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Quaternion Hyperbolic Fourier Transforms and Uncertainty Principles

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Abstract

The present study introduces the two-sided and right-sided Quaternion Hyperbolic Fourier Transforms (QHFTs) for analyzing two-dimensional quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure. The different forms of these transforms are defined by replacing the Euclidean plane waves with the corresponding hyperbolic plane waves in one dimension, giving the hyperbolic counterpart of the corresponding Euclidean Quaternion Fourier Transforms. Using hyperbolic geometry tools, we study the main operational and mapping properties of the QHFTs, such as linearity, shift, modulation, dilation, symmetry, inversion, and derivatives. Emphasis is placed on novel hyperbolic derivative and hyperbolic primitive concepts, which lead to the differentiation and integration properties of the QHFTs. We further prove the Riemann–Lebesgue Lemma and Parseval's identity for the two-sided QHFT. Besides, we establish the Logarithmic, Heisenberg–Weyl, Donoho–Stark, and Benedicks' uncertainty principles associated with the two-sided QHFT by invoking hyperbolic counterparts of the convolution, Pitt's inequality, and the Poisson summation formula. This work is motivated by the potential applications of the QHFTs and the analysis of the corresponding hyperbolic quaternionic signals.

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1 Introduction

In recent years, the *Quaternion Fourier Transforms* (QFTs), which are generalizations of the Fourier transform, have been the focus of many research papers because of their applicability to signal and image processing. Much progress has been made on this topic and applying QFTs in theoretical and applied mathematics. These results can be found in image diffusion, electromagnetism, multi-channel processing, vector field processing, shape representation, linear scale-invariant filtering, fast vector pattern matching, phase correlation, analysis of nonstationary improper complex signals, flow analysis, partial differential systems, disparity estimation, and texture segmentation, as well as spectral representations for hypercomplex wavelet analysis (see [2–5, 9, 10, 20, 42, 45, 46] and elsewhere).

Recent literature advocates hyperbolic manifolds as embedding spaces for machine learning and computer vision tasks (see [1, 35, 41] and references therein). Hyperbolic embeddings also have profound connections to visual data due to latent hierarchical structures in vision data sets. These works use the Poincaré model of hyperbolic geometry as embedding space. The analogue of classical Fourier analysis for Riemannian symmetric spaces of noncompact type was developed by Helgason (see, e.g., [26–28]). The kernel of the Helgason–Fourier transform in hyperbolic space consists of scalar-valued eigenfunctions of a second-order differential operator. In recent years, Petrov [43] defined the Fourier transform and the convolution of functions on the interval $(-1, 1)$ by employing the diffeomorphism between $\mathbb{R}$ and $(-1, 1)$. This transform is a particular case of the Helgason–Fourier transform for the one-dimensional case, which found applications in the study of differential and integro-differential type equations, including Prandtl, Tricomi, Lavrentjev–Bitsadze, and Laplace–Beltrami equations on the sphere [43, 44]. In [24], Ferreira used the algebraic structure of the Möbius gyrogroup to study hyperbolic harmonic analysis on the Poincaré ball model of hyperbolic geometry.

This paper aims to extend the QFTs to spaces of quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure. We call these new transforms the *Quaternion Hyperbolic Fourier Transforms* (QHFTs). Although this can be accomplished in different ways, as explained below, we shall confine our attention to the two-sided and right-sided QHFTs. For this purpose, we consider two quaternionic hyperbolic exponential kernels, where two generators of the Quaternion Algebra take over the role of the imaginary unit. The hyperbolic geometry of the Poincaré disk model encoded in the transforms considered in [28] and [24] differs from the one we propose since we will work in an open rectangle endowed with a hyperbolic measure involving two different directions.

The works of Ernst et al. [22] and Delsuc [16] in the late 80 s, as based upon Sommen's definition of a Clifford Fourier Transform (CFT) [47, 48], were the historical starting point from which a significant part of the development of QFTs originated.

Ernst and Delsuc's two-dimensional QFTs were put forward and applied to nuclear magnetic resonance imaging. The QFTs in question were of the following form:

$$\mathcal{F}(f)(\omega_1, \omega_2) = \int_{\mathbb{R}^2} f(x_1, x_2) e^{\mathbf{i}x_1\omega_1} e^{\mathbf{j}x_2\omega_2} dx_1 dx_2; \quad f : \mathbb{R}^2 \to \mathbb{H}, \tag{1}$$

where (x_1, x_2), (ω_1, ω_2) are points in $\mathbb{R}^2$, and the product $e^{\mathbf{i}x_1\omega_1} e^{\mathbf{j}x_2\omega_2}$ is a two-dimensional quaternion Fourier kernel. Another often-used convention for the QFT is to split in (1) the factor $(2\pi)^{-2}$ asymmetrically or equivalently, replacing it with a factor 2π in the exponents. This version of the QFT is merely a particular case of the CFT introduced by Brackx et al. [8]. In [10], Bülow et al. followed a different approach to the CFT. In [36], Li et al. extended the complex Fourier Transform holomorphically to a function of several complex variables. A discussion of the main properties of the QFT of the form (1), about linearity, shift, modulation, dilation, moments, inversion, derivatives, Plancherel and Parseval identities, and investigation of a convolution theorem can be found in [30]. Specific studies relating to those particular cases of QFTs were discussed in [21, 31] and [25, Ch. 11]. In [39], Mawardi et al. derived an uncertainty principle for the QFT of the form (1), which prescribes a lower bound on the product of the effective widths of quaternionic signals in the spatial and frequency domains; cf. also [29, 32, 38]. Recently, two novel uncertainty principles were proposed in [49], commencing with a QFT in the form (1). Generalized sampling expansions of band-limited quaternionic signals associated with (1) were established in [14]. An account of the essential recent investigations originating in the QFT can be found in [11] and [34].

Because the exponentials in (1) do not commute, nor with the signal f, different formulations are possible for the two-dimensional QFTs. In the meantime, an indication of a QFT with the two exponentials positioned on each side of the quaternion signal was given by Ell [18, 19]:

$$\mathcal{F}(f)(\omega_1, \omega_2) = \int_{\mathbb{R}^2} e^{\mathbf{i}x_1\omega_1} f(x_1, x_2) e^{\mathbf{j}x_2\omega_2} dx_1 dx_2. \tag{2}$$

The study of Pitt's inequality and the uncertainty principles associated with the two-sided QFT (2) can be found in [12]. Zou et al. [50] used this version of the QFT to study a new class of two-dimensional quaternionic signals whose energy concentration is maximal in both space and frequency. For a given finite energy quaternionic signal, the authors found the possible proportions of its energy in a bounded spatial domain and a bounded frequency domain, including the signals that do the best job of simultaneous space and frequency concentration.

The paper is organized as follows. Section 2 provides some basic concepts and notations of quaternionic analysis and introduces the hyperbolic tools essential in the sequel. Section 3 presents the two-sided QHFT and establishes its main properties, including linearity, shift, modulation, dilation, symmetry, inversion, derivatives, the Riemann–Lebesgue Lemma, Plancherel Theorem, and Parseval's identity, representing hyperbolic counterparts of the corresponding properties for the Euclidean QFTs. Emphasis is placed on hyperbolic derivative and hyperbolic primitive concepts, which lead to differentiation properties of the QHFTs. In Sect. 4.1, we prove Pitt's inequality for the two-sided QHFT, which plays an essential role in establishing the quaternionic versions of the Logarithmic, Heisenberg–Weyl, Donoho–Stark, and Benedicks' uncertainty principles on hyperbolic spaces in Sects. 4.2 and 4.3. These results provide an impetus regarding the potential applications of the QHFTs. The approach requires introducing the hyperbolic convolution operation for the two-sided QHFT that works well with both the QHFT and its inverse, leading to a hyperbolic analogue of the convolution and product formulas of the QFTs. Section 4.3 establishes as well the hyperbolic analogue of the Poisson summation formula for the two-sided QHFT and some related proper identities. Section 5 uses the steerable orthogonal 2D planes split of quaternions [30, 33] to decompose the two-sided QHFT into two complex transforms. The new general form of the two-sided QHFT allows us to prove Hausdorff–Young and Pitt's inequalities. Section 6 studies the right-sided QHFT and its main properties. Section 7 shows the concluding remarks. These results are done here for the first time to the best of our knowledge.

2 Preliminaries

2.1 The Hamilton's Quaternion Algebra and Quaternion Modules

Let $\mathbb{H}$ denote the Hamilton's Quaternion Algebra over $\mathbb{R}$ defined by

$$\mathbb{H} := \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \colon q_i \in \mathbb{R},\ i = 0, 1, 2, 3\},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternionic imaginary units satisfying the multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. A quaternion q can be written as $q = q_0 + \underline{q}$, where the scalar and vector parts of q are defined, respectively, by $\mathrm{Sc}(q) = q_0$ and $\underline{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. Even though the multiplication of two quaternions is noncommutative, we have $\mathrm{Sc}(pq) = \mathrm{Sc}(qp)$, for all $p, q \in \mathbb{H}$. The conjugate of a quaternion q is defined by $\overline{q} = \mathrm{Sc}(q) - \underline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$, and the (algebraic) norm of q is defined by $|q|^2 = q\overline{q} = \overline{q}q = \sum_{i=0}^{3} q_i^2$. The following properties hold:

$$\overline{\overline{q}} = q, \quad \overline{p+q} = \overline{p} + \overline{q}, \quad \overline{pq} = \overline{q}\,\overline{p}, \quad |pq| = |p||q|, \qquad \forall p, q \in \mathbb{H}. \tag{3}$$

Due to the tensorial nature of the QFTs, we are concerned with $\mathbb{H}$-valued functions defined in an open rectangle

$$\mathbb{R}^2_{t_1,t_2} = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < t_1,\ |x_2| < t_2\},$$

where $t_1, t_2 \in \mathbb{R}^+$; that is, functions $f \colon \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ of the form

$$f(x) = f_0(x) + \mathbf{i}f_1(x) + \mathbf{j}f_2(x) + \mathbf{k}f_3(x) \tag{4}$$

with $x = (x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$ or, equivalently, in a symmetric form as (see [30])

$$f(x) = f_0(x) + \mathbf{i} f_1(x) + f_2(x)\mathbf{j} + \mathbf{i} f_3(x)\mathbf{j}, \tag{5}$$

where the f_i's ($i = 0, 1, 2, 3$) are real-valued functions defined in $\mathbb{R}^2_{t_1,t_2}$.

For $q = q_0 + \underline{q} \in \mathbb{H}$, the quaternion exponential function e^q is defined employing an infinite series as $e^q := \sum_{n=0}^{\infty} q^n/n!$ (see, e.g., [40]). Using the Cauchy product of e^{q_0} and $e^{\underline{q}}$, one obtains $e^q = e^{q_0} e^{\underline{q}}$ with

$$e^{\underline{q}} = \cos(|\underline{q}|) + \frac{\underline{q}}{|\underline{q}|} \sin(|\underline{q}|). \tag{6}$$

Definition 1 A function $f = f_0 + \mathbf{i} f_1 + \mathbf{j} f_2 + \mathbf{k} f_3$ defined on $\mathbb{R}^2_{t_1,t_2}$ is said to be h-measurable if and only if each of the f_i's are measurable on $\mathbb{R}^2_{t_1,t_2}$ with respect to the hyperbolic measure defined by

$$d\mu(x) = d\mu(x_1) d\mu(x_2) = \frac{dx_1}{\left(1 - \frac{x_1^2}{t_1^2}\right)} \frac{dx_2}{\left(1 - \frac{x_2^2}{t_2^2}\right)}. \tag{7}$$

Further, we say that f is h-integrable over $\mathbb{R}^2_{t_1,t_2}$ with respect to the given measure (7) if and only if each of the f_i's is integrable on $\mathbb{R}^2_{t_1,t_2}$, i.e., the f_i's are measurable functions on $\mathbb{R}^2_{t_1,t_2}$ and $\int_{\mathbb{R}^2_{t_1,t_2}} f_i(x)\, d\mu(x) < \infty$ for every i.

Definition 2 Let $1 \le p < \infty$. The space $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is defined to be the collection of all equivalence classes of all h-measurable $\mathbb{H}$-valued functions f defined on $\mathbb{R}^2_{t_1,t_2}$ such that $|f|^p \in L^1(\mathbb{R}^2_{t_1,t_2})$, i.e.,

$$\begin{aligned} &L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) \\ &= \left\{ f : \mathbb{R}^2_{t_1,t_2} \to \mathbb{H} \text{ measurable}, \ \|f\|_p := \left(\int_{\mathbb{R}^2_{t_1,t_2}} |f(x)|^p d\mu(x) \right)^{1/p} < \infty \right\}. \end{aligned}$$

For $p = \infty$, the space $L^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ contains essentially the bounded h-measurable functions $f : \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ with norm $\|f\|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^2_{t_1,t_2}} |f(x)|$.

It is clear that if $f \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then αf is also in $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ for all $\alpha \in \mathbb{H}$. Since $|f + g|^p \le 2^p (|f|^p + |g|^p)$, $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is also closed under addition. Accordingly, $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is a left-linear module over $\mathbb{H}$.

We will consider the primary space $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ endowed with the left-quaternionic inner product

$$\langle f, g \rangle := \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, \overline{g(x)}\, d\mu(x) \tag{8}$$

for all $f, g \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. It is a (left) quaternionic Hilbert space with the associated norm $\|f\|_2 = (\langle f, f\rangle)^{1/2}$, which coincides with the usual $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$-norm for f, viewed as a vector-valued function in $\mathbb{R}^2_{t_1,t_2}$.

2.2 The 1D Hyperbolic Plane Waves Revisited

Let us consider the open interval $(-t, t)$, with $t \in \mathbb{R}^+$, endowed with the binary operation

$$a \oplus b = \frac{a+b}{1+\frac{ab}{t^2}}, \quad a, b \in (-t, t). \tag{9}$$

Then $((-t, t), \oplus)$ forms an abelian group.

It is possible to introduce a relativistic scalar multiplication in $(-t, t)$ given by

$$r \otimes b = t \tanh(r \tanh^{-1}(b/t)), \quad r \in \mathbb{R},\ b \in (-t, t) \tag{10}$$

turning $((-t, t), \oplus, \otimes)$ into a vector space. The following distributive laws hold:

1. $r \otimes (a \oplus b) = (r \otimes a) \oplus (r \otimes b)$
2. $(r + s) \otimes a = (r \otimes a) \oplus (s \otimes a)$

for all $r, s \in \mathbb{R}$ and $a, b \in (-t, t)$.

In this way, the open interval $(-t, t)$ has an algebraic structure similar to $\mathbb{R}$, and in the limit $t \to +\infty$, the hyperbolic structure agrees with the Euclidean structure. There exists indeed an isomorphism between $(\mathbb{R}, +, \times)$ and $((-t, t), \oplus, \otimes)$ through the mapping $f(x) = t \tanh(x/t)$, $x \in \mathbb{R}$.

We will now introduce the 1D hyperbolic plane waves on $(-t, t)$, which have similarities with the 1D Euclidean plane waves.

Definition 3 (cf. [24, 43]) Let $t \in \mathbb{R}^+$. For $\omega \in \mathbb{R}$ and $x \in (-t, t)$, the 1D hyperbolic plane waves $e_{\omega,t}(x)$ are defined by

$$e_{\omega,t}(x) = \left(\frac{1+\frac{x}{t}}{1-\frac{x}{t}}\right)^{\frac{i\omega t}{2}}. \tag{11}$$

It turns out that we can write (11) as

$$e_{\omega,t}(x) = e^{\frac{i\omega t}{2}\ln\left(\frac{1+\frac{x}{t}}{1-\frac{x}{t}}\right)} = e^{i\omega t \tanh^{-1}\left(\frac{x}{t}\right)}. \tag{12}$$

The following proposition shows the main properties of the function $e_{\omega,t}(x)$ defined by (11).

