Complex Analysis and Operator Theory (2023) 17:99
https://doi.org/10.1007/s11785-023-01400-5

Complex Analysis and Operator Theory

A Note on C^*-Algebra of Toeplitz Operators with $\mathcal{L}$-Invariant Symbols

Shubham R. Bais[1] · D. Venku Naidu[1]

Received: 6 July 2022 / Accepted: 27 July 2023 / Published online: 13 August 2023

Abstract

Let $\mathcal{L} \subset \mathbb{C}^n$ be a Lagrangian plane. In this article, we give structure for $\mathcal{L}$-invariant operators on the Fock space $F^2(\mathbb{C}^n)$. With the help of this structure, we study Toeplitz operators $T_{\mathbf{a}}$ on $F^2(\mathbb{C}^n)$ with $\mathcal{L}$-invariant symbols $\mathbf{a} \in L^\infty(\mathbb{C}^n)$. We show that every operator in the C^*-algebra generated by Toeplitz operators with $\mathcal{L}$-invariant symbols, denoted by $\mathcal{T}_{\mathcal{L}}(L^\infty)$, can be represented as an integral operator of the form

$$(H_\varphi^X f)(z) = \int_{\mathbb{C}^n} f(w)\varphi(z + X^*\overline{Xw})e^{z\overline{w}} d\lambda(w)$$

for some $\varphi \in F^2(\mathbb{C}^n)$ and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. In fact, we prove that $H_\varphi^X \in \mathcal{T}_{\mathcal{L}}(L^\infty)$ if and only if there exists $m \in \mathcal{C}_{b,u}(\mathbb{R}^n)$ such that

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{Xz}{2})^2} dx, \quad z \in \mathbb{C}^n.$$

Here $\mathcal{C}_{b,u}(\mathbb{R}^n)$ denotes all functions on $\mathbb{R}^n$ which are bounded uniformly continuous with respect to the standard metric on $\mathbb{R}^n$.

Keywords Fock space · Bargmann transform · $\mathcal{L}$-invariant operator · Toeplitz operator · Multiplication operator

---

Communicated by Ilan Hirshberg.

This article is part of the topical collection "Spectral Theory and Operators in Mathematical Physics" edited by Jussi Behrndt, Fabrizio Colombo and Sergey Naboko.

✉ D. Venku Naidu
venku@math.iith.ac.in

Shubham R. Bais
ma18resch11003@iith.ac.in; shubhambais007@gmail.com

[1] Department of Mathematics, Indian Institute of Technology - Hyderabad, Kandi, Sangareddy 502 284, Telangana, India

Mathematics Subject Classification 30H20 · 47G10 · 47B35

1 Introduction and Preliminaries

The Fock space $F^2 := F^2(\mathbb{C}^n)$ consists of all entire functions f on $\mathbb{C}^n$ which are square integrable with respect to the Gaussian measure

$$d\lambda(z) = \pi^{-n} e^{-|z|^2} dz.$$

The space $F^2(\mathbb{C}^n)$ is a closed subspace of the Hilbert space $L^2(\lambda) := L^2(\mathbb{C}^n, d\lambda)$ of all Lebesgue measurable functions f over $\mathbb{C}^n$ such that

$$\int_{\mathbb{C}^n} |f(z)|^2 d\lambda(z) < \infty.$$

The norm and inner product on $F^2(\mathbb{C}^n)$ are inherited from $L^2(\mathbb{C}^n, d\lambda)$ and they are respectively denoted by $\|\cdot\|_{F^2}$ and $\langle\cdot,\cdot\rangle_{F^2}$. This space is a reproducing kernel Hilbert space (in short, RKHS) with reproducing kernel given by

$$K(z, \overline{w}) = K_w(z) = e^{z\overline{w}}, \quad \forall z, w \in \mathbb{C}^n.$$

For each fixed $w \in \mathbb{C}^n$, the function K_w is called reproducing kernel at the point w and it belongs to $F^2(\mathbb{C}^n) := F^2$. For $\varphi \in L^\infty(\mathbb{C}^n) := L^\infty$, the Toeplitz operator T_φ on $F^2(\mathbb{C}^n)$ is defined by $T_\varphi f = P\varphi f$, where P is the orthogonal projection on $L^2(\mathbb{C}^n, d\lambda)$ with range $F^2(\mathbb{C}^n)$ and it is given by

$$(Pf)(z) = \int_{\mathbb{C}^n} f(w) e^{z\overline{w}} d\lambda(w)$$

for all $f \in L^2(\mathbb{C}^n, d\lambda)$ and $z \in \mathbb{C}^n$.

Since few decades, Toeplitz operators on holomorphic function spaces (Hardy space, Bergman space, Fock space, etc.) have been widely studied. To obtain deeper results, these operators are studied by restricting the defining symbols to specific subset of L^∞. We observe that the Berezin symbols of these Toeplitz operators also belong to a specific subset of L^∞. We refer to [4, 5, 7–11, 13] and references therein for similar problems studied in Fock space, Bergman spaces and weighted Bergman spaces.

If S is a bounded linear operator on $F^2(\mathbb{C}^n)$, then it's Berezin symbol (also called as Berezin transform), denoted by $\widetilde{S}$, is a bounded function in $L^\infty(\mathbb{C}^n)$ given by

$$\widetilde{S}(z) = \langle Sk_z, k_z\rangle_{F^2}, \tag{1.1}$$

where $k_z = K_z/\|K_z\|_{F^2}$ is called normalized reproducing kernel at z. Let $\mathcal{B}(F^2(\mathbb{C}^n)) := \mathcal{B}(F^2)$ denote the space of all bounded linear operators on $F^2(\mathbb{C}^n)$. For every $S \in \mathcal{B}(F^2)$, there exists a unique operator $S^* \in \mathcal{B}(F^2)$ such that $\langle Sf, g\rangle_{F^2} = \langle f, S^*g\rangle_{F^2}$ for all $f, g \in F^2$. This operator S^* is known as adjoint of S. Due to the existence of the reproducing kernel, every operator $S \in \mathcal{B}(F^2)$ can be uniquely written as an integral operator as shown below.

