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Norm of the Hilbert Matrix on Logarithmically Weighted Bergman Spaces

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Abstract

It is well known that the Hilbert matrix $\mathcal{H}$ is bounded from the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ into Bergman space A^2 when $\alpha > 2$. In this paper, we calculate lower bound and upper bound for the norm of the Hilbert matrix operator $\mathcal{H}$ from the logarithmically weighted Bergman space $A^2_{\log^\alpha}$ into Bergman space A^2 when $\alpha > 2$. We also calculate lower bound and upper bound for the norm of the Hilbert matrix operator from $A^p_{\log^\alpha}$ into A^p, for $2 < p < \infty$ and $\alpha > 1$.

Keywords Operator norm · Hilbert matrix · Logarithmically weighted Bergman spaces

Mathematics Subject Classification 47B38 · 30H20

1 Introduction and Preliminaries

In recent years, the study for Hilbert matrix operator $\mathcal{H}$'s boundedness and norm on different analytic function spaces has been under active investigation (see [1–8]). Diamantopoulus and Siskakis [7] studied $\mathcal{H}$ is bounded on Hardy space $H^p(1 < p < \infty)$ and also obtained an upper bound estimate for its norm. In [6], Diamantopoulus began consider the boundedness of $\mathcal{H}$ on the Bergman spaces $A^p(2 < p < \infty)$, and obtained the upper bound estimate for the norm of $\mathcal{H}$. Then Dostanić, Jevtić, Vukotić

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[8] established the precise value of the norm of $\mathcal{H}$ in the Hardy space $H^p(1 < p < \infty)$ and also gave exact value of the norm of $\mathcal{H}$ in the Bergman space $A^p(4 < p < \infty)$. In 2017, Božin and Karapetrović [3] solved the question of exact value of the norm of $\mathcal{H}$ in the Bergman space A^p, for $2 < p < 4$. The norm of Hilbert matrix operator $\mathcal{H}$ has also been studied in other analytic function spaces like Korenblum spaces H_α^∞ [5, 19].

The study of boundedness of $\mathcal{H}$ on A_α^p was initiated in [10] and some partial results were obtained. The boundedness of the Hilbert matrix on A_α^p for $1 < 2+\alpha < p$ was also studied by Jevtić M and Karapetrović B in [12]. In [13], Karapetrović obtained the exact norm of $\mathcal{H}$ on A_α^p when $4 \le 2(2+\alpha) \le p < \infty$. Additionally, he showed that the same lower bound holds for all $p > 2+\alpha > 1$. He also conjectured that the upper bound for the norm of $\mathcal{H}$ is the same as above also for the case $1 < 2+\alpha < p < 2(2+\alpha)$ in [13]. In [18] Lindström, Miihkinen and Wikman confirmed the conjecture in the positive for $2+\alpha+\sqrt{\alpha^2+\frac{7}{2}\alpha+3} \le p < 2(2+\alpha)$. Recently Karapetrović generalized the work of [18] by showing that the conjecture holds for $2+\alpha+\sqrt{(2+\alpha)^2-(\sqrt{2}-\frac{1}{2})(2+\alpha)} \le p < 2(2+\alpha)$. In [2], Bralović and Karapetrović provide a new upper bound for the norm of the Hilbert matrix $\mathcal{H}$ on the weighted Bergman spaces A_α^p when $-1 < \alpha < 0$, which represents an improvement.

We also realized the Hilbert matrix operator is unbounded on A^2 in [6]. And the situation is actually even worse: the series defining $\mathcal{H}f(0)$ is divergent in [8]. Then in [16] Łanucha, Nowak and Pavlović considered $\mathcal{H}$ acts as a bounded operator from $A^2_{\log^\alpha}$ to A^2 for $\alpha > 3$ and this was improved in [11, Theorem 4.5], where it is proved that $\mathcal{H}$ maps $A^2_{\log^\alpha}$ into A^2 for $\alpha > 2$. The last result is also improved in [15, Theorem 3.2] by Karapetrović, where it is proved that $\mathcal{H}$ maps $A^2_{\log^\alpha}$ into $A^2_{\log^{\alpha-2-\varepsilon}}$ for $\alpha > 2$ and $0 < \varepsilon \le \alpha - 2$. In this paper, we obtained the upper bound for the norm from logarithmically weighted Bergman spaces $A^2_{\log^\alpha}$ into Bergman spaces A^2 for $\alpha > 2$. We also find $\mathcal{H}$ acts as a bounded operator from $A^p_{\log^\alpha}$ into A^p for $2 < p < \infty$ and $\alpha > 0$, and also calculate the upper bound for the norm of $\mathcal{H}$.

Let $\mathbb{D}$ denote the open unit disk of the complex plane $\mathbb{C}$, and let $H(\mathbb{D})$ denote the set of all analytic functions in $\mathbb{D}$.