Proposition 1 *For $\omega, \xi \in \mathbb{R}$ and $x, y \in (-t, t)$, we have*

1. $e_{\omega,t}(x)\, e_{\xi,t}(x) = e_{\omega+\xi,t}(x)$,
2. $e_{\omega,t}(x \oplus y) = e_{\omega,t}(x)\, e_{\omega,t}(y)$,

3. $\lim_{t \to +\infty} e_{\omega,t}(x) = e^{i\omega x}$.

Proof By (12), Property 1 follows from

$$\begin{aligned} e_{\omega,t}(x)\, e_{\xi,t}(x) &= e^{i\omega\, t \tanh^{-1}(x/t)}\, e^{i\xi\, t \tanh^{-1}(x/t)} \\ &= e^{i(\omega+\xi)\, t \tanh^{-1}(x/t)} \\ &= e_{\omega+\xi,t}(x). \end{aligned}$$

To prove Property 2, we consider $x/t = \tanh(\theta_1) \in (-1, 1)$ and $y/t = \tanh(\theta_2) \in (-1, 1)$. Using the addition formula

$$\frac{\tanh(\theta_1) + \tanh(\theta_2)}{1 + \tanh(\theta_1)\tanh(\theta_2)} = \tanh(\theta_1 + \theta_2),$$

we obtain

$$\begin{aligned} e_{\omega,t}(x \oplus y) &= e^{i\omega\, t(\theta_1+\theta_2)} \\ &= e^{i\omega\, t\theta_1}\, e^{i\omega\, t\theta_2} \\ &= e^{i\omega\, t \tanh^{-1}(x/t)}\, e^{i\omega\, t \tanh^{-1}(y/t)} \\ &= e_{\omega,t}(x)\, e_{\omega,t}(y). \end{aligned}$$

Property 3 is based on the following limit computed using L'Hôpital's rule:

$$\lim_{t \to +\infty} t \tanh^{-1}(x/t) = \lim_{t \to +\infty} \frac{x}{1 - \frac{x^2}{t^2}} = x.$$

□

From Property 3, we can see that in the large limit of t, $t \to +\infty$, the hyperbolic plane waves converge to the corresponding Euclidean plane waves.

3 The Two-Sided Quaternion Hyperbolic Fourier Transform

3.1 Definition and Properties

In this section, the definition of the two-sided QHFT and a discussion of its main properties for functions in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ will be provided. In addition, a theorem will be proven that gives conditions under which the inverse of the QHFT can be calculated. The treatment given here is a generalization of that considered by Ernst et al. [22] and Delsuc [16], employing tools of analytic hyperbolic geometry.

Throughout this paper, we found it convenient to introduce a special symbol to denote the extension of the hyperbolic variable $x = (x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$ to the whole of the space $\mathbb{R}^2$ by setting

$$\underline{x} = (\underline{x}_1, \underline{x}_2) = \left(t_1 \tanh^{-1}\left(x_1/t_1\right),\ t_2 \tanh^{-1}\left(x_2/t_2\right)\right) \in \mathbb{R}^2. \tag{13}$$

The hyperbolic addition and the scalar multiplication in $\mathbb{R}^2_{t_1,t_2}$ are defined componentwise by

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus y_1, x_2 \oplus y_2)$$

and

$$\lambda \otimes (x_1, x_2) = (\lambda \otimes x_1, \lambda \otimes x_2)$$

for all $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_{t_1,t_2}$ and all $\lambda \in \mathbb{R}$.

Definition 4 The steerable two-sided QHFT of $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is the function $\mathcal{F}_{QH}(f) \colon \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ defined as

$$\mathcal{F}_{QH}(f)(\omega) = \widehat{f}(\omega) := \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} f(x) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x), \tag{14}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$. We refer to (x_1, x_2) as hyperbolic-space variables and (ω_1, ω_2) as angular-frequency variables.

Since $|e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} f(x) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}| = |f(x)|$ for $x \in \mathbb{R}^2_{t_1,t_2}$ and $\omega \in \mathbb{R}^2$, it is clear that if f is absolutely integrable in $\mathbb{R}^2_{t_1,t_2}$, then the two-sided QHFT given as (14) is defined, and the corresponding integral converges absolutely for $\omega \in \mathbb{R}^2$. We shall observe that the order of the factors in (14) has to be written in a fixed order since the quaternion hyperbolic Fourier kernels $e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1}, e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}$ do not generally commute with every element of the quaternion algebra.

It is well to observe that with (1), the four QFT-components separate four symmetry cases for real signals f in the form

$$\begin{aligned}\mathcal{F}_{QH}(f)(\omega) = &\int_{\mathbb{R}^2_{t_1,t_2}} \cos(2\pi \underline{x}_1 \omega_1) \cos(2\pi \underline{x}_2 \omega_2) f(x)\, d\mu(x)\\ &- \mathbf{i} \int_{\mathbb{R}^2_{t_1,t_2}} \sin(2\pi \underline{x}_1 \omega_1) \cos(2\pi \underline{x}_2 \omega_2) f(x)\, d\mu(x)\\ &- \mathbf{j} \int_{\mathbb{R}^2_{t_1,t_2}} \cos(2\pi \underline{x}_1 \omega_1) \sin(2\pi \underline{x}_2 \omega_2) f(x)\, d\mu(x)\\ &+ \mathbf{k} \int_{\mathbb{R}^2_{t_1,t_2}} \sin(2\pi \underline{x}_1 \omega_1) \sin(2\pi \underline{x}_2 \omega_2) f(x)\, d\mu(x).\end{aligned} \tag{15}$$

Figure 1 shows some of the basis functions of the QHFT in the spatial domain. The frequency parameter is modified from image to image.

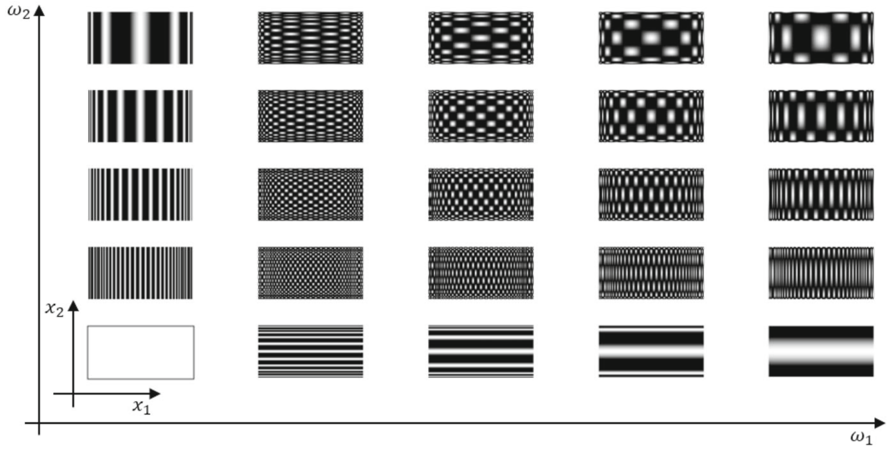

Fig. 1 The small images are intensity images of the real part of the basis function in (15), for $t_1 = 20$ and $t_2 = 10$

From (5), it is easy to see that $\mathcal{F}_{QH}(f)$ has a symmetric representation:

$$\widehat{f}(\omega) = \widehat{f_0}(\omega) + \mathbf{i}\widehat{f_1}(\omega) + \widehat{f_2}(\omega)\mathbf{j} + \mathbf{i}\widehat{f_3}(\omega)\mathbf{j}, \tag{16}$$

where $\widehat{f_i} = \mathcal{F}_{QH}(f_i)(\omega)$ $(i = 0, 1, 2, 3)$. Similarly, as in [12], we define a new modulus of $\widehat{f}$ that depends on ω by

$$|\widehat{f}(\omega)|_Q := \left(\sum_{i=0}^{3} |\widehat{f_i}(\omega)|^2\right)^{1/2}. \tag{17}$$

By the modulus $|\widehat{f}|_Q$ of a quaternion-valued function $\widehat{f}$, we understand the function whose value at any point ω equals the sum of the (algebraic) norm of each component $\widehat{f_i}$ of $\widehat{f}$ at that point. Thus, $|\widehat{f}|_Q$ is always a real-valued nonnegative function.

The above discussion motivates the following definition [12]. By (17), the L^p-norm of $\widehat{f}$ is defined by

$$\|\widehat{f}\|_{Q,p} := \left(\int_{\mathbb{R}^2} |\widehat{f}(\omega)|_Q^p \, d\omega\right)^{1/p}, \tag{18}$$

where $d\omega = d\omega_1 d\omega_2$ is the Lebesgue measure on $\mathbb{R}^2$. This allows to define the following L^p-space for the two-sided QHFT:

$$L^p(\mathbb{R}^2, \mathbb{H}) = \{\widehat{f} \colon \mathbb{R}^2 \to \mathbb{H} \text{ measurable}, \ \|\widehat{f}\|_{Q,p} < \infty\}.$$

We remark that the norms $\|\widehat{f}\|_p$ and $\|\widehat{f}\|_{Q,p}$ do not coincide when f is a quaternionic (non-real) function.

The following proposition shows the elementary operational properties of the proposed two-sided QHFT defined in (14).

Proposition 2 *Let* $f, g \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2_{t_1,t_2}$, $\theta = (\theta_1, \theta_2) \in \mathbb{R}^2$, *and* $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$. *Then*

1. *(Linearity)*

$$\mathcal{F}_{QH}(\alpha f + \beta g)(\omega) = \alpha \widehat{f}(\omega) + \beta \widehat{g}(\omega), \quad \forall \alpha, \beta \in \mathbb{R}, \tag{19}$$

2. *(Hyperbolic translation)*

$$\mathcal{F}_{QH}(f(x_1 \oplus y_1, x_2 \oplus y_2))(\omega) = e^{2\pi \mathbf{i} \underline{y}_1 \omega_1} \widehat{f}(\omega) e^{2\pi \mathbf{j} \underline{y}_2 \omega_2}, \tag{20}$$

3. *(Modulation)*

$$\mathcal{F}_{QH}(e^{2\pi \mathbf{i} \underline{x}_1 \theta_1} f(x)\, e^{2\pi \mathbf{j} \underline{x}_2 \theta_2})(\omega) = \widehat{f}(\omega - \theta), \tag{21}$$

4. *(Hyperbolic dilation/scaling)*

$$\mathcal{F}_{QH}\big(f\big(\lambda_1 \otimes x_1, \lambda_2 \otimes x_2\big)\big)(\omega) = \frac{1}{|\lambda_1 \lambda_2|}\, \widehat{f}\Big(\frac{\omega_1}{\lambda_1}, \frac{\omega_2}{\lambda_2}\Big), \tag{22}$$

5. *(Symmetry)*

$$\mathcal{F}_{QH}(f(\pm x_1, \pm x_2))(\omega) = \widehat{f}(\pm \omega_1, \pm \omega_2). \tag{23}$$

Proof The first property is immediate by definition (14) of the two-sided QHFT, and the last property follows since $\tanh^{-1}$ is an odd function. Property 2 follows using the change of variables $x_i \oplus y_i = z_i$, $i = 1, 2$, which are equivalent to $x_i = z_i \ominus y_i$, $i = 1, 2$, together with the hyperbolic translation invariance property of the hyperbolic metric (7) and Property 2 in Proposition 1.

Now, Property 3 follows from the equalities

$$e^{2\pi \mathbf{i} \underline{x}_1 \theta_1} e^{-2\pi \mathbf{i} \underline{x}_1 \omega_1} = e^{-2\pi \mathbf{i} \underline{x}_1 (\omega_1 - \theta_1)}$$

and

$$e^{2\pi \mathbf{j} \underline{x}_2 \theta_2} e^{-2\pi \mathbf{j} \underline{x}_2 \omega_2} = e^{-2\pi \mathbf{j} \underline{x}_2 (\omega_2 - \theta_2)}.$$

To prove Property 4, we make the change of variables $\lambda_i \otimes x_i = y_i$, $i = 1, 2$, which are equivalent to $x_i = (1/\lambda_i) \otimes y_i$, $i = 1, 2$. Thus, it follows that

$$e^{2\pi \mathbf{i}\, (1/\lambda_1) \otimes \underline{y}_1 \omega_1} = e^{-2\pi \mathbf{i}\, \underline{y}_1 (\omega_1/\lambda_1)}$$

and

$$e^{2\pi \mathbf{j}\, (1/\lambda_2) \otimes \underline{y}_2 \omega_2} = e^{-2\pi \mathbf{j}\, \underline{y}_2 (\omega_2/\lambda_2)}.$$

Since

$$\prod_{i=1}^{2} \frac{|\lambda_i|}{\cosh^2\left(\lambda_i \tanh^{-1}(y_i/t_i)\right)} \frac{1}{1-\frac{y_i^2}{t_i^2}}$$

gives the Jacobian of the change of variables, then by straightforward computations, we obtain

$$d\mu\Big(\frac{1}{\lambda_1}\otimes y_1, \frac{1}{\lambda_2}\otimes y_2\Big) = \frac{1}{|\lambda_1\lambda_2|}\, d\mu(y_1, y_2).$$

Therefore, Property 4 follows. This completes the proof of the proposition. □

In the remainder of this section, we shall establish the continuity and differentiability properties of the two-sided QHFT defined in (14). To facilitate the motive, as a preliminary step, we shall describe the concepts of hyperbolic continuity and hyperbolic derivative within our context.

Definition 5 Let $t \in \mathbb{R}^+$, $f : D \subseteq (-t, t) \to \mathbb{R}$, and a an interior point of D. We say that f is h-continuous at the point a if for any real number $\epsilon > 0$ there exists some real number $\delta > 0$ such that for every $x \in D$ with $|x \ominus a| < \delta$, it holds that $|f(x) - f(a)| < \epsilon$.

Definition 6 Let $t \in \mathbb{R}^+$, $f : I \to \mathbb{R}$, where I is an interval in $(-t, t)$, and a, x interior points of I. We say that f has a hyperbolic derivative (hereafter referred to as h-derivative) or is h-differentiable at the point $x = a$ if the following limit

$$\lim_{\epsilon\to 0} \frac{f(a\oplus\epsilon) - f(a)}{\epsilon} \tag{24}$$

exists and is finite. We call the h-derivative of f at $x = a$ to the limit value and denote it by $f_h'(a)$. If the h-derivative exists and is finite for all points x in I, we denote the h-derivative of f by $f_h'(x)$ and say that f is h-differentiable at every point of I.

As a consequence of the above definition, we have the following result.

Proposition 3 *If $f : I \to \mathbb{R}$ is h-differentiable, then*

$$f_h'(x) = f'(x)\Big(1 - \frac{x^2}{t^2}\Big), \tag{25}$$

where $f'(x)$ denotes the standard (or Euclidean) derivative of f.

Proof By using L'Hôpital's rule, we have

$$
\begin{aligned}
f_h'(x) &= \lim_{\epsilon\to 0} \frac{f\left(\frac{x+\epsilon}{1+\frac{x\epsilon}{t^2}}\right) - f(x)}{\epsilon} \\
&= \lim_{\epsilon\to 0} f'\left(\frac{x+\epsilon}{1+\frac{x\epsilon}{t^2}}\right) \frac{t^2 - x^2}{t^2\left(1+\frac{x\epsilon}{t^2}\right)^2} \\
&= f'(x)\left(1 - \frac{x^2}{t^2}\right).
\end{aligned}
$$

□

It is easily seen that $f_h'(x) = f'(x)$ when $t \to \infty$. The following properties are immediate consequences of Definition 6 and Proposition 3 and, therefore, their proofs will not be given.

