of Proposition 3.2. Using (3.3) and the identity

$$\mathcal{F}_{k,a}(e^{-\frac{|.|^a}{a}})(\lambda) = e^{-\frac{|\lambda|^a}{a}},$$

we derive that for every $\delta > 0$ and for every $x \in \mathbb{R}$, we have

$$\tau_x^{k,a}[e^{-\frac{|s|^a}{\delta a^2}}](y) = (\delta\,a)^{\frac{2k+a-1}{a}}\int_{\mathbb{R}}\overline{B_{k,a}(\xi,x)B_{k,a}(\xi,y)}e^{-\delta|\xi|^a}d\gamma_{k,a}(\xi).$$

Thus, the result is obtained by using Lemma 3.1 and the formula (2.5). □

Recently, the author in [1] has obtained the product formula for the one-dimensional (k,a)-generalized Fourier kernel, his result extending the special case of [10] when $a = \frac{2}{n}$, $n \in \mathbb{N}$. More precisely, the author has proved the following:

Theorem 3.1 *The (k,a)-generalized kernel $B_{k,a}$ satisfies the product formula*

$$B_{k,a}(\lambda,x)B_{k,a}(\lambda,y) = \int_{\mathbb{R}} B_{k,a}(\lambda,z)d\zeta_{x,y}^{k,a}(z),\tag{3.8}$$

where

$$d\zeta_{x,y}^{k,a}(z) = \begin{cases} \Delta_{k,a}(x,y,z)d\gamma_{k,a}(z) & \text{if } xy \neq 0 \\ \delta_x(z) & \text{if } y = 0 \\ \delta_y(z) & \text{if } x = 0. \end{cases} \tag{3.9}$$

Here

$$\begin{aligned} \Delta_{k,\,a}(x,y,z) := a^{\frac{2k-1+a}{a}} 2^{\frac{2k-1-a}{a}} \big(\Gamma(\frac{2k-1}{a}+1)\big)^2 \Bigg\{ & \frac{R_{\frac{2k-1}{a},\frac{2k-1}{a}}(|x|^{\frac{a}{2}},|y|^{\frac{a}{2}},|z|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} \\ & + e^{\frac{-2i\pi}{a}} \operatorname{sgn}(xy) \frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|x|^{\frac{a}{2}},|y|^{\frac{a}{2}},|z|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} \\ & + \operatorname{sgn}(xz) \frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|x|^{\frac{a}{2}},|z|^{\frac{a}{2}},|y|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} \\ & + \operatorname{sgn}(yz) \frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|y|^{\frac{a}{2}},|z|^{\frac{a}{2}},|x|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} \Bigg\}, \end{aligned} \tag{3.10}$$

and $R_{\mu,\nu}$ is the Macdonal integral, given by

$$\begin{aligned} \forall\, x,y,z > 0, \quad R_{\mu,\nu}(x,y,z) \; &:= \frac{1}{2^{2\nu+\mu}(\Gamma(\nu+1))^2\Gamma(\mu+1)} \\ & \int_0^\infty j_\nu(xt) j_\nu(yt) j_\mu(zt) t^{2\nu+1} dt, \end{aligned}$$

provided $Re\,\mu > \frac{-1}{2}$ and $Re\,\nu > \frac{-1}{2}$.

Corollary 3.1 *For all f in $\mathcal{W}_{k,a}(\mathbb{R})$ or for all f in $L^2_{k,a}(\mathbb{R})$ such that $\mathcal{F}_{k,a}(f)$ belongs to $L^1_{k,a}(\mathbb{R})$ and $x \in \mathbb{R}$, we have for almost every $y \in \mathbb{R}$*

$$\tau_x^{k,a} f(y) = \int_{\mathbb{R}} f(z) \overline{d\zeta_{x,y}^{k,a}(z)}.$$

Proof When $x = 0$ or $y = 0$, the result is trivial from the definition of the measure $d\zeta_{x,y}^{k,a}$. Let $x, y \in \mathbb{R}^*$, involving the relations (3.3), (3.8), Fubini's theorem and the inversion formula, we derive that

$$\begin{aligned} \tau_x^{k,a} f(y) &= \int_{\mathbb{R}} \overline{\Delta_{k,\,a}(x,y,z)} \Big(\int_{\mathbb{R}} \overline{B_{k,a}(\xi,z)} \mathcal{F}_{k,a}(f)(\xi) d\gamma_{k,a}(\xi) \Big) d\gamma_{k,a}(z) \\ &= \int_{\mathbb{R}} \overline{\Delta_{k,\,a}(x,y,z)} f(z) d\gamma_{k,a}(z). \end{aligned}$$

Thus, the proof is finished. □

Profiting the fact that for all $x, y \in \mathbb{R}$, $\int_{\mathbb{R}} |d\zeta^{k,a}_{x,y}(z)|$ is finite and uniformly bounded, we extend the (k, a)-generalized translation operator on the space of functions locally integrable as follow:

Definition 3.2 Let $x \in \mathbb{R}$ and $f \in L^1_{\text{loc}}(d\gamma_{k,a})$. For $k \geq \max(\frac{2-a}{2}, 0)$, we define the (k, a)-generalized translation operator $\tau^{k,a}_x$ by

$$\tau^{k,a}_x f(y) = \int_{\mathbb{R}} f(z)\overline{d\zeta^{k,a}_{x,y}(z)}. \tag{3.11}$$

Remark 3.1 For all $x, y, \lambda \in \mathbb{R}$, we have the product formula

$$\tau^{k,a}_x \overline{B_{k,a}(\lambda, y)} = \overline{B_{k,a}(\lambda, x)B_{k,a}(\lambda, y)}. \tag{3.12}$$

Notation. We denote by P^μ_ν and Q^μ_ν the Legendre functions given in term of hypergeometric function as follow

$$P^\mu_\nu(x) = \frac{1}{\Gamma(1-\mu)}(\tfrac{1+x}{1-x})^{\frac{\mu}{2}}\ {}_2F_1(\nu+1, -\nu, 1-\mu, \tfrac{1-x}{2}), \quad -1 < x < 1,$$
$$Q^\mu_\nu(x) = e^{i\pi\mu}\frac{\sqrt{\pi}\Gamma(1+\nu+\mu)(x^2-1)^{\frac{\mu}{2}}}{2^{\nu+1}x^{\nu+\mu+1}\Gamma(\frac{3}{2}+\nu)}(\tfrac{1+x}{1-x})^{\frac{\mu}{2}}\ {}_2F_1(\tfrac{\nu+\mu+2}{2}, \tfrac{\nu+\mu+1}{2}, \nu+\tfrac{3}{2}, x^{-2}), \quad 1 < x.$$

On the follows we will prove the "trigonometric-hyperbolic" form of the (k, a)-generalized translation operator.

Theorem 3.2 *For all f in $C_{b,e}(\mathbb{R})$, we have*

$$\tau^{k,a}_x f(y) = \int_0^\pi f\big(\langle\!\langle x, y\rangle\!\rangle_{\phi,a}\big)\mathcal{N}_{k,a}(x, y, \phi)(\sin\phi)^{\frac{2(2k-1)+a}{2a}}d\phi$$
$$+\int_0^\infty f\big(\langle\!\langle x, y\rangle\!\rangle^{\phi,a}\big)\mathcal{M}_{k,a}(x, y, \phi)(\sinh\phi)^{\frac{2(2k-1)+a}{2a}}d\phi,$$

where

$$\mathcal{N}_{k,a}(x, y, \phi) = \frac{a\Gamma(\frac{2k-1}{a}+1)}{2^{1-\frac{2k-1}{a}}\sqrt{2\pi}}\left\{\frac{2^{\frac{1}{2}-\frac{2k-1}{a}}}{\Gamma(\frac{2k-1}{a}+\frac{1}{2})}(\sin\phi)^{\frac{2(2k-1)-a}{2a}} + \frac{e^{\frac{2i\pi}{a}}\operatorname{sgn}(xy)}{\sqrt{2\pi}}P^{\frac{1}{2}-\frac{2k-1}{a}}_{\frac{2k+1}{a}-\frac{1}{2}}\big(\cos\phi\big)\right\},$$

$$\mathcal{M}_{k,a}(x, y, \phi) = \frac{a\Gamma(\frac{2k-1}{a}+1)}{2^{1-\frac{2k-1}{a}}}\frac{e^{-(\frac{4k-2-a}{2a})i\pi}\sin(\frac{2\pi}{a})\operatorname{sgn}(xy)}{\sqrt{(\frac{1}{2}\pi)^3}}Q^{\frac{1}{2}-\frac{2k-1}{a}}_{\frac{2k+1}{a}-\frac{1}{2}}\big(\cosh\phi\big),$$

$$\langle\!\langle x, y\rangle\!\rangle_{\phi,a} := \Big(|x|^a + |y|^a - 2|xy|^{a/2}\cos\phi\Big)^{1/a} \tag{3.13}$$

and

$$\langle\!\langle x, y\rangle\!\rangle^{\phi,a} := \Big(|x|^a + |y|^a + 2|xy|^{a/2}\cosh\phi\Big)^{1/a}. \tag{3.14}$$

Proof By (3.10), the even and odd parts of the function $\Delta_{k,a}(x,y,\cdot)$ are given respectively by

$$\Delta_{k,a,\mathrm{e}}(x,y,z) := a^{\frac{2k-1+a}{a}} 2^{\frac{2k-1-a}{a}} \left(\Gamma(\frac{2k-1}{a}+1)\right)^2 \left\{ \frac{R_{\frac{2k-1}{a},\frac{2k-1}{a}}(|x|^{\frac{a}{2}},|y|^{\frac{a}{2}},|z|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} + e^{\frac{-2i\pi}{a}} \operatorname{sgn}(xy) \frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|x|^{\frac{a}{2}},|y|^{\frac{a}{2}},|z|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} \right\},$$

$$\Delta_{k,a,\mathrm{o}}(x,y,z) := a^{\frac{2k-1+a}{a}} 2^{\frac{2k-1-a}{a}} (\Gamma(\frac{2k-1}{a}+1))^2 \left\{ \operatorname{sgn}(xz) \frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|x|^{\frac{a}{2}},|z|^{\frac{a}{2}},|y|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} + \operatorname{sgn}(yz) \frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|y|^{\frac{a}{2}},|z|^{\frac{a}{2}},|x|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}} \right\}.$$

Hence, equation (3.11) turns into

$$\tau_x^{k,a} f(y) = \frac{a\Gamma(\frac{2k-1}{a}+1)}{2^{1-\frac{2k-1}{a}}} \left[\int_0^\infty f(z) \left\{ \overline{\frac{R_{\frac{2k-1}{a},\frac{2k-1}{a}}(|x|^{\frac{a}{2}},|y|^{\frac{a}{2}},|z|^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}}} + e^{\frac{2i\pi}{a}} \operatorname{sgn}(xy) \overline{\frac{R_{\frac{2k-1}{a},\frac{2k+1}{a}}(|x|^{\frac{a}{2}},|y|^{\frac{a}{2}},z^{\frac{a}{2}})}{|xyz|^{\frac{2k-1}{2}}}} \right\} dz \right].$$

Noting that the Macdonal integral can be written as

$$\forall x,y,z>0, \quad R_{\mu,\nu}(x,y,z) = \begin{cases} 0, & \text{if } z<|x-y| \\ \frac{(xy)^{\mu-1}\sin^{\mu-\frac{1}{2}}\phi}{\sqrt{2\pi}z^\mu} P_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}\big(\cos\phi\big), & \text{if } |x-y|<z<|x+y| \\ \frac{e^{(\mu-\frac{1}{2})i\pi}\sin((\nu-\mu)\pi)(xy)^{\mu-1}\sinh^{\frac{\mu-1}{2}}\phi}{\sqrt{(\frac{1}{2}\pi)^3}z^\mu} Q_{\nu-\frac{1}{2}}^{\frac{1}{2}-\mu}(\cosh\phi), & \text{if } x+y<z, \end{cases}$$

where $|x|^2+|y|^2-|z|^2 = 2xy\cos\phi$ if $|x-y|<z<|x+y|$ and $|z|^2-|x|^2-|y|^2 = 2xy\cosh\phi$ if $x+y<z$.

Moreover, that if $\nu-\mu=n$ is a nonnegative integer we have for all $x,y,z>0$,

$$R_{\mu,\nu}(x,y,z) = \begin{cases} \frac{2^{\frac{1}{2}-\mu}\Gamma(2\mu)n!(xy)^{\mu-1}\sin^{2\mu-1}\phi}{\Gamma(\nu+\mu)\Gamma(\frac{\mu+1}{2})\sqrt{2\pi}z^\mu} C_n^\mu\big(\cos\phi\big), & \text{if } |x-y|<z<|x+y| \\ 0, & \text{if } x+y<z \quad \text{or } z<|x-y|, \end{cases}$$

where C_n^μ is the Gegenbauer polynomial.