For $0 < p \le \infty$, the Hardy space H^p is the space of all functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p} = \sup_{0 \le r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}, \quad 0 < p < \infty;$$
$$M_\infty(r, f) = \sup_{0 \le t < 2\pi} |f(re^{it})|.$$

For $0 < p < \infty$ the Bergman space A^p consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p} \stackrel{def}{=} \left(\int_{\mathbb{D}} |f(z)|^p dA(z)\right)^{1/p} < \infty.$$

Here, dA stands for the area measure on $\mathbb{D}$, normalized so that the total area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta.$

Then for $0 < p < \infty$ and $\alpha > 0$ the logarithmically weighted Bergman space $A^p_{\log^\alpha}$ consists of those $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_{\log^\alpha}} \stackrel{def}{=} \left(\int_{\mathbb{D}} |f(z)|^p \left(\log \frac{2}{1-|z|^2}\right)^\alpha dA(z)\right)^{1/p} < \infty.$$

The relation between these spaces we introduced above is well known that $A^p_{\log^\alpha} \subset A^p$.

The Hilbert matrix is an infinite matrix $\mathcal{H}$ whose entries are $a_{n,k} = \frac{1}{n+k+1}$, $n, k \geq 0$. The Hilbert matrix $\mathcal{H}$ can be also viewed as an operator on spaces of analytic functions by its action on their Taylor coefficients. Hence for those $f \in H(\mathbb{D})$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then we define a transformation $\mathcal{H}$ by

$$\mathcal{H}f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right) z^n.$$

As usual, throughout this paper, C denotes a positive constant which depends only on the displayed parameters but not necessarily the same from one occurrence to the next.

2 Norm Estimates of the Hilbert Matrix $\|\mathcal{H}\|_{A^2_{\log^\alpha} \to A^2}$

In this section, we drive norm estimates for Hilbert matrix operator acting from $A^2_{\log^\alpha}$ into A^2 for $\alpha > 2$.

According to [16, Lemma 4.2], we obtain that there exists a constant $C > 0$ such that

$$\sum_{k=0}^{\infty} \frac{|a_k|}{k+1} \leq C\|f\|_{A^2_{\log^\alpha}}$$

for every $f(z) = \sum_{k=0}^{\infty} a_k z^k$ that belongs to $A^2_{\log^\alpha}$, $\alpha > 2$. Then we obtain a well-defined analytic function $\mathcal{H}f(z)$ on $\mathbb{D}$. Hence, we have that

$$\mathcal{H}f(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right) z^n$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \int_0^1 t^{n+k} dt \right) z^n \\
&= \int_0^1 \sum_{k=0}^{\infty} a_k t^k \sum_{n=0}^{\infty} t^n z^n dt \\
&= \int_0^1 \frac{f(t)}{1-tz} dt. \qquad (2.1)
\end{aligned}
$$

2.1 Upper Bound for the Norm $\|\mathcal{H}\|_{A^2_{\log^\alpha} \to A^2}$

We know that the Hilbert matrix operator $\mathcal{H}$ has an integral representation in terms of weighted composition operators T_t (see [6]):

$$
\mathcal{H}f(z) = \int_0^1 T_t f(z) dt, \qquad (2.2)
$$

where

$$
T_t f(z) = w_t(z) f(\phi_t(z)), \quad w_t(z) = \frac{1}{1-(1-t)z}, \quad \phi_t(z) = \frac{t}{1-(1-t)z}.
$$

Theorem 2.1 *Let $\alpha > 2$. Then the norm of the Hilbert matrix operator acting from $A^2_{\log^\alpha}$ into A^2 satisfies the upper estimate*

$$
\|\mathcal{H}\|_{A^2_{\log^\alpha} \to A^2} \le \int_0^1 \frac{2}{\left(\log \frac{2}{1-x^2}\right)^{\frac{\alpha}{2}}} \left(\frac{1}{1-x} + \frac{1}{(x(1-x))^{\frac{1}{2}}} \right) dx.
$$

Proof By Minkowski's inequality, we have

$$
\begin{aligned}
\|\mathcal{H}f\|_{A^2} &= \left(\int_{\mathbb{D}} |\mathcal{H}f(z)|^2 dA(z) \right)^{\frac{1}{2}} \\
&= \left(\int_{\mathbb{D}} \left| \int_0^1 T_t(z) dt \right|^2 dA(z) \right)^{\frac{1}{2}} \\
&\le \int_0^1 \left(\int_{\mathbb{D}} |T_t f(z)|^2 dA(z) \right)^{\frac{1}{2}} dt \\
&= \int_0^1 \|T_t f\|_{A^2} dt. \qquad (2.3)
\end{aligned}
$$

Using linear fractional change of variable $w = \phi_t(z)$, $z \in \mathbb{D}$, we obtain that

$$\begin{aligned}\|T_t f\|_{A^2}^2 &= \int_{\mathbb{D}} |w_t(z)|^2 |f(\phi_t(z))|^2 dA(z) \\ &= \int_{\phi_t(\mathbb{D})} |w_t(\phi_t^{-1}(w))|^2 \frac{|f(w)|^2}{|\phi_t'(\phi_t^{-1}(w))|^2} dA(w) \\ &= \frac{1}{(1-t)^2} \int_{\phi_t(\mathbb{D})} |w|^{-2} |f(w)|^2 dA(w).\end{aligned}$$