Proposition 4 *Let $f, g\colon D \subseteq (-t, t) \to \mathbb{R}$ be h-differentiable functions in D. Then*

1. $(f \pm g)_h' = f_h' \pm g_h'$,
2. $(fg)_h' = f_h' g + f g_h'$,
3. $\left(\frac{f}{g}\right)_h' = \frac{f_h' g - f g_h'}{g^2}$, $g \neq 0$,
4. $(f \circ g)_h' = (f' \circ g) \times g_h'$, *whenever the composition* $\circ$ *is well-defined.*

The hyperbolic second derivative of f is given by

$$
f_h''(x) = \left(1 - \frac{x^2}{t^2}\right)\left(\left(1 - \frac{x^2}{t^2}\right) f''(x) - \frac{2x}{t^2} f'(x)\right), \tag{26}
$$

which is linked with the Laplace–Beltrami operator in the Möbius gyrovector space (see [24]) when we restrict it to the one-dimensional case. By (26), one can introduce the hyperbolic Laplace operator in $\mathbb{R}^2_{t_1,t_2}$ given by

$$
\Delta_{t_1,t_2} := \sum_{i=1}^{2} \left(1 - \frac{x_i^2}{t_i^2}\right)\left(\left(1 - \frac{x_i^2}{t_i^2}\right)\frac{\partial^2}{\partial x_i^2} - \frac{2x_i}{t_i^2}\frac{\partial}{\partial x_i}\right), \tag{27}
$$

whose fundamental solution is given by $\phi(x_1, x_2) = -(1/2\pi)\ln(|\underline{x}|)$. In the limit of large of t_1 and t_2, $t_1, t_2 \to \infty$, we recover the Euclidean Laplacian in $\mathbb{R}^2$.

After introducing the concept of h-derivative, we can now define the notion of h-primitive and present the Fundamental Theorem of Calculus within our context.

Definition 7 An h-differentiable function $F\colon I \subseteq (-t, t) \to \mathbb{R}$ is called an h-primitive of f in I if $F_h'(x) = f(x)$, for all $x \in I$.

Two h-primitives F_1 and F_2 of f defined in $[a, b]$ differ only by a constant; that is, there exists $C \in \mathbb{R}$ such that $F_1(x) = F_2(x) + C$, for all $x \in [a, b]$.

Theorem 5 *Let f be an h-continuous function in $[a, b] \subset (-t, t)$. Then the function*

$$
F(x) = \int_a^x f(y)\, d\mu(y) \tag{28}
$$

is an h-primitive of f, i.e., $F'_h(x) = f(x)$ *for all* $x \in (a, b)$. *Further,*

$$\int_a^b f(y)\, d\mu(y) \;=\; G(b) - G(a), \tag{29}$$

where G is an h-primitive of f (i.e., $G'_h = f$*).*

Proof Suppose x and $x \oplus \epsilon$ are in $[a, b]$. Without loss of generality, we assume $\epsilon > 0$. By (28), we have

$$\frac{F(x \oplus \epsilon) - F(x)}{\epsilon} \;=\; \frac{1}{\epsilon} \int_x^{x\oplus\epsilon} f(y)\, \frac{dy}{1 - \frac{y^2}{t^2}}. \tag{30}$$

Since f is h-continuous on $[x, x \oplus \epsilon]$, then $f(y)/(1 - y^2/t^2)$ is also h-continuous on $[x, x \oplus \epsilon]$. Therefore, by the Weierstrass Extreme Value Theorem, there exists $v, w \in [x, x \oplus \epsilon]$ such that $m = f(v)/(1 - v^2/t^2)$ and $M = f(w)/(1 - w^2/t^2)$, where m and M denote, respectively, the infimum and supremum of $f(y)/(1 - y^2/t^2)$ in the interval $[x, x \oplus \epsilon]$. Thus, we have

$$\begin{aligned}
& m(x \oplus \epsilon - x) \;\le\; \int_x^{x\oplus\epsilon} f(y)\, \frac{dy}{1 - \frac{y^2}{t^2}} \;\le\; M(x \oplus \epsilon - x) \\
\Leftrightarrow\;& m\, \frac{\epsilon\left(1 - \frac{x^2}{t^2}\right)}{1 + \frac{\epsilon x}{t^2}} \;\le\; \int_x^{x\oplus\epsilon} f(y)\, \frac{dy}{1 - \frac{y^2}{t^2}} \;\le\; M\, \frac{\epsilon\left(1 - \frac{x^2}{t^2}\right)}{1 + \frac{\epsilon x}{t^2}} \\
\Leftrightarrow\;& m\, \frac{1 - \frac{x^2}{t^2}}{1 + \frac{\epsilon x}{t^2}} \;\le\; \frac{F(x \oplus \epsilon) - F(x)}{\epsilon} \;\le\; M\, \frac{1 - \frac{x^2}{t^2}}{1 + \frac{\epsilon x}{t^2}}.
\end{aligned} \tag{31}$$

Letting $\epsilon \to 0$ then $v \to x$, $w \to x$, and so it follows that $f(x) \le F'_h(x) \le f(x)$. Hence, we conclude that $F'_h(x) = f(x)$ for all $x \in (a, b)$. Finally, to prove (29), if G is another h-primitive of f, then there exists $C \in \mathbb{R}$ such that $G(x) = F(x) + C$ for all $x \in [a, b]$. Since $G(a) = F(a) + C = C$, then (29) follows. □

The following result presents the method of integration by parts within the hyperbolic context. The proof relies on using the product derivation rule (2) given in Proposition 4.

Proposition 6 *Let f and g be continuously h-differentiable functions defined in* $I \subseteq (-t, t)$. *The formula for integrating by parts is as follows:*

$$\int f'_h(x)\, g(x)\, d\mu(x) \;=\; f(x)\, g(x) \;-\; \int f(x)\, g'_h(x)\, d\mu(x).$$

Let us introduce the following quaternionic spaces, which will be of use in further discussion.

Definition 8 We denote by

1. $C(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions that are h-continuous in $\mathbb{R}^2_{t_1,t_2}$;
2. $C^m(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ the space of all $\mathbb{H}$-valued functions f such that $\partial_h^\beta f = \frac{\partial_h^{\beta_1+\beta_2} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \in C(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ whenever $|\beta| \leq m$;
3. $C^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ the space of all smooth $\mathbb{H}$-valued functions that belong to $C^m(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ for every $m \in \mathbb{N}$.

The hyperbolic partial derivatives $\partial_h f/\partial x_i$ $(i = 1, 2)$ in Definition 8 are constructed from (24) by

$$\frac{\partial_h f}{\partial x_1}(x_1, x_2) = \lim_{\epsilon \to 0} \frac{f(x_1 \oplus \epsilon, x_2) - f(x_1, x_2)}{\epsilon}$$

and

$$\frac{\partial_h f}{\partial x_2}(x_1, x_2) = \lim_{\epsilon \to 0} \frac{f(x_1, x_2 \oplus \epsilon) - f(x_1, x_2)}{\epsilon}.$$

Definition 9 We say that an $\mathbb{H}$-valued function f on $\mathbb{R}^2_{t_1,t_2}$ is rapidly decreasing if for every $m \geq 0$, $\sup_{x \in \mathbb{R}^2_{t_1,t_2}} |\underline{x}|^m |f(x)| < \infty$. We further denote by $\mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ the Schwartz space of all rapidly decreasing $\mathbb{H}$-valued smooth functions on $\mathbb{R}^2_{t_1,t_2}$, defined by

$$\mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) = \left\{ f \in C^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) : \forall \alpha, \beta \in \mathbb{N}^2, \sup_{x \in \mathbb{R}^2_{t_1,t_2}} |\underline{x}^\alpha \, \partial_h^\beta f(x)| < \infty \right\},$$

where $\underline{x}^\alpha = \underline{x}_1^{\alpha_1} \underline{x}_2^{\alpha_2}$.

Using Proposition 6, we can now establish the differential properties of the two-sided QHFT.

Proposition 7 *Let $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ and $m, n \in \mathbb{N}$. Then*

1. *(Partial h-derivatives)*

$$\mathcal{F}_{QH}\Big(\frac{\partial_h^{m+n}}{\partial_{x_1}^m \, \partial_{x_2}^n} f(x)\Big)(\omega) = (2\pi \mathbf{i}\, \omega_1)^m \, \widehat{f}(\omega) \, (2\pi \mathbf{j}\, \omega_2)^n, \tag{32}$$

2. *(Powers of $\underline{x}_1$ and $\underline{x}_2$)*

$$\mathcal{F}_{QH}\big((-2\pi \mathbf{i}\, \underline{x}_1)^m \, f(x) \, (-2\pi \mathbf{j}\, \underline{x}_2)^n\big)(\omega) = \frac{\partial^{m+n}}{\partial_{\omega_1}^m \, \partial_{\omega_2}^n} \, \widehat{f}(\omega). \tag{33}$$

Proof Bearing in mind that

$$\frac{\partial_h}{\partial x_1}\, e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} = -2\pi \mathbf{i}\omega_1\, e^{2\pi \mathbf{i}\, \underline{x}_1 \omega_1}$$

and

$$\frac{\partial_h}{\partial x_2}\, e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2} = -2\pi \mathbf{j}\omega_2\, e^{2\pi \mathbf{j}\, \underline{x}_2 \omega_2}$$

then for $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ it follows, by induction, that

$$\begin{aligned}
&\int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} \left(\frac{\partial_h^{m+n}}{\partial x_1^m \partial x_2^n}\, f(x) \right) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\
&= (-1)^{m+n} \int_{\mathbb{R}^2_{t_1,t_2}} \left(\frac{\partial_h^m}{\partial x_1^m} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} \right) f(x) \left(\frac{\partial_h^n}{\partial x_2^n} e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2} \right) d\mu(x) \\
&= (2\pi \mathbf{i}\, \omega_1)^m\, \widehat{f}(\omega)\, (2\pi \mathbf{j}\, \omega_2)^n\,.
\end{aligned}$$

Identity (33) is proved analogously. $\square$

The following proposition describes the important mapping properties of the two-sided QHFT.

Proposition 8 *Let* $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. *Then*

1. $\widehat{f} \in L^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ *and* $\|\widehat{f}\|_{Q,\infty} \leq 2\|f\|_1$.
2. $\widehat{f}$ *is a continuous function and hence a measurable function.*
3. $\widehat{f}(\omega) \to 0$ *as* $|\omega| \to \infty$ *(Riemann–Lebesgue Lemma).*

Proof Property 1 is a consequence of absolute inequality. To prove Property 2, we use Definition 4 and Property 3 in Proposition 2 to obtain

$$\begin{aligned}
&\widehat{f}(\omega) - \widehat{f}(\omega - \theta) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} \big(f(x) - e^{2\pi \mathbf{i}\, \underline{x}_1 \theta_1} f(x) e^{2\pi \mathbf{j}\, \underline{x}_2 \theta_2} \big) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x).
\end{aligned}$$

Since

$$\left| \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} \big(f(x) - e^{2\pi \mathbf{i}\, \underline{x}_1 \theta_1} f(x) e^{2\pi \mathbf{j}\, \underline{x}_2 \theta_2} \big) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \right| \leq 2\|f\|_1$$

and f is integrable, by Lebesgue's Dominated Convergence Theorem, we conclude that $|\widehat{f}(\omega) - \widehat{f}(\omega - \theta)| \to 0$ as $|\theta| \to 0$.

To prove the Riemann–Lebesgue Lemma, we use a density argument as in the classical case and restrict to the case where f is a function in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Let

$y = (y_1, y_2)$ be such that $y_1 = t_1 \tanh\left(\frac{1}{2t_1\omega_1}\right)$ and $y_2 = t_2 \tanh\left(\frac{1}{t_2\omega_2}\right)$. By (6), it follows that $e^{-2\pi \mathbf{i}\, \underline{y}_1 \omega_1} = -1$ and $e^{-2\pi \mathbf{j}\, \underline{y}_2 \omega_2} = 1$. Therefore, by Property 2 in Proposition 2, we find

$$\begin{aligned} \widehat{f}(\omega) &= \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} f(x) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\ &= \frac{1}{2} \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} [f(x) - f(x \ominus y)] e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x). \end{aligned} \tag{34}$$

Hence,

$$|\widehat{f}(\omega)| \leq \frac{1}{2} \int_{\mathbb{R}^2_{t_1,t_2}} |f(x) - f(x \ominus y)|\, d\mu(x).$$

Now, for any $\epsilon > 0$, let $g \in C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ be such that $\|f - g\|_1 \leq \epsilon$, where $C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ denotes the space of h-continuous and compactly supported functions from $\mathbb{R}^2_{t_1,t_2}$ into $\mathbb{R}$. We note that this choose is possible since $C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ is dense in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Now, putting $f_y(x) = f(x \ominus y)$, then

$$f - f_y = (f - g) + (g - g_y) + (g_y - f_y).$$

When ω is sufficiently large, y becomes very small; thus, $\|f_y - g_y\|_1 = \|f - g\|_1 \leq \epsilon$, because g is h-continuous and has compact support. Finally, we have

$$\|g - g_y\|_1 = \int_{\mathbb{R}^2_{t_1,t_2}} |g(x) - g(x \ominus y)|\, d\mu(x) \;\to\; 0 \quad \text{as} \quad |y| \to 0.$$

Then $\|f - f_y\|_1 \to 0$ as $|y| \to 0$. Since this holds for any $\epsilon > 0$, it follows that $\widehat{f}(\omega) \to 0$ as $|\omega| \to \infty$. The representation formula (5) allows to extend the result to $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. Thus the proposition is established. □

3.2 Inverse and Parseval's Relation

To prove the inversion formula and Parseval's Theorem for the two-sided QHFT, we will first define a hyperbolic Gaussian function in our context and compute its QHFT.

Definition 10 Let $\epsilon > 0$. The Gaussian function defined in $\mathbb{R}^2_{t_1,t_2}$ using the hyperbolic variable $x = (x_1, x_2)$ is given by

$$g_\epsilon(x) = \exp\left[-\pi \left(\left(t_1 \tanh^{-1}\left(\sqrt{\epsilon} \otimes \frac{x_1}{t_1} \right) \right)^2 + \left(t_2 \tanh^{-1}\left(\sqrt{\epsilon} \otimes \frac{x_2}{t_2} \right) \right)^2 \right) \right]. \tag{35}$$

Remark 1 It is possible to rewrite (35) using the extended variable $\underline{x} = (\underline{x}_1, \underline{x}_2) \in \mathbb{R}^2$. Since $x_i/t_i \in (-1, 1)$, $i = 1, 2$, by (10), it follows that $\sqrt{\epsilon} \otimes (x_i/t_i)$ represents a relativistic multiplication in $(-1, 1)$ given by

$$\sqrt{\epsilon} \otimes \frac{x_i}{t_i} = \tanh\left(\sqrt{\epsilon}\, \tanh^{-1}\left(\frac{x_i}{t_i}\right)\right), \quad i = 1, 2.$$

Thus, the hyperbolic Gaussian function (35) can be written as

$$g_\epsilon(x) = \exp\big[-\pi\, \epsilon\, (\underline{x}_1^2 + \underline{x}_2^2)\big]. \tag{36}$$

We now show that the QHFT of the hyperbolic Gaussian function (35) is again a Gaussian function in the Fourier transformation domain.