Involving these properties of the Macdonal integral, we will derive the "trigonometric-hyperbolic" form of the (k, a)-generalized translation operator. Indeed, for

$$||x|^{\frac{a}{2}} - |y|^{\frac{a}{2}}| < z^{\frac{a}{2}} < |x|^{\frac{a}{2}} + |y|^{\frac{a}{2}},$$

we substitute

$$\cos\phi := \frac{|x|^a + |y|^a - |z|^a}{2|xy|^{\frac{a}{2}}} \tag{3.15}$$

with $\phi \in [0, \pi]$.
For

$$|x|^{\frac{a}{2}} + |y|^{\frac{a}{2}} < z^{\frac{a}{2}},$$

we may substitute

$$\cosh\phi := \frac{|z|^a - |x|^a - |y|^a}{2|xy|^{\frac{a}{2}}} \tag{3.16}$$

with $\phi \in [0, \infty)$. Using, for $z > 0$, $\mathrm{sgn}(xz) = \mathrm{sgn}(x)$ and $\mathrm{sgn}(yz) = \mathrm{sgn}(y)$, the (k, a)-generalized translation operator takes the desired form. □

Involving the previous Theorem, we infer the following expression for the generalized heat kernel associated with $\Delta_{k,a}$:

Lemma 3.2 *For every $\lambda > 0$ and for every $x \in \mathbb{R}$, we have*

$$\tau_x^{k,a}(e^{-\lambda|.|^a})(y) = e^{-\lambda\left(|x|^a + |y|^a\right)} V_{k,a}(\lambda; x, y),$$

where

$$V_{k,a}(\lambda; x, y) := \int_0^\pi e^{2\lambda|xy|^{\frac{a}{2}}\cos\phi} \mathcal{N}_{k,a}(x, y, \phi)(\sin\phi)^{\frac{2k-1+a}{2a}} d\phi + \int_0^\infty e^{-2\lambda|xy|^{\frac{a}{2}}\cosh\phi} \mathcal{M}_{k,a}(x, y, \phi)(\sinh\phi)^{\frac{2k-1+a}{2a}} d\phi.$$

Remark 3.2 Involving the previous lemma, the properties of the Gegenbauer polynomials and by simple calculations we infer that there exist a positive constant $\mathfrak{C}(k, a)$ such that

$$|\tau_x^{k,a}(e^{-\lambda|.|^a})(y)| \le \mathfrak{C}(k, a) e^{-\lambda\left(|x|^{\frac{a}{2}} - |y|^{\frac{a}{2}}\right)^2}.$$

Proposition 3.3 *(i) For all $x, y \in \mathbb{R}^*$, we have*

$$\int_{\mathbb{R}} \Delta_{k,a}(x, y, z) d\gamma_{k,a}(z) = 1. \tag{3.17}$$

(ii) For all $x, y, z \in \mathbb{R}^*$, we have

$$\Delta_{k,a}(x, y, z) = \Delta_{k,a}(y, x, z). \tag{3.18}$$

(iii) For all $x, y, z \in \mathbb{R}^*$, we have

$$\Delta_{k,a}(x, y, z) = \overline{\Delta_{k,a}(e^{\frac{2i\pi}{a}} x, z, y)}. \tag{3.19}$$

(iv) There exist a positive constant $C(k, a)$ independent of x, y such that

$$\int_{\mathbb{R}} |\Delta_{k,a}(x, y, z)| d\gamma_{k,a}(z) \leq C(k, a). \tag{3.20}$$

Proof (i) By taking $\lambda = 0$ in the product formula (3.8), we derive the result.
(ii) Involving the symmetry of the Macdonald function with respect to the first two variables, the relation (3.18) is immediate.
(iii) For all $x, y, z \in \mathbb{R}^*$, we write

$$\operatorname{sgn}(xy) = \frac{xy}{|xy|}, \quad \operatorname{sgn}(xz) = \frac{xz}{|xz|} \quad \text{and} \quad \operatorname{sgn}(yz) = \frac{yz}{|yz|}$$

and as the modulus of the complex number $e^{\frac{-2i\pi}{a}}$ is equal to 1, we write the term $e^{\frac{-2i\pi}{a}} \operatorname{sgn}(xy)$ as $\frac{e^{\frac{-2i\pi}{a}} xy}{|e^{\frac{-2i\pi}{a}} xy|}$, where $|e^{\frac{-2i\pi}{a}} xy|$ denote the modulus of the complex number which equal the absolute value of the real number xy. On the other hand as $|e^{\frac{-2i\pi}{a}}| = 1$, we deduce that the modulus $|e^{\frac{-2i\pi}{a}} x|^{\frac{a}{2}}$ and $|e^{\frac{-2i\pi}{a}} x|^{\frac{2k-1}{2}}$ are respectively equal $|x|^{\frac{a}{2}}$ and $|x|^{\frac{2k-1}{2}}$. Thus by simple calculus we derive (3.19).
(iv) The result is proved in [1]. □

Remark 3.3 (i) When $\frac{2}{a} \in \mathbb{N}$, the authors in [10] have proved that $C(k, a) \leq 4$.

(ii) Involving the definition of the function $\Delta_{k,a}$, it is easy to see that for any $x, y \in \mathbb{R}^*$ we have

$$\int_{\mathbb{R}} |\Delta_{k,a}(x, z, y)| d\gamma_{k,a}(z) \leq C(k, a), \tag{3.21}$$

where $C(k, a)$ is the constant given by the formula (3.20).

(iii) Using (3.19) and proceeding as in [1], we prove that

$$\overline{B_{k,a}(\lambda, x)} B_{k,a}(\lambda, y) = \int_{\mathbb{R}} B_{k,a}(\lambda, z) \Delta_{k,a}(e^{\frac{2i\pi}{a}} x, z, y) d\gamma_{k,a}(z). \tag{3.22}$$

Now, let us go back to the properties of the (k,a)-generalized translation product.

Theorem 3.3 *Let* $k \geq \max(\frac{2-a}{2}, 0)$, *then*

(i) For all $f \in L^1_{\mathrm{loc}}(d\gamma_{k,a})$ *and for all* $x, y \in \mathbb{R}$, *we have*

$$\tau_x^{k,a} f(y) = \tau_y^{k,a} f(x) \quad \text{and} \quad \tau_0^{k,a} f = f.$$

(ii) For all $1 \leq p \leq \infty$ *and* $f \in L^p_{k,a}(\mathbb{R})$,

$$\|\tau_x^{k,a} f\|_{L^p_{k,a}(\mathbb{R})} \leq C(k,a)\|f\|_{L^p_{k,a}(\mathbb{R})} \tag{3.23}$$

with $x \in \mathbb{R}$ *and* $C(k,a)$ *is the constant given by the formula (3.20).*

(iii) If $f \in L^1_{k,a}(\mathbb{R})$, *and* $x \in \mathbb{R}$, *then* $\mathcal{F}_{k,a}(\tau_x^{k,a} f)(\lambda) = \overline{B_{k,a}(\lambda, x)}\mathcal{F}_{k,a}(f)(\lambda)$ *for every* $\lambda \in \mathbb{R}$.

(iv) If $f \in L^2_{k,a}(\mathbb{R})$, *and* $x \in \mathbb{R}$, *then* $\mathcal{F}_{k,a}(\tau_x^{k,a} f)(\lambda) = \overline{B_{k,a}(\lambda, x)}\mathcal{F}_{k,a}(f)(\lambda)$ *for almost every* $\lambda \in \mathbb{R}$.

(v) If $f \in L^p_{k,a}(\mathbb{R})$, $1 \leq p \leq 2$ *and* $x \in \mathbb{R}$, *then*

$$\mathcal{F}_{k,a}(\tau_x^{k,a} f)(\lambda) = \overline{B_{k,a}(\lambda, x)}\mathcal{F}_k(f)(\lambda), \quad a.e.\ \lambda \in \mathbb{R}. \tag{3.24}$$

(vi) Let $x, y \in \mathbb{R}$. *For all* f *in* $L^p_{k,a}(\mathbb{R})$, $1 \leq p \leq \infty$, *we have*

$$\tau_x^{k,a}\tau_y^{k,a}(f) = \tau_y^{k,a}\tau_x^{k,a}(f). \tag{3.25}$$

Proof (i) When $x = 0$ or $y = 0$, the result is trivial from the definition of the measure $d\zeta^{k,a}_{x,y}$. Let $x, y \in \mathbb{R}^*$, we have

$$\tau_x^{k,a} f(y) = \int_{\mathbb{R}} \overline{\Delta_{k,a}(x,y,z)} f(z) d\gamma_{k,a}(z).$$

Involving (3.18), we derive the result.

(ii) Using Hölder's inequality and (3.20), we obtain

$$\Big|\int_{\mathbb{R}} \overline{\Delta_{k,a}(x,y,z)} f(z) d\gamma_{k,a}(z)\Big|^p \leq \big(C(k,a)\big)^{\frac{p}{p'}} \int_{\mathbb{R}} |\Delta_{k,a}(x,y,z)|\,|f(z)|^p\, d\gamma_{k,a}(z).$$

Thus by Fubini–Tonelli's theorem and (3.21), we deduce that

$$\int_{\mathbb{R}} \Big|\int_{\mathbb{R}} \overline{\Delta_{k,a}(x,y,z)} f(z) d\gamma_{k,a}(z)\Big|^p d\gamma_{k,a}(y) \leq \big(C(k,a)\big)^p ||f||_{L^p_{k,a}(\mathbb{R})}.$$

(iii) The result is trivial if $x = 0$. If $f \in L^1_{k,a}(\mathbb{R})$, and $x \in \mathbb{R}^*$, we have

$$\mathcal{F}_{k,a}(\tau_x^{k,a} f)(\lambda) = \int_{\mathbb{R}} \tau_x^{k,a} f(y) B_{k,a}(\lambda, y) d\gamma_{k,a}(y).$$

Using the relations (3.11), (3.19) and product formula (3.22), we get

$$\begin{aligned}\mathcal{F}_{k,a}(\tau_x^{k,a} f)(\lambda) &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \overline{\Delta_{k,a}(x, y, z)} f(z) d\gamma_{k,a}(z) \Big) B_{k,a}(\lambda, y) d\gamma_{k,a}(y) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \Delta_{k,a}(e^{\frac{2i\pi}{a}} x, z, y) f(z) d\gamma_{k,a}(z) B_{k,a}(\lambda, y) d\gamma_{k,a}(y) \\ &= \int_{\mathbb{R}} f(z) \Big(\int_{\mathbb{R}} B_{k,a}(\lambda, y) \Delta_{k,a}(e^{\frac{2i\pi}{a}} x, z, y) d\gamma_{k,a}(y) \Big) d\gamma_{k,a}(z) \\ &= \int_{\mathbb{R}} f(z) \overline{B_{k,a}(\lambda, x)} B_{k,a}(\lambda, z) d\gamma_{k,a}(z) \\ &= \overline{B_{k,a}(\lambda, x)} \mathcal{F}_{k,a}(f)(\lambda).\end{aligned}$$

Thus the assertion is proved.

(iv) Using the fact that the mappings $f \longmapsto \mathcal{F}_{k,a}(\tau_x^{k,a} f)$ and $f \longmapsto \overline{B_{k,a}(\lambda, x)} \mathcal{F}_{k,a}(f)$ are continuous from $L^2_{k,a}(\mathbb{R})$ into itself, and from (iii) these mappings are equal on $L^1_{k,a}(\mathbb{R}) \bigcap L^2_{k,a}(\mathbb{R})$, we derive the result by the density of $L^1_{k,a}(\mathbb{R}) \bigcap L^2_{k,a}(\mathbb{R})$ in $L^2_{k,a}(\mathbb{R})$.

(v) The cases $p = 1$ and $p = 2$ are proved in above. For $1 < p < 2$, we have

$$\|\mathcal{F}_{k,a}(\tau_x^{k,a} f) - \overline{B_{k,a}(\lambda, x)} \mathcal{F}_{k,a}(f)\|_{L^{p'}_{k,a}(\mathbb{R})} \leq \|\mathcal{F}_{k,a}(\tau_x^{k,a} f)\|_{L^{p'}_{k,a}(\mathbb{R})} + \|\mathcal{F}_{k,a}(f)\|_{L^{p'}_{k,a}(\mathbb{R})}.$$

Involving (2.17) and (3.23), we derive that

$$\|\mathcal{F}_{k,a}(\tau_x^{k,a} f) - \overline{B_{k,a}(\lambda, x)} \mathcal{F}_{k,a}(f)\|_{L^{p'}_{k,a}(\mathbb{R})} \leq (C(k, a) + 1) \|f\|_{L^{p}_{k,a}(\mathbb{R})}.$$

Thus by a density argument we prove the assertion.

Now we will to prove (vi). Using (3.24) and the injectivity for the (k, a)-generalized Fourier transform, we derive the result for $p \in [1, 2]$. Next, by duality we deduce the result for any $p \in [2, \infty]$. □

Notation. Let us denote by $\mathbb{R}_a$ be the set defined by

$$\mathbb{R}_a := \mathbb{R} \cup e^{\frac{-2i\pi}{a}} \mathbb{R} \cup e^{\frac{2i\pi}{a}} \mathbb{R}, \quad a > 0.$$

Remark 3.4 (i) Using Proposition 3.3, we can extend the formula (3.11) of the generalized translation operator $\tau_x^{k,a}$ for $x \in \mathbb{R}_a$.

By means of the generalized translation operator, we define the generalized convolution product of two suitable functions f and g by

$$f *_{k,a} g(x) = \int_{\mathbb{R}} \tau^{k,a}_{e^{\frac{2i\pi}{a}} y} f(x) g(y) d\gamma_{k,a}(y). \tag{3.26}$$

Now, let us go back to the properties of the generalized convolution product.