For $S \in \mathcal{B}(F^2)$ and $z \in \mathbb{C}^n$, we have

$$\begin{aligned}(Sf)(z) &= \langle Sf, K_z\rangle_{F^2} \\ &= \langle f, S^* K_z\rangle_{F^2} \\ &= \int_{\mathbb{C}^n} f(w)\overline{(S^*K_z)}(w)d\lambda(w).\end{aligned}$$

Let $K_S(z, w) := \overline{(S^*K_z)(w)} = \overline{\langle S^*K_z, K_w\rangle_{F^2}} = \overline{\langle K_z, SK_w\rangle_{F^2}} = \langle SK_w, K_z\rangle_{F^2}$ for all $z, w \in \mathbb{C}^n$. Then we have

$$(Sf)(z) = \int_{\mathbb{C}^n} f(w)K_S(z, w)d\lambda(w), \quad \forall z, w \in \mathbb{C}^n. \tag{1.2}$$

In this article, we consider various classes of integral operators of the form (1.2) such that Berezin symbols of operators in each class belong to a specific subset of $L^\infty(\mathbb{C}^n)$. In Sect. 2, we study operators with Berezin symbols invariant under imaginary translations. Such operators are called horizontal operators. We show that every horizontal operator on $F^2(\mathbb{C}^n)$ can be represented as an integral operator of the form

$$(H_\varphi f)(z) = \int_{\mathbb{C}^n} f(w)\varphi(z + \overline{w})e^{z\overline{w}}d\lambda(w), \tag{1.3}$$

where $\varphi \in F^2$, $f \in F^2$ and $z \in \mathbb{C}^n$ (See Theorem 2.12). Let $\mathfrak{B}$ consists of all bounded operators of the form (1.3). We show that $\mathfrak{B}$ is a maximal commutative C^*-subalgebra of $\mathcal{B}(F^2(\mathbb{C}^n))$. Let $\mathcal{T}_{hor}(L^\infty)$ denote the C^*-algebra generated by Toeplitz operators $T_{\mathbf{a}}$ with horizontal symbols $\mathbf{a} \in L^\infty(\mathbb{C}^n)$. As every Toeplitz operator $T_{\mathbf{a}}$ with horizontal symbol $\mathbf{a} \in L^\infty(\mathbb{C}^n)$ is horizontal operator, we get $\mathcal{T}_{hor}(L^\infty) \subseteq \mathfrak{B}$ (See Lemma 2.15). We give explicit representation of operators in $\mathcal{T}_{hor}(L^\infty)$ in the form H_φ.

Let $\mathcal{L}$ be a Lagrangian plane of $\mathbb{C}^n$. In Sect. 3, we consider operators on the Fock space having Berezin symbols invariant under translations over the Lagrangian plane $\mathcal{L}$. These operators are called $\mathcal{L}$-invariant operators. We show that every $\mathcal{L}$-invariant operator on F^2 is of the form

$$(H_\varphi^X f)(z) = \int_{\mathbb{C}^n} f(w)\varphi(z + X^*\overline{Xw})e^{z\overline{w}}d\lambda(w), \tag{1.4}$$

where $\varphi \in F^2$, X is unitary matrix of order n over $\mathbb{C}$ such that $X\mathcal{L} = i\mathbb{R}^n$. Let $\mathfrak{B}^X$ be the collection of all bounded linear operators of the form (1.4). As every Toeplitz operator with $\mathcal{L}$-invariant symbol is an $\mathcal{L}$-invariant operator, the C^*-algebra generated by these operators, denoted by $\mathcal{T}_{\mathcal{L}}(L^\infty)$, is a subalgebra of $\mathfrak{B}^X$. We give explicit integral representation of the form (1.4) for operators in the collection $\mathcal{T}_{\mathcal{L}}(L^\infty)$.

2 Horizontal Toeplitz Operators

We first give some basic notations, definitions and results which will be used throughout the section. Let $L^2(\mathbb{R}^n) := L^2$ denote the space of all complex valued measurable functions such that

$$\|f\|_{L^2}^2 = \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty,$$

where $dx = dx_1 dx_2 \ldots dx_n$ is the Lebesgue measure on $\mathbb{R}^n$. If f is a suitable measurable function on $\mathbb{R}^n$, then it's Fourier transform is defined by

$$(\mathcal{F}f)(x) = \frac{1}{(\pi)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-2ixy} dy.$$

The Fourier transform $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a unitary operator and the inverse Fourier transform is given by

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(\pi)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{2ixy} dy.$$

Let $a, b \in \mathbb{R}^n$ and f be a measurable function on $\mathbb{R}^n$. Then the translation and modulation of f are given respectively by

$$(T_a f)(x) = f(x-a), \quad (M_{e^{2\pi i b(\cdot)}} f)(x) = e^{2\pi i b x} f(x), \quad x \in \mathbb{R}^n. \tag{2.1}$$

The operators T_a and $M_{e^{2\pi i b(\cdot)}}$ defined above are unitary operators on $L^2(\mathbb{R}^n)$. The following theorem is well known.

Theorem 2.1 *For any real numbers $a, b \in \mathbb{R}^n$, we have*

$$\mathcal{F} T_a \mathcal{F}^{-1} = M_{e^{-2\pi i \frac{a}{\pi}(\cdot)}}, \quad \mathcal{F} M_{e^{2\pi i b(\cdot)}} \mathcal{F}^{-1} = T_{-\pi b}$$

Thus, the Fourier transform intertwines the operators T_a and $M_{e^{-2\pi i \frac{a}{\pi}(\cdot)}}$ for all $a \in \mathbb{R}^n$.