Proposition 9 *Let $\epsilon > 0$. The QHFT of the hyperbolic Gaussian function (35) is given by*

$$\widehat{g_\epsilon}(\omega) = \epsilon^{-1} \exp[-\pi\, \epsilon^{-1} |\omega|^2]. \tag{37}$$

Proof Using (36), we have

$$\widehat{g_\epsilon}(\omega) = \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x_1} \omega_1}\, e^{-\pi \epsilon\, \underline{x}_1^2}\, e^{-\pi \epsilon\, \underline{x}_2^2}\, e^{-2\pi \mathbf{j}\, \underline{x_2} \omega_2}\, d\mu(x).$$

Considering the change of variables $\underline{x}_i = t_i \tanh^{-1}(x_i/t_i) = y_i$, $i = 1, 2$, we obtain

$$\begin{aligned}
\widehat{g_\epsilon}(\omega) &= \int_{\mathbb{R}} e^{-2\pi \mathbf{i}\, y_1 \omega_1}\, e^{-\pi \epsilon\, y_1^2}\, dy_1 \int_{\mathbb{R}} e^{-2\pi \mathbf{j}\, y_2 \omega_2}\, e^{-\pi \epsilon\, y_2^2}\, dy_2 \\
&= \epsilon^{-1/2} e^{-\pi \epsilon^{-1} \omega_1^2}\, \epsilon^{-1/2} e^{-\pi \epsilon^{-1} \omega_2^2} \\
&= \epsilon^{-1} e^{-\pi \epsilon^{-1} (\omega_1^2 + \omega_2^2)}.
\end{aligned}$$

The result follows. ◻

We now derive and prove the inversion formula of the two-sided QHFT, which shows that the original signal f can be recovered from the transformed domain $L^1(\mathbb{R}^2, \mathbb{H})$.

Theorem 10 (Inversion formula) *Let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ be of the form (5) with $\mathcal{F}_{QH}(f) \in L^1(\mathbb{R}^2, \mathbb{H})$. Then, for a.e. $x \in \mathbb{R}^2_{t_1,t_2}$, the components of f can be reconstructed as*

$$f_i(x) = \mathcal{F}_{QH}^{-1}(\widehat{f_i})(x) = \int_{\mathbb{R}^2} e^{2\pi \mathbf{i}\, \underline{x}_1 \omega_1}\, \widehat{f_i}(\omega)\, e^{2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\omega, \quad i = 0, 1, 2, 3. \tag{38}$$

Proof For each $i = 0, 1, 2, 3$, we consider the integral

$$\int_{\mathbb{R}^2_{t_1,t_2}} f_i(y)\, \widehat{g_\epsilon}(\underline{y} - \underline{x})\, d\mu(y), \tag{39}$$

where $\underline{y} = (\underline{y}_1, \underline{y}_2) = (t_1 \tanh^{-1}(y_1/t_1),\ t_2 \tanh^{-1}(y_2/t_2))$, $\underline{x}$ is given by (13), and $\widehat{g_\epsilon}$ is defined as in (37). By Hölder's inequality, the previous integral is absolutely convergent and is equal to

$$\int_{\mathbb{R}^2_{t_1,t_2}} f_i(y) \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{z}_1 (\underline{y}_1 - \underline{x}_1)}\, g_\epsilon(z)\, e^{-2\pi \mathbf{j}\, \underline{z}_2 (\underline{y}_2 - \underline{x}_2)}\, d\mu(z)\, d\mu(y),$$

where $z = (z_1, z_2) \in \mathbb{R}^2_{t_1,t_2}$ and $(\underline{z}_1, \underline{z}_2) = (t_1 \tanh^{-1}(z_1/t_1), t_2 \tanh^{-1}(z_2/t_2))$. Making the change of variables $t_i \tanh^{-1}(z_i/t_i) = \omega_i$, $i = 1, 2$ and then using Fubini's Theorem, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2_{t_1,t_2}} f_i(y) \int_{\mathbb{R}^2} e^{-2\pi \mathbf{i}\, (\underline{y}_1 - \underline{x}_1)\omega_1}\, g_\epsilon\left(t_1 \tanh\left(\frac{\omega_1}{t_1}\right), t_2 \tanh\left(\frac{\omega_2}{t_2}\right)\right) \\
&\qquad\qquad \times\, e^{-2\pi \mathbf{j}\, (\underline{y}_2 - \underline{x}_2)\omega_2}\, d\omega\, d\mu(y) \\
&= \int_{\mathbb{R}^2} e^{2\pi \mathbf{i}\, \underline{x}_1 \omega_1}\, g_\epsilon\left(t_1 \tanh\left(\frac{\omega_1}{t_1}\right), t_2 \tanh\left(\frac{\omega_2}{t_2}\right)\right) \widehat{f_i}(\omega)\, e^{2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\omega.
\end{aligned} \tag{40}$$

Since the family $\{\widehat{g_\epsilon},\ \epsilon > 0\}$ is an approximation to the identity, letting $\epsilon \to 0$ then (39) tends to

$$\int_{\mathbb{R}^2_{t_1,t_2}} f_i(y)\delta(\underline{y} - \underline{x})\, d\mu(y) \;=\; \int_{\mathbb{R}^2_{t_1,t_2}} f_i(y)\, \delta(y - x)\, d\mu(y) \;=\; f_i(x),$$

where δ is the Dirac delta function. Accordingly, letting $\epsilon \to 0$ in (40) then $g_\epsilon\, \widehat{f_i} \to \widehat{f_i}$ in $L^2(\mathbb{R}^2, \mathbb{H})$, by Lebesgue's Dominated Convergence Theorem. Thus, we finally get

$$f_i(x) = \int_{\mathbb{R}^2} e^{2\pi \mathbf{i}\, \underline{x}_1 \omega_1}\, \widehat{f_i}(\omega)\, e^{2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\omega,$$

which is the inversion formula for the two-sided QHFT. □

Using the previous result, we can obtain Parseval's Theorem for the two-sided QHFT.

Theorem 11 (Parseval's Theorem) *If $f \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, then $\widehat{f} \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\|\widehat{f}\|_2 = \|f\|_2$. Further, the map $f \mapsto \widehat{f}$ has a unique extension to a continuous linear map from $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ into $L^2(\mathbb{R}^2, \mathbb{H})$ and $\|\widehat{f}\|_{Q,2} = \|f\|_2$, whenever $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.*

Proof Suppose that $f \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Since $\widehat{f}$ is bounded then the integral

$$\int_{\mathbb{R}^2} |\widehat{f}(\omega)|^2 \, e^{-\pi\epsilon|\omega|^2} d\omega \tag{41}$$

is well-defined. We can rewrite (41) as

$$\begin{aligned}\int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2_{t_1,t_2}} & e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} \, f(x) \, e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2} \, d\mu(x) \Big) \\ & \times \Big(\int_{\mathbb{R}^2_{t_1,t_2}} e^{2\pi \mathbf{j}\, \underline{y}_2 \omega_2} \, f(y) \, e^{2\pi \mathbf{i}\, \underline{y}_1 \omega_1} \, d\mu(y) \Big) \, e^{-\pi\epsilon|\omega|^2} \, d\omega.\end{aligned}$$

By applying Fubini's Theorem and similar computations as in the proof of Proposition 9, we obtain

$$\begin{aligned}& \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \Big(\int_{\mathbb{R}} e^{-2\pi \mathbf{i}\, \omega_1 \underline{x}_1} \, e^{-\epsilon\pi\omega_1^2} \\ & \quad \times \Big(\int_{\mathbb{R}} e^{2\pi \mathbf{j}\, \omega_2 (-\underline{x}_2 + \underline{y}_2)} \, e^{-\epsilon\pi\omega_2^2} \, d\omega_2 \Big) \, e^{2\pi \mathbf{i}\, \underline{y}_1 \omega_1} \, d\omega_1 \Big) f(x) \, f(y) d\mu(x) \, d\mu(y) \\ & = \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \frac{1}{\epsilon} e^{-\pi\,\epsilon^{-1}((\underline{y}_1^2 - \underline{x}_1^2) + (\underline{y}_2^2 - \underline{x}_2^2))} \, f(x) \, f(y) \, d\mu(x) \, d\mu(y),\end{aligned} \tag{42}$$

where $\underline{x} = (\underline{x}_1, \underline{x}_2)$, $\underline{y} = (\underline{y}_1, \underline{y}_2)$, and

$$\epsilon^{-1} e^{-\pi\,\epsilon^{-1}((\underline{y}_1^2 - \underline{x}_1^2) + (\underline{y}_2^2 - \underline{x}_2^2))} = \widehat{g}_\epsilon(|\underline{y} - \underline{x}|^2).$$

Since the family $\{\widehat{g}_\epsilon, \ \epsilon > 0\}$ is an approximation to the identity and bearing in mind the equality $\delta(\tanh^{-1}(y_i/t_i) - \tanh^{-1}(x_i/t_i)) = \delta(y_i - x_i), \ i = 1, 2$, letting $\epsilon \to 0$, the integral (42) reduces to

$$\int_{\mathbb{R}^2_{t_1,t_2}} |f(x)|^2 d\mu(x).$$

This shows that (41) is uniformly bounded in ϵ. By the Monotone Convergence Theorem, (41) is equal to $\int_{\mathbb{R}^2} |\widehat{f}(\omega)|^2 \, d\omega$. Therefore, we have proved that $\widehat{f} \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\|\widehat{f}\|_2 = \|f\|_2$.

Now, let f be in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ but not in $L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Since $L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ is dense in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, there exists a sequence $f^j \in L^1 \cap L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ such that $\|f^j - f\|_2 \to 0$. By Parseval's relation $\|\widehat{f^j} - \widehat{f^m}\|_2 = \|f^j - f^m\|_2$, and hence $\widehat{f^j}$ is a Cauchy sequence in $L^2(\mathbb{R}^2, \mathbb{H})$ that converges to some function in $L^2(\mathbb{R}^2, \mathbb{H})$, which we still denote by $\widehat{f}$. Then we have

$$\|\widehat{f}\|_2 = \lim_{j\to\infty} \|\widehat{f^j}\|_2 = \lim_{j\to\infty} \|f^j\|_2 = \|f\|_2.$$

Finally, let $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. The previous argument can be extended to every component f_i, $i = 0, 1, 2, 3$ of f in (5) and, hence, $\widehat{f} \in L^2(\mathbb{R}^2, \mathbb{H})$. Moreover, it holds

$$\|\widehat{f}\|^2_{Q,2} = \sum_{i=0}^{3} \int_{\mathbb{R}^2} |\widehat{f_i}(\omega)|^2 \, d\omega = \sum_{i=0}^{3} \int_{\mathbb{R}^2_{t_1,t_2}} |f_i(x)|^2 \, d\mu(x) = \|f\|_2^2.$$

The proof is now completed. □

It should be remarked that the above theorem establishes that the two-sided QHFT is a norm-preserving map from $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ into $L^2(\mathbb{R}^2, \mathbb{H})$.

Remark 2 We have proved the following mapping properties of the two-sided QHFT so far:

1. $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) \Rightarrow \widehat{f} \in L^\infty(\mathbb{R}^2, \mathbb{H})$ and $\|\widehat{f}\|_{Q,\infty} \leq 2\|f\|_1$,
2. $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H}) \Rightarrow \widehat{f} \in L^2(\mathbb{R}^2, \mathbb{H})$ and $\|\widehat{f}\|_{Q,2} = \|f\|_2$.

The boundedness of the QHFT $\widehat{f}\colon L^q \to L^p$ for $1 \leq p \leq 2$ with $1/p + 1/q = 1$, particularly $\|\widehat{f}\|_{Q,q} \leq \|f\|_p$, follows from its boundedness for $\widehat{f}\colon L^1 \to L^\infty$ and $\widehat{f}\colon L^2 \to L^2$ using Riesz-Thorin's Interpolation Theorem. Thus, $\widehat{f}$ exists in the L^p-norm and $\widehat{f} \in L^q(\mathbb{R}^2, \mathbb{H})$ whenever $f \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.

4 Uncertainty Principles

4.1 Pitt's Inequality and the Logarithmic Uncertainty Principle

Weighted norm inequalities for the Fourier transform play a central role in harmonic analysis, providing a natural measure to characterize uncertainty. In [6], Beckner proved Pitt's inequality in $\mathbb{R}^n$ by applying the sharp L^1 Young's inequality for the convolution on $\mathbb{R}^+$ and showed that the logarithmic uncertainty principle follows from Pitt's inequality. In this subsection, we shall prove Pitt's inequality for the two-sided QHFT, which describes a fundamental relationship between a sufficiently smooth quaternion function and the corresponding QHFT. We will then derive a logarithmic uncertainty principle associated with the two-sided QHFT.

We first turn our attention to some auxiliary results involving the power function $|\underline{x}|^\alpha$. (Although this is proved in a similar manner as in Chen et al. [12], we include the proof for completeness since it is necessary to employ the definitions (38) and (9).)

Proposition 12 *Let $f \in C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, $0 < \alpha < 2$, and $c_\alpha := \pi^{-\alpha/2}\,\Gamma(\alpha/2)$. Then*

$$\mathcal{F}^{-1}_{QH}(c_\alpha(|\omega|^{-\alpha}\, \widehat{f}(\omega)))(x) = c_{2-\alpha} \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x} - \underline{y}|^{\alpha-2} \, f(y) \, d\mu(y). \qquad (43)$$

Proof Combining Theorem 10, (5), and (16) is sufficient to prove the result for $f \in C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. To prove (43), we need to apply the formula

$$c_\alpha \, |\omega|^{-\alpha} = \int_0^\infty e^{-\pi \, \lambda |\omega|^2} \, \lambda^{\alpha/2-1} \, d\lambda = \int_0^\infty e^{-\pi \, \lambda^{-1} |\omega|^2} \, (\lambda^{-1})^{1+\alpha/2} \, d\lambda, \qquad (44)$$

where we made the change of variable $\lambda \mapsto \lambda^{-1}$ in the last equality. Since $|\omega|^{-\alpha} \, \widehat{f}(\omega)$ is integrable, by employing (38), (44), Fubini's Theorem, and Proposition 9, direct computations show that

$$\begin{aligned}
&\mathcal{F}_{QH}^{-1}(c_\alpha(|\omega|^{-\alpha} \, \widehat{f}(\omega)))(x) \\
&= \int_{\mathbb{R}^2} e^{2\pi \, \mathbf{i} \, \underline{x}_1 \omega_1} \Big(\int_0^\infty e^{-\pi \, \lambda |\omega|^2} \, \lambda^{\frac{\alpha}{2}-1} \, d\lambda \Big) \Big(\int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i} \, \underline{y}_1 \omega_1} \, f(y) \, e^{-2\pi \mathbf{j} \, \underline{y}_2 \omega_2} \, d\mu(y) \Big) \\
&\quad \times \, e^{2\pi \mathbf{j} \, \underline{x}_2 \omega_2} \, d\omega \\
&= \int_0^\infty \lambda^{\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^2_{t_1,t_2}} \Big(\int_{\mathbb{R}^2} e^{2\pi \mathbf{i} \, (\underline{x_1 \ominus y_1}) \omega_1} \, e^{-\pi \, \lambda |\omega|^2} \, e^{2\pi \mathbf{j} \, (\underline{x_2 \ominus y_2}) \omega_2} \, d\omega \Big) f(y) \, d\mu(y) \Big) d\lambda \\
&= \int_0^\infty \lambda^{\frac{\alpha}{2}-1} \Big(\int_{\mathbb{R}^2_{t_1,t_2}} \lambda^{-1} e^{-\pi \, \lambda^{-1} \, ((\underline{x_1 \ominus y_1})^2 + (\underline{x_2 \ominus y_2})^2)} \, f(y) \, d\mu(y) \Big) d\lambda \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} \Big(\int_0^\infty e^{-\pi \, \lambda^{-1} ((\underline{x_1 \ominus y_1})^2 + (\underline{x_2 \ominus y_2})^2)} \, (\lambda^{-1})^{1+\frac{2-\alpha}{2}} \, d\lambda \Big) f(y) \, d\mu(y) \\
&= c_{2-\alpha} \int_{\mathbb{R}^2_{t_1,t_2}} ((\underline{x_1 \ominus y_1})^2 + (\underline{x_2 \ominus y_2})^2)^{-(2-\alpha)} \, f(y) \, d\mu(y).
\end{aligned}$$

Moreover, since

$$\begin{aligned}
\underline{x_i \ominus y}_i &= t_i \, \tanh^{-1} \Big(\frac{1}{t_i} (x_i \ominus y_i) \Big) = t_i \, \tanh^{-1} \left(\frac{\frac{x_i}{t_i} - \frac{y_i}{t_i}}{1 - \frac{x_i y_i}{t_i^2}} \right) \\
&= t_i \, \tanh^{-1} \Big(\frac{x_i}{t_i} \Big) - t_i \, \tanh^{-1} \Big(\frac{y_i}{t_i} \Big), \quad i = 1, 2
\end{aligned}$$

it follows that

$$(\underline{x_1 \ominus y_1})^2 + (\underline{x_2 \ominus y_2})^2 = |\underline{x} - \underline{y}|^2,$$

which completes the proof. $\square$

We now introduce a convolution structure in $\mathbb{R}^2_{t_1,t_2}$ that allows computing the two-sided QHFT of a convolution of two quaternion functions under suitable conditions. Let us introduce the following definition.