Theorem 3.4 *Let* $k \geq \max(\frac{2-a}{2}, 0)$, *then*

(i) For two suitable functions f *and* g *we have*

$$f *_{k,a} g = g *_{k,a} f. \tag{3.27}$$

(ii) (Young inequality). *For* p, q, r *such that* $1 \leq p, q, r \leq \infty$ *and* $1/p + 1/q = 1 + 1/r$, *and for* $f \in L^q_{k,a}(\mathbb{R})$ *and* $g \in L^p_{k,a}(\mathbb{R})$, *the convolution product* $f *_{k,a} g$ *is a well defined element in* $L^r_{k,a}(\mathbb{R})$ *and*

$$\|f *_{k,a} g\|_{L^r_{k,a}(\mathbb{R})} \leq C(k, a) \|f\|_{L^q_{k,a}(\mathbb{R})} \|g\|_{L^p_{k,a}(\mathbb{R})}, \tag{3.28}$$

where $C(k, a)$ *is the constant given by the formula (3.20).*

(iii) For p, q, r *such that* $1 \leq p, q, r \leq 2$ *and* $1/p + 1/q = 1 + 1/r$, *and for* $f \in L^q_{k,a}(\mathbb{R})$ *and* $g \in L^p_{k,a}(\mathbb{R})$, *we have*

$$\mathcal{F}_{k,a}(f *_{k,a} g) = \mathcal{F}_{k,a}(f)\mathcal{F}_{k,a}(g). \tag{3.29}$$

(iv) For p, q, r *such that* $1 \leq p, q, r \leq 2$ *and* $1/p + 1/q + 1/r = 2$, *and for* $f \in L^q_{k,a}(\mathbb{R})$, $g \in L^p_{k,a}(\mathbb{R})$ *and* $h \in L^r_{k,a}(\mathbb{R})$, *we have*

$$\Big| \int_{\mathbb{R}} f(x) \big(g *_{k,a} h(x)\big) d\gamma_{k,a}(y) \Big| \leq C(k, a) \|f\|_{L^q_{k,a}(\mathbb{R})} \|g\|_{L^p_{k,a}(\mathbb{R})} \|h\|_{L^r_{k,a}(\mathbb{R})}.$$

Proof (i) Involving the relations (3.26), (3.11), the fact that

$$\Delta_{k,a}(e^{\frac{2i\pi}{a}} z, x, y) = \Delta_{k,a}(e^{\frac{2i\pi}{a}} y, x, z),$$

we derive that

$$\begin{aligned}
f *_{k,a} g(x) &= \int_{\mathbb{R}} \tau^{k,a}_{e^{\frac{2i\pi}{a}} y} f(x) g(y) d\gamma_{k,a}(y) \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\Delta_{k,a}(e^{\frac{2i\pi}{a}} y, x, z)} f(z) d\gamma_{k,a}(z) g(y) d\gamma_{k,a}(y) \\
&= \int_{\mathbb{R}} f(z) \Big(\int_{\mathbb{R}} \overline{\Delta_{k,a}(e^{\frac{2i\pi}{a}} y, x, z)} g(y) d\gamma_{k,a}(y) \Big) d\gamma_{k,a}(z) \\
&= \int_{\mathbb{R}} f(z) \Big(\int_{\mathbb{R}} \overline{\Delta_{k,a}(e^{\frac{2i\pi}{a}} z, x, y)} g(y) d\gamma_{k,a}(y) \Big) d\gamma_{k,a}(z)
\end{aligned}$$

$$= \int_{\mathbb{R}} f(z)\tau^{k,a}_{e^{\frac{2i\pi}{a}}z} g(x) d\gamma_{k,a}(z).$$

Thus the assertion is proved.

For part (ii), if $r = \infty$ then $1/p + 1/q = 1$. Hence, by Hölder's inequality and (3.23), $f *_{k,a} g$ exists and

$$\|f *_{k,a} g\|_\infty \leq C(k,a)\|f\|_{L^q_{k,a}(\mathbb{R})}\|g\|_{L^p_{k,a}(\mathbb{R})}.$$

Assume $r < \infty$, which implies $p, q \leq r$. Let $s = p(1 - 1/q) = 1 - p/r$ and note that $0 \leq s < 1$. We have

$$\begin{aligned}|f *_{k,a} g(x)| &\leq \int_{\mathbb{R}} |f(y)|\,|\tau^{k,a}_x g(e^{\frac{2i\pi}{a}}y)|^{1-s}\,|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}y)|^s\, d\gamma_{k,a}(y)\\ &\leq \Big(\int_{\mathbb{R}} |f(y)|^q\,|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}y)|^{q(1-s)}\, d\gamma_{k,a}(y)\Big)^{1/q} \|\,|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}.)|^s\|_{L^{q'}_{k,a}(\mathbb{R})}.\end{aligned}$$

If $s = 0$ then $q = 1$. If $s \neq 0$ then $sq' = p$. In either cases taking the q^{th} power we obtain

$$\begin{aligned}|f *_{k,a} g(x)|^q &\leq \Big(\int_{\mathbb{R}} |f(y)|^q\,|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}y)|^{q(1-s)}\, d\gamma_{k,a}(y)\Big)\|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}.)\|^{sq}_{L^p_{k,a}(\mathbb{R})}\\ &\leq \big(C(k,a)\big)^{sq}\,\|g\|^{sq}_{L^p_{k,a}(\mathbb{R})}\Big(\int_{\mathbb{R}} |f(y)|^q\,|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}y)|^{q(1-s)}\, d\gamma_{k,a}(y)\Big).\end{aligned}$$

Thus, for $t := r/q$, by the generalized Minkowski inequality we have

$$\begin{aligned}\|f *_{k,a} g\|^q_{L^{qt}_{k,a}(\mathbb{R})} &= \|\,|f *_{k,a} g|^q\|_{L^t_{k,a}(\mathbb{R})}\\ &\leq \big(C(k,a)\big)^{sq}\,\|g\|^{sq}_{L^p_{k,a}(\mathbb{R})}\\ &\quad \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} |f(y)|^{qt}\,|\tau^{k,a}_x(g)(e^{\frac{2i\pi}{a}}y)|^{qt(1-s)} d\gamma_{k,a}(x)\Big)^{1/t} d\gamma_{k,a}(y)\\ &\leq \big(C(k,a)\big)^q\|f\|^q_{L^q_{k,a}(\mathbb{R})}\|g\|^q_{L^p_{k,a}(\mathbb{R})}\end{aligned}$$

since $qt = r$ and $(1 - s)r = p$, we derive the result.

On the follow we prove the assertion (iii) for $p = q = 1$. Indeed, we have

$$\mathcal{F}_{k,a}(f *_{k,a} g)(\lambda) = \int_{\mathbb{R}} f *_{k,a} g(x) B_{k,a}(\lambda, x) d\gamma_{k,a}(x).$$

Involving the relations (3.26), (3.8), Fubini's theorem, (3.19) and (2.10), we get

$$\begin{aligned}&\mathcal{F}_{k,a}(f *_{k,a} g)(\lambda)\\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} \tau^{k,a}_{e^{\frac{2i\pi}{a}}y} f(x) g(y) d\gamma_{k,a}(y)\Big) B_{k,a}(\lambda, x) d\gamma_{k,a}(x)\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \overline{\Delta_{k,a}(e^{\frac{2i\pi}{a}} y, x, z) f(z) d\gamma_{k,a}(z)\Big)} g(y) d\gamma_{k,a}(y)) B_{k,a}(\lambda, x) d\gamma_{k,a}(x) \\
&= \int_{\mathbb{R}} f(z) \Big(\int_{\mathbb{R}} \int_{\mathbb{R}} \Delta_{k,a}(y, z, x) B_{k,a}(\lambda, x) d\gamma_{k,a}(x) \Big) d\gamma_{k,a}(z) g(y) d\gamma_{k,a}(y) \\
&= \int_{\mathbb{R}} f(z) \int_{\mathbb{R}} B_{k,a}(\lambda, y) B_{k,a}(\lambda, z) d\gamma_{k,a}(z)) g(y) d\gamma_{k,a}(y) \\
&= \Big(\int_{\mathbb{R}} f(z) B_{k,a}(\lambda, z) d\gamma_{k,a}(z) \Big) \Big(\int_{\mathbb{R}} B_{k,a}(\lambda, y) g(y) d\gamma_{k,a}(y) \Big) \\
&= \mathcal{F}_{k,a}(f)(\lambda) \mathcal{F}_{k,a}(g)(\lambda).
\end{aligned}
$$

Thus the result is proved when $p = q = 1$. For $p \neq 1$ and/or $q \neq 1$, using Hölder's inequality, (2.17) and (3.28), we get

$$
\begin{aligned}
&\|\mathcal{F}_{k,a}(f *_{k,a} g) - \mathcal{F}_{k,a}(f)\mathcal{F}_{k,a}(g)\|_{L^{r'}_{k,a}(\mathbb{R})} \\
&\quad \le \|\mathcal{F}_{k,a}(f *_{k,a} g)\|_{L^{r'}_{k,a}(\mathbb{R})} + \|\mathcal{F}_{k,a}(f)\mathcal{F}_{k,a}(g)\|_{L^{r'}_{k,a}(\mathbb{R})} \\
&\quad \le \|f *_{k,a} g\|_{L^{r}_{k,a}(\mathbb{R})} + \|\mathcal{F}_{k,a}(f)\|_{L^{q'}_{k,a}(\mathbb{R})} \|\mathcal{F}_{k,a}(g)\|_{L^{p'}_{k,a}(\mathbb{R})} \\
&\quad \le (C(k,a) + 1)\|f\|_{L^{q}_{k,a}(\mathbb{R})} \|g\|_{L^{p}_{k,a}(\mathbb{R})},
\end{aligned}
$$

so that a density argument proves (iii) for $p \neq 1$ and/or $q \neq 1$.

Now we will to prove (iv). Using Hölder's inequality, we get

$$
\Big| \int_{\mathbb{R}} f(x)(g *_{k,a} h(x)) d\gamma_{k,a}(y) \Big| \le \|f\|_{L^{q}_{k,a}(\mathbb{R})} \|g *_{k,a} h\|_{L^{q'}_{k,a}(\mathbb{R})}.
$$

Therefore using (3.28), we have

$$
\begin{aligned}
&| \int_{\mathbb{R}} f(x)(g *_{k,a} h(x)) | d\gamma_{k,a}(y \le C(k,a) \|f\|_{L^{q}_{k,a}(\mathbb{R})} \|g\|_{L^{p}_{k,a}(\mathbb{R})} \|h\|_{L^{r}_{k,a}(\mathbb{R})}, \\
&\frac{1}{q'} = \frac{1}{p} + \frac{1}{r} - 1.
\end{aligned}
$$

Thus the assertion is proved. □

Theorem 3.5 Inversion formula. *If f belongs in $L^1_{k,a}(\mathbb{R})$ such that $\mathcal{F}_{k,a}(f)$ belongs to $L^1_{k,a}(\mathbb{R})$, then we have the following inversion formula*

$$
\forall x \in \mathbb{R}, \quad f(x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(y) \overline{B_{k,a}(x, y)} d\gamma_{k,a}(y). \tag{3.30}
$$

Proof For $\varepsilon > 0$, we introduce the function $F_\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by

$$
\forall (y, x) \in \mathbb{R} \times \mathbb{R}, \quad F_\varepsilon(y, x) := e^{-\varepsilon |y|^a} \overline{B_{k,a}(x, y)} \mathcal{F}_{k,a}(f)(y).
$$

Then we have

- For all $x \in \mathbb{R}$, the functions $F_\varepsilon(., x)$ belong to $L^1_{k,a}(\mathbb{R})$;
- $\forall\, x \in \mathbb{R}, \quad \lim_{\varepsilon\to 0} F_\varepsilon(y, x) = \overline{B_{k,a}(x, y)}\mathcal{F}_{k,a}(f)(y)$;
- $\forall\, (y, x) \in \mathbb{R} \times \mathbb{R}$, $|F_\varepsilon(y, x)| \le |\mathcal{F}_{k,a}(f)(y)| \in L^1_{k,a}(\mathbb{R})$.

Then from the dominated convergence theorem we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} F_\varepsilon(y, x)d\gamma_{k,a}(y) = \int_{\mathbb{R}} \overline{B_{k,a}(x, y)}\mathcal{F}_{k,a}(f)(y)d\gamma_{k,a}(y). \tag{3.31}$$

We note that for all $x \in \mathbb{R}$, we have

$$\begin{aligned}\int_{\mathbb{R}} F_\varepsilon(y, x)d\gamma_{k,a}(y) &= \int_{\mathbb{R}} e^{-\varepsilon|y|^a}\overline{B_{k,a}(x, y)}\mathcal{F}_{k,a}(f)(y)d\gamma_{k,a}(y)\\ &= \int_{\mathbb{R}} e^{-\varepsilon|y|^a}\Big(\int_{\mathbb{R}} f(s)B_{k,a}(s, y)d\gamma_{k,a}(s)\Big)\overline{B_{k,a}(x, y)}d\gamma_{k,a}(y).\end{aligned}$$

Let $x \in \mathbb{R}$. We put $G_\varepsilon(x, y, s) = e^{-\varepsilon|y|^a} f(s)B_{k,a}(s, y)\overline{B_{k,a}(x, y)}$ where $y, s \in \mathbb{R}$.