Therefore

$$\|T_t f\|_{A^2} = \frac{1}{1-t} \left(\int_{D_t} |w|^{-2} |f(w)|^2 dA(w) \right)^{\frac{1}{2}},$$

here $D_t = \phi_t(\mathbb{D})$. It is easy to find that $D_t = D\left(\frac{1}{2-t}, \frac{1-t}{2-t}\right)$, i.e. D_t is the Euclidean disc with center on $\frac{1}{2-t}$ and of radius $\frac{1-t}{2-t}$. It is easy to see that $|w| \geq \frac{t}{2-t}$, for $w \in D_t$, and $D_t \subset E_t$, where $E_t = \{w \in \mathbb{C} : \frac{t}{2-t} < |w| < 1\}$. Hence, we obtain

$$\|T_t f\|_{A^2} \leq \frac{1}{1-t} \left(\int_{E_t} |w|^{-2} |f(w)|^2 dA(w) \right)^{\frac{1}{2}}. \tag{2.4}$$

On the other hand, we also have

$$\left(\int_{E_t} |w|^{-2} |f(w)|^2 dA(w) \right)^{\frac{1}{2}} = \left(2 \int_{\frac{t}{2-t}}^{1} \frac{1}{r^2} \cdot r M_2^2(r, f) dr \right)^{\frac{1}{2}}.$$

It is easy to find function $r \to \frac{1}{r^2}$ is decreasing and function $r \to r M_2^2(r, f)$ is increasing, by using Chebyshev's inequality, we get

$$\begin{aligned}\left(\int_{E_t} |w|^{-2} |f(w)|^2 dA(w) \right)^{\frac{1}{2}} &\leq \left(\frac{2}{1 - \frac{t}{2-t}} \int_{\frac{t}{2-t}}^{1} \frac{1}{r^2} dr \int_{\frac{t}{2-t}}^{1} r M_2^2(r, f) dr \right)^{\frac{1}{2}} \\ &= \left(\frac{2-t}{t} \cdot 2 \int_{\frac{t}{2-t}}^{1} r M_2^2(r, f) dr \right)^{\frac{1}{2}} \\ &= \left(\frac{2-t}{t} \int_{E_t} |f(w)|^2 dA(w) \right)^{\frac{1}{2}}.\end{aligned} \tag{2.5}$$

And by using (2.5), we have that

$$\int_0^1 \|T_t f\|_{A^2} dt \leq \int_0^1 \left(\frac{1}{1-t} \cdot \frac{(2-t)^{\frac{1}{2}}}{t^{\frac{1}{2}}} \right) \left(\int_{E_t} |f(w)|^2 dA(w) \right)^{\frac{1}{2}} dt$$

$$
\begin{aligned}
&= \int_0^1 \left(\frac{1}{1-t}\cdot\left(1+\frac{2(1-t)}{t}\right)^{\frac{1}{2}}\right)\left(\int_{E_t}|f(w)|^2 dA(w)\right)^{\frac{1}{2}} dt\\
&\le \int_0^1 \left(\frac{1}{1-t}\cdot\left(1+\left(\frac{2(1-t)}{t}\right)^{\frac{1}{2}}\right)\right)\left(\int_{E_t}|f(w)|^2 dA(w)\right)^{\frac{1}{2}} dt\\
&= \int_0^1 \left(\frac{1}{1-t}+\frac{\sqrt{2}}{(t(1-t))^{\frac{1}{2}}}\right)\left(\int_{E_t}|f(w)|^2 dA(w)\right)^{\frac{1}{2}} dt\\
&= \int_0^1 \left(\frac{1}{1-t}+\frac{\sqrt{2}}{(t(1-t))^{\frac{1}{2}}}\right)\left(\int_{E_t}\left(\log\frac{2}{1-|w|^2}\right)^{-\alpha}|f(w)|^2\right.\\
&\times\left.\left(\log\frac{2}{1-|w|^2}\right)^{\alpha} dA(w)\right)^{\frac{1}{2}} dt. \qquad (2.6)
\end{aligned}
$$

Since function $w \to \left(\log \frac{2}{1-|w|^2}\right)^{-\alpha}$ is decreasing, by simple calculation we found that,

$$
\begin{aligned}
\int_0^1 \|T_t f\|_{A^2} dt &\le \int_0^1 \frac{\left(\frac{1}{1-t}+\frac{\sqrt{2}}{(t(1-t))^{\frac{1}{2}}}\right)}{\left(\log\frac{2}{1-(\frac{t}{2-t})^2}\right)^{\frac{\alpha}{2}}}\left(\int_{E_t}|f(w)|^2\left(\log\frac{2}{1-|w|^2}\right)^{\alpha} dA(w)\right)^{\frac{1}{2}} dt\\
&\le \int_0^1 \left(\frac{1}{(1-t)\left(\log\frac{2}{1-(\frac{t}{2-t})^2}\right)^{\frac{\alpha}{2}}}+\frac{\sqrt{2}}{(t(1-t))^{\frac{1}{2}}\left(\log\frac{2}{1-(\frac{t}{2-t})^2}\right)^{\frac{\alpha}{2}}}\right) dt \|f\|_{A^2_{\log^\alpha}}, \qquad (2.7)
\end{aligned}
$$