Definition 11 For any pair of h-measurable quaternionic functions, f and g in $\mathbb{R}^2_{t_1,t_2}$, the hyperbolic convolution of f and g is the function $f *_h g : \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ defined by

$$(f *_h g)(x) = \int_{\mathbb{R}^2_{t_1,t_2}} f(x \ominus y)\, g(y)\, d\mu(y), \quad x \in \mathbb{R}^2_{t_1,t_2} \tag{45}$$

where $x \ominus y = (x_1 \ominus y_1, x_2 \ominus y_2)$, provided that the integral exists.

It can be easily seen that if one of the functions is in $L^\infty(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ and the other is in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, or one of the functions is bounded with compact support while the other is h-continuous, then the convolution defined in (45) exists for all $x \in \mathbb{R}^2_{t_1,t_2}$. The properties of the hyperbolic convolution are analogous to those corresponding to the Euclidean convolution and, therefore, will not be given. Next, we shall utilize the hyperbolic convolution to derive the quaternionic versions of Young's inequality and the convolution theorem associated with the two-sided QHFT.

Proposition 13 (Young's inequality for the hyperbolic convolution) *Let* $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ *and* $g \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ *with* $1 \leq p < \infty$. *The hyperbolic convolution* (45) *is defined a.e. for* $x \in \mathbb{R}^2_{t_1,t_2}$ *and also belongs to* $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. *Further,*

$$\|f *_h g\|_p \leq \|f\|_1 \, \|g\|_p.$$

In [7], Beckner proved the following lemma for the Euclidean convolution of functions $|x|^\alpha$.

Lemma 1 *For* $0 < \alpha < n,\ 0 < \beta < n$, *and* $0 < \alpha + \beta < n$, *holds*

$$(|x|^{\alpha-n} * |x|^{\beta-n})(y) = \frac{c_{n-\alpha-\beta}\, c_\alpha\, c_\beta}{c_{\alpha+\beta}\, c_{n-\alpha}\, c_{n-\beta}} |y|^{\alpha+\beta-n}, \tag{46}$$

where $x, y \in \mathbb{R}^n$ *and* $c_\alpha = \pi^{-\alpha/2}\, \Gamma(\alpha/2)$.

The following result gives the analogous statement for the hyperbolic convolution within our context.

Lemma 2 *For* $0 < \alpha < n,\ 0 < \beta < n$, *and* $0 < \alpha + \beta < n$, *holds*

$$(|\underline{x}|^{\alpha-n} *_h |\underline{x}|^{\beta-n})(\underline{y}) = \frac{c_{n-\alpha-\beta}\, c_\alpha\, c_\beta}{c_{\alpha+\beta}\, c_{n-\alpha}\, c_{n-\beta}} |\underline{y}|^{\alpha+\beta-n}, \tag{47}$$

where $\underline{x}, \underline{y} \in \mathbb{R}^2$.

Proof Using the definition (45) of the hyperbolic convolution, the change of variables $t_i \ \tanh^{-1}(x_i/t_i) = z_i$, $i = 1, 2$, and (46) lead to

$$(|\underline{x}|^{\alpha-n} *_h |\underline{x}|^{\beta-n})(\underline{y}) = \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x \ominus y}|^{\alpha-n} |\underline{x}|^{\beta-n}\, d\mu(x)$$

$$= \int_{\mathbb{R}^2} |z|^{\alpha-n} |z - \underline{y}|^{\beta-n} \, dz$$
$$= \frac{c_{n-\alpha-\beta}\, c_\alpha\, c_\beta}{c_{\alpha+\beta}\, c_{n-\alpha}\, c_{n-\beta}} \, |\underline{y}|^{\alpha+\beta-n}.$$

□

Now, we extend Proposition 12 to the space $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.

Proposition 14 *If $0 \le \alpha < 2$ and $f \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ with $p = 2/(1+\alpha)$, then $\widehat{f}$ defined by (14) exists. Further, let*

$$g(x) = c_{2-\alpha} \, (|\underline{y}|^{\alpha-2} *_h f(y))(x), \tag{48}$$

where c_α is defined as in Proposition 12. Then g belongs to $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ and hence has a QHFT, $\widehat{g}$, given by

$$\widehat{g}(\omega) \;=\; c_\alpha \, |\omega|^{-\alpha} \, \widehat{f}(\omega). \tag{49}$$

For $f \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, we have the equality

$$\begin{aligned} & c_{2\alpha} \int_{\mathbb{R}^2} |\omega|^{-2\alpha} \, |\widehat{f}(\omega)|^2 \, d\omega \\ & = c_{2-2\alpha} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, f(y) \, |\underline{x} - \underline{y}|^{2\alpha-2} \, d\mu(x)\, d\mu(y) \end{aligned} \tag{50}$$

and for $f \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ holds

$$\begin{aligned} & c_{2\alpha} \int_{\mathbb{R}^2} |\omega|^{-2\alpha} \, |\widehat{f}(\omega)|^2_{Q,2} \, d\omega \\ & = c_{2-2\alpha} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \sum_{i=0}^{3} f_i(x)\, f_i(y) \, |\underline{x} - \underline{y}|^{2\alpha-2} \, d\mu(x)\, d\mu(y). \end{aligned} \tag{51}$$

Proof As in Proposition 12, we consider first $f \in L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Since $C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ is dense in $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ there exists a sequence of functions f^j in $C_c(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ such that $\|f^j - f\|_p \to 0$ as $j \to \infty$. Setting $g := c_{2-\alpha} \, |\underline{y}|^{\alpha-2} *_h f$ and $g^j := c_{2-\alpha} \, |\underline{y}|^{\alpha-2} *_h f^j$, by Fubini's Theorem and Lemma 2, we have

$$\begin{aligned} & \int_{\mathbb{R}^2_{t_1,t_2}} |g(x)|^2 \, d\mu(x) \\ & = \int_{\mathbb{R}^2_{t_1,t_2}} \Big(\int_{\mathbb{R}^2_{t_1,t_2}} c_{2-\alpha} \, |\underline{x} - \underline{y}|^{\alpha-2} \, f(y) \, d\mu(y) \Big) \end{aligned}$$

$$\times \left(\int_{\mathbb{R}^2_{t_1,t_2}} c_{2-\alpha} \, |\underline{x} - \underline{z}|^{\alpha-2} \, f(z) \, d\mu(z) \right) d\mu(x)$$
$$= (c_{2-\alpha})^2 \int_{\mathbb{R}^2_{t_1,t_2}} f(z) \int_{\mathbb{R}^2_{t_1,t_2}} f(y)$$
$$\times \left(\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x} - \underline{y}|^{\alpha-2} |\underline{x} - \underline{z}|^{\alpha-2} \, d\mu(x) \right) d\mu(y) \, d\mu(z).$$

Making the change of variables $\underline{x} - \underline{z} = \underline{v}$, where

$$\underline{v} = (t_1 \, \tanh^{-1}(v_1/t_1), \; t_2 \, \tanh^{-1}(v_2/t_2)),$$

which is equivalent to $x_i \ominus z_i = v_i, \; i = 1, 2$, we then get

$$\int_{\mathbb{R}^2_{t_1,t_2}} |g(x)|^2 \, d\mu(x)$$
$$= (c_{2-\alpha})^2 \int_{\mathbb{R}^2_{t_1,t_2}} f(z) \int_{\mathbb{R}^2_{t_1,t_2}} f(y) \left(\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{v} - (\underline{y} - \underline{z})|^{\alpha-2} |\underline{v}|^{\alpha-2} \, d\mu(v) \right)$$
$$= \frac{c_{2-2\alpha} \, (c_\alpha)^2}{c_{2\alpha}} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f(y) \, |\underline{y} - \underline{z}|^{2\alpha-2} \, f(z) \, d\mu(y) \, d\mu(z). \tag{52}$$

Since $f^j \to f$ in $L^p(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, by Remark 2, we have that $\widehat{f^j} \to \widehat{f}$ in $L^q(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Now, by the Hardy–Littlewood–Sobolev inequality, it follows that g, g^j are in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ and $g^j \to g$ in $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Consequently, by Parseval's Theorem, $\widehat{g}^j \to \widehat{g}$ in $L^2(\mathbb{R}^2, \mathbb{R})$. From Proposition 12, we can conclude that

$$\widehat{g}^j(\omega) \; = \; c_\alpha \, |\omega|^{-\alpha} \, \widehat{f}^j(\omega).$$

By the completeness of the L^p space and the pointwise convergence $\widehat{f}^j \to \widehat{f}$ a.e., we conclude that

$$\widehat{g}(\omega) \; = \; \lim_{j \to \infty} c_\alpha \, |\omega|^{-\alpha} \, \widehat{f}^j(\omega) \; = \; c_\alpha \, |\omega|^{-\alpha} \, \widehat{f}(\omega) \;\; \text{a.e.}.$$

By (49) and Parseval's Theorem, we have

$$\int_{\mathbb{R}^2_{t_1,t_2}} |g(x)|^2 \, d\mu(x) = c_\alpha^2 \int_{\mathbb{R}^2} |\omega|^{-2\alpha} \, |\widehat{f}(\omega)|^2 \, d\omega \, . \tag{53}$$

From (52) and (53), we obtain equality (50). The equality (51) follows from (5), (16), and (18). □

We are now ready to establish Pitt's inequality for the two-sided QHFT.

Theorem 15 (Pitt's inequality) *For any* $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$*, Pitt's inequality for the two-sided QHFT is given by*

$$\int_{\mathbb{R}^2} |\omega|^{-\alpha}\, |\widehat{f}(\omega)|_Q^2\, d\omega \;\leq\; C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^{\alpha}\, |f(x)|^2\, d\mu(x), \quad 0 \leq \alpha < 2 \tag{54}$$

where $C_\alpha = \pi^\alpha \big(\Gamma(\frac{2-\alpha}{4})/\,\Gamma(\frac{2+\alpha}{4})\big)^2$.

Proof We first prove the inequality for $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. Considering the function $F(x) = |\underline{x}|^{\alpha/2}\, f(x)$ then by Proposition 14, we can see that the left-hand side of (54) is

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\omega|^{-\alpha}\, |\widehat{f}(\omega)|^2\, d\omega \\
&= \frac{c_{2-\alpha}}{c_\alpha} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, |\underline{x} - \underline{y}|^2\, f(y)\, d\mu(x)\, d\mu(y) \\
&= \frac{c_{2-\alpha}}{c_\alpha} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \frac{F(x)}{|\underline{x}|^{\alpha/2}}\, |\underline{x} - \underline{y}|^{\alpha-2}\, \frac{F(y)}{|\underline{y}|^{\alpha/2}}\, d\mu(x)\, d\mu(y),
\end{aligned}$$

and the right-hand side of (54) is

$$\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^{\alpha}\, |f(x)|^2\, d\mu(x) \;=\; \int_{\mathbb{R}^2_{t_1,t_2}} |F(x)|^2\, d\mu(x).$$

Then, to prove (54) is equivalent to showing that

$$\begin{aligned}
&\int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \frac{F(x)}{|\underline{x}|^{\alpha/2}}\, |\underline{x} - \underline{y}|^{\alpha-2}\, \frac{F(y)}{|\underline{y}|^{\alpha/2}}\, d\mu(x)\, d\mu(y) \\
&= C_\alpha\, \frac{c_\alpha}{c_{2-\alpha}} \int_{\mathbb{R}^2_{t_1,t_2}} |f(x)|^2\, d\mu(x).
\end{aligned} \tag{55}$$

Without loss of generality, we may assume that f is nonnegative. Considering

$$\underline{x} = r\xi, \quad \underline{y} = s\mu, \quad r, s \in \mathbb{R}^+, \quad \xi, \mu \in S^1$$

which means that $x_i = t_i\, \tanh(r\xi_i/t_i)$ and $y_i = t_i\, \tanh(s\mu_i/t_i)$, $i = 1, 2$, we define the following functions:

$$\begin{aligned}
\varphi(r\xi) \;&=\; r\, f(t_1 \tanh(r\xi_1/t_1)\,,\; t_2\, \tanh(r\xi_2/t_2)), \\
\phi(r, \xi\cdot\mu) \;&=\; r^{\alpha/4}\, \big(r + \frac{1}{r} - 2\xi\cdot\mu\big)^{-(1-\alpha/4)}.
\end{aligned}$$

Then (55) can be formulated as a convolution estimate on $\mathbb{R}_+ \times S^1$:

$$\|\phi * \varphi\|_{L^2(\mathbb{R}_+\times S^1)} \;\leq\; \|\phi\|_{L^1(\mathbb{R}_+\times S^1)}\, \|\varphi\|_{L^2(\mathbb{R}_+\times S^1)}.$$

To see this, we first observe that

$$\begin{aligned}\|f\|_2^2 &= \int_{\mathbb{R}^2_{t_1,t_2}} |f(x)|^2\, d\mu(x)\\ &= \int_{\mathbb{R}_+\times S^1} |f(t_1\ \tanh(r\xi_1/t_1)\,,\ t_2 \tanh(r\xi_2/t_2))|^2\, r\, dr\, d\xi\\ &= \int_{\mathbb{R}_+\times S^1} |r\ f(t_1\ \tanh(r\xi_1/t_1)\,,\ t_2 \tanh(r\xi_2/t_2))|^2\, d\xi\, \frac{dr}{r}\\ &= \|\varphi\|^2_{L^2(\mathbb{R}_+\times S^1)}.\end{aligned}$$