From the Fubini–Tonelli theorem, we have

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} |G_\varepsilon(x, y, s)|d\gamma_{k,a}(y)d\gamma_{k,a}(s) &\le \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\varepsilon|y|^a}|f(s)|d\gamma_{k,a}(y)d\gamma_{k,a}(s)\\ &\le \Big(\int_{\mathbb{R}} |f(s)|d\gamma_{k,a}(s)\Big)\Big(\int_{\mathbb{R}} e^{-\varepsilon|y|^a}d\gamma_{k,a}(y)\Big).\end{aligned}$$

As $f \in L^1_{k,a}(\mathbb{R})$, then $\int_{\mathbb{R}} |f(s)|d\gamma_{k,a}(s) < \infty$. On the other hand using the fact that

$$\int_{\mathbb{R}} e^{-\varepsilon|y|^a}d\gamma_{k,a}(y) = C(k, \varepsilon, a),$$

we derive that

$$\int_{\mathbb{R}\times\mathbb{R}} |G_\varepsilon(x, y, s)|d\gamma_{k,a}(y)d\gamma_{k,a}(s) < \infty.$$

Thus we can apply the Fubini's theorem for the function $G_\varepsilon(x, ., .)$ on the space $\mathbb{R}\times\mathbb{R}$ and we deduce that

$$\begin{aligned}\int_{\mathbb{R}\times\mathbb{R}} G_\varepsilon(x, y, s)d\gamma_{k,a}(y)d\gamma_{k,a}(s) &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} G_\varepsilon(x, y, s)d\gamma_{k,a}(y)\Big)d\gamma_{k,a}(s)\\ &= \int_{\mathbb{R}}\Big(\int_{\mathbb{R}} G_\varepsilon(x, y, s)d\gamma_{k,a}(s)\Big)d\gamma_{k,a}(y).\end{aligned}$$

On the other hand it is easy to see that

$$\int_{\mathbb{R}} F_\varepsilon(y, x)d\gamma_{k,a}(y) = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} G_\varepsilon(x, y, s)d\gamma_{k,a}(s)\right)d\gamma_{k,a}(y).$$

Thus, from above we deduce that

$$\begin{aligned}\int_{\mathbb{R}} F_\varepsilon(y,x)d\gamma_{k,a}(y) &= \int_{\mathbb{R}\times\mathbb{R}} G_\varepsilon(x,y,s)d\gamma_{k,a}(y)d\gamma_{k,a}(s)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\varepsilon|y|^a} f(s)B_{k,a}(s,y)\overline{B_{k,a}(x,y)}d\gamma_{k,a}(y)d\gamma_{k,a}(s)\\ &= \int_{\mathbb{R}} f(s)\Big(\int_{\mathbb{R}} e^{-\varepsilon|y|^a} B_{k,a}(s,y)\overline{B_{k,a}(x,y)}d\gamma_{k,a}(y)\Big)d\gamma_{k,a}(s).\end{aligned} \tag{3.32}$$

Moreover using the Hecke identity proved in [4], we derive that

$$\int_{\mathbb{R}} e^{-\varepsilon|y|^a} B_{k,a}(s,y)\overline{B_{k,a}(x,y)}d\gamma_{k,a}(y) = (\tfrac{1}{a\varepsilon})^{\frac{2k-1+a}{a}} \tau_x^{k,a}(e^{-\frac{1}{a^2\varepsilon}|.|^a})(e^{\frac{2i\pi}{a}} s). \tag{3.33}$$

Combining the relations (3.32) and (3.33), we get

$$\int_{\mathbb{R}} F_\varepsilon(y,x)d\gamma_{k,a}(y) = (\tfrac{1}{a\varepsilon})^{\frac{2k-1+a}{a}} \int_{\mathbb{R}} f(s)\tau_x^{k,a}(e^{-\frac{1}{a^2\varepsilon}|.|^a})(e^{\frac{2i\pi}{a}} s)d\gamma_{k,a}(s). \tag{3.34}$$

Using the change of variable $s = (a^2\varepsilon)^{\frac{1}{a}} v$ in the second member of the formula (3.34) we obtain

$$\int_{\mathbb{R}} F_\varepsilon(y,x)d\gamma_{k,a}(y) = a^{\frac{2k-1+a}{a}} \int_{\mathbb{R}} \tau_x^{k,a} f(e^{\frac{2i\pi}{a}}(a^2\varepsilon)^{\frac{1}{a}} v)e^{-|v|^a} d\gamma_{k,a}(v).$$

Therefore by applying the dominated convergence theorem we get

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} F_\varepsilon(y,x)d\gamma_{k,a}(y) = a^{\frac{2k-1+a}{a}} f(x)\int_{\mathbb{R}} e^{-|v|^a} d\gamma_{k,a}(v).$$

Moreover using the formula

$$\int_{\mathbb{R}} e^{-|v|^a} d\gamma_{k,a}(v) = (\frac{1}{a})^{\frac{2k-1+a}{a}},$$

we deduce that

$$\forall x\in\mathbb{R},\quad \lim_{\varepsilon\to 0}\int_{\mathbb{R}} F_\varepsilon(y,x)d\gamma_{k,a}(y) = f(x). \tag{3.35}$$

Finally combining the relations (3.31) and (3.35), we derive that

$$\forall x \in \mathbb{R}, \quad f(x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(y)\overline{B_{k,a}(x,y)}d\gamma_{k,a}(y)$$

which achieves the proof. □

On the follow we give another proof of the Proposition 2.3.

Corollary 3.2 *(i) If $f \in L^1_{k,a}(\mathbb{R}) \cap L^2_{k,a}(\mathbb{R})$ then $\mathcal{F}_{k,a}(f)$ belongs to $L^2_{k,a}(\mathbb{R})$ and*

$$||\mathcal{F}_{k,a}(f)||_{L^2_{k,a}(\mathbb{R})} = ||f||_{L^2_{k,a}(\mathbb{R})}.$$

(ii) There exists a unique isometry on $L^2_{k,a}(\mathbb{R})$ that coincides with on $\mathcal{F}_{k,a}$ on $L^1_{k,a}(\mathbb{R}) \cap L^2_{k,a}(\mathbb{R})$.

Proof (i) From the previous theorem, we have

$$f(x)\overline{g(x)} = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(y)\overline{B_{k,a}(x,y)g(x)}d\gamma_{k,a}(y).$$

The application of Fubini's theorem yields

$$\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(y)\Big(\int_{\mathbb{R}} \overline{B_{k,a}(x,y)g(x)}d\gamma_{k,a}(x)\Big)d\gamma_{k,a}(y)$$
$$= \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(y)\overline{\mathcal{F}_{k,a}(g)(y)}d\gamma_{k,a}(y).$$

Thus, we derive the result.

(ii) Let $f \in L^2_{k,a}(\mathbb{R})$, then $f_j = f\chi_{[-j,j]}$ belongs to $L^1_{k,a}(\mathbb{R})$ and

$$\lim_{j\to\infty} ||f - f_j||_{L^2_{k,a}(\mathbb{R})} = \lim_{j\to\infty} \int_{|x|\geq j} |f(x)|^2 d\gamma_{k,a}(x) = 0.$$

From (i) we deduce that the operator

$$\mathcal{F}_{k,a} : L^1_{k,a}(\mathbb{R}) \cap L^2_{k,a}(\mathbb{R}) \to L^2_{k,a}(\mathbb{R})$$

is continuous for the norm $||.||_{L^2_{k,a}(\mathbb{R})}$. As the space $L^1_{k,a}(\mathbb{R}) \cap L^2_{k,a}(\mathbb{R})$ is a dense part of $L^2_{k,a}(\mathbb{R})$ and $L^2_{k,a}(\mathbb{R})$ is complete, so by the theorem of extension of uniformly continuous applications there exists a unique extension of $\mathcal{F}_{k,a}$ in $L^2_{k,a}(\mathbb{R})$. The extension is still an isometry of the norm $||.||_{L^2_{k,a}(\mathbb{R})}$ by passing through the limit in the equality of (i). If we also know that the image is dense, then the operator is surjective and the inverse is continuous because $\mathcal{F}_{k,a}$ is an isometry. □

We close this section by giving the following new results:

Proposition 3.4 *Let $\phi \in \mathcal{S}_e(\mathbb{R})$. Then there exist a positive constant C, such that for all $x, y \in \mathbb{R}$ we have*

$$||\tau_x^{k,a}\phi - \tau_y^{k,a}\phi||_{L^1_{k,a}(\mathbb{R})} \leq C|x-y|||\phi||_{L^1_{k,a}(\mathbb{R})}.$$

Proof Involving Theorem 3.2, mean value theorem, the properties of the Gegenbauer polynomials and by simple calculations we derive the result. □

Proposition 3.5 *Let $f \in L^1_{k,a}(\mathbb{R})$ and $t > 0$ such that $supp(\mathcal{F}_{k,a}(f)) \subset [-t, t]$. Then there exist a positive constant $\mathrm{M}(k, a)$, such that for all $x, y \in \mathbb{R}$ we have*

$$||\tau_x^{k,a} f - \tau_y^{k,a} f||_{L^1_{k,a}(\mathbb{R})} \leq \mathrm{M}(k,a)t|x-y|||f||_{L^1_{k,a}(\mathbb{R})}.$$

Proof Choose $\phi \in \mathcal{S}_e(\mathbb{R})$ such that $\mathcal{F}_{k,a}(\phi) = 1$ in $[-1, 1]$ and put $\phi_t(x) = t^{2k+a-1}\phi(tx)$.
Thus

$$\mathcal{F}_{k,a}(\phi_t)(x) = \mathcal{F}_{k,a}(\phi)\left(\frac{x}{t}\right) = 1 \quad \text{on} \quad [-t, t],$$

and we can write

$$\begin{aligned}\tau_x^{k,a} f(z) - \tau_y^{k,a} f(z) &= \phi_t *_{k,a} (\tau_x^{k,a} f - \tau_y^{k,a} f)(z)\\ &= f *_{k,a} (\tau_x^{k,a}(\phi_t) - \tau_y^{k,a}(\phi_t))(z).\end{aligned}$$

Involving (3.28) and the previous proposition, we derive that

$$\begin{aligned}||\tau_x^{k,a} f - \tau_y^{k,a} f||_{L^1_{k,a}(\mathbb{R})} &\leq C(k,a)||f||_{L^1_{k,a}(\mathbb{R})}||\tau_x^{k,a}(\phi_t) - \tau_y^{k,a}(\phi_t)||_{L^1_{k,a}(\mathbb{R})}\\ &\leq C(k,a)||f||_{L^1_{k,a}(\mathbb{R})}||\tau_{tx}^{k,a}\phi - \tau_{ty}^{k,a}\phi||_{L^1_{k,a}(\mathbb{R})}\\ &\leq CC(k,a)t|x-y|||f||_{L^1_{k,a}(\mathbb{R})}\end{aligned}$$

which finishes the proof of the proposition. □

4 Applications

4.1 (k, a)-Generalized Wavelet Transform in $L^p_{k,a}(\mathbb{R})$

In this subsection, we shall study the basic results on the (k, a)-generalized wavelet transform. For typographical convenience, we fix some notations as under:

- $\mathbb{R}^2_+ = \left\{(b, x) \in \mathbb{R}^2 : b > 0\right\}$.
- $L^p_{\mu_{k,a}}(\mathbb{R}^2_+)$, $p \in [1, \infty]$, the space of measurable functions f on $\mathbb{R}^2_+$ such that

$$\|f\|_{L^p_{\mu_{k,a}}(\mathbb{R}^2_+)} := \left(\int_{\mathbb{R}^2_+} |f(b,x)|^p d\mu_{k,a}(b,x)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty,$$

$$\|f\|_{L^\infty_{\mu_{k,a}}(\mathbb{R}^2_+)} := \underset{(b,x)\in\mathbb{R}^2_+}{ess\sup} |f(b,x)| < \infty,$$

where the measure $\mu_{k,a}$ is defined by

$$\forall (b,x) \in \mathbb{R}^2_+, \quad d\mu_{k,a}(b,x) = \frac{d\gamma_{k,a}(x)db}{b^{2k+a}}.$$

Definition 4.1 ([34]). A (k,a)-generalized wavelet on $\mathbb{R}$ is a measurable function h on $\mathbb{R}$ satisfying for almost all $x \in \mathbb{R}$, the condition

$$0 < C_h := \int_0^\infty |\mathcal{F}_{k,a}(\overline{h})(\lambda x)|^2 \frac{d\lambda}{\lambda} < \infty. \tag{4.1}$$

Let $b > 0$ and h be a measurable function. We consider the function h_b defined by

$$\forall x \in \mathbb{R}, \quad h_b(x) := \frac{1}{b^{2k+a-1}} h(\frac{x}{b}). \tag{4.2}$$

Proposition 4.1 *(i) For every $h \in L^p_{k,a}(\mathbb{R})$, $p \in [1,\infty]$. The function h_b belongs to $L^p_{k,a}(\mathbb{R})$ and we have*

$$||h_b||_{L^p_{k,a}(\mathbb{R})} = b^{(2k+a-1)(\frac{1}{p}-1)}||h||_{L^p_{k,a}(\mathbb{R})}. \tag{4.3}$$

(ii) Let $b > 0$ and h be in $L^1_{k,a}(\mathbb{R}) \bigcup L^2_{k,a}(\mathbb{R})$. We have

$$\forall y \in \mathbb{R}, \quad \mathcal{F}_{k,a}(h_b)(y) = \mathcal{F}_{k,a}(h)(by). \tag{4.4}$$

Let $b > 0$ and h be in $L^p_{k,a}(\mathbb{R})$, $p \in [1,\infty]$. We consider the family $h_{b,x}$, $x \in \mathbb{R}$, of (k,a)-generalized wavelets on $\mathbb{R}$ in $L^p_{k,a}(\mathbb{R})$, $p \in [1,\infty]$ defined by

$$\forall y \in \mathbb{R}, \quad h_{b,x}(y) := b^{\frac{2k+a-1}{2}} \tau^{k,a}_x \overline{h_b}(e^{\frac{2i\pi}{a}} y), \tag{4.5}$$

where $\tau^{k,a}_x$, $x \in \mathbb{R}$, are the generalized translation operators.

Definition 4.2 Let h be a (k,a)-generalized wavelet on $\mathbb{R}$ in $L^2_{k,a}(\mathbb{R})$. The (k,a)-generalized continuous wavelet transform $\Phi^{k,a}_h$ on $\mathbb{R}$ is defined for regular functions f on $\mathbb{R}$ by

$$\forall (b,x) \in \mathbb{R}^2_+, \quad \Phi^{k,a}_h(f)(b,x) = \int_{\mathbb{R}} f(y)h_{b,x}(y)d\gamma_{k,a}(y). \tag{4.6}$$

This transform can also be written in the form

$$\Phi_h^{k,a}(f)(b,x) = b^{\frac{2k+a-1}{2}} f *_{k,a} \overline{h_b}(x), \tag{4.7}$$

where $*_{k,a}$ is the generalized convolution product given by (3.26).