Making the change of variable in (2.7), we obtain that

$$
\begin{aligned}
\int_0^1 \|T_t f\|_{A^2} dt &\le \int_0^1 \frac{2}{(1-t^2)\left(\log\frac{2}{1-t^2}\right)^{\frac{\alpha}{2}}} dt \|f\|_{A^2_{\log^\alpha}}\\
&+ \int_0^1 \frac{2}{(t+1)(t(1-t))^{\frac{1}{2}}\left(\log\frac{2}{1-t^2}\right)^{\frac{\alpha}{2}}} dt \|f\|_{A^2_{\log^\alpha}}\\
&= \int_0^1 \frac{2}{(1+t)\left(\log\frac{2}{1-t^2}\right)^{\frac{\alpha}{2}}}\left(\frac{1}{1-t}+\frac{1}{(t(1-t))^{\frac{1}{2}}}\right) dt \|f\|_{A^2_{\log^\alpha}}. \qquad (2.8)
\end{aligned}
$$

From (2.3) and (2.8), we have that

$$\begin{aligned}\|\mathcal{H}f\|_{A^2} &\le \int_0^1 \frac{2}{(1+t)\left(\log\frac{2}{1-t^2}\right)^{\frac{\alpha}{2}}}\left(\frac{1}{1-t}+\frac{1}{(t(1-t))^{\frac{1}{2}}}\right)dt\|f\|_{A^2_{\log^\alpha}}\\ &\le \int_0^1 \frac{2}{\left(\log\frac{2}{1-t^2}\right)^{\frac{\alpha}{2}}}\left(\frac{1}{1-t}+\frac{1}{(t(1-t))^{\frac{1}{2}}}\right)dt\|f\|_{A^2_{\log^\alpha}}.\end{aligned}$$

It means that

$$\|\mathcal{H}\|_{A^2_{\log^\alpha}\to A^2} \le \int_0^1 \frac{2}{\left(\log\frac{2}{1-x^2}\right)^{\frac{\alpha}{2}}}\left(\frac{1}{1-x}+\frac{1}{(x(1-x))^{\frac{1}{2}}}\right)dx,$$

and the last integral converges for $\alpha > 2$.

This finishes the proof of the theorem. □

This result improves Theorem 4.3 in [16], and we also give a new proof method of Theorem 4.5 in [11].

2.2 Lower Bound for the Norm $\|\mathcal{H}\|_{A^2_{\log^\alpha}\to A^2}$

Before we get lower bound for the norm $\|\mathcal{H}\|_{A^2_{\log^\alpha}\to A^2}$, we need find a special function in $A^2_{\log^\alpha}$.

Lemma 2.1 *Let $\alpha > 2$, $b \ge 1$ and $1 < \gamma < 2$. Then the function*

$$f(z) = \left(\frac{1}{z}\log\frac{b}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}},$$

belongs to $A^2_{\log^\alpha}$.

Proof First we recall a well known result of Littlewood [17, pp.93–96]: Shows the function has an an integral mean with growth [9, p49]

$$\begin{aligned}M_2(r,f)^2 &= \frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{1}{re^{i\theta}}\log\frac{b}{1-re^{i\theta}}\right)^{-\alpha}(1-re^{i\theta})^{-\gamma}\right|d\theta\\ &\le \frac{1}{2\pi}\int_0^{2\pi}\left|\left(\frac{1}{re^{i\theta}}\log\frac{1}{1-re^{i\theta}}\right)^{-\alpha}(1-re^{i\theta})^{-\gamma}\right|d\theta\\ &\sim C\frac{1}{(1-r)^{\gamma-1}}\left(\log\frac{1}{1-r}\right)^{-\alpha}, \quad r\to 1.\end{aligned}$$

Since

$$(a+b)^p \le 2^p(a^p+b^p), \quad a \ge 0, \quad b \ge 0,$$

we have

$$\begin{aligned} M_2(r,f)^2\left(\log\frac{2}{1-r^2}\right)^{\alpha} &\le C\frac{1}{(1-r)^{\gamma-1}}\left(\log\frac{1}{1-r}\right)^{-\alpha}2^{\alpha}\left(\log 2^{\alpha}+\left(\log\frac{1}{1-r}\right)^{\alpha}\right)\\ &= C\frac{2^{\alpha}\log 2^{\alpha}}{(1-r)^{\gamma-1}}\left(\log\frac{1}{1-r}\right)^{-\alpha}+C\frac{2^{\alpha}}{(1-r)^{\gamma-1}}, \quad r\to 1. \end{aligned}$$

Thus we can find the integral $\int_0^1 M_2(r,f)^2\left(\log\frac{2}{1-r^2}\right)^{\alpha}dr$ converges while $\alpha > 2$, $b \ge 1$ and $1 < \gamma < 2$, this shows that $f(z) \in A^2_{\log^{\alpha}}$. It is also easy to see that

$$\lim_{\gamma\to 2}\|f\|_{A^2_{\log^{\alpha}}} = \infty.$$

□

Corollary 2.1 *Let $\alpha > 1$, $b \ge 1$ and $1 < \gamma < 2$. Then the function*

$$f(z) = \left(\frac{1}{z}\log\frac{b}{1-z}\right)^{-\frac{\alpha}{p}}(1-z)^{-\frac{\gamma}{p}},$$

belongs to $A^p_{\log^{\alpha}}(p > 2)$.