In the second equality we made the change of variables $x_1 = t_1 \tanh(r\cos\theta/t_1)$ and $x_2 = t_2 \tanh(r \sin\theta/t_2)$. The corresponding Jacobian is

$$\frac{r}{\cosh^2\left(\frac{r\cos\theta}{t_1}\right)\,\cosh^2\left(\frac{r\sin\theta}{t_2}\right)}$$

and hence $d\mu(x) = r\, dr\, d\theta = r\, dr\, d\xi$. Secondly, we have

$$\begin{aligned}&\|\phi * \varphi\|^2_{L^2(\mathbb{R}_+\times S^1)}\\ &= \int_{\mathbb{R}_+\times S^1} |\phi * \varphi|^2\, d\zeta\, \frac{da}{a}\\ &= \int_{\mathbb{R}_+\times S^1} \left(\int_{\mathbb{R}_+\times S^1} h(r\xi)\,\phi\Big(\frac{a}{r}, \xi\cdot\zeta\Big)\, d\xi\, \frac{dr}{r}\right)\\ &\quad\times\left(\int_{\mathbb{R}_+\times S^1} h(s\mu)\,\phi\Big(\frac{a}{s}, \mu\cdot\zeta\Big)\, d\mu\, \frac{ds}{s}\right) d\xi\, \frac{da}{a}\\ &= \int_{\mathbb{R}_+\times S^1}\int_{\mathbb{R}_+\times S^1} h(r\xi)\, h(s\mu)\, K(r,s,\xi\cdot\zeta,\mu\cdot\zeta)\, d\xi\, \frac{dr}{r}\, d\mu\, \frac{ds}{s},\end{aligned} \tag{56}$$

where the kernel is given by

$$K(r,s,\xi\cdot\zeta,\mu\cdot\zeta) = \int_{\mathbb{R}_+\times S^1} \phi\Big(\frac{a}{r}, \xi\cdot\zeta\Big)\,\phi\Big(\frac{a}{s}, \mu\cdot\zeta\Big)\, d\zeta\, \frac{da}{a}.$$

To compute $K(r, s, \xi\cdot\zeta, \mu\cdot\zeta)$, we use Lemma 2:

$$\begin{aligned}&\int_{\mathbb{R}_+\times S^1} \phi\left(\frac{a}{r}, \xi\cdot\zeta\right)\phi\left(\frac{a}{s}, \mu\cdot\zeta\right) d\zeta\, \frac{da}{a}\\ &= \int_{\mathbb{R}_+\times S^1} \left(\frac{a}{r}\right)^{\alpha/4}\left(\frac{a}{r}+\frac{r}{a}-2\xi\cdot\zeta\right)^{-\frac{2-\alpha/2}{2}}\left(\frac{a}{s}\right)^{\alpha/4}\\ &\quad\times\left(\frac{a}{s}+\frac{s}{a}-2\mu\cdot\zeta\right)^{-\frac{2-\alpha/2}{2}} d\xi\, \frac{da}{a}\end{aligned}$$

$$
\begin{aligned}
&= s^{1-\alpha/2}\, r^{1-\alpha/2} \int_{\mathbb{R}_+\times S^1} |a\zeta - r\xi|^{-2-\alpha/2}\, |a\zeta - s\mu|^{-2-\alpha/2}\, a\, d\xi\, da \\
&= |\underline{x}|^{1-\alpha/2}\, |\underline{y}|^{1-\alpha/2} \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{z} - \underline{x}|^{\alpha/2-2}\, |\underline{z} - \underline{y}|^{\alpha/2-2}\, d\mu(z) \\
&= |\underline{x}|^{1-\alpha/2}\, |\underline{y}|^{1-\alpha/2}\, \frac{c_{2-\alpha}}{c_\alpha} \left(\frac{c_{\alpha/2}}{c_{2-\alpha/2}}\right)^2 |\underline{x} - \underline{y}|^{\alpha-2}.
\end{aligned}
$$

Then (56) is equal to

$$
\begin{aligned}
&\|\phi * \varphi\|_{L^2(\mathbb{R}_+\times S^1)} \\
&= \frac{c_{2-\alpha}}{c_\alpha} \left(\frac{c_{\alpha/2}}{c_{2-\alpha/2}}\right)^2 \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} \frac{F(x)}{|\underline{x}|^{\alpha/2}}\, |\underline{x} - \underline{y}|^{\alpha-2}\, \frac{F(y)}{|\underline{y}|^{\frac{\alpha}{2}}}\, d\mu(x)\, d\mu(y).
\end{aligned}
$$

Hence the best constant C_α in (55) is

$$
\left(\frac{c_{2-\alpha/2}}{c_{\alpha/2}}\right)^2 \|\phi\|^2_{L^1(\mathbb{R}_+\times S^1)}.
$$

Now, we compute $\|\phi\|_{L^1}$ using Lemma 2:

$$
\begin{aligned}
\|\phi\|_{L^1(\mathbb{R}_+\times S^1)} &= \int_{\mathbb{R}_+\times S^1} r^{\alpha/4}\left(r + \frac{1}{r} - 2\xi\cdot\mu\right)^{-(1-\alpha/4)} d\xi\, \frac{dr}{r} \\
&= \int_{\mathbb{R}_+\times S^1} r^{-1}(r^2 + 1 - 2r\xi\cdot\mu)^{-(1-\alpha/4)}\, r\, d\xi\, dr \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x} - \mu|^{-(2-\alpha/2)}\, |\underline{x}|^{-1}\, d\mu(x), \\
&= \frac{c_{1-\alpha/2}\, c_{\alpha/2}}{c_{1+\alpha/2}\, c_{2-\alpha/2}},
\end{aligned}
$$

where $|\mu| = 1$. Therefore,

$$
C_\alpha = \left(\frac{c_{1-\alpha/2}}{c_{1+\alpha/2}}\right)^2 = \pi^\alpha \left(\Gamma\left(\frac{2-\alpha}{4}\right) / \Gamma\left(\frac{2+\alpha}{4}\right)\right)^2,
$$

which proves (54) for $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$. To prove (54) for $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, we only need to apply the previous result for each component, due to (5) and (17). □

The logarithmic uncertainty principle for the two-sided QHFT follows now from the sharp Pitt's inequality in Theorem 15. Since the inequality (54) is an equation for α, we will differentiate Pitt's inequality with respect to α at $\alpha = 0$.

Corollary 1 (Logarithmic uncertainty principle) *For any nontrivial function $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, the following uncertainty inequality holds:*

$$\int_{\mathbb{R}^2_{t_1,t_2}} \ln|\underline{x}|\,|f(x)|^2\,d\mu(x) + \int_{\mathbb{R}^2} \ln|\omega|\,|\widehat{f}(\omega)|_Q^2\,d\omega \geq D \int_{\mathbb{R}^2_{t_1,t_2}} |f(x)|^2\,d\mu(x), \tag{57}$$

where $D := \psi(1/2) - \ln\pi$ and ψ is the digamma function defined for $x > 0$ by $\psi(x) = \frac{d}{dx}\ln(\Gamma(x)) = \Gamma'(x)/\Gamma(x)$.

Proof Putting $|\omega|^{-\alpha} = e^{-\alpha\ln|\omega|}$ and $|\underline{x}|^\alpha = e^{\alpha\ln|\underline{x}|}$ in (54) and differentiating with respect to α, we find

$$\begin{aligned} &-\int_{\mathbb{R}^2} \ln|\omega|\,e^{-\alpha\ln|\omega|}\,|\widehat{f}(\omega)|_Q^2\,d\omega \\ &\leq (C_\alpha)' \int_{\mathbb{R}^2_{t_1,t_2}} e^{\alpha\ln|\underline{x}|}\,|f(x)|^2\,d\mu(x) + C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} \ln|\underline{x}|\,e^{\alpha\ln|\underline{x}|}\,|f(x)|^2\,d\mu(x), \end{aligned}$$

where

$$(C_\alpha)' = \frac{\pi^\alpha\,\Gamma\left(\frac{2-\alpha}{4}\right)^2\left(2\ln(\pi) - \psi\left(\frac{2-\alpha}{4}\right) - \psi\left(\frac{2+\alpha}{4}\right)\right)}{2\,\Gamma\left(\frac{2+\alpha}{4}\right)^2}.$$

Now, setting $\alpha = 0$, we then obtain (57). This completes the proof. □

The qualitative nature of this result underlines the relationships connecting entropy, the Hardy–Littlewood–Sobolev inequality, and the logarithmic Sobolev inequality (see [6]).

In the large limits of t_1 and t_2, i.e., $t_1, t_2 \to \infty$, Corollary 1 yields the uncertainty inequality for the two-sided QFT, obtained by Chen et al. [12].

4.2 Heisenberg–Weyl's Uncertainty Principle

Note, in passing, that the logarithmic uncertainty principle (57) implies Heisenberg–Weyl's inequality. To see this, we recall that the logarithm is a concave function, and then by Jensen's inequality, for any $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ such that $\|f\|_2 = 1$, we have

$$\ln\left[\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^2\,|f(x)|^2\,d\mu(x)\right]^{1/2} \geq \int_{\mathbb{R}^2_{t_1,t_2}} \ln|\underline{x}|\,|f(x)|^2\,d\mu(x) \tag{58}$$

and

$$\ln\left[\int_{\mathbb{R}^2} |\omega|^2\,|\widehat{f}(\omega)|_Q^2\,d\omega\right]^{1/2} \geq \int_{\mathbb{R}^2} \ln|\omega|\,|\widehat{f}(\omega)|_Q^2\,d\omega. \tag{59}$$

Now, by (57), (58), and (59), it follows that

$$\begin{aligned}
&\ln\Big[\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^2\,|f(x)|^2\,d\mu(x)\ \int_{\mathbb{R}^2} |\omega|^2\,|\widehat{f}(\omega)|_Q^2\,d\omega\Big]^{1/2}\\
&= \ln\Big[\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^2\,|f(x)|^2\,d\mu(x)\Big]^{1/2} + \ln\Big[\int_{\mathbb{R}^2} |\omega|^2\,|\widehat{f}(\omega)|_Q^2\,d\omega\Big]^{1/2}\\
&\geq \int_{\mathbb{R}^2_{t_1,t_2}} \ln|\underline{x}|\,|f(x)|^2\,d\mu(x) + \int_{\mathbb{R}^2} \ln|\omega|\,|\widehat{f}(\omega)|_Q^2\,d\omega\\
&\geq D
\end{aligned}$$

with $D = \psi(1/2) - \ln\pi$. Now, since the logarithm is an increasing function, we obtain Heisenberg–Weyl's uncertainty principle in the form

$$\Big(\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^2\,|f(x)|^2\,d\mu(x)\Big)\Big(\int_{\mathbb{R}^2} |\omega|^2\,|\widehat{f}(\omega)|_Q^2\,d\omega\Big) \geq e^{2D}. \tag{60}$$

By considering the n-dimensional Heisenberg–Weyl's uncertainty principle presented in [23, Cor. 2.8], we obtain, for the two-dimensional hyperbolic setting the inequality

$$\Big(\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^2\,|f(x)|^2\,d\mu(x)\Big)\Big(\int_{\mathbb{R}^2} |\omega|^2\,|\widehat{f}(\omega)|_Q^2\,d\omega\Big) \geq \Big(\frac{2}{4\pi}\Big)^2 \tag{61}$$

where we put $\|f\|_2 = 1$. Inequality (61) provides the best constant for the two-dimensional hyperbolic Heisenberg–Weyl's uncertainty principle. The equality is obtained for the normalized quaternionic hyperbolic function

$$f(x) = K\sqrt{\frac{2}{\pi}}\,e^{-|\underline{x}|^2},$$

where K is a quaternionic constant such that $|K| = 1$.

Introducing the standard deviations

$$\Delta_f x = \Big(\int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x} - \underline{x}_0|^2\,|f(x)|^2\,d\mu(x)\Big)^{1/2}$$

and

$$\Delta_f \omega = \Big(\int_{\mathbb{R}^2} |\omega - \omega_0|^2\,|\widehat{f}(\omega)|_Q^2\,d\omega\Big)^{1/2}$$

we can state the Heisenberg–Weyl uncertainty principle associated with the two-sided QHFT in the following form.

Theorem 16 (Heisenberg–Weyl's uncertainty principle) *Let* $f \in L^2(\mathbb{R}^2_{t_1,t_2}\mathbb{H})$ *with* $\|f\|_2 = 1$. *Then*

$$\Delta_f x \cdot \Delta_f \omega \geq \frac{1}{2\pi}. \tag{62}$$

4.3 Donoho–Stark and Benedicks' Uncertainty Principles

Donoho–Stark's uncertainty principle is an inequality giving local information about a function and its Fourier transform since the support is not fixed a priori. More precisely, it asserts that a signal and its Fourier transform cannot both be well-concentrated around their respective means: narrowing one broadens necessarily the other. A multi-dimensional generalization of this theorem in which the QFT is defined by (2) and the sets are measurable was proved, as in [17], by Chen et al. in [12]. Global and local uncertainty principles for the Fourier transform in $\mathbb{R}^n$ and groups with Plancherel measure are surveyed in [23].

The following definition is adapted to our context from [17].

Definition 12 Let $\epsilon_T \geq 0$. We say that a function $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is ϵ_T-concentrated on a measurable set $T \subseteq \mathbb{R}^2_{t_1,t_2}$ if

$$\left(\int_{T^c} |f(x)|\, d\mu(x) \right)^{1/2} \leq \epsilon_T \, \|f\|_2, \tag{63}$$

where T^c denotes the complement of T.

If $0 \leq \epsilon_T \leq 1/2$, then the most energy of f is concentrated on T, and T is indeed the essential support of f. If $\epsilon_T = 0$, then T is the exact support of f.

According to Definition 12, we extend Donoho-Stark's uncertainty principle within our context as follows:

Theorem 17 (Donoho-Stark's uncertainty principle) *Suppose that* $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, *not identically zero, is* ϵ_T*-concentrated on* $T \subseteq \mathbb{R}^2_{t_1,t_2}$ *and* $\widehat{f}$ *is* ϵ_Ω*-concentrated on* $\Omega \subseteq \mathbb{R}^2$. *Then*

$$|T|\,|\Omega| \geq (1 - \epsilon_T - \epsilon_\Omega)^2.$$

We omit the proof since it is analogous to the one given in [12] for the Euclidean case and is based on the space-limiting and band-limiting operators P_T and Q_Ω, which we define on $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ by

$$P_T f := \chi_T(f)$$

and

$$Q_\Omega f := \mathcal{F}_{QH}^{-1}(\chi_\Omega(\mathcal{F}_{QH}(f))) = \int_\Omega e^{2\pi \mathbf{i}\, \underline{x}_1 \omega_1}\, \widehat{f}(\omega)\, e^{2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\omega,$$

where χ_T is the characteristic function of T. It can be proved that these operators are orthogonal projections on $L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$.

It is not in the scope of the present work to thoroughly discuss the applications of Theorem 17. Nevertheless, to understand the importance of this result in a quaternionic context, set $\epsilon_T = \epsilon_\Omega = 0$ in the theorem and observe that f is concentrated on T if and only if supp $f \subseteq T$ and $\widehat{f}$ is concentrated on Ω if and only if supp $\widehat{f} \subseteq \Omega$. Hence, we obtain the following result.

Corollary 2 *Let $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ be non identically zero with* supp $f \subseteq T$ *and* supp $\widehat{f} \subseteq \Omega$. *Then $|T|\,|\Omega| \geq 1$.*

The following result is the hyperbolic counterpart of [23] for signals defined in the n-dimensional Euclidean space. It is a qualitative uncertainty principle meaning that, without giving quantitative estimates for f and $\widehat{f}$, a nonzero quaternion function and its two-sided QHFT cannot be highly concentrated unless $f = 0$, independent of the chosen concentration sets T and Ω. This result was first stated by Benedicks for the Fourier transform in $\mathbb{R}^n$.