Definition 2.2 (*Weyl operator*) For $a \in \mathbb{C}^n$, the Weyl operator, denoted by W_a, is a unitary operator on $F^2(\mathbb{C}^n)$ given by

$$(W_a f)(z) = f(z-a) e^{z\overline{a}} e^{-\frac{|a|^2}{2}}, \quad z \in \mathbb{C}^n, \ f \in F^2(\mathbb{C}^n). \tag{2.2}$$

Bargmann transform: In [2], V. Bargmann introduced a transform B, known as Bargmann transform, which is an isometric isomorphism from $L^2(\mathbb{R}^n)$ onto the Fock space $F^2(\mathbb{C}^n)$ and it is defined by

$$(Bf)(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{R}^n} f(x) e^{2xz - x^2 - \frac{z^2}{2}} dx, \quad z \in \mathbb{C}^n, \ f \in L^2(\mathbb{R}^n).$$

The inverse of the Bargamann transform is given by

$$(B^{-1}f)(x) = \left(\frac{2}{\pi}\right)^{\frac{n}{4}} \int_{\mathbb{R}^n} f(z) e^{2x\bar{z} - x^2 - \frac{\bar{z}^2}{2}} d\lambda(z), \quad x \in \mathbb{R}^n, \ \ f \in F^2(\mathbb{C}^n).$$

We refer to [1] for recent application of Bargamann transform on $L^2(\mathbb{R}^{2n})$. We refer to [6, 14] for more information about the Bargmann transform and it's various applications in mathematics. The following properties of the Bargmann transform are well known and can be found in [15].

Theorem 2.3 *For any real numbers $a, b \in \mathbb{R}^n$ we have*

1. $BT_aB^{-1} = W_a$.
2. $BM_{e^{2\pi ib(\cdot)}}B^{-1} = W_{-\pi bi}$.
3. $B(M_{e^{2\pi ib(\cdot)}}T_a)B^{-1} = e^{\pi abi}W_{a-\pi bi}$.
4. $(B\mathcal{F}B^{-1}f)(z) = f(-iz)$, $f \in F^2(\mathbb{C}^n)$, $z \in \mathbb{C}^n$.
5. $(B\mathcal{F}^{-1}B^{-1}f)(z) = f(iz)$, $f \in F^2(\mathbb{C}^n)$, $z \in \mathbb{C}^n$.

The following lemma gives a necessary condition for boundedness of linear operator on the space F^2.

Lemma 2.4 [14, Proposition 3.1] *The linear mapping $T \to \widetilde{T}$ is one-to-one, order preserving and bounded operator from $\mathcal{B}(F^2(\mathbb{C}^n))$ to $L^\infty(\mathbb{C}^n)$ with $\|\widetilde{T}\|_{L^\infty(\mathbb{C}^n)} \leq \|T\|_{F^2 \to F^2}$.*

Let m be a measurable function on $\mathbb{R}^n$ and M_m be a multiplication operator on $L^2(\mathbb{R}^n)$ defined by $M_m f = m \cdot f$ for all $f \in L^2(\mathbb{R}^n)$. Then the operator M_m is bounded on $L^2(\mathbb{R}^n)$ if and only if $m \in L^\infty(\mathbb{R}^n)$. Moreover, $\|M_m\|_{L^2 \to L^2} = \|m\|_{L^\infty(\mathbb{R}^n)}$. The following theorem is well known.

Theorem 2.5 [10, Lemma 2.1] *Let $M \in \mathcal{B}(L^2(\mathbb{R}^n))$. Then $MM_{e^{2\pi ia(\cdot)}} = M_{e^{2\pi ia(\cdot)}}M$ for all $a \in \mathbb{R}^n$ if and only if $M = M_m$ for some $m \in L^\infty(\mathbb{R}^n)$.*

Now we give definitions of horizontal function and horizontal operator.

Definition 2.6 (*Horizontal function* [5]) A function $\varphi \in L^\infty(\mathbb{C}^n)$ is said to be horizontal if it is invariant under imaginary translations. That is, for every $h \in \mathbb{R}^n$, we have $\varphi((\cdot) - ih) = \varphi(\cdot)$ almost everywhere on $\mathbb{C}^n$.

Definition 2.7 (*Horizontal operator*) Let $T \in \mathcal{B}(F^2(\mathbb{C}^n))$. Then T is said to be horizontal operator if it's Berezin symbol is a horizontal function on $\mathbb{C}^n$.

We have the following criteria for a bounded operator on $F^2(\mathbb{C}^n)$ to be horizontal.

Theorem 2.8 [5, Theorem 3.7] *Let $T \in \mathcal{B}(F^2(\mathbb{C}^n))$. Then the following are equivalent:*

(1) *T is horizontal operator.*

(2) *T commutes with Weyl operators W_{ia} for all $a \in \mathbb{R}^n$. That is,*

$$TW_{ia} = W_{ia}T, \quad \forall a \in \mathbb{R}^n.$$

(3) *$(B^{-1}TB)$ commutes with modulations on $L^2(\mathbb{R}^n)$. That is,*

$$(B^{-1}TB)M_{e^{2\pi ib(\cdot)}} = M_{e^{2\pi ib(\cdot)}}(B^{-1}TB), \quad \forall b \in \mathbb{R}^n.$$

(4) *There exists $m \in L^\infty(\mathbb{R}^n)$ such that $BM_mB^{-1} = T$.*

Remark 2.9 Let $\sigma \in L^\infty(\mathbb{R}^n)$. Then a straight forward calculation shows that

$$(BM_\sigma B^{-1}f)(z) = \int_{\mathbb{C}^n} f(w)\left(\int_{\mathbb{R}^n} \sigma(x)e^{-2(x-\frac{z+\overline{w}}{2})^2}dx\right)e^{z\bar{w}}d\lambda(w), \quad z \in \mathbb{C}^n.$$

Motivated by the Remark 2.9, we consider the following class of integral operators on the Fock space $F^2(\mathbb{C}^n)$.