Remark 4.1 (i) Let h be a (k,a)-generalized wavelet in $L^2_{k,a}(\mathbb{R})$. Then from (3.28) and (4.7), for all f in $L^2_{k,a}(\mathbb{R})$ we have

$$\|\Phi_h^{k,a}(f)\|_{L^\infty_{\mu_{k,a}}(\mathbb{R}^2_+)} \leq C(k,a)\|f\|_{L^2_{k,a}(\mathbb{R})}\|h\|_{L^2_{k,a}(\mathbb{R})}. \tag{4.8}$$

(ii) For any $f \in L^2_{k,a}(\mathbb{R})$, we have

$$\mathcal{F}_{k,a}\Big(\Phi_h^{k,a}(f)(b,.)\Big)(\xi) = b^{\frac{2k+a-1}{2}}\mathcal{F}_{k,a}(\overline{h})(b\xi)\mathcal{F}_{k,a}(f)(\xi). \tag{4.9}$$

Proposition 4.2 *Let h be a (k,a)-generalized wavelet. Then for all f and g in $L^2_{k,a}(\mathbb{R})$, there holds*

$$\int_{\mathbb{R}^2_+}\Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}d\mu_{k,a}(b,x) = C_h\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x), \tag{4.10}$$

where

$$C_h := \int_0^\infty |\mathcal{F}_{k,a}(\overline{h})(b\xi)|^2\frac{db}{b}. \tag{4.11}$$

Proof Using Fubini's Theorem, relation (4.7), Parseval's formula (2.15) and (3.29), we get

$$\begin{aligned}
&\int_{\mathbb{R}^2_+}\Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}d\mu_{k,a}(b,x)\\
&= \int_0^\infty b^{2k+a-1}\int_{\mathbb{R}}\big(f *_{k,a}\overline{h}_b(x)\big)\overline{g *_{k,a}\overline{h}_b(x)}d\mu_{k,a}(b,x)\\
&= \int_0^\infty\int_{\mathbb{R}}\big(f *_{k,a}\overline{h}_b(x)\big)\overline{g *_{k,a}\overline{h}_b(x)}d\gamma_{k,a}(x)\frac{db}{b}\\
&= \int_0^\infty\int_{\mathbb{R}}\mathcal{F}_{k,a}(f)(\xi)\overline{\mathcal{F}_{k,a}(g)(\xi)}|\mathcal{F}_{k,a}(\overline{h}_b)(\xi)|^2 d\gamma_{k,a}(\xi)\frac{db}{b}\\
&= \int_{\mathbb{R}}\mathcal{F}_{k,a}(f)(\xi)\overline{\mathcal{F}_{k,a}(g)(\xi)}\Big(\int_0^\infty|\mathcal{F}_{k,a}(\overline{h}_b)(\xi)|^2\frac{db}{b}\Big)d\gamma_{k,a}(\xi).
\end{aligned}$$

On the other hand using the relations (4.1) and (4.4) we deduce that

$$\int_{\mathbb{R}^2_+} \Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}d\mu_{k,a}(b,x) = C_h \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\xi)\overline{\mathcal{F}_{k,a}(g)(\xi)}d\gamma_{k,a}(\xi).$$

Finally using Proposition 2.3 ii) we obtain the result. □

Remark 4.2 If h is a (k,a)-generalized wavelet and $f = g$ we obtain the following Plancherel's formula

$$\int_{\mathbb{R}^2_+} |\Phi_h^{k,a}(f)(b,x)|^2 d\mu_{k,a}(b,x) = C_h \int_{\mathbb{R}} |f(x)|^2 d\gamma_{k,a}(x). \tag{4.12}$$

We generalize the notion of the (k,a)-generalized wavelet as follows.

Definition 4.3 Let u and v be in $L^2_{k,a}(\mathbb{R})$. We say that the pair (u,v) is a (k,a)-generalized two-wavelet on $\mathbb{R}$ if the following integral, noted by $C_{u,v}$,

$$\int_0^\infty \overline{\mathcal{F}_{k,a}(\overline{v})(\lambda x)}\mathcal{F}_{k,a}(\overline{u})(\lambda x)\frac{d\lambda}{\lambda} \tag{4.13}$$

is constant for almost all $x \in \mathbb{R}$. We call the number $C_{u,v}$ the (k,a)-generalized two-wavelet constant associated with the functions u and v.

Theorem 4.1 *Let (u,v) be a (k,a)-generalized two-wavelet. Then for all f and g in $L^2_{k,a}(\mathbb{R})$*

$$\int_{\mathbb{R}^2_+} \Phi_u^{k,a}(f)(b,x)\overline{\Phi_v^{k,a}(g)(b,x)}d\mu_{k,a}(b,x) = C_{u,v} \int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x), \tag{4.14}$$

where

$$C_{u,v} := \int_0^\infty \mathcal{F}_{k,a}(\overline{u})(b\xi)\overline{\mathcal{F}_{k,a}(\overline{v})(b\xi)}\frac{db}{b}. \tag{4.15}$$

Proof Using Fubini's Theorem, relation (4.7), Parseval's formula (2.15) and (3.29), we get

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} \Phi_u^{k,a}(f)(b,x)\overline{\Phi_v^{k,a}(g)(b,x)}d\mu_{k,a}(b,x)\\
&= \int_0^\infty b^{2k+a-1}\int_{\mathbb{R}} f *_{k,a} \bar{u}_b(x)\overline{g *_{k,a} \bar{v}_b(x)}d\mu_{k,a}(b,x)\\
&= \int_0^\infty \int_{\mathbb{R}} f *_{k,a} \bar{u}_b(x)\overline{g *_{k,a} \bar{v}_b(x)}d\gamma_{k,a}(x)\frac{db}{b}
\end{aligned}$$

$$= \int_0^\infty \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\xi)\overline{\mathcal{F}_{k,a}(g)(\xi)}$$
$$\mathcal{F}_{k,a}(\bar{u}_b)(\xi)\overline{\mathcal{F}_{k,a}(\overline{v_b})(\xi)}d\gamma_{k,a}(\xi)\frac{db}{b}.$$

Using the definition of the (k, a)-generalized two-wavelet and relation (4.4), we deduce that

$$\int_{\mathbb{R}^2_+} \Phi_u^{k,a}(f)(b,x)\overline{\Phi_v^{k,a}(g)(b,x)}d\mu_{k,a}(b,x)$$
$$= \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\xi)\overline{\mathcal{F}_{k,a}(g)(\xi)}$$
$$\overline{\mathcal{F}_{k,a}(\overline{v})(b\xi)}\mathcal{F}_{k,a}(\overline{u})(b\xi)\frac{db}{b}\Big)d\gamma_{k,a}(\xi)$$
$$= C_{u,v}\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\xi)\overline{\mathcal{F}_{k,a}(g)(\xi)}d\gamma_{k,a}(\xi).$$

Finally using Parseval's formula (2.15) we obtain the result. □

Proposition 4.3 *Let h be a (k, a)-generalized wavelet, $f \in L^2_{k,a}(\mathbb{R})$ and p belongs in* $[2, \infty]$. *We have*

$$\|\Phi_h^{k,a}(f)\|_{L^p_{\mu_{k,a}}(\mathbb{R}^2_+)} \le (C_h)^{\frac{1}{p}}(C(k,a)\|h\|_{L^2_{k,a}(\mathbb{R})})^{\frac{p-2}{p}}\|f\|_{L^2_{k,a}(\mathbb{R})}. \tag{4.16}$$

Proof Using relations (4.12) and (4.8) the result follows by applying the Riesz–Thorin interpolation theorem. □

Theorem 4.2 (Inversion formula) *Let h be a (k, a)-generalized wavelet. For all f in $L^1_{k,a}(\mathbb{R})$ (resp. $L^2_{k,a}(\mathbb{R})$) such that $\mathcal{F}_{k,a}(f)$ belongs to $L^1_{k,a}(\mathbb{R})$ (resp. $L^1_{k,a}(\mathbb{R})\bigcap L^\infty_{k,a}(\mathbb{R})$) we have*

$$C_h\, f(y) = \int_{\mathbb{R}} \Big(\int_0^\infty \Phi_h^{k,a}(f)(b,x)\overline{h_{b,x}(y)}\frac{db}{b^{\frac{2k+a+1}{2}}}\Big)d\gamma_{k,a}(x),\ \ a.e. \tag{4.17}$$

where for each $y \in \mathbb{R}$, both the inner integral and the outer integral are absolutely convergent, but eventually not the double integral.

Proof Using similar ideas as in the proof for Theorem 6.III.3 of [48] page 99, we obtain the relation (4.17). □

Composition of wavelet transforms have been studied by Pathak [40] for the Fourier transform and later Prasad and Kumar [41] has been studied the same subject for the fractional Fourier transform. On the follow we will study the composition of the (k, a)-generalized wavelet transforms. Indeed, if h_1 and $h_2 \in L^2_{k,a}(\mathbb{R})$ are two (k, a)-generalized wavelets and $\Phi_{h_1}^{k,a}(f)(b, x)$, $\Phi_{h_2}^{k,a}(f)(c, y)$ are the (k, a)-generalized wavelet transforms of $f \in L^2_{k,a}(\mathbb{R})$, respectively, then from (4.9) we derive that the composition of these wavelet transforms is given by

$$\begin{aligned}\mathfrak{W}^{k,a}_{h_1,h_2}(f)(b,x,c) &= \Phi^{k,a}_{h_1}\big(\Phi^{k,a}_{h_2}(f)(.,c)\big)(b,x)\\ &= \int_{\mathbb{R}} \overline{B_{k,a}(x,\lambda)}\mathcal{F}_{k,a}\big(\Phi^{k,a}_{h_2}(f)(.,c)\big)(\lambda)(\mathcal{F}_{k,a}(\overline{h}_1))(b\lambda)d\gamma_{k,a}(\lambda)\\ &= \int_{\mathbb{R}} \overline{B_{k,a}(x,\lambda)}\mathcal{F}_{k,a}(f)(\lambda)(\mathcal{F}_{k,a}(\overline{h}_1))(b\lambda)(\mathcal{F}_{k,a}(\overline{h}_2))(c\lambda)d\gamma_{k,a}(\lambda).\end{aligned}$$

Thus, formally we can write

$$(\mathfrak{W}^{k,a}_{h_1,h_2}f)(b,x,c) = (f *_{k,a} \overline{h}_{1,b} *_{k,a} \overline{h}_{2,c})(x), \tag{4.18}$$

where $\overline{h}_{j,t}$, $j=1,2$, is defined by $\mathcal{F}_{k,a}(\overline{h}_{j,t})(\lambda) = t^{\frac{2k+a-1}{2}}\mathcal{F}_{k,a}(\overline{h}_j)(\lambda t)$.

Admissibility condition: Let $h_1,\ h_2 \in L^2_{k,a}(\mathbb{R})$. Consequently, the following definition serves as the admissibility requirement for the composition of two (k,a)-generalized wavelet transforms:

$$C_{h_1,h_2} = \int_0^\infty \int_0^\infty |\mathcal{F}_{k,a}(h_1)(b\lambda)|^2 |\mathcal{F}_{k,a}(h_2)(c\lambda)|^2 \frac{db}{b}\frac{dc}{c} < \infty. \tag{4.19}$$

Theorem 4.3 (Parseval's relation) *Let $h_1,\ h_2 \in L^2_{k,a}(\mathbb{R})$ be two (k,a)-generalized wavelets. Then, given $f,\ g \in L^2_{k,a}(\mathbb{R})$, we have*

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^\infty\int_0^\infty \mathfrak{W}^{k,a}_{h_1,h_2}f(b,x,c)\overline{\mathfrak{W}^{k,a}_{h_1,h_2}g(b,x,c)}d\gamma_{k,a}(x)\frac{db}{b^{\frac{2k+a+1}{2}}}\frac{dc}{c^{\frac{2k+a+1}{2}}}\\ &= C_{h_1,h_2}\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x),\end{aligned}$$

where C_{h_1,h_2} is defined as (4.19).

Proof Using (4.18), Parseval's formula for the (k,a)-generalized Fourier transform and (3.29), we get

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^\infty\int_0^\infty \mathfrak{W}^{k,a}_{h_1,h_2}f(b,x,c)\overline{\mathfrak{W}^{k,a}_{h_1,h_2}g(b,x,c)}d\gamma_{k,a}(x)\frac{db}{b^{\frac{2k+a+1}{2}}}\frac{dc}{c^{\frac{2k+a+1}{2}}}\\ &= \int_{\mathbb{R}}\int_0^\infty\int_0^\infty \mathcal{F}_{k,a}(f)(\lambda)\mathcal{F}_{k,a}(\overline{h}_1)(b\lambda)\mathcal{F}_{k,a}(\overline{h}_2)(c\lambda)\\ &\quad \overline{\mathcal{F}_{k,a}(g)}(\lambda)\overline{\mathcal{F}_{k,a}(\overline{h}_1)(b\lambda)\mathcal{F}_{k,a}(\overline{h}_2)(c\lambda)}d\gamma_{k,a}(\lambda)\frac{db}{b}\frac{dc}{c}.\end{aligned}$$

Now from (4.19) the above equality is written as

$$C_{h_1,h_2}\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\overline{\mathcal{F}_{k,a}(g)(\lambda)}d\gamma_{k,a}(\lambda) = C_{h_1,h_2}\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x). \tag{4.20}$$

This completes the proof. □

Remark 4.3 When $f = g$, we derive the following Plancherel's formula

$$\int_{\mathbb{R}} \int_0^\infty \int_0^\infty |\mathfrak{W}_{h_1,h_2}^{k,a} f(b,x,c)|^2 d\gamma_{k,a}(x) \frac{db}{b^{\frac{2k+a+1}{2}}} \frac{dc}{c^{\frac{2k+a+1}{2}}} = C_{h_1,h_2} ||f||^2_{L^2_{k,a}(\mathbb{R})}.$$

The characterization of $L^p_{k,a}(\mathbb{R})$, for $1 < p < \infty$, by the mean of the (k,a)-generalized wavelet transform is given by the following theorem. A function $h \in L^2_{k,a}(\mathbb{R})$ is said to satisfy (H_1) if

$$\int_{||y|-|x||\geq 2|x-x_0|} |\tau_x^{k,a} h(y) - \tau_{x_0}^{k,a} h(y)| d\gamma_{k,a}(y) \leq C, \quad x, x_0 \in \mathbb{R}.$$

In the rest of this subsection, we assume that the (k,a)-generalized wavelet h satisfy (H_1).