Theorem 18 (Benedicks' uncertainty principle) *Let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ be non identically zero with* supp $f \subseteq T$ *and* supp $\widehat{f} \subseteq \Omega$. *If $|T|\,|\Omega| < \infty$, then $f = 0$.*

This result is proved using the following quaternionic hyperbolic version of the Poisson summation formula.

Lemma 3 (Hyperbolic Poisson summation formula in $\mathbb{R}^2_{t_1,t_2}$) *If $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, then the series*

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} f(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2})$$

with $\theta_{n_1} = t_1\ \tanh(n_1/t_1)$ and $\beta_{n_2} = t_2\ \tanh(n_2/t_2)$, converges in $L^1([0,\theta_1]\times[0,\beta_1])$ and

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} f(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2}) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} e^{2\pi \mathbf{i}\, \underline{x}_1 k_1}\, \widehat{f}(k_1,k_2)\, e^{2\pi \mathbf{j}\, \underline{x}_2 k_2}.$$

Proof We first observe that the function $\phi(x) = \sum_{(n_1,n_2)\in\mathbb{Z}^2} f(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2})$ is θ_1 h-periodic in the first variable and β_1 h-periodic in the second variable. Bearing in mind that

$$\theta_1 \oplus \theta_m = \frac{t\ \tanh(\frac{1}{t}) + t\ \tanh(\frac{m}{t})}{1 + \tanh(\frac{1}{t})\ \tanh(\frac{m}{t})} = t\ \tanh\left(\frac{1+m}{t}\right) = \theta_{m+1}$$

then we have

$$\phi(x_1 \oplus \theta_1, x_2 \oplus \beta_1) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} f((x_1 \oplus \theta_1) \oplus \theta_{k_1}, (x_2 \oplus \beta_1) \oplus \beta_{k_2})$$

$$
\begin{aligned}
&= \sum_{(k_1,k_2)\in\mathbb{Z}^2} f(x_1 \oplus (\theta_1 \oplus \theta_{k_1}), x_2 \oplus (\beta_1 \oplus \beta_{k_2}))\\
&= \sum_{(k_1,k_2)\in\mathbb{Z}^2} f((x_1 \oplus \theta_{k_1+1}, x_2 \oplus \beta_{k_2+1})\\
&= \sum_{(k_1',k_2')\in\mathbb{Z}^2} f(x_1 \oplus \theta_{k_1'}, x_2 \oplus \beta_{k_2'})\\
&= \phi(x_1, x_2).
\end{aligned}
$$

Since f is integrable then ϕ converges in $L^1([0,\theta_1]\times[0,\beta_1],\mathbb{H})$. Therefore, we can take the quaternion hyperbolic fourier series of ϕ :

$$
\phi(x) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} e^{2\pi\mathbf{i}\,\underline{x}_1 k_1} a_{(k_1,k_2)}\, e^{2\pi\mathbf{j}\,\underline{x}_2 k_2}, \tag{64}
$$

where the coefficients $a_{(k_1,k_2)}$ are computed by

$$
\begin{aligned}
a_{(k_1,k_2)} &= \int_0^{\theta_1}\int_0^{\beta_1} e^{-2\pi\mathbf{i}\,\underline{x}_1 k_1}\phi(x) e^{-2\pi\mathbf{j}\,\underline{x}_2 k_2}\, d\mu(x)\\
&= \int_0^{\theta_1}\int_0^{\beta_1} e^{-2\pi\mathbf{i}\,\underline{x}_1 k_1} \sum_{(n_1,n_2)\in\mathbb{Z}^2} f(x_1\oplus\theta_{n_1}, x_2\oplus\beta_{n_2}) e^{-2\pi\mathbf{j}\,\underline{x}_2 k_2}\, d\mu(x)\\
&= \sum_{(n_1,n_2)\in\mathbb{Z}^2} \int_0^{\theta_1}\int_0^{\beta_1} e^{-2\pi\mathbf{i}\,\underline{x}_1 k_1} f(x_1\oplus\theta_{n_1}, x_2\oplus\beta_{n_2}) e^{-2\pi\mathbf{j}\,\underline{x}_2 k_2}\, d\mu(x).
\end{aligned}
$$

Making the change of variables $x_1 \oplus \theta_{n_1} = y_1$ and $x_2 \oplus \beta_{n_2} = y_2$, then using Property 2 in Proposition 1 and the hyperbolic invariance of the measure, we get

$$
\begin{aligned}
&a_{(k_1,k_2)}\\
&= \sum_{(n_1,n_2)\in\mathbb{Z}^2} \int_{\theta_{n_1}}^{\theta_{n_1+1}}\int_{\beta_{n_2}}^{\beta_{n_2+1}} e^{-2\pi\mathbf{i}\,\underline{y}_1 k_1}\, e^{2\pi\mathbf{i}\,\underline{\theta_{n_1}} k_1} f(y) e^{-2\pi\mathbf{j}\,\underline{y}_2 k_2}\, e^{2\pi\mathbf{j}\,\underline{\beta_{n_2}} m_2}\, d\mu(y).
\end{aligned}
$$

By using the facts that $\lim_{n_1\to\pm\infty}\theta_{n_1} = \pm t_1$ and $\lim_{n_2\to\pm\infty}\beta_{n_2} = \pm t_2$, $\underline{\theta_{n_1}} = n_1$ and $\underline{\beta_{n_2}} = n_2$, and

$$
e^{2\pi\mathbf{i}n_1k_1} = e^{2\pi\mathbf{j}n_2k_2} = 1,\ \forall n_1, n_2, k_1, k_2 \in \mathbb{Z},
$$

it follows that

$$
\begin{aligned}
a_{(k_1,k_2)} &= \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi\mathbf{i}\,\underline{y}_1 k_1}\; e^{2\pi\mathbf{i}\,n_1 k_1}\, f(y)\, e^{-2\pi\mathbf{j}\,\underline{y}_2 k_2}\; e^{2\pi\mathbf{i}\,n_2 k_2}\, d\mu(y)\\
&= \widehat{f}(k_1, k_2).
\end{aligned}
$$

Finally, we get from (64)

$$\phi(x) = \sum_{(k_1,k_2)\in\mathbb{Z}^2} e^{2\pi \mathbf{i} \underline{x}_1 k_1} \widehat{f}(k_1, k_2) e^{2\pi \mathbf{j} \underline{x}_2 k_2},$$

which gives the desired result. □

Proof of Theorem 18 To prove the theorem, we may assume that $|T| < 1$ by replacing $f(x)$ by $f(\lambda \otimes x)$, for some $\lambda > 0$. We have

$$\begin{aligned}\int_{[0,1]^2} \sum_{(k_1,k_2)\in\mathbb{Z}^2} \chi_\Omega(\omega_1 + k_1, \omega_2 + k_2)\, d\omega &= \sum_{(k_1,k_2)\in\mathbb{Z}^2} \int_{k_1}^{k_1+1} \int_{k_2}^{k_2+1} \chi_\Omega(\xi_1, \xi_2)\, d\xi \\ &= \int_{\mathbb{R}^2} \chi_\Omega(\xi_1, \xi_2)\, d\xi \\ &= |\Omega| < \infty\end{aligned}$$

and

$$\begin{aligned}&\int_0^{\theta_1} \int_0^{\beta_1} \sum_{(n_1,n_2)\in\mathbb{Z}^2} \chi_T(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2})\, d\mu(x) \\ &= \sum_{(n_1,n_2)\in\mathbb{Z}^2} \int_{\theta_{n_1}}^{\theta_{n_1+1}} \int_{\beta_{n_2}}^{\beta_{n_2+1}} \chi_T(y_1, y_2)\, d\mu(y) \\ &= \int_{\mathbb{R}^2_{t_1,t_2}} \chi_T(y_1, y_2)\, d\mu(y) \\ &= |T| < 1,\end{aligned}$$

where $\theta_{n_1}, \beta_{n_2}$ are defined as in Lemma 3.

These inequalities imply, respectively, that

(i) There exists $E \subseteq [0, 1]^2$ with $|E| = 1$ such that

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \chi_\Omega(a_1 \oplus \theta_{n_1}, a_2 \oplus \beta_{n_2}) < \infty$$

for $(a_1, a_2) \in E$, and hence $\widehat{f}(a_1+k_1, a_2+k_2) \neq 0$ for only finitely many (k_1, k_2) if $(a_1, a_2) \in E$.

(ii) There exists $F \subseteq [0, \theta_1] \times [0, \beta_1]$ with $|F| > 0$ such that

$$\sum_{(n_1,n_2)\in\mathbb{Z}^2} \chi_T(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2}) = 0$$

for $(x_1, x_2) \in F$, and hence $f(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2}) = 0$ for all (n_1, n_2) if $(x_1, x_2) \in F$.

Given $a \in E$, let

$$\phi_a(x) = \sum_{(n_1,n_2)\in\mathbb{Z}^2} e^{-2\pi \mathbf{i}\,\underline{(x_1 \oplus \theta_{n_1})} a_1} f(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2})\, e^{-2\pi \mathbf{j}\,\underline{(x_2 \oplus \beta_{n_2})} a_2}.$$

Since $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ then, by Lemma 3, $\phi_a(x) \in L^1([0,\theta_1]\times[0,\beta_1], \mathbb{H})$ and the quaternionic hyperbolic Fourier series of ϕ_a is

$$\sum_{(k_1,k_2)\in\mathbb{Z}^2} e^{2\pi \mathbf{i}\,\underline{x}_1 k_1}\, \widehat{f}(k_1 + a_1, k_2 + a_2)\, e^{2\pi \mathbf{j}\,\underline{x}_2 k_2}.$$

Since $a \in E$, ϕ_a is a trigonometric polynomial that cannot vanish on a set of positive measure, unless it vanishes identically. On the other hand, $|\phi_a(x)| \leq \sum_{(n_1,n_2)} |f(x_1 \oplus \theta_{n_1}, x_2 \oplus \beta_{n_2})| = 0$ for $(x_1, x_2) \in F$. We conclude that $\phi_a = 0$ for all $(a_1, a_2) \in E$; thus, $\widehat{f}(a_1 + k_1, a_2 + k_2) = 0$ for all $(a_1, a_2) \in E$ and $(k_1, k_2) \in \mathbb{Z}^2$. This means that $\widehat{f} = 0$ a.e. and so $f = 0$. □

As a consequence of Theorem 18, we can conclude that either $f \equiv 0$ or $|\operatorname{supp} f|\,|\operatorname{supp} \widehat{f}| = \infty$.

5 Orthogonal Two-dimensional Plane Split of the Quaternion Signal

The two-dimensional split of a quaternion signal allows rewriting the two-sided QHFT as the sum of two complex transforms. Some results presented in the preceding subsections may be improved by exploiting this.

Following [30], we recall that each quaternion q can be split into

$$q = q_+ + q_-, \qquad q_\pm = \frac{1}{2}(q \pm \mathbf{i} q \mathbf{j}). \tag{65}$$

Explicitly, in real coordinates $q_0, q_1, q_2, q_3 \in \mathbb{R}$, we have

$$q_\pm = \Big(q_0 \pm q_3 + \mathbf{i}(q_1 \mp q_2)\Big)\frac{1 \pm \mathbf{k}}{2} = \frac{1 \pm \mathbf{k}}{2}\Big(q_0 \pm q_3 + \mathbf{j}(q_2 \mp q_1)\Big), \tag{66}$$

where $q_\pm$ is orthogonal in the sense that $\mathrm{Sc}(q_+\overline{q_-}) = 0$. Note that q_+ lives in the plane spanned by $\{\mathbf{i} - \mathbf{j}, 1 + \mathbf{ij}\}$ and q_- lives in the plane spanned by $\{\mathbf{i} + \mathbf{j}, 1 - \mathbf{ij}\}$. Since these two planes are orthogonal, they span the whole quaternion. Consequently, we have the modulus identity:

$$|q|^2 = |q_+|^2 + |q_-|^2. \tag{67}$$

In view of (65), the following relations are immediate:

$$\mathbf{i} q_\pm = \mp q_\pm \mathbf{j}, \quad q_\pm \mathbf{j} = \mp \mathbf{i} q_\pm. \tag{68}$$

By splitting the two-dimensional quaternion signal $f(x) = f_+(x) + f_-(x)$, with

$$f_\pm(x) = \Big(f_0(x) \pm f_3(x) + \mathbf{i}(f_1(x) \mp f_2(x))\Big)\frac{1 \pm \mathbf{k}}{2}, \quad x = (x_1, x_2) \in \mathbb{R}^2_{t_1,t_2} \tag{69}$$

and using (68), the two-sided QHFT defined by (14) can be written as

$$\mathcal{F}_{QH}(f)(\omega) = \mathcal{F}_{QH}(f_+ + f_-)(\omega) = \mathcal{F}_{QH}(f_+)(\omega) + \mathcal{F}_{QH}(f_-)(\omega), \tag{70}$$

where

$$\mathcal{F}_{QH}(f_\pm)(\omega) = \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi\mathbf{i}\,(\underline{x}_1\omega_1 \mp \underline{x}_2\omega_2)} f_\pm(x)\, d\mu(x)$$

or, similarly,

$$\mathcal{F}_{QH}(f_\pm)(\omega) = \int_{\mathbb{R}^2_{t_1,t_2}} f_\pm(x)\, e^{-2\pi\mathbf{j}\,(\mp\underline{x}_1\omega_1 + \underline{x}_2\omega_2)}\, d\mu(x) \tag{71}$$

are complex HFTs.

For $f : \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ in the form (69), by the modulus identity (67), it immediately follows that

$$|f(x)|^2 = |f_+(x)|^2 + |f_-(x)|^2, \tag{72}$$

$$\|\mathcal{F}_{QH}(f)\|_p^p = \|\mathcal{F}_{QH}(f_+)\|_p^p + \|\mathcal{F}_{QH}(f_-)\|_p^p, \quad p \geq 1. \tag{73}$$

Putting all these facts together, we are in a position to derive a Plancherel theorem for the two-sided QHFT.