Theorem 2.2 *Let $\alpha > 2$. Then the norm of the Hilbert matrix operator acting from $A^2_{\log^{\alpha}}$ into A^2 satisfies the lower estimate*

$$\|\mathcal{H}\|_{A^2_{\log^{\alpha}}\to A^2} \ge C_{\alpha}\int_0^1 \frac{x^{\frac{\alpha}{2}}}{(1-x)\left(\log\frac{1}{1-x}\right)^{\frac{\alpha}{2}}}dx.$$

where

$$C_{\alpha} = \limsup_{\gamma\to 2}\frac{\left\|(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2_{\log^{\alpha}}}}.$$

Proof Let $\alpha > 2$, we begin by selecting a family of test functions. Choose an arbitrary γ such that $1 < \gamma < 2$. It is a standard exercise to check that the function

$$f_{\gamma}(z) = \left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}},$$

and Lemma 2.1 shows that $f_\gamma(z) \in A^2_{\log^\alpha}$. It is also easy to see that

$$\lim_{\gamma\to 2} \|f_\gamma\|_{A^2_{\log^\alpha}} = \infty.$$

And we let

$$F_\gamma(z) = (1-z)^{-\frac{\gamma}{2}},$$

belong to A^2, and

$$\lim_{\gamma\to 2} \|F_\gamma\|_{A^2} = \infty,$$

we obtain a relationship between $f_\gamma(z)$ and $F_\gamma(z)$ by the proof of Lemma 2.1, that

$$\begin{aligned}
\limsup_{\gamma\to 2} \frac{\|F_\gamma\|_{A^2}}{\|f_\gamma\|_{A^2_{\log^\alpha}}} &= \limsup_{\gamma\to 2} \frac{\left\|(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2_{\log^\alpha}}} \\
&\le C \limsup_{\gamma\to 2} \frac{\int_0^1 \frac{1}{(1-r)^{\gamma-1}}dr}{\int_0^1 M_2(r,f)^2\left(\log\frac{1}{1-r}\right)^\alpha dr} \\
&\le C \limsup_{\gamma\to 2} \frac{\int_0^1 \frac{1}{(1-r)^{\gamma-1}}dr}{\int_0^1 \frac{1}{(1-r)^{\gamma-1}}dr} = C < \infty. \qquad (2.9)
\end{aligned}$$

Thus, we let $C_\alpha = \limsup_{\gamma\to 2} \frac{\left\|(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2_{\log^\alpha}}}$, and C_α is a constant, depending only on α.

Using (2.1), we find that

$$\begin{aligned}
\mathcal{H}f_\gamma(z) &= \int_0^1 \frac{f_\gamma(t)}{1-tz}dt \\
&= \int_0^1 \frac{t^{\frac{\alpha}{2}}dt}{\left(\log\frac{1}{1-t}\right)^{\frac{\alpha}{2}}(1-t)^{\frac{\gamma}{2}}(1-tz)}. \qquad (2.10)
\end{aligned}$$

Then making the change of variable $w = (1-tz)/(1-t)$, we calculate that

$$\mathcal{H}f_\gamma(z) = (1-z)^{-\frac{\gamma}{2}} \int_1^\infty \frac{(w-1)^{\frac{\alpha}{2}}dw}{\left(\log\frac{1}{1-(\frac{w-1}{w-z})}\right)^{\frac{\alpha}{2}} w(w-z)^{\frac{\alpha}{2}+1-\frac{\gamma}{2}}}, \qquad (2.11)$$

we define

$$\phi_\gamma(z) = \int_1^\infty \frac{(w-1)^{\frac{\alpha}{2}} dw}{\left(\log \frac{1}{1-(\frac{w-1}{w-z})^2}\right)^{\frac{\alpha}{2}} w(w-z)^{\frac{\alpha}{2}+1-\frac{\gamma}{2}}}.$$

for every z in $\mathbb{D}$, which shows $\mathcal{H} f_\gamma(z) = F_\gamma(z)\phi_\gamma(z)$.

Knowing that in the definition (2.10) of the function ϕ_γ is similar to the function ϕ_γ defined in the [8, Theorm 4], we can let w to be a real number $s \geq 1$. Thus we obtained that ϕ_γ belongs to the disk algebra whenever $\gamma \leq 2$, (the case $\gamma = 2$ will also be useful to us although $f_2 \notin A^2_{\log^\alpha}$), we can view ϕ_γ is an analytic function of z that

$$\phi_\gamma(z) = \int_1^\infty \frac{(s-1)^{\frac{\alpha}{2}} ds}{\left(\log \frac{1}{1-(\frac{s-1}{s-z})^2}\right)^{\frac{\alpha}{2}} s(s-z)^{\frac{\alpha}{2}+1-\frac{\gamma}{2}}}. \tag{2.12}$$

We will use the test function $g_\gamma(z) = \frac{f_\gamma(z)}{\|F_\gamma\|_{A^2}}$ and $G_\gamma(z) = \frac{F_\gamma(z)}{\|F_\gamma\|_{A^2}}$, we obtain that

$$\|\mathcal{H}\|_{A^2_{\log^\alpha} \to A^2} \geq \frac{\|\mathcal{H}(g_\gamma)\|_{A^2}}{\|g_\gamma\|_{A^2_{\log^\alpha}}} = \frac{\|F_\gamma\|_{A^2}}{\|f_\gamma\|_{A^2_{\log^\alpha}}} \frac{\|\mathcal{H}(f_\gamma)\|_{A^2}}{\|F_\gamma\|_{A^2}} = \frac{\|F_\gamma\|_{A^2}}{\|f_\gamma\|_{A^2_{\log^\alpha}}} \frac{\|F_\gamma(z)\phi_\gamma(z)\|_{A^2}}{\|F_\gamma\|_{A^2}}.$$