Theorem 4.4 *Let h be a real (k,a)-generalized wavelet. Then the (k,a)-generalized wavelet transform $\Phi_h^{k,a}$ is a bounded linear operator*

$$\begin{array}{lll} L^p_{k,a}(\mathbb{R}) \to & L^2(\mathbb{R}^+, \frac{db}{b^{2k+a}}) \times L^p_{k,a}(\mathbb{R}), \\ f & \longmapsto \Phi_h^{k,a}(f), \end{array}$$

moreover, for any $f \in L^p_{k,a}(\mathbb{R}),\ 1 < p < \infty$

$$\|f\|_{L^p_{k,a}(\mathbb{R})} \simeq \Big(\int_{\mathbb{R}} \Big(\int_0^\infty |\Phi_h^{k,a}(f)(b,x)|^2 \frac{db}{b^{2k+a}} \Big)^{\frac{p}{2}} d\gamma_{k,a}(x) \Big)^{\frac{1}{p}}. \qquad (4.21)$$

Proof Let $W^p_{k,a}$ denote the space $L^2(\mathbb{R}^+, \frac{db}{b^{2k+a}}) \times L^p_{k,a}(\mathbb{R})$ associated to the norm

$$\|f\|_{W^p_{k,a}} = \Big\{ \int_{\mathbb{R}} \Big(\int_0^\infty |f(b,x)|^2 \frac{db}{b^{2k+a}} \Big)^{\frac{p}{2}} d\gamma_{k,a}(x) \Big\}^{\frac{1}{p}}.$$

If we take $p = 2$, then from Plancherel's formula (4.12):

$$\|\Phi_h^{k,a}(f)\|_{W^2_{k,a}} = \Big\{ \int_{\mathbb{R}} \Big(\int_0^\infty |\Phi_h^{k,a}(f)(b,x)|^2 \frac{db}{b^{2k+a}} \Big) d\gamma_{k,a}(x) \Big\}^{\frac{1}{2}} = \sqrt{C_h} \|f\|_{L^2_{k,a}(\mathbb{R})}.$$

Then the singular integral theorem, (see [45]), applied to operators with values in $L^2(\mathbb{R}^+, \frac{db}{b^{2k+a}})$, leads to the:

$$\|\Phi_h^{k,a}(f)\|_{W^p_{k,a}} \leq A(p,h) \|f\|_{L^p_{k,a}(\mathbb{R})}, \quad 1 < p \leq 2,$$

where the constant $A(p,h)$ depends only on p and h. Due to duality the inequality is also valid for $1 < p < \infty$. It follows that

$$\Big(\int_{\mathbb{R}}\Big(\int_0^{\infty}|\Phi_h^{k,a}(f)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p}{2}}d\gamma_{k,a}(x)\Big)^{\frac{1}{p}} \leq A(p,h)\|f\|_{L^p_{k,a}(\mathbb{R})}. \quad (4.22)$$

Conversely suppose that $f \in L^2_{k,a}(\mathbb{R}) \cap L^p_{k,a}(\mathbb{R})$. Since the (k,a)-generalized wavelet transform being an isometry, for every $g \in L^2_{k,a}(\mathbb{R}) \cap L^{p'}_{k,a}(\mathbb{R})$, we can write

$$\frac{1}{C_h}\int_{\mathbb{R}}\int_0^{\infty}\Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}\frac{db}{b^{2k+a}}d\gamma_{k,a}(x) = \int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x). \quad (4.23)$$

Now,

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x)\Big| \\
&= \frac{1}{C_h}\Big|\int_{\mathbb{R}}\int_0^{\infty}\Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}\frac{db}{b^{2k+a}}d\gamma_{k,a}(x)\Big| \\
&\leq \frac{1}{C_h}\int_{\mathbb{R}}\int_0^{\infty}|\Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}|\frac{db}{b^{2k+a}}d\gamma_{k,a}(x).
\end{aligned}$$

Involving Cauchy-Schwarz's inequality, Hölder's inequality and (4.22), we get

$$\begin{aligned}
&\Big|\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x)\Big| \\
&\leq \frac{1}{C_h}\Big(\int_{\mathbb{R}}\Big(\int_0^{\infty}|\Phi_h^{k,a}(f)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p}{2}}d\gamma_{k,a}(x)\Big)^{\frac{1}{p}} \\
&\Big(\int_{\mathbb{R}}\Big(\int_0^{\infty}|\Phi_h^{k,a}(g)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p'}{2}}d\gamma_{k,a}(x)\Big)^{\frac{1}{p'}} \\
&\leq \frac{A(p,h)}{C_h}\Big(\int_{\mathbb{R}}\Big(\int_0^{\infty}|\Phi_h^{k,a}(f)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p}{2}}d\gamma_{k,a}(x)\Big)^{\frac{1}{p}}\|g\|_{L^{p'}_{k,a}(\mathbb{R})}.
\end{aligned}$$

By density the inequality is valid for all $g \in L^{p'}_{k,a}(\mathbb{R})$, remember that:

$$\|f\|_{L^p_{k,a}(\mathbb{R})} = \sup_{g\in L^{p'}_{k,a}(\mathbb{R})}\frac{|\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x)|}{||g||_{L^{p'}_{k,a}(\mathbb{R})}}$$

then, we get

$$\|f\|_{L^p_{k,a}(\mathbb{R})} \leq \frac{A(p,h)}{C_h}\Big(\int_0^{\infty}\Big(\int_0^{\infty}|\Phi_h^{k,a}(f)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p}{2}}d\gamma_{k,a}(x)\Big)^{\frac{1}{p}}.$$

□

In the rest of this subsection, we assume that $a = \frac{2}{n}$.

Theorem 4.5 *(*Parseval's formula) *Let us assume* $f \in L^p_{k,a}(\mathbb{R})$, $g \in L^{p'}_{k,a}(\mathbb{R})$ *with* $1 < p < \infty$. *If h is a real* (k, a)*-generalized wavelet, then*

$$\frac{1}{C_h}\int_{\mathbb{R}}\int_0^\infty \Phi_h^{k,a}(f)(b,x)\overline{\Phi_h^{k,a}(g)(b,x)}\frac{db}{b^{2k+a}}d\gamma_{k,a}(x) = \int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,a}(x). \tag{4.24}$$

Proof Consider the bilinear transform L:

$$\begin{aligned} L : L^p_{k,a}(\mathbb{R}) \times L^{p'}_{k,a}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ (f, g) &\longmapsto \langle \Phi_h^{k,a}(f), \Phi_h^{k,a}(g)\rangle_{d\mu_{k,a}}. \end{aligned}$$

where $\langle ., .\rangle_{d\mu_{k,a}}$ is defined by

$$\langle u, \chi\rangle_{d\mu_{k,a}(b,x)} = \int_{\mathbb{R}}\int_0^\infty u(b,x)\chi(b,x)d\mu_{k,a}(b,x), \quad (u,\chi) \in W^p_{k,a} \times W^{p'}_{k,a}.$$

Involving Hölder's inequality two times, we get

$$\begin{aligned} |L(f,g)| &= |\langle \Phi_h^{k,a}(f), \Phi_h^{k,a}(g)\rangle_{d\mu_{k,a}}| \\ &\le \int_{\mathbb{R}}\Big(\int_0^\infty |\Phi_h^{k,a}(f)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{1}{2}}\Big(\int_0^\infty |\Phi_h^{k,a}(g)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{1}{2}} d\gamma_{k,a}(x) \\ &\le \Big(\int_{\mathbb{R}}\Big(\int_0^\infty |\Phi_h^{k,a}(f)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p}{2}} d\gamma_{k,a}(x)\Big)^{\frac{1}{p}} \\ &\quad \Big(\int_{\mathbb{R}}\Big(\int_0^\infty |\Phi_h^{k,a}(g)(b,x)|^2\frac{db}{b^{2k+a}}\Big)^{\frac{p'}{2}} d\gamma_{k,a}(x)\Big)^{\frac{1}{p'}}. \end{aligned}$$

Using Theorem 4.4, we have

$$|L(f,g)| \le C\|f\|_{L^p_{k,a}(\mathbb{R})}\|g\|_{L^{p'}_{k,a}(\mathbb{R})}. \tag{4.25}$$

Moreover for all $f \in L^2_{k,a}(\mathbb{R}) \cap L^p_{k,a}(\mathbb{R})$ and $g \in L^2_{k,a}(\mathbb{R}) \cap L^{p'}_{k,a}(\mathbb{R})$ we get

$$L(f,g) = \langle \Phi_h^{k,a}(f), \Phi_h^{k,a}(g)\rangle_{d\mu_{k,a}} = C_h\langle f, g\rangle_{L^2_{k,a}(\mathbb{R})}. \tag{4.26}$$

From equations (4.25), (4.26) and density of spaces $L^2_{k,a}(\mathbb{R}) \cap L^p_{k,a}(\mathbb{R})$ in $L^p_{k,a}(\mathbb{R})$ gives the result. □

Corollary 4.1 (An inversion formula) *Let h be a real (k,a)-generalized wavelet. For all f in $L^p_{k,a}(\mathbb{R})$, $1 < p < \infty$, we have*

$$f(y) = \frac{1}{C_h} \int_{\mathbb{R}} \int_0^\infty \Phi_h^{k,a}(f)(b,x) h_{b,x}(y) d\mu_{k,a}(b,x). \tag{4.27}$$

The equality holds in $L^p_{k,a}(\mathbb{R})$ sense and the integral of right hand side have to be taken in the sense of distributions.

Proof The proof follows from Theorem 4.5. □

Remark 4.4 We note that the authors in [31], have studied the analogues of the results presented in this section in the setting of a class of singular differential-difference operators on the real line.

4.2 Time-Invariant Filter

Definition 4.4 Let f be any signal and $b \in \mathbb{R}$. Then the linear operator L is said to be time-invariant filter if it satisfies

$$L(\tau_b^{k,a} f)(t) = \tau_b^{k,a}(Lf)(t), \quad \text{for all } t \in \mathbb{R}.$$

As follows theorem, we will show that the (k,a)-generalized convolution operator is time-invariant filter.

Lemma 4.1 *Let L be a linear time-invariant filter. Then, there exist a function $g \in L^1_{k,a}(\mathbb{R})$ such that*

$$L(B_{k,a}(\lambda,t)) = \overline{\mathcal{F}_{k,a}(g)(\lambda)} B_{k,a}(\lambda,t). \tag{4.28}$$

Proof Set

$$g^\lambda(t) = L(B_{k,a}(\lambda,t)). \tag{4.29}$$

Applying operator L to (3.8) and using (4.29), we have

$$\begin{aligned} L(B_{k,a}(\lambda,t)B_{k,a}(\lambda,y)) &= \int_{\mathbb{R}} L(B_{k,a}(\lambda,x)) d\zeta_{t,y}^{k,a}(x) \\ &= \int_{\mathbb{R}} g^\lambda(x) d\zeta_{t,y}^{k,a}(x). \end{aligned} \tag{4.30}$$

As, L is linear, then by (4.29), we get

$$L(B_{k,a}(\lambda,t)B_{k,a}(\lambda,y)) = B_{k,a}(\lambda,y) g^\lambda(t). \tag{4.31}$$

Thus (4.30) and (4.31) give us

$$\int_{\mathbb{R}} g^{\lambda}(x)d\zeta_{t,y}^{k,a}(x) = B_{k,a}(\lambda, y)g^{\lambda}(t). \tag{4.32}$$

Put $t = 0$ in (4.32) and using (3.9), we obtain

$$g^{\lambda}(y) = g^{\lambda}(0)B_{k,a}(\lambda, y).$$

Now replacing y by t and assume $g^{\lambda}(0) = \overline{\mathcal{F}_{k,a}(g)(\lambda)}$ for some function $g \in L^1_{k,a}(\mathbb{R})$, we obtain

$$g^{\lambda}(t) = \overline{\mathcal{F}_{k,a}(g)(\lambda)}B_{k,a}(\lambda, t). \tag{4.33}$$

Thus, involving (4.29) and (4.33) the result is proved. □

Theorem 4.6 *Let L be a linear time-invariant filter. For each signal f, we have*

$$Lf(t) = f *_{k,a} g(t),$$

where g is the function given by (4.28).