For $\varphi \in F^2(\mathbb{C}^n)$, consider the operator H_φ on $F^2(\mathbb{C}^n)$ formally defined by

$$(H_\varphi f)(z) = \int_{\mathbb{C}^n} f(w)\varphi(z+\overline{w})e^{z\overline{w}}d\lambda(w), \quad z \in \mathbb{C}^n.$$

We observe that if H_φ is bounded operator then it's Berezin transform, denoted by $\widetilde{H}_\varphi$, is a horizontal function given by

$$\widetilde{H}_\varphi(z) = \varphi(z+\overline{z}), \quad \forall z \in \mathbb{C}^n.$$

Hence, every bounded operator H_φ is horizontal operator.

Lemma 2.10 *Let $m \in L^\infty(\mathbb{R}^n)$. Define*

$$\varphi(z) = \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z}{2})^2}dx$$

for all $z \in \mathbb{C}^n$. Then $\varphi \in F^2(\mathbb{C}^n)$.

Proof Let $z = u + iv \in \mathbb{C}^n$. Then we have

$$\begin{aligned}
\varphi(z) &= \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z}{2})^2}dx \\
&= \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{u+iv}{2})^2}dx \\
&= \int_{\mathbb{R}^n} m\left(x+\frac{u}{2}\right)e^{-2x^2+2ixv+\frac{v^2}{2}}dx \\
&= \pi^{n/2}e^{\frac{v^2}{2}}\mathcal{F}^{-1}\left(m\left(x+\frac{u}{2}\right)e^{-2x^2}\right)(v).
\end{aligned}$$

Therefore

$$\begin{aligned}\|\varphi\|_{F^2}^2 &= \pi^{n/2}\int_{\mathbb{R}^n} e^{-u^2}\int_{\mathbb{R}^n}\left|\mathcal{F}^{-1}\Big(m\Big(x+\frac{u}{2}\Big)e^{-2x^2}\Big)(v)\right|^2 dv du\\ &= \pi^{n/2}\int_{\mathbb{R}^n} e^{-u^2}\int_{\mathbb{R}^n}\left|m\Big(x+\frac{u}{2}\Big)e^{-2x^2}\right|^2 dx du\\ &\leq \|m\|_\infty\Big(\int_{\mathbb{R}^n} e^{-u^2}du\Big)\Big(\int_{\mathbb{R}^n} e^{-4x^2}dx\Big)\\ &< \infty.\end{aligned}$$

□

Lemma 2.11 *Let $\varphi_1, \varphi_2 \in F^2(\mathbb{C}^n)$ such that the operators $H_{\varphi_1}, H_{\varphi_2}$ are bounded on $F^2(\mathbb{C}^n)$. Then $H_{\varphi_1} = H_{\varphi_2}$ if and only if $\varphi_1 = \varphi_2$.*

Proof If $\varphi_1 = \varphi_2$, then it is trivial to check that $H_{\varphi_1} = H_{\varphi_2}$. Conversely, suppose $H_{\varphi_1} = H_{\varphi_2}$. Then for every $f \in F^2(\mathbb{C}^n)$, we have $(H_{\varphi_1} f)(0) = (H_{\varphi_2} f)(0)$. Define $\varphi_1^*(z) = \overline{\varphi_1(\overline{z})}$ and $\varphi_2^*(z) = \overline{\varphi_2(\overline{z})}$ for all $z \in \mathbb{C}^n$. Clearly, $\varphi_1^*, \varphi_2^* \in F^2(\mathbb{C}^n)$. We observe that

$$\langle f, \varphi_1^*\rangle_{F^2} = (H_{\varphi_1} f)(0) = (H_{\varphi_2} f)(0) = \langle f, \varphi_2^*\rangle_{F^2}, \quad \forall f \in F^2(\mathbb{C}^n).$$

So we have $\langle f, \varphi_1^* - \varphi_2^*\rangle_{F^2} = 0$ for all $f \in F^2(\mathbb{C}^n)$. In particular, if $f = \varphi_1^* - \varphi_2^*$ then we get $\varphi_1^* = \varphi_2^*$ and hence $\varphi_1 = \varphi_2$. □

Let $\mathfrak{A} = \{T : T \text{ is horizontal operator on } F^2\}$ and $\mathfrak{B} = \{H_\varphi \in \mathcal{B}(F^2) : \varphi \in F^2\}$. Now we show that $\mathfrak{A} = \mathfrak{B}$.

Theorem 2.12 *Let T be a bounded linear operator on the Fock space $F^2(\mathbb{C}^n)$. Then T is horizontal if and only if there exists a unique $\varphi \in F^2(\mathbb{C}^n)$ such that $T = H_\varphi$.*

Proof Suppose $T \in \mathfrak{A}$. Then by Theorem 2.8, we have that there exists $\sigma \in L^\infty(\mathbb{R}^n)$ such that $T = BM_\sigma B^{-1}$. Define

$$\varphi(z) = \int_{\mathbb{R}^n}\sigma(x)e^{-2(x-\frac{z}{2})^2}dx, \quad \forall z \in \mathbb{C}^n.$$

By Lemma 2.10, we get $\varphi \in F^2(\mathbb{C}^n)$ and Remark 2.9 implies that $T = H_\varphi$. The uniqueness of φ follows from Lemma 2.11.