Letting $\gamma \to 2$, and by [8, Theorem 4] we get,

$$\begin{aligned}
\|\mathcal{H}\|_{A^2_{\log^\alpha} \to A^2} &\geq \limsup_{\gamma \to 2} \frac{\|F_\gamma\|_{A^2}}{\|f_\gamma\|_{A^2_{\log^\alpha}}} \frac{\|F_\gamma(z)\phi_\gamma(z)\|_{A^2}}{\|F_\gamma\|_{A^2}} = C_\alpha \lim_{\gamma \to 2} \|G_\gamma \phi_\gamma\|_{A^2} = C_\alpha \|\phi_2\|_\infty \\
&= C_\alpha \sup_{z \in \mathbb{D}} \int_1^\infty \frac{(s-1)^{\frac{\alpha}{2}}}{\left(\log \frac{1}{1-(\frac{s-1}{s-z})^2}\right)^{\frac{\alpha}{2}} s(s-z)^{\frac{\alpha}{2}}} ds \\
&\geq C_\alpha \sup_{0 \leq r \leq 1} \int_1^\infty \frac{(s-1)^{\frac{\alpha}{2}}}{\left(\log \frac{1}{1-(\frac{s-1}{s-z})^2}\right)^{\frac{\alpha}{2}} s(s-z)^{\frac{\alpha}{2}}} ds \\
&\geq C_\alpha \int_1^\infty \frac{(s-1)^{\frac{\alpha}{2}}}{s^{\frac{\alpha}{2}+1} \left(\log \frac{1}{1-(\frac{s-1}{s})^2}\right)^{\frac{\alpha}{2}}} ds.
\end{aligned}$$

Then making the change of variable $x = (s-1)/s$, we calculate that

$$\int_1^\infty \frac{(s-1)^{\frac{\alpha}{2}}}{s^{\frac{\alpha}{2}+1} \left(\log \frac{1}{1-(\frac{s-1}{s})^2}\right)^{\frac{\alpha}{2}}} ds = \int_0^1 \frac{x^{\frac{\alpha}{2}}}{(1-x)\left(\log \frac{1}{1-x}\right)^{\frac{\alpha}{2}}} dx.$$

Thus we obtained that,

$$\|\mathcal{H}\|_{A^2_{\log^\alpha}\to A^2} \geq C_\alpha \int_0^1 \frac{x^{\frac{\alpha}{2}}}{(1-x)\left(\log \frac{1}{1-x}\right)^{\frac{\alpha}{2}}} dx,$$

where

$$C_\alpha = \limsup_{\gamma\to 2} \frac{\left\|(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2_{\log^\alpha}}}.$$

and the last integral converges for $\alpha > 2$.

This concludes the proof. □

Corollary 2.2 *Let $\alpha > 2$. Then the norm of the Hilbert matrix operator acting from $A^2_{\log^\alpha}$ into A^2 satisfies*

$$\begin{aligned} C_\alpha \int_0^1 \frac{x^{\frac{\alpha}{2}}}{(1-x)\left(\log \frac{1}{1-x}\right)^{\frac{\alpha}{2}}} dx &\leq \|\mathcal{H}\|_{A^2_{\log^\alpha}\to A^2} \\ &\leq \int_0^1 \frac{2}{\left(\log\frac{2}{1-x^2}\right)^{\frac{\alpha}{2}}}\left(\frac{1}{1-x}+\frac{1}{(x(1-x))^{\frac{1}{2}}}\right)dx. \end{aligned}$$

where

$$C_\alpha = \limsup_{\gamma\to 2} \frac{\left\|(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{2}}\right\|_{A^2_{\log^\alpha}}}.$$

3 Norm Estimates of the Hilbert Matrix $\|\mathcal{H}\|_{A^p_{\log^\alpha}\to A^p}$

Then we consider the boundedness of Hilbert matrix from into $A^p_{\log^\alpha}$ into A^p, for $p > 2$ and $\alpha > 0$. That can easy obtain Lemma 3.1.

Lemma 3.1 *If $p > 2$ and $\alpha > 0$, then $\mathcal{H}$ acts as a bounded operator from $A^p_{\log^\alpha}$ into A^p.*

Since $A^p_{\log^\alpha} \subset A^p$, that the lemma is obviously established.

Lemma 3.2 [3, 6] *Let $2 < p < \infty$. Then the norm of the Hilbert matrix operator acting on A^p satisfies*

$$\|\mathcal{H}\|_{A^p \to A^p} = \frac{\pi}{\sin \frac{2\pi}{p}}.$$

Then we give the upper bound for the norm estimates of the Hilbert matrix $\|\mathcal{H}\|_{A^p_{\log^\alpha} \to A^p}$.