Proof We have

$$f(t) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\overline{B_{k,a}(\lambda, t)}d\gamma_{k,a}(\lambda).$$

Applying operator L to both sides of the last identity

$$Lf(t) = L\Big[\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\overline{B_{k,a}(\lambda, t)}d\gamma_{k,a}(\lambda)\Big].$$

Using (4.28), we obtain

$$\begin{aligned} Lf(t) &= \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\mathcal{F}_{k,a}(g)(\lambda)\overline{B_{k,a}(\lambda, t)}d\gamma_{k,a}(\lambda) \\ &= \int_{\mathbb{R}} \mathcal{F}_{k,a}(f *_{k,a} g)(\lambda)\overline{B_{k,a}(\lambda, t)}d\gamma_{k,a}(\lambda). \end{aligned}$$

Using inversion formula for the (k, a)-generalized Fourier transform, we derive that

$$Lf(t) = f *_{k,a} g(t).$$

□

Theorem 4.7 *Let $f \in L^2_{k,a}(\mathbb{R})$ and $h \in L^2_{k,a}(\mathbb{R})$ be a (k,a)-generalized wavelet. Let $(b,x) \in \mathbb{R}^2_+$, there exist a linear time-invariant filter L_b such that we can write the (k,a)-generalized wavelet transform as*

$$\Phi_h^{k,a}(f)(b,x) = L_b f(x). \tag{4.34}$$

Proof Involving (4.9) and the inversion formula for the (k,a)-generalized Fourier transform, we get

$$\Phi_h^{k,a}(f)(b,x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\mathcal{F}_{k,a}(\overline{h})(b\lambda)\overline{B_{k,a}(\lambda,x)}d\gamma_{k,a}(\lambda).$$

From above, we can prove that there exist a linear time-invariant filter L_b satisfying

$$L_b[\overline{B_{k,a}(\lambda,x)}] = \mathcal{F}_{k,a}(\overline{h})(b\lambda)\overline{B_{k,a}(\lambda,x)}.$$

So, we derive that

$$\Phi_h^{k,a}(f)(b,x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)L_b[\overline{B_{k,a}(\lambda,x)}]d\gamma_{k,a}(\lambda).$$

By linearity property of L_b, we have

$$\Phi_h^{k,a}(f)(b,x) = L_b\Big[\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\overline{B_{k,a}(\lambda,x)}d\gamma_{k,a}(\lambda)\Big].$$

Hence

$$\Phi_h^{k,a}(f)(b,x) = L_b f(x).$$

This completes the Theorem. □

Example 4.1 Let h be a function with a finite support. For a signal f, we put

$$Lf(t) = h *_{k,a} f(t). \tag{4.35}$$

The operator L is a time-invariant filter.

Proof On the follow we will to prove that the operator L is time-invariant filter. Indeed, for any $b \in \mathbb{R}$, we have

$$\begin{aligned}\tau_b^{k,a}(Lf)(t) &= \int_{\mathbb{R}} Lf(z)\overline{\Delta_{k,a}(b,t,z)}d\gamma_{k,a}(z)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} h(x)\tau^{k,a}_{e^{\frac{2i\pi}{a}}x} f(z)\overline{\Delta_{k,a}(b,t,z)}d\gamma_{k,a}(x)d\gamma_{k,a}(z)\end{aligned}$$

$$= \int_{\mathbb{R}} h(x)\Big(\int_{\mathbb{R}} \tau^{k,a}_{e^{\frac{2i\pi}{a}}x} f(z)\overline{\Delta_{k,a}(b,t,z)}d\gamma_{k,a}(z)\Big)d\gamma_{k,a}(x)$$

$$= \int_{\mathbb{R}} h(x)\tau^{k,a}_b \tau^{k,a}_{e^{\frac{2i\pi}{a}}x} f(t)d\gamma_{k,a}(x)$$

$$= \int_{\mathbb{R}} h(x)\tau^{k,a}_{e^{\frac{2i\pi}{a}}x} (\tau^{k,a}_b f)(t)d\gamma_{k,a}(x)$$

$$= \tau^{k,a}_b f *_{k,a} h(t)$$

$$= L(\tau^{k,a}_b f)(t).$$

This completes the proof. □

Remark 4.5 We note that Prasad and Kumar [41], have studied the time-filter for the fractional wavelet transform.

4.3 (k, a)-Generalized Hausdorff Operator

The main purpose of this subsection is to extend some results of the classical Hausdorff operator given in [19, 30] in the (k, a)-generalized Fourier setting.

Definition 4.5 Let $\varphi \in L^1(\mathbb{R})$ and $f \in L^1_{k,a}(\mathbb{R})$. The (k, a)-generalized Hausdorff operator is defined as:

$$\mathcal{H}^{k,a}_{\varphi} f(x) := \int_{\mathbb{R}} \frac{\varphi(t)}{|t|^{2k+a-1}} f(\frac{x}{t})dt, \quad x \in \mathbb{R}. \tag{4.36}$$

By simple calculations we prove the following:

Lemma 4.2 *If f belongs to $L^1_{k,a}(\mathbb{R})$ and φ belongs to $L^1(\mathbb{R})$, we have*

$$\mathcal{F}_{k,a}(\mathcal{H}^{k,a}_{\varphi} f)(t) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(tx)\varphi(x)dx. \tag{4.37}$$

Definition 4.6 For $f \in L^2_{k,a}(\mathbb{R})$, the Riesz transform $\mathcal{R}_{k,a}$ is defined as:

$$\mathcal{R}_{k,a}(f)(x) = C_{k,a} \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} \frac{\tau^{k,a}_x f(y)}{|y|^{2k+a}} d\gamma_{k,a}(y), \tag{4.38}$$

where $C_{k,a} = a^{\frac{2k+a-1}{a}} \frac{\Gamma(\frac{2k+a}{a})}{\Gamma(\frac{1}{a})}$.

The Hardy type space $H^1_{k,a}(\mathbb{R})$ associated with the (k, a)-generalized Fourier transform is defined by

$$H^1_{k,a}(\mathbb{R}) := \Big\{ f \in L^1_{k,a}(\mathbb{R});\ \mathcal{R}_{k,a}(f) \in L^1_{k,a}(\mathbb{R}) \Big\},$$

endowed with the norm

$$\|f\|_{H^1_{k,a}(\mathbb{R})} := \|f\|_{L^1_{k,a}(\mathbb{R})} + \|\mathcal{R}_{k,a}(f)\|_{L^1_{k,a}(\mathbb{R})}. \tag{4.39}$$

We note that in [2], we have proved that $\mathcal{R}_{k,a}$ is a multiplier operator given by

$$\mathcal{F}_{k,a}(\mathcal{R}_{k,a} f)(t) = (e^{\frac{-i\pi}{a}} \operatorname{sgn} t)\mathcal{F}_{k,a}(f)(t). \tag{4.40}$$

Thus, it follows immediately that if $f \in H^1_{k,a}(\mathbb{R})$, then

$$\mathcal{F}_{k,a}(f)(0) = 0 \tag{4.41}$$

and by uniqueness of the (k, a)-generalized Fourier transform

$$\mathcal{R}_{k,a}(\mathcal{R}_{k,a} f)(t) = e^{\frac{-2i\pi}{a}} f(t). \tag{4.42}$$

In particular, if $f \in H^1_{k,a}(\mathbb{R})$, then $\mathcal{R}_{k,a} f \in H^1_{k,a}(\mathbb{R})$ and

$$\|\mathcal{R}_{k,a} f\|_{H^1_{k,a}(\mathbb{R})} = \|f\|_{H^1_{k,a}(\mathbb{R})}.$$

Theorem 4.8 *Let $p \in [1, \infty]$ and φ be a measurable function on $\mathbb{R}$ such that*

$$C(p, k, a, \varphi) = \int_{\mathbb{R}} |\varphi(x)||t|^{(2k+a-1)(\frac{1}{p}-1)} dt < \infty.$$

The Hausdorff operator $\mathcal{H}^{k,a}_{\varphi}$ is bounded linear operator on $L^p_{k,a}(\mathbb{R})$, with

$$\|\mathcal{H}^{k,a}_{\varphi} f\|_{L^p_{k,a}(\mathbb{R})} \le C(p, k, a, \varphi)\|f\|_{L^p_{k,a}(\mathbb{R})}.$$

Proof Let us note $f_t(x) := f(x/t),\ d\gamma_{k,a}(t) := |t|^{-2k-a+1}|\varphi(t)|dt$ and let's consider the integral

$$I = \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f_t(x)| d\gamma_{k,a}(t)\Big)^p d\gamma_{k,a}(x)\Big)^{1/p}.$$

Using Minkowski's inequality for the measure $d\gamma_{k,a}$, we get

$$\begin{aligned} I &= \| \int_{\mathbb{R}} |f_t(.)| d\gamma_{k,a}(t)\|_{L^p_{k,a}(\mathbb{R})} \le \int_{\mathbb{R}} \|f_t(.)\|_{L^p_{k,a}(\mathbb{R})} d\gamma_{k,a}(t) \\ &= \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |f_t(x)|^p d\gamma_{k,a}(x)\Big)^{1/p} d\gamma_{k,a}(t). \end{aligned}$$

For $t \neq 0$ fixed, the change of variable $x \mapsto u = x/t$ in the last integral gives

$$\int_{\mathbb{R}} |f_t(x)|^p d\gamma_{k,a}(x) = \int_{\mathbb{R}} |f(x/t)|^p d\gamma_{k,a}(x) = |t|^{2k+a-1} \int_{\mathbb{R}} |f(u)|^p d\gamma_{k,a}(u) = |t|^{2k+a-1} \|f\|^p_{L^p_{k,a}(\mathbb{R})}.$$

Thus

$$I \leq \|f\|_{L^p_{k,a}(\mathbb{R})} \int_{\mathbb{R}} |\varphi(t)||t|^{(2k+a-1)(\frac{1}{p}-1)} dt = C(p,k,a,\varphi)\|f\|_{L^p_{k,a}(\mathbb{R})}.$$

Going back to the definition of I, we deduce that the integral $\mathcal{H}^{k,a}_\varphi f(x) = \int_{\mathbb{R}} f_t(x) d\gamma_{k,a}(t)$ is absolutely convergent for almost all $x \in \mathbb{R}$, and defines a function $\mathcal{H}^{k,a}_\varphi f \in L^p_{k,a}(\mathbb{R})$ with

$$\|\mathcal{H}^{k,a}_\varphi f\|_{L^p_{k,a}(\mathbb{R})} \leq I \leq C(p,k,a,\varphi)\|f\|_{L^p_{k,a}(\mathbb{R})}.$$

□

Remark 4.6 When $p = 1$, $C(p,k,a,\varphi) = ||\varphi||_{L^1(\mathbb{R})}$.

Lemma 4.3 *Let $\varphi \in L^1(\mathbb{R})$ and $\widetilde{\varphi}(t) := (\operatorname{sgn} t)\varphi(t),\ t \in \mathbb{R}$. Then for $f \in L^1_{k,a}(\mathbb{R})$*

$$\mathcal{R}_{k,a}\mathcal{H}^{k,a}_\varphi f = \mathcal{H}^{k,a}_{\widetilde{\varphi}}\mathcal{R}_{k,a} f.$$

Proof Let $\varphi \in L^1(\mathbb{R})$. By Theorem 4.8, for $p = 1$, we have $\mathcal{H}^{k,a}_\varphi f \in L^1(\mathbb{R})$. From (4.40) and (4.37) it follows that

$$\mathcal{F}_{k,a}\mathcal{R}_{k,a}\mathcal{H}^{k,a}_\varphi f(y) = e^{\frac{-i\pi}{a}}(\operatorname{sgn} y)\mathcal{F}_{k,a}\mathcal{H}^{k,a}_\varphi f(y) = e^{\frac{-i\pi}{a}}(\operatorname{sgn} y)\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(ty)\varphi(t)dt$$

and analogously,

$$\begin{aligned}
\mathcal{F}_{k,a}\mathcal{H}^{k,a}_{\widetilde{\varphi}}\mathcal{R}_{k,a} f(y) &= \int_{\mathbb{R}} \mathcal{F}_{k,a}(\mathcal{R}_{k,a} f)(ty)\widetilde{\varphi}(t)dt \\
&= e^{\frac{-i\pi}{a}} \int_{\mathbb{R}} (\operatorname{sgn} ty)\mathcal{F}_{k,a}(f)(ty)\widetilde{\varphi}(t)dt \\
&= e^{\frac{-i\pi}{a}}(\operatorname{sgn} y) \int_{\mathbb{R}} (\operatorname{sgn} t)\mathcal{F}_{k,a}(f)(ty)\widetilde{\varphi}(t)dt \\
&= e^{\frac{-i\pi}{a}}(\operatorname{sgn} y) \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(ty)\varphi(t)dt.
\end{aligned}$$

Involving the injectivity of the generalized Fourier transform $\mathcal{F}_{k,a}$, we derive the result. □

On the follow we will to prove the boundedness of the (k, a)-generalized Hausdorff operators on the Hardy space $H^1_{k,a}(\mathbb{R})$.