Conversely, suppose $T = H_\varphi$ for some $H_\varphi \in \mathfrak{B}$. We know that every bounded H_φ is horizontal. Hence T is horizontal operator. □

Corollary 2.13 *Let $\varphi \in F^2(\mathbb{C}^n)$. Then the operator H_φ given by (1.3) is bounded on $F^2(\mathbb{C}^n)$ if and only if there exists $m \in L^\infty(\mathbb{R}^n)$ such that*

$$\varphi(z) = \Big(\frac{2}{\pi}\Big)^{n/2}\int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z}{2})^2}dx \tag{2.3}$$

for all $z \in \mathbb{C}^n$. Moreover, we have

$$\|H_\varphi\|_{F^2\to F^2} = \|m\|_{L^\infty(\mathbb{R}^n)}.$$

Proof Let $\varphi \in F^2(\mathbb{C}^n)$ such that H_φ is bounded operator. Then H_φ is horizontal operator. By Theorem 2.8, it follows that there exists $m \in L^\infty(\mathbb{R}^n)$ such that

$$H_\varphi = BM_m B^{-1}. \tag{2.4}$$

By Remark 2.9, we get

$$(BM_m B^{-1} f)(z) = \int_{\mathbb{C}^n} f(w)\left(\int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z+\overline{w}}{2})^2}dx\right)e^{z\bar{w}}d\lambda(w).$$

Define

$$\psi(z) = \int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z+\overline{w}}{2})^2}dx, \quad z \in \mathbb{C}^n.$$

It follows from Lemma 2.10 that $\psi \in F^2(\mathbb{C}^n)$. This implies that

$$BM_m B^{-1} = H_\psi. \tag{2.5}$$

Combining (2.4) and (2.5), we get $H_\varphi = H_\psi$. Then, using Lemma 2.11, we get $\varphi = \psi$. That is,

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{n/2}\int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z}{2})^2}dx, \quad z \in \mathbb{C}^n.$$

Conversely, suppose $\varphi \in F^2(\mathbb{C}^n)$ and it satisfies (2.3) for some $m \in L^\infty(\mathbb{R}^n)$. Then $M_m \in \mathcal{B}(L^2(\mathbb{R}^n))$ and, by Remark 2.9, it follows that $H_\varphi = BM_m B^{-1} \in \mathcal{B}(F^2(\mathbb{C}^n))$.

Also, $\|H_\varphi\|_{F^2\to F^2} = \|M_m\|_{L^2\to L^2} = \|m\|_{L^\infty(\mathbb{R}^n)}$. □

Corollary 2.14 *The collection $\mathfrak{B}$ is a maximal commutative C^*-subalgebra of $\mathcal{B}(F^2)$.*

Proof Consider the map $\eta : L^\infty(\mathbb{R}^n) \to \mathfrak{B}$ defined by $m \to H_\varphi$, where

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{n/2}\int_{\mathbb{R}^n} m(x)e^{-2(x-\frac{z}{2})^2}dx, \quad z \in \mathbb{C}^n.$$

Notice that $\eta(\overline{m}) = H_{\widetilde{\varphi}}$, where $\widetilde{\varphi} \in F^2(\mathbb{C}^n)$ and it is given by

$$\widetilde{\varphi}(z) = \left(\frac{2}{\pi}\right)^{n/2}\int_{\mathbb{R}^n} \overline{m}(x)e^{-2(x-\frac{z}{2})^2}dx, \quad z \in \mathbb{C}^n.$$

Let $m_1, m_2 \in L^\infty(\mathbb{R}^n)$. Then $m_1 m_2 \in L^\infty(\mathbb{R}^n)$ and $\eta(m_1 m_2) = H_\varphi$, where $\varphi \in F^2(\mathbb{C}^n)$ and it is given by

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} m_1(x) m_2(x) e^{-2(x-\frac{z}{2})^2} dx, \quad z \in \mathbb{C}^n.$$

By (2.4), we get

$$\eta(m_1 m_2) = H_\varphi = B M_{m_1 m_2} B^{-1} = (B M_{m_1} B^{-1})(B M_{m_2} B^{-1}) = \eta(m_1)\eta(m_2).$$

This implies that the map η is well-defined $*$-preserving onto isometric isomorphism. Since $L^\infty(\mathbb{R}^n)$ is a maximal commutative C^*-algebra (see [12, Proposition 1.14]), it follows that $\mathfrak{B}$ is also maximal commutative C^*-subalgebra of $\mathcal{B}(F^2)$. □

For horizontal Toeplitz operator $T_{\mathbf{a}} (\mathbf{a} \in L^\infty(\mathbb{C}^n))$, the following two results are proved in [5].

Lemma 2.15 [5, Lemma 3.6] *Let* $\mathbf{a} \in L^\infty(\mathbb{C}^n)$. *Then* $T_{\mathbf{a}}$ *is horizontal Toeplitz operator on* $F^2(\mathbb{C}^n)$ *if and only if* $\mathbf{a}$ *is horizontal function.*

Combining [5, Theorem 3.7(iv)] and [5, Theorem 3.8] we have the following.

Lemma 2.16 *Let* $T_{\mathbf{a}}$ *be Toeplitz operator on* $F^2(\mathbb{C}^n)$ *with horizontal symbol then there exists* $\gamma_{\mathbf{a}} \in L^\infty(\mathbb{R}^n)$ *such that*

$$\widetilde{T_{\mathbf{a}}}(z) = \pi^{-n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(x) e^{-(x-\frac{z+\bar{z}}{\sqrt{2}})^2} dx,$$

where

$$\gamma_{\mathbf{a}}(x) = \pi^{-n/2} \int_{\mathbb{R}^n} \mathbf{a}\left(\frac{y}{\sqrt{2}}\right) e^{-(x-y)^2} dy, \quad x \in \mathbb{R}^n.$$

In the next theorem, we give an alternative representation of the form (1.3) for Toeplitz operator on $F^2(\mathbb{C}^n)$ with horizontal symbol.