Theorem 19 *Let $f, g \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ be such that $f(-x_1, x_2) = f(x_1, x_2)$ or $g(-x_1, x_2) = g(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2_{t_1,t_2}$. Then*

$$\langle \mathcal{F}_{QH}(f), \mathcal{F}_{QH}(g)\rangle_{L^2(\mathbb{R}^2,\mathbb{H})} = \langle f, g\rangle_{L^2(\mathbb{R}^2_{t_1,t_2},\mathbb{H})}. \tag{74}$$

Proof Using (70), we have

$$\begin{aligned}&\langle \mathcal{F}_{QH}(f), \mathcal{F}_{QH}(g)\rangle_{L^2(\mathbb{R}^2,\mathbb{H})}\\ &= \int_{\mathbb{R}^2} \Big(\mathcal{F}_{QH}(f_+)(\omega) + \mathcal{F}_{QH}(f_-)(\omega)\Big)\Big(\overline{\mathcal{F}_{QH}(g_+)(\omega)} + \overline{\mathcal{F}_{QH}(g_-)(\omega)}\Big)\, d\omega.\end{aligned}$$

We split the above integral into four integrals and compute each one. For the first integral, by (71) and Fubini's Theorem, we have

$$I_1 = \int_{\mathbb{R}^2} \mathcal{F}_{QH}(f_+)(\omega)\, \overline{\mathcal{F}_{QH}(g_+)(\omega)}\, d\omega$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(x)\, e^{-2\pi \mathbf{j}(-\underline{x}_1\omega_1+\underline{x}_2\omega_2)} e^{2\pi \mathbf{j}(-\underline{y}_1\omega_1+\underline{y}_2\omega_2)} \\
&\quad \times \overline{g_+(y)}\, d\mu(y)\, d\mu(x)\, d\omega \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(x) \int_{\mathbb{R}^2} e^{2\pi \mathbf{j}(\underline{x}_1-\underline{y}_1)\omega_1} e^{2\pi \mathbf{j}(-\underline{x}_2+\underline{y}_2)\omega_2}\, d\omega\, \overline{g_+(y)}\, d\mu(y)\, d\mu(x) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(x)\delta(\underline{x}_1-\underline{y}_1)\delta(-\underline{x}_2+\underline{y}_2)\, \overline{g_+(y)}\, d\mu(y)\, d\mu(x) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} f_+(x)\, \overline{g_+(x)}\, d\mu(x)\,.
\end{aligned}
$$

For the second integral, by combining (71) and Fubini's Theorem, we obtain

$$
\begin{aligned}
I_2 &= \int_{\mathbb{R}^2} \mathcal{F}_{QH}(f_+)(\omega)\, \overline{\mathcal{F}_{QH}(g_-)(\omega)}\, d\omega \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(x)\, e^{-2\pi \mathbf{j}(-\underline{x}_1\omega_1+\underline{x}_2\omega_2)} e^{2\pi \mathbf{j}(\underline{y}_1\omega_1+\underline{y}_2\omega_2)} \\
&\quad \times \overline{g_-(y)}\, d\mu(y)\, d\mu(x)\, d\omega \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(-x_1,x_2)\, e^{-2\pi \mathbf{j}(-\underline{x}_1\omega_1+\underline{x}_2\omega_2)} e^{2\pi \mathbf{j}(\underline{y}_1\omega_1+\underline{y}_2\omega_2)} \\
&\quad \times \overline{g_-(y)}\, d\mu(y)\, d\mu(x)\, d\omega \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(u_1,u_2) \int_{\mathbb{R}^2} e^{2\pi \mathbf{j}(-\underline{u}_1+\underline{y}_1)\omega_1} e^{2\pi \mathbf{j}(\underline{y}_2-\underline{u}_2)\omega_2}\, d\omega \\
&\quad \times \overline{g_-(y)}\, d\mu(y)\, d\mu(u) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} \int_{\mathbb{R}^2_{t_1,t_2}} f_+(u)\delta(-\underline{u}_1+\underline{y}_1)\delta(\underline{y}_2-\underline{u}_2)\, \overline{g_-(y)}\, d\mu(y)\, d\mu(u) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} f_+(u)\, \overline{g_-(u)}\, d\mu(u)\,.
\end{aligned}
$$

Similarly, for the third and fourth integrals, we get

$$
I_3 = \int_{\mathbb{R}^2} \mathcal{F}_{QH}(f_-)(\omega)\, \overline{\mathcal{F}_{QH}(g_+)(\omega)}\, d\omega \; = \int_{\mathbb{R}^2_{t_1,t_2}} f_-(x)\overline{g_+(x)}\, d\mu(x)
$$

and

$$
I_4 = \int_{\mathbb{R}^2} \mathcal{F}_{QH}(f_-)(\omega)\, \overline{\mathcal{F}_{QH}(g_-)(\omega)}\, d\omega \; = \int_{\mathbb{R}^2_{t_1,t_2}} f_-(x)\overline{g_-(x)}\, d\mu(x).
$$

This completes the proof of the theorem. $\square$

Remark 3 Using the quaternion-valued inner product (8), Plancherel's Theorem generally holds by imposing an even partial symmetry for the quaternion signals f or g. In the particular case $f(x) = f_0(x) + \mathbf{i} f_1(x)$ with $f_0, f_1 \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$ and $g(x) = g_0(x) + \mathbf{i} g_1(x)$, with $g_0, g_1 \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{R})$, Plancherel's Theorem holds without the assumption of even partial symmetry for f and g since in this case, $\mathbf{i} f = f \mathbf{i}$ and $\mathbf{i} g = g \mathbf{i}$.

Further, if we use the scalar inner product

$$\langle f, g \rangle_0 := \mathrm{Sc} \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, \overline{g(x)}\, d\mu(x)$$

then Plancherel's Theorem holds for every $f, g \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, as in the Euclidean case (see [30, Thm. 2.2]).

Theorem 20 (Hausdorff–Young inequality) *For* $1 \leq p \leq 2$ *and* $1/p + 1/q = 1$, *we have*

$$\|\mathcal{F}_{QH}(f)\|_q \leq \|f\|_p.$$

Proof Let $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$. We have

$$\|\mathcal{F}_{QH}(f)\|_\infty = \|\mathcal{F}_{QH}(f_+)\|_\infty + \|\mathcal{F}_{QH}(f_-)\|_\infty \leq \|f_+\|_1 + \|f_-\|_1 = \|f\|_1.$$

For $f \in L^2(\mathbb{R}^2, \mathbb{H})$, by Parseval's Theorem, it follows that

$$\|\mathcal{F}_{QH}(f)\|_2 = \|\mathcal{F}_{QH}(f_+)\|_2 + \|\mathcal{F}_{QH}(f_-)\|_2 = \|f_+\|_2 + \|f_-\|_2 = \|f\|_2.$$

Now, by the Riesz–Thorin Interpolation Theorem, we further obtain

$$\|\mathcal{F}_{QH}(f)\|_q \leq \|f\|_p$$

provided that $1 \leq p \leq 2$ and $1/p + 1/q = 1$. □

We end this subsection by providing an alternative proof of Pitt's inequality for the two-sided QHFT based on its decomposition into two complex HFTs.

Theorem 21 *For any* $f \in \mathcal{S}(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$, *Pitt's inequality for the two-sided QHFT is given by*

$$\int_{\mathbb{R}^2} |\omega|^{-\alpha} |\mathcal{F}_{QH}(f)(\omega)|^2\, d\omega \leq C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^\alpha |f(x)|^2 d\mu(x), \quad 0 \leq \alpha < 2$$

where $C_\alpha = \pi^\alpha \big(\Gamma(\frac{2-\alpha}{4}) / \Gamma(\frac{2+\alpha}{4})\big)^2$.

Proof By Pitt's inequality for the complex HFT and the split of a quaternion, we have

$$\int_{\mathbb{R}^2} |\omega|^{-\alpha} |\mathcal{F}_{QH}(f_-)(\omega)|^2 \, d\omega \ \leq \ C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^\alpha \, |f_-(x)| \, d\mu(x)$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^2} |(\omega_1, -\omega_2)|^{-\alpha} |\mathcal{F}_{QH}(f_+)(\omega_1, -\omega_2)|^2 \, d\omega \\
&= \int_{\mathbb{R}^2} |(\omega_1, \omega_2)|^{-\alpha} |\mathcal{F}_{QH}(f_+)(\omega_1, \omega_2)|^2 \, d\omega \\
&\leq C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^\alpha \, |f_+(x)| \, d\mu(x).
\end{aligned}$$

Now, by the modulus identity, we get

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\omega|^{-\alpha} |\mathcal{F}_{QH}(f)(\omega)|^2 \, d\omega \\
&\leq \int_{\mathbb{R}^2} |\omega|^{-\alpha} (|\mathcal{F}_{QH}(f_+)(\omega))|^2 + |\mathcal{F}_{QH}(f_-)(\omega))|^2) \, d\omega \\
&\leq C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^\alpha \, (|f_+(x)|^2 + |f_-(x)|^2) \, d\mu(x) \\
&\leq C_\alpha \int_{\mathbb{R}^2_{t_1,t_2}} |\underline{x}|^\alpha \, |f(x)|^2 \, d\mu(x).
\end{aligned}$$

□

We call the attention that Hausdorff–Young's inequality was proved for the Euclidean case in [37]. Unfortunately, the constant C_α is not given correctly in [37, Thm. 3]; also, the constant in the logarithmic uncertainty principle [37, Thm. 4] is incorrect.

6 The Right-Sided QHFT: Definition and Properties

As mentioned in the introduction, there are different ways to define a QFT for each quaternion signal $f \in L^1(\mathbb{R}^2, \mathbb{H})$. This is due to the non-commutativity of the quaternions and the fact that many elements in $\mathbb{H}$ serve as an imaginary unit. The most studied examples are the left-sided, the right-sided, and the two-sided QFTs.

This section presents the hyperbolic counterpart of the right-sided QFT (1) of a two-dimensional quaternionic signal in $L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ and establishes its main properties. The treatment given here is a generalization of that provided by Ernst et al. [22] and Delsuc [16]. All results can be performed straightforwardly to the left-sided QFT, but we do not dwell further on this structure.

Definition 13 The steerable right-sided QHFT of $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ is the function $\mathcal{F}^r_{QH}(f) : \mathbb{R}^2_{t_1,t_2} \to \mathbb{H}$ defined as the quaternion-valued (Lebesgue) integral

$$\mathcal{F}^r_{QH}(f)(\omega) = \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x), \tag{75}$$

where $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$.

To distinguish the right-sided QHFT from the two-sided QHFT, we use a superscript r.

We now proceed to find a relationship between these two transforms. In view of (75) and (14), a straightforward calculation shows that

$$\begin{aligned}
\mathcal{F}^r_{QH}(f)(\omega) &= \int_{\mathbb{R}^2_{t_1,t_2}} f(x)\, e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} (f_0(x) + \mathbf{i} f_1(x)) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\
&\quad + \int_{\mathbb{R}^2_{t_1,t_2}} e^{2\pi \mathbf{i}\, \underline{x}_1 \omega_1} (\mathbf{j} f_2(x) + \mathbf{k} f_3(x)) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\
&= \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} (f_0(x) + \mathbf{i} f_1(x)) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\
&\quad + \int_{\mathbb{R}^2_{t_1,t_2}} e^{-2\pi \mathbf{i}\, \underline{x}_1 \omega_1} (\mathbf{j} f_2(-x_1, x_2) + \mathbf{k} f_3(-x_1, x_2)) e^{-2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\mu(x) \\
&= \mathcal{F}_{QH}(\widetilde{f})(\omega),
\end{aligned}$$

where

$$\widetilde{f}(x) = f_0(x_1, x_2) + \mathbf{i} f_1(x_1, x_2) + \mathbf{j} f_2(-x_1, x_2) + \mathbf{k} f_3(-x_1, x_2).$$

The following table contains the main operational properties of the right-sided QHFT (Table 1).

Remark 4 For quaternion functions $f \in L^2(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ such that $\mathbf{i} f = f \mathbf{i}$, i.e., $f = f_0 + \mathbf{i} f_1$ then we get

$$\begin{aligned}
\mathcal{F}^r_{QH}(f(x \ominus y))(\omega) &= e^{-2\pi\, \mathbf{i}\, \underline{y}_1 \omega_1} \mathcal{F}^r_{QH}(f)(\omega) e^{-2\pi\, \mathbf{j}\, \underline{y}_2 \omega_2} \\
\mathcal{F}^r_{QH}\Big(\frac{\partial_h^{m+n}}{\partial_{x_1}^m \partial_{x_2}^n} f\Big)(\omega) &= (2\pi \mathbf{i} \omega_1)^m \mathcal{F}^r_{QH}(f)(\omega) (2\pi \mathbf{j} \omega_2)^n.
\end{aligned}$$

The following theorems give the inversion formula, Plancherel and Parseval's relations for the right-sided QHFT. We state these results without proof for simplicity, but the techniques employed can be adapted from those used in [13, 15, 30]. We accordingly give a minimum of detail.

Table 1 Operational properties of the right-sided QHFT

Property	Quat. Signal	$\mathcal{F}^r_{QH}$
Linearity	$\alpha f(x) + \beta g(x), \ \ \alpha, \beta \in \mathbb{H}$	$\alpha \mathcal{F}^r_{QH}(f) + \beta \mathcal{F}^r_{QH}(g)$
Shift	$f(x \ominus y)$	$\mathcal{F}^r_{QH}(f(x) e^{-2\pi \mathbf{i}\, \underline{y}_1 \omega_1})(\omega) e^{-2\pi \mathbf{j}\, \underline{y}_2 \omega_2}$
Dilation	$f(\lambda_1 \otimes x_1, \lambda_2 \otimes x_2)$	$\frac{1}{\lvert\lambda_1 \lambda_2\rvert} \mathcal{F}^r_{QH}(f)(\frac{1}{\lambda_1}\omega_1, \frac{1}{\lambda_2}\omega_2)$
Part. Deriv	$\frac{\partial_h^{m+n}}{\partial x_1^m \partial x_2^n} f(x) \mathbf{i}^{-m}$	$(2\pi \omega_1)^m \mathcal{F}^r_{QH}(f)(\omega)(2\pi \mathbf{j} \omega_2)^n$
Powers of $\underline{x}_1, \underline{x}_2$	$(-2\pi \underline{x}_1)^m (-2\pi \underline{x}_2)^n f(x) \mathbf{i}^m$	$\frac{\partial^{m+n}}{\partial \omega_1^m \partial \omega_2^n} \mathcal{F}^r_{QH}(f)(\omega)\, \mathbf{j}^{-n}$

Theorem 22 *Suppose that $f \in L^1(\mathbb{R}^2_{t_1,t_2}, \mathbb{H})$ satisfies $\mathcal{F}^r_{QH}(f) \in L^1(\mathbb{R}^2, \mathbb{H})$. Then, the inversion formula for the right-sided QHFT given by (75) is*

$$f(x) = \int_{\mathbb{R}^2} \mathcal{F}^r_{QH}(f)(\omega)\, e^{2\pi \mathbf{j}\, \underline{x}_1 \omega_1}\, e^{2\pi \mathbf{j}\, \underline{x}_2 \omega_2}\, d\omega \tag{76}$$

for a.e. $x \in \mathbb{R}^2_{t_1,t_2}$.

Theorem 23 *If $f \in L^1 \cap L^2(\mathbb{R}^2, \mathbb{H})$, then $\mathcal{F}^r_{QH}(f) \in L^2(\mathbb{R}^2, \mathbb{H})$. The right-sided QHFT satisfies the following Plancherel identity:*

$$\int_{\mathbb{R}^2_{t_1,t_2}} f(x)\overline{g(x)}\, d\mu(x) \ = \ \int_{\mathbb{R}^2} \mathcal{F}^r_{QH}(f)(\omega)\, \overline{\mathcal{F}^r_{QH}(g)(\omega)}\, d\omega.$$

Further, it satisfies Parseval's identity, namely $\|\mathcal{F}^r_{QH}\|_2^2 = \|f\|_2^2$.

It is interesting to explore other properties of the right-sided QHFT (75) in more detail. Further research on this topic is now under investigation and will be reported in a forthcoming paper.

7 Concluding Remarks and Perspectives

In this paper, we constructed the hyperbolic counterpart of the two-sided and right-sided QFTs of two-dimensional quaternion-valued signals defined in an open rectangle of the Euclidean plane endowed with a hyperbolic measure. The different forms of these transforms were prescribed by replacing the Euclidean plane waves with the corresponding hyperbolic plane waves in one dimension. Furthermore, we also presented their main operational and mapping properties, including an inversion formula, Parseval's relations, and uncertainty principles. It should be pointed out that in the large limits of t_1 and t_2, i.e., $t_1, t_2 \to +\infty$, our results yield the corresponding results for the Euclidean two-sided and right-sided QFTs. Some new concepts were introduced, such as the hyperbolic derivative and the hyperbolic primitive, which led to the differentiation properties of the QHFTs. Applications of these results will be presented elsewhere.

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Declarations

Conflict on interest The authors declare no conflict of interest.

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