Theorem 3.1 *Let $2 < p < \infty$ and $\alpha > 1$. Then the norm of the Hilbert matrix operator acting from $A^p_{\log^\alpha}$ into A^p satisfies*

$$C_{\alpha,p} \int_0^1 \frac{x^{\frac{\alpha}{p}}}{(1-x)^{\frac{2}{p}} \left(\log \frac{1}{1-x}\right)^{\frac{\alpha}{p}}} dx \le \|\mathcal{H}\|_{A^p_{\log^\alpha} \to A^p} \le \int_0^1 \frac{2^{\frac{2}{p}} x^{\frac{2}{p}-1}}{(1-x)^{\frac{2}{p}} \left(\log \frac{2}{1-x^2}\right)^{\frac{\alpha}{p}}} dt,$$

where

$$C_{\alpha,p} = \limsup_{\gamma \to 2} \frac{\left\| (1-z)^{-\frac{\gamma}{p}} \right\|_{A^p}}{\left\| \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-\frac{\alpha}{2}} (1-z)^{-\frac{\gamma}{p}} \right\|_{A^2_{\log^\alpha}}}.$$

Proof First, we establish the lower bound for the norm $\|\mathcal{H}\|_{A^p_{\log^\alpha} \to A^p}$. We also construct a family of test functions like Theorem 2.2. Choose an arbitrary γ such that $1 < \gamma < 2$. It is a standard exercise to check that the function

$$f_\gamma^1(z) = \left(\frac{1}{z} \log \frac{1}{1-z}\right)^{-\frac{\alpha}{p}} (1-z)^{-\frac{\gamma}{p}},$$

After some elementary calculations, we can also establish that $f_\gamma^1(z)$ belongs to $A^p_{\log^\alpha}$. It is also easy to observe that

$$\lim_{\gamma \to 2} \|f_\gamma^1\|_{A^p_{\log^\alpha}} = \infty.$$

And we let

$$F_\gamma^1(z) = (1-z)^{-\frac{\gamma}{p}},$$

belong to A^p, have that

$$\lim_{\gamma \to 2} \|F_\gamma^1\|_{A^p} = \infty.$$

We also obtain a relationship between $f_\gamma^1(z)$ and $F_\gamma^1(z)$ by the Theorem 2.2 and Corollary 2.1, that

$$\limsup_{\gamma\to 2}\frac{\|F_\gamma^1\|_{A^p}}{\|f_\gamma^1\|_{A^p_{\log^\alpha}}}\le C<\infty.$$

Thus, we let $C_{\alpha,p}=\limsup_{\gamma\to 2}\frac{\left\|(1-z)^{-\frac{\gamma}{p}}\right\|_{A^p}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{p}}\right\|_{A^2_{\log^\alpha}}}$ and $C_{\alpha,p}$ is a constant, depending only on α and p.

It can be seen from the proof of Theorem 2.2, we define

$$\phi_\gamma^1(z)=\int_1^\infty\frac{(w-1)^{\frac{\alpha}{p}}}{\left(\log\frac{1}{1-(\frac{w-1}{w-z})}\right)^{\frac{\alpha}{p}}w(w-z)^{\frac{\alpha}{p}+1-\frac{\gamma}{p}}}dw$$

for every z in $\mathbb{D}$, which shows $\mathcal{H}f_\gamma^1(z)=F_\gamma^1(z)\phi_\gamma^1(z)$.

According the proof of Theorem 2.2, we can let w to be a real number $s\ge 1$. And we view ϕ_γ^1 is an analytic function of z that

$$\phi_\gamma^1(z)=\int_1^\infty\frac{(s-1)^{\frac{\alpha}{p}}}{\left(\log\frac{1}{1-(\frac{s-1}{s-z})^2}\right)^{\frac{\alpha}{p}}s(s-z)^{\frac{\alpha}{p}+1-\frac{\gamma}{p}}}ds.$$

We will use the test function $g_\gamma^1(z)=\frac{f_\gamma^1(z)}{\|F_\gamma\|_{A^p}}$ and $G_\gamma^1(z)=\frac{F_\gamma^1(z)}{\|F_\gamma^1\|_{A^p}}$, we have that

$$\|\mathcal{H}\|_{A^p_{\log^\alpha}\to A^p}\ge\frac{\|\mathcal{H}(g_\gamma^1)\|_{A^p}}{\|g_\gamma^1\|_{A^p_{\log^\alpha}}}=\frac{\|F_\gamma^1\|_{A^p}}{\|f_\gamma^1\|_{A^p_{\log^\alpha}}}\frac{\|\mathcal{H}(f_\gamma^1)\|_{A^p}}{\|F_\gamma^1\|_{A^p}}=\frac{\|F_\gamma^1\|_{A^p}}{\|f_\gamma^1\|_{A^p_{\log^\alpha}}}\frac{\|F_\gamma^1(z)\phi_\gamma^1(z)\|_{A^p}}{\|F_\gamma^1\|_{A^p}}.$$