Theorem 2.17 *For every Toeplitz operator* $T_{\mathbf{a}}$ *with horizontal symbol* $\mathbf{a}$, *there exists* $\varphi \in F^2(\mathbb{C}^n)$ *such that* $T_{\mathbf{a}} = H_\varphi$, *where*

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathbf{a}(y) e^{-2(x-y)^2} dy\right) e^{-2(x-\frac{z}{2})^2} dx.$$

Proof Let $T_{\mathbf{a}}$ be a Toeplitz operator with horizontal symbol $\mathbf{a}$. By Lemma 2.16, Berezin transform of $T_{\mathbf{a}}$ is given by

$$\widetilde{T_{\mathbf{a}}}(z) = \pi^{-n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(x) e^{-(x-\frac{z+\bar{z}}{\sqrt{2}})^2} dx, \quad z \in \mathbb{C}^n,$$

where $\gamma_{\mathbf{a}} \in L^\infty(\mathbb{R}^n)$ and it is given by

$$\gamma_{\mathbf{a}}(x) = \pi^{-n/2} \int_{\mathbb{R}^n} \mathbf{a}\left(\frac{y}{\sqrt{2}}\right) e^{-(x-y)^2} dy, \quad x \in \mathbb{R}^n.$$

We observe that

$$\begin{aligned}\widetilde{T_{\mathbf{a}}}(z) &= \pi^{-n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(\sqrt{2}x) e^{-2(x-\frac{z+\bar{z}}{2})^2} 2^{n/2} dx \\ &= \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(\sqrt{2}x) e^{-2(x-\frac{z+\bar{z}}{2})^2} dx, \quad z \in \mathbb{C}^n.\end{aligned}$$

Define

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(\sqrt{2}x) e^{-2(x-\frac{z}{2})^2} dx, \quad z \in \mathbb{C}^n. \tag{2.6}$$

By Lemma 2.10, we have that $\varphi \in F^2(\mathbb{C}^n)$. Also, Corollary 2.13 implies that the operator $H_\varphi \in \mathcal{B}(F^2(\mathbb{C}^n))$ and it's Berezin transform is given by

$$\begin{aligned}\widetilde{H_\varphi}(z) &= \varphi(z+\bar{z}) \\ &= \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(\sqrt{2}x) e^{-2(x-\frac{z+\bar{z}}{2})^2} dx.\end{aligned}$$

Therefore, we get $\widetilde{T_{\mathbf{a}}} = \widetilde{H_\varphi}$ and using Theorem 2.4 we get $T_{\mathbf{a}} = H_\varphi$. Thus, we proved that every Toeplitz operator $T_{\mathbf{a}}$ with horizontal symbol $\mathbf{a} \in L^\infty(\mathbb{C}^n)$ is of the form H_φ for some $\varphi \in F^2(\mathbb{C}^n)$. The functions $\mathbf{a}$ and φ are related as follows.

$$\begin{aligned}\varphi(z) &= \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} \gamma_{\mathbf{a}}(\sqrt{2}x) e^{-2(x-\frac{z}{2})^2} dx \\ &= \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} \left(\pi^{-n/2} \int_{\mathbb{R}^n} \mathbf{a}\left(\frac{y}{\sqrt{2}}\right) e^{-(\sqrt{2}x-y)^2} dy\right) e^{-2(x-\frac{z}{2})^2} dx \\ &= \left(\frac{2}{\pi}\right)^{n} \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathbf{a}(y) e^{-2(x-y)^2} dy\right) e^{-2(x-\frac{z}{2})^2} dx.\end{aligned}$$

□

Let f_1, f_2 be suitable measurable functions on $\mathbb{R}^n$. Then the convolution of f_1 and f_2, denoted by $f_1 * f_2$, is given by

$$(f_1 * f_2)(x) = \int_{\mathbb{R}^n} f_1(x-y) f_2(y) dy, \quad x \in \mathbb{R}^n.$$

By [5, Lemma 3.6], we have that every horizontal function $\mathbf{a} \in L^\infty(\mathbb{C}^n)$ is of the form $\mathbf{a}(z) = b(\Re z)$ for some $b \in L^\infty(\mathbb{R}^n)$ and vice versa. In fact, it can be easily seen that

$b = \mathbf{a}|_{\mathbb{R}}^{n}$. So hereafter we identify horizontal functions in $L^\infty(\mathbb{C}^n)$ by functions in $L^\infty(\mathbb{R}^n)$. Consider the function

$$g(x) = \left(\frac{2}{\pi}\right)^{n/2} e^{-2x^2}, \quad x \in \mathbb{R}^n. \tag{2.7}$$

Clearly $g \in L^1(\mathbb{R}^n)$. Let $\widehat{g}$ denote the Fourier transform of g. It is easy to see that $\widehat{g}(y) \neq 0$ for all $y \in \mathbb{R}^n$.

Let $\mathcal{C}_{b,u}(\mathbb{R}^n) := \mathcal{C}_{b,u}$ denote the collection of all Lebesgue measurable functions m on $\mathbb{R}^n$ that are bounded uniformly continuous with respect to the standard metric on $\mathbb{R}^n$. Clearly, $\mathcal{C}_{b,u}(\mathbb{R}^n) \subseteq L^\infty(\mathbb{R}^n)$. The following result is well known and gives some of the dense subsets of $\mathcal{C}_{b,u}(\mathbb{R}^n)$.

Lemma 2.18 [5, Proposition 5.4] *Let $h \in L^1(\mathbb{R}^n)$ such that $\widehat{h}(x) \neq 0$ for all $x \in \mathbb{R}^n$. Then the collection $\left\{h * f : f \in L^\infty(\mathbb{R}^n)\right\}$ is dense in $\mathcal{C}_{b,u}(\mathbb{R}^n)$.*

Theorem 2.19 *The C^*-algebra generated by Toeplitz operators on $F^2(\mathbb{C}^n)$ with horizontal symbols, denoted by $\mathcal{T}_{hor}(L^\infty)$, is given by*

$$\mathcal{T}_{hor}(L^\infty) = \left\{H_\varphi : \varphi(z) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} m(x) e^{-2(x-\frac{z}{2})^2} dx \textit{ for some } m \in \mathcal{C}_{b,u}(\mathbb{R}^n)\right\}.$$