Theorem 4.9 *Let $\varphi \in L^1(\mathbb{R})$. Then for $f \in H^1_{k,a}(\mathbb{R})$*

$$\|\mathcal{H}^{k,a}_\varphi f\|_{H^1_{k,a}(\mathbb{R})} \le \|\varphi\|_{L^1(\mathbb{R})}\|f\|_{H^1_{k,a}(\mathbb{R})}.$$

Proof Let $f \in H^1_{k,a}(\mathbb{R})$. Then $\mathcal{R}_{k,a} f \in L^1_{k,a}(\mathbb{R})$ hence by Theorem 4.8, for $p = 1$, and Lemma 4.3 we have

$$\begin{aligned}
\|\mathcal{H}^{k,a}_\varphi f\|_{H^1_{k,a}(\mathbb{R})} &= \|\mathcal{H}^{k,a}_\varphi f\|_{L^1_{k,a}(\mathbb{R})} + \|\mathcal{R}_{k,a}\mathcal{H}^{k,a}_\varphi f\|_{L^1_{k,a}(\mathbb{R})}\\
&= \|\mathcal{H}^{k,a}_\varphi f\|_{L^1_{k,a}(\mathbb{R})} + \|\mathcal{H}^{k,a}_{\widetilde{\varphi}}\mathcal{R}_{k,a} f\|_{L^1_{k,a}(\mathbb{R})}\\
&\le \|\varphi\|_{L^1(\mathbb{R})}\|f\|_{L^1_{k,a}(\mathbb{R})} + \|\widetilde{\varphi}\|_{L^1(\mathbb{R})}\|\mathcal{R}_{k,a} f\|_{L^1_{k,a}(\mathbb{R})}.
\end{aligned}$$

This shows the theorem, since $\|\varphi\|_{L^1(\mathbb{R})} = \|\widetilde{\varphi}\|_{L^1(\mathbb{R})}$. □

Proposition 4.4 *Let $h \in L^2_{k,a}(\mathbb{R})$ be a (k, a)-generalized wavelet and $f \in L^1_{k,a}(\mathbb{R})$, then the generalized convolution $f *_{k,a} h$ is a (k, a)-generalized wavelet.*

Proof Using Theorem 3.4, we derive that $f *_{k,a} h \in L^2_{k,a}(\mathbb{R})$ and we have

$$\|f *_{k,a} h\|_{L^2_{k,a}(\mathbb{R})} \le C(k, a)\|f\|_{L^1_{k,a}(\mathbb{R})}\|h\|_{L^2_{k,a}(\mathbb{R})}.$$

Next, we have to show that

$$\int_0^\infty |\mathcal{F}_{k,a}(f *_{k,a} h)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty.$$

Involving Plancherel's formula (2.15) and (2.13), we get

$$\begin{aligned}
\int_0^\infty |\mathcal{F}_{k,a}(f *_{k,a} h)(\lambda)|^2 \frac{d\lambda}{\lambda} &= \int_0^\infty |\mathcal{F}_{k,a}(f)(\lambda)\mathcal{F}_{k,a}(h)(\lambda)|^2 \frac{d\lambda}{\lambda} < \infty\\
&\le \|\mathcal{F}_{k,a}(f)\|_{L^\infty_{k,a}(\mathbb{R})} \int_0^\infty |\mathcal{F}_{k,a}(h)(\lambda)|^2 \frac{d\lambda}{\lambda}\\
&\le \|f\|_{L^1_{k,a}(\mathbb{R})} \int_0^\infty |\mathcal{F}_{k,a}(h)(\lambda)|^2 \frac{d\lambda}{\lambda}.
\end{aligned}$$

This shows that $f *_{k,a} h$ is a (k, a)-generalized wavelet. □

We obtain a relation between the (k, a)-generalized wavelet transformation and the (k, a)-generalized Hausdorff operator.

Theorem 4.10 *Let $f \in L^2_{k,a}(\mathbb{R})$, $h \in L^1_{k,a}(\mathbb{R}) \cap L^2_{k,a}(\mathbb{R})$ a (k,a)-generalized wavelet and a measurable function φ on $\mathbb{R}$ such that $\int_{\mathbb{R}} |\varphi(t)||t|^{-\frac{2k+a-1}{2}} dt < \infty$. Then, we have*

$$\Phi_h^{k,a}(\mathcal{H}_\varphi^{k,a} f)(b,x) = \int_{\mathbb{R}} \Phi_h^{k,a}(f)\left(\frac{b}{t}, \frac{x}{t}\right) t^{-2k-a+1}\varphi(t)dt.$$

Proof Let $h \in L^2_{k,a}(\mathbb{R})$ be a (k,a)-generalized wavelet. By Parseval's formula (2.15) and (4.4) we have

$$\Phi_h^{k,a}(f)(b,x) = \int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda)\mathcal{F}_{k,a}(\overline{h})(b\lambda)\overline{B_{k,a}(x,\lambda)}d\gamma_{k,a}(\lambda).$$

Therefore,

$$\begin{aligned}\Phi_h^{k,a}(\mathcal{H}_\varphi^{k,a} f)(b,x) &= \int_{\mathbb{R}} \mathcal{F}_{k,a}(\mathcal{H}_\varphi^{k,a} f)(\lambda)\mathcal{F}_{k,a}(\overline{h})(b\lambda)\overline{B_{k,a}(x,\lambda)}d\gamma_{k,a}(\lambda)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(\lambda t)\varphi(t)dt\mathcal{F}_{k,a}(\overline{h})(b\lambda)\overline{B_{k,a}(x,\lambda)}d\gamma_{k,a}(\lambda).\end{aligned}$$

Putting $\lambda t = u$, we obtain

$$\begin{aligned}\Phi_h^{k,a}(\mathcal{H}_\varphi^{k,a} f)(b,x) &= \int_{\mathbb{R}} \frac{\varphi(t)}{t^{2k+a-1}}\Big(\int_{\mathbb{R}} \mathcal{F}_{k,a}(f)(u)\mathcal{F}_{k,a}(\overline{h})(\frac{b}{t}u)\overline{B_{k,a}(\frac{x}{t},u)}\Big)d\gamma_{k,a}(u)\\ &= \int_{\mathbb{R}} \Phi_h^{k,a}(f)\left(\frac{b}{t}, \frac{x}{t}\right) t^{-2k-a+1}\varphi(t)dt.\end{aligned}$$

Thus the theorem is proved. □

5 Conclusion

In the present paper we have successfully studied the generalized translation operator and the generalized convolution product associated with the (k,a)-generalized Fourier transform. Profiting of the harmonic analysis presented in this paper, we have studied the analogues of the results of [35]. In the forthcoming papers, we will focus on some problems of time-frequency analysis, harmonic analysis and PDE.

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

References

1. Amri, B.: Product formula for one-dimensional (k, a)-generalized Fourier kernel. arXiv:2301.06587
2. Amri, B., Mejjaoli H.: (k, a)-Generalized Riesz transforms and applications. Preprint (2023)
3. Ben Hamadi, N., Omri, S.: Uncertainty principles for the continuous wavelet transform in the Hankel setting. Appl. Anal. **97**, 513–527 (2018)
4. Ben Saïd, S., Kobayashi, T., Ørsted, B.: Laguerre semigroup and Dunkl operators. Compos. Math. **148**(4), 1265–1336 (2012)
5. Ben Saïd, S.: A product formula and a convolution structure for a k-Hankel transform on $\mathbb{R}$. J. Math. Anal. Appl. **463**(2), 1132–1146 (2018)
6. Ben Saïd, S., Deleaval, L.: A Hardy–Littlewood maximal operator for the generalized Fourier transform on $\mathbb{R}$. J. Geol. Anal. **30**, 2273–2289 (2020)
7. Ben Saïd, S., Deleaval, L.: Translation operator and maximal function for the $(k, 1)$-generalized Fourier transform. J. Funct. Anal. **279**(8), 108706 (2020)
8. Ben Saïd, S., Negzaoui, S.: Flett potentials associated with differential-difference Laplace operators. J. Math. Phys. **63**, 033504 (2022). https://doi.org/10.1063/5.0063053
9. Ben Saïd, S., Negzaoui, S.: Norm inequalities for maximal operators. J. Inequal. Appl. **1**, 1–18 (2022)
10. Boubatra, M.A., Negzaoui, S., Sifi, M.: A new product formula involving Bessel functions. Integr. Transforms Spec. Funct. **33**(3), 247–263 (2022)
11. Chettaoui, C., Othmani, Y.: Real Paley–Wiener theorems for the multivariable Bessel transform. Int. J. Open Probl. Complex Anal. **6**(1), 90–110 (2014)
12. Constales, D., De Bie, H., Lian, P.: Explicit formulas for the (κ, a)-generalized dihedral kernel and the (κ, a)-generalized Fourier kernel. J. Math. Anal. Appl. **460**(2), 900–926 (2018)
13. Debnath, L., Shah, F.A.: Wavelet Transforms and Their Applications. Birkhäuser, Boston (2015)
14. Debnath, L., Shah, F.A.: Lecture Notes on Wavelet Transforms. Birkhäuser, Boston (2017)
15. De Bie, H., Xu, Y.: On the Clifford–Fourier transform. Int. Math. Res. Not. **22**, 5123–5163 (2011)
16. De Bie, H.: Clifford algebras, Fourier transforms, and quantum mechanics. Math. Methods Appl. Sci. **35**(18), 2198–2228 (2012)
17. Dunkl, C.F.: Differential–difference operators associated to reflection groups. Trans. Am. Math. Soc. **311**, 167–183 (1989)
18. Dunkl, C.F.: Hankel transforms associated to finite reflection groups. Contemp. Math. **138**, 123–138 (1992)
19. Giang, D.V., Moricz, F.: The Cesaro operator is bounded on the Hardy space $H^1(\mathbb{R})$. Acta Sci. Math. **61**, 535–544 (1995)
20. Ghobber, S.: Some results on wavelet scalograms. Int. J. Wavelets Multiresol. Inf. Process. **15**(3), 1750019 (2017)
21. Ghobber, S., Hkimi, S., Omri, S.: Localization operators and uncertainty principles for the Hankel wavelet transform. Stud. Sci. Math. Hungar. **58**(3), 335–358 (2021)
22. Ghobber, S., Mejjaoli, H.: Deformed wavelet transform and related uncertainty principles. Symmetry **15**(3), 675 (2023). https://doi.org/10.3390/sym15030675
23. Gorbachev, D., Ivanov, V., Tikhonov, S.: Sharp Pitt inequality and logarithmic uncertainty principle for (k, a)-generalized Fourier transform in L^2. J. Approx. Theory **202**, 109–118 (2016)
24. Gorbachev, D., Ivanov, V., Tikhonov S.: On the kernel of the (k, a)-generalized Fourier transform. arXiv:2210.15730
25. Howe R.: The oscillator semigroup. In: The Mathematical Heritage of Hermann Weyl (Durham, NC, 1987), Proceedings of the Symposium Pure Mathematics, vol. 48, pp. 61–132. American Mathematical Society, Providence (1988)
26. Jafarov, E.I., Stoilova, N.I., Van der Jeugt, J.: The $su(2)_\alpha$ Hahn oscillator and a discrete Hahn–Fourier transform. J. Phys. A Math. Theor **44**, 355205 (2011)
27. Johansen, T.R.: Weighted inequalities and uncertainty principles for the (k, a)-generalized Fourier transform. Int. J. Math. **27**(3), 1650019 (2016)

28. Kobayashi, T., Mano, G.: The inversion formula an holomorphic extension of the minimal representation of the conformal group, harmonic analysis, group representations, automorphic forms and invariant theory: in honor of Roger Howe. Word Sci. **2007**, 159–223 (2007)
29. Kumar, V., Ruzhansky, M.: $L^p - L^q$ boundedness of (k, a)-Fourier multipliers with applications to nonlinear equations. arXiv:2101.03416v1 (2021)
30. Liflyand, E., Moricz, F.: The Hausdorff operator is bounded on the real Hardy space $H^1(\mathbb{R})$. Proc. Am. Math. Soc. **128**, 1391–1396 (1999)
31. Mejjaoli, H., Sraeib, N.: The continuous wavelet transform associated with a differential–difference operator and applications. Commun. Math. Anal. **9**(1), 48–65 (2010)
32. Mejjaoli, H., Jelassi, M., Othmani, Y.: Multivariable Bessel Gabor transform and applications. Oper. Matrices **9**(3), 637–657 (2015)
33. Mejjaoli, H.: Spectral theorems associated with the (k, a)-generalized wavelet multipliers. J. Pseudo-Differ. Oper. Appl. **9**, 735–762 (2018)
34. Mejjaoli, H.: (k, a)-generalized wavelet transform and applications. J. Pseudo-Differ. Oper. Appl. **11**, 55–92 (2020)
35. Mejjaoli, H., Trimèche, K.: k-Hankel two-wavelet theory and localization operators. Integr. Transforms Spec. Funct. **31**(8), 620–644 (2020)
36. Mejjaoli, H.: k-Hankel Gabor transform on $\mathbb{R}^d$ and its applications to the reproducing kernel theory. Complex Anal. Oper. Theory **15**(14), 1–54 (2021)
37. Mejjaoli, H.: Time-frequency analysis associated with k-Hankel Gabor transform on $\mathbb{R}^d$. J. Pseudo-Differ. Oper. Appl. **12**(41), 1–58 (2021). https://doi.org/10.1007/s11868-021-00399-7
38. Mejjaoli, H.: New uncertainty principles for the (k, a)-generalized wavelet transform. Revista de la Unión Matemàtica Argentina **63**(1), 239–279 (2022)
39. Mejjaoli, H.: Generalized translation operator and uncertainty principles associated with the deformed Stockwell transform. Revista de la Unión Matemàtica Argentina **65**(2), 375–423 (2023)
40. Pathak, R.S.: The Wavelet Transform, vol. 4. Springer, Berlin (2009)
41. Prasad, A., Kumar, P.: Composition of the continuous fractional wavelet transform. Nat. Acad. Sci. Lett. **39**(2), 115–120 (2016)
42. Rösler, M.: Bessel-type signed hypergroups on $\mathbb{R}$. In: Heyer, H., Mukherjea, A. (Eds.) Probability Measures on Groups and Related Structures XI, Proceedings of the Oberwolfach, vol. 1994, pp. 292–304. World Scientific, Singapore (1995)
43. Shah, F.A., Tantary, A.Y.: Polar wavelet transform and the associated uncertainty principles. Int. J. Theor. Phys. **57**(6), 1774–1786 (2018)
44. Sraeib, N.: k-Hankel Wigner transform and its applications to the Localization operators theory. J. Pseudo-Differ. Oper. Appl. (2022). https://doi.org/10.1007/s11868-022-00467-6
45. Stein, E.M.: Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals. Series: Princeton Mathematical Series. Princeton University Press, Princeton (1993)
46. Thangavelu, S., Xu, Y.: Convolution operator and maximal functions for Dunkl transform. J. d'Analyse Mathematique **97**, 25–56 (2005)
47. Teng, W.: Hardy inequalities for fractional (k, a)-generalized harmonic oscillator (2020), arXiv:2008.00804
48. Trimèche, K.: Generalized Wavelets and Hypergroups. Gordon and Breach Science (1997)
49. Wong, M.W.: Wavelet Transforms and Localization Operators, vol. 136. Springer, London (2002)