Letting $\gamma\to 2$, and by [8, Theorem 4] we get,

$$\begin{aligned}\|\mathcal{H}\|_{A^p_{\log^\alpha}\to A^p}&\ge\limsup_{\gamma\to 2}\frac{\|F_\gamma^1\|_{A^p}}{\|f_\gamma^1\|_{A^p_{\log^\alpha}}}\frac{\|F_\gamma^1(z)\phi_\gamma(z)\|_{A^p}}{\|F_\gamma^1\|_{A^p}}\\&=C_{\alpha,p}\lim_{\gamma\to 2}\|G_\gamma^1\phi_\gamma^1\|_{A^p}=C_{\alpha,p}\|\phi_2^1\|_\infty\\&=C_{\alpha,p}\sup_{z\in\mathbb{D}}\int_1^\infty\frac{(s-1)^{\frac{\alpha}{p}}}{\left(\log\frac{1}{1-(\frac{s-1}{s-z})}\right)^{\frac{\alpha}{p}}s(s-z)^{\frac{\alpha-2}{p}+1}}ds\end{aligned}$$

$$\geq C_{\alpha,p}\sup_{0\leq r\leq 1}\int_1^\infty \frac{(s-1)^{\frac{\alpha}{p}}}{\left(\log\frac{1}{1-(\frac{s-1}{s-r})}\right)^{\frac{\alpha}{p}} s(s-r)^{\frac{\alpha-2}{p}+1}}ds$$

$$\geq C_{\alpha,p}\int_1^\infty \frac{(s-1)^{\frac{\alpha}{p}}}{s^{\frac{\alpha-2}{p}+2}\left(\log\frac{1}{1-(\frac{s-1}{s})}\right)^{\frac{\alpha}{p}}}ds.$$

Then making the change of variable $x=(s-1)/s$, we calculate that

$$\int_1^\infty \frac{(s-1)^{\frac{\alpha}{p}}}{s^{\frac{\alpha-2}{p}+2}\left(\log\frac{1}{1-(\frac{s-1}{s})}\right)^{\frac{\alpha}{p}}}ds=\int_0^1\frac{x^{\frac{\alpha}{p}}}{(1-x)^{\frac{2}{p}}\left(\log\frac{1}{1-x}\right)^{\frac{\alpha}{p}}}dx.$$

Thus we obtained that,

$$\|\mathcal{H}\|_{A^p_{\log^\alpha}\to A^p}\geq C_{\alpha,p}\int_0^1\frac{x^{\frac{\alpha}{p}}}{(1-x)^{\frac{2}{p}}\left(\log\frac{1}{1-x}\right)^{\frac{\alpha}{p}}}dx.$$

where

$$C_{\alpha,p}=\limsup_{\gamma\to 2}\frac{\left\|(1-z)^{-\frac{\gamma}{p}}\right\|_{A^p}}{\left\|\left(\frac{1}{z}\log\frac{1}{1-z}\right)^{-\frac{\alpha}{2}}(1-z)^{-\frac{\gamma}{p}}\right\|_{A^2_{\log^\alpha}}}.$$

On the other hand, we give the upper bound for the norm $\|\mathcal{H}\|_{A^p_{\log^\alpha}\to A^p}$. We using the method in the proof of Theorem 2.1 and Lemma 3.2, by simple calculation we found that

$$\|\mathcal{H}f\|_{A^p}\leq\int_0^1\|T_tf\|_{A^p}dt=\int_0^1\frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}}\left(\int_{D_t}|w|^{p-4}|f(w)|^pdA(w)\right)^{\frac{1}{p}}dt$$

$$\leq\int_0^1\frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}\left(\log\frac{2}{1-(\frac{t}{2-t})^2}\right)^{\frac{\alpha}{p}}}dt\left(\int_{D_t}|w|^{p-4}|f(w)|^p\left(\log\frac{2}{1-|w|^2}\right)^\alpha dA(w)\right)^{\frac{1}{p}}.$$

By Theorem 3.2 in [3] and Lemma 2 in [6] we can find when $2<p<4$ and $4\leq p<\infty$, we have that

$$\|\mathcal{H}f\|_{A^p}\leq\int_0^1\frac{t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}}\left(\log\frac{2}{1-(\frac{t}{2-t})^2}\right)^{\frac{\alpha}{p}}}dt\left(\int_{\mathbb{D}}|f(w)|^p\left(\log\frac{2}{1-|w|^2}\right)^\alpha dA(w)\right)^{\frac{1}{p}}$$

$$= \int_0^1 \frac{2^{\frac{2}{p}} t^{\frac{2}{p}-1}}{(1+t)(1-t)^{\frac{2}{p}} \left(\log \frac{2}{1-t^2}\right)^{\frac{\alpha}{p}}} dt \|f\|_{A^p_{\log^\alpha}}.$$
$$\leq \int_0^1 \frac{2^{\frac{2}{p}} t^{\frac{2}{p}-1}}{(1-t)^{\frac{2}{p}} \left(\log \frac{2}{1-t^2}\right)^{\frac{\alpha}{p}}} dt \|f\|_{A^p_{\log^\alpha}}.$$

Hence, in this case, we conclude that

$$\|\mathcal{H}\|_{A^p_{\log^\alpha} \to A^p} \leq \int_0^1 \frac{2^{\frac{2}{p}} x^{\frac{2}{p}-1}}{(1-x)^{\frac{2}{p}} \left(\log \frac{2}{1-x^2}\right)^{\frac{\alpha}{p}}} dt.$$

and this concludes the proof. □

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Data Availability The authors declare that all data and material in this paper are available.

Declarations

Conflict of interest The authors declare that there are no conflicts of interest regarding the publication of this paper.

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