Proof Let $\mathfrak{J} = \left\{a * g : \mathbf{a} \in L^\infty(\mathbb{C}^n) \text{ is a horizontal function and } \mathbf{a}|_{\mathbb{R}^n} = a\right\}$. By Theorem 2.17 and Corollary 2.14, we have $\eta(\mathfrak{J})$ is equal to the collection of all Toeplitz operator with horizontal symbols. Therefore, $\mathcal{T}_{hor}(L^\infty)$ is equal to the image of C^*-algebra generated by the collection $\mathfrak{J}$ under the map η. But, by Lemma 2.18, the C^*-algebra generated by $\mathfrak{J}$ is equal to $\mathcal{C}_{b,u}(\mathbb{R}^n)$. Therefore, we get $\mathcal{T}_{hor}(L^\infty) = \eta(\mathcal{C}_{b,u}(\mathbb{R}^n))$. That is,

$$\mathcal{T}_{hor}(L^\infty) = \left\{H_\varphi : \varphi(z) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} m(x) e^{-2(x-\frac{z}{2})^2} dx \text{ for some } m \in \mathcal{C}_{b,u}(\mathbb{R}^n)\right\}.$$

This proves the theorem. □

Now, we recall the example of horizontal Toeplitz operator $T_{\mathbf{a}}$ given in [5, Example 5.7] which does not belong to the C^*-algebra $\mathcal{T}_{hor}(L^\infty)$. By Theorem 2.19, we have that the defining symbol $\mathbf{a}$ of such Toeplitz operator is unbounded. We now find the function $\varphi \in F^2(\mathbb{C})$ so that the operator $T_{\mathbf{a}}$ can be represented as an integral operator H_φ given by (1.3).

Example 2.20 Let $\mathbf{a}(x) = e^{(i+1)x^2}$, $x \in \mathbb{R}$. Clearly, $\mathbf{a} \notin L^\infty(\mathbb{R})$. Also, we have

$$\gamma_{\mathbf{a}}(x) = \sqrt{1+i}\, e^{ix^2}, \quad x \in \mathbb{R}.$$

Let $x_n = n$ and $y_n = n + \frac{\pi}{2n}$ for each $n \in \mathbb{N}$. Then it is easy to see that $\lim_{n\to\infty} |x_n - y_n| = 0$ but $\lim_{n\to\infty} |\gamma_{\mathbf{a}}(x_n) - \gamma_{\mathbf{a}}(y_n) = 2^{5/4} \neq 0$.

Since $\|\gamma_{\mathbf{a}}\|_{L^\infty(\mathbb{R})} \leq \left|\sqrt{1+i}\right|$, we define

$$\begin{aligned}\varphi(z) &= \left(\frac{2}{\pi}\right)^{1/2} \int_{\mathbb{R}} \gamma_{\mathbf{a}}(\sqrt{2}x) e^{-2(x-\frac{z}{2})^2} dx \\ &= \left(\frac{2}{\pi}\right)^{1/2} \int_{\mathbb{R}} \sqrt{1+i} e^{2ix^2} e^{-2(x-\frac{z}{2})^2} dx.\end{aligned}$$

By Lemma 2.10, we get $\varphi \in F^2(\mathbb{C})$ and proceeding as in the proof of Theorem 2.17 it follows that $\widetilde{T}_{\mathbf{a}} = \widetilde{H}_\varphi$. Then Theorem 2.4 implies that $T_{\mathbf{a}} = H_\varphi$.

3 $\mathcal{L}$-Invariant Toeplitz Operators

In this section, we consider slight change in our notations. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the mapping $(z_1, z_2, \ldots, z_n) \to (x, y)$, where $x = (\Re z_1, \ldots, \Re z_n)$ and $y = (\Im z_1, \ldots, \Im z_n)$. Thus, $i\mathbb{R}^n$ is identified with $\{0\} \times \mathbb{R}^n$. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The set $\mathcal{M}(n, \mathbb{K})$ denote the collection of all $n \times n$ square matrices with entries in $\mathbb{K}$. Let $J \in \mathcal{M}(2n, \mathbb{R})$ be such that

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where 0 and I_n are $n \times n$ zero and identity matrices respectively. Let $S \in \mathcal{M}(2n, \mathbb{R})$. Then S is said to be *symplectic matrix* if it satisfies

$$S^T J S = S J S^T = J, \tag{3.1}$$

where the matrix J is as above. The set of all symplectic matrices is denoted by $Sp(2n, \mathbb{R})$. The matrix J given above is also a symplectic matrix and it is known as *standard symplectic matrix*.

Let ω denote a bilinear form on $\mathbb{R}^{2n}$. Then ω is said to be *symplectic form* if it is antisymmetric and non-degenerate. The standard symplectic form on $\mathbb{R}^{2n}$, denoted by ω_0, is given by

$$\omega_0(z, w) = Jz \cdot w = u \cdot y - v \cdot x$$

for all $z = (x, y),\ w = (u, v) \in \mathbb{R}^{2n}$ and J is the standard symplectic matrix. A symplectic space (V, ω) is a vector space V equipped with a symplectic form ω. We now define Lagrangian planes of the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$.

Definition 3.1 (*Lagrangian plane* [5]) An n-dimensional linear subspace $\mathcal{L}$ of $\mathbb{R}^{2n}$ is said to be a *Lagrangian plane* of the symplectic space $(\mathbb{R}^{2n}, \omega_0)$ if for every $z, w \in \mathcal{L}$ we have $\omega_0(z, w) = 0$. We denote the set of all Lagrangian planes in $(\mathbb{R}^{2n}, \omega_0)$ by $Lag(2n, \mathbb{R})$.

For $\mathcal{L} \in Lag(2n, \mathbb{R})$, we define below $\mathcal{L}$-invariant functions on $\mathbb{R}^{2n}$.

Definition 3.2 ($\mathcal{L}$*-invariant functions* [5]) Let $\mathcal{L} \in Lag(2n, \mathbb{R})$. A function $\varphi \in L^\infty(\mathbb{R}^{2n})$ is said to be $\mathcal{L}$-invariant if for every $a \in \mathcal{L}$ we have $\varphi((\cdot) - a) = \varphi(\cdot)$ almost everywhere on $\mathbb{R}^{2n}$.

A matrix $A \in \mathcal{M}(2n, \mathbb{R})$ is said to preserve the standard symplectic form ω_0 if $\omega_0(Az, Aw) = \omega_0(z, w)$ for all $z, w \in \mathbb{R}^{2n}$. Let $\mathcal{U}(2n, \mathbb{R}) = Sp(2n, \mathbb{R}) \cap \mathcal{O}(2n, \mathbb{R})$ denote the set of all *symplectic rotations* which is a group and every matrix in it preserves the standard symplectic form, where $\mathcal{O}(2n, \mathbb{R})$ is the collection of all orthogonal matrices in $\mathcal{M}(2n, \mathbb{R})$. Let $\mathcal{U}(n, \mathbb{C})$ be the set of all unitary matrices in $\mathcal{M}(n, \mathbb{C})$. From [3, Proposition 30], we have that $\mathcal{U}(2n, \mathbb{R})$ is isomorphic to the unitary group $\mathcal{U}(n, \mathbb{C})$ via the isomorphism $\eta : \mathcal{U}(n, \mathbb{C}) \to \mathcal{U}(2n, \mathbb{R})$ given by

$$\eta(U + iV) = \begin{bmatrix} U & -V \\ V & U \end{bmatrix}$$

for all $U, V \in \mathcal{M}(n, \mathbb{R})$ such that $U + iV \in \mathcal{U}(n, \mathbb{C})$. Using the isomorphism $\eta : \mathcal{U}(n, \mathbb{C}) \to \mathcal{U}(2n, \mathbb{R})$, we now identify each Lagrangian plane $\mathcal{L}$ of $\mathbb{R}^{2n}$ with a subset of $\mathbb{C}^n$, which will also be denoted by $\mathcal{L}$.

Let $\mathcal{L}$ be any Lagrangian plane of $\mathbb{R}^{2n}$. Then the transitive property of $\mathcal{U}(2n, \mathbb{R})$ and the isomorphism $\eta : \mathcal{U}(n, \mathbb{C}) \to \mathcal{U}(2n, \mathbb{R})$ implies that there exists a unitary matrix $X \in \mathcal{U}(n, \mathbb{C})$ such that

$$X\mathcal{L} = i\mathbb{R}^n.$$

We refer to [3] for more information about the symplectic forms and the symplectic matrices. From now on, for simplicity of calculations, we use this fact and rewrite the definition of $\mathcal{L}$-invariant function as follows:

A function $\varphi \in L^\infty(\mathbb{C}^n)$ is said to be $\mathcal{L}$-invariant if for every $h \in \mathcal{L}$ we have

$$\varphi((\cdot) - h) = \varphi(\cdot)$$

almost everywhere on $\mathbb{C}^n$.

The horizontal case corresponds to $\mathcal{L} = i\mathbb{R}^n$. Now we give definition of $\mathcal{L}$-invariant operators in $\mathcal{B}(F^2(\mathbb{C}^n))$.

Definition 3.3 ($\mathcal{L}$*-invariant operator*) Let $\mathcal{L}$ be a Lagrangian plane. A bounded operator T on the Fock space $F^2(\mathbb{C}^n)$ is said to be $\mathcal{L}$-invariant if it's Berezin transform is an $\mathcal{L}$-invariant function on $\mathbb{C}^n$. That is, for each $h \in \mathcal{L}$, the Berezin transfrom $\widetilde{T}$ satisfies

$$\widetilde{T}(z - h) = \widetilde{T}(z)$$

for almost all $z \in \mathbb{C}^n$.

Let $\mathcal{O}(n, \mathbb{R})$ be the collection of all othogonal matrices in $\mathcal{M}(n, \mathbb{R})$ and $\mathcal{O}(n)$ is the image of $\mathcal{O}(n, \mathbb{R})$ by the restriction of the embedding $\eta : \mathcal{U}(n, \mathbb{C}) \to \mathcal{U}(2n, \mathbb{R})$. Then we have the follwing:

Proposition 3.4 [3, Proposition 46] *The collection $Lag(2n, R)$ is homeomorphic to the coset space $\mathcal{U}(2n, \mathbb{R})/\mathcal{O}(n)$.*

For $\varphi \in F^2(\mathbb{C}^n)$ and a Lagrangian plane $\mathcal{L}$, we define formally an integral operator $H_\varphi^X : F^2(\mathbb{C}^n) \to F^2(\mathbb{C}^n)$ as

$$H_\varphi^X f(z) = \int_{\mathbb{C}^n} f(w)\varphi(z + X^*\overline{Xw})e^{z\overline{w}}d\lambda(w),$$

where $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Note that the operator defined by (1.3) corresponds to $X = I_n$.

Remark 3.5 If $\mathcal{L} \in Lag(2n, \mathbb{R})$ and $X, Y \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$ and $Y\mathcal{L} = i\mathbb{R}^n$, then it follows from Proposition 3.4 that $Y = XO$ for some $O \in \mathcal{O}(n, \mathbb{R})$. This gives us $H_\varphi^X = H_\varphi^{XO} = H_\varphi^Y$. Hence it is enough to study the operator H_φ^X by fixing $X \in \mathcal{U}(n, \mathbb{C})$ satisfying $X\mathcal{L} = i\mathbb{R}^n$.

Lemma 3.6 *Let $\varphi \in F^2(\mathbb{C}^n)$, $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Then the Berezin transform of $H_\varphi^X \in \mathcal{B}(F^2(\mathbb{C}^n))$, denoted by $\widetilde{H}_\varphi^X$, is given by*

$$\widetilde{H}_\varphi^X(a) = \varphi(a + X^*\overline{Xa})$$

for all $a \in \mathbb{C}^n$. Moreover, $\widetilde{H}_\varphi^X$ is an $\mathcal{L}$-invariant function on $\mathbb{C}^n$. That is,

$$\widetilde{H}_\varphi^X(a - l) = \widetilde{H}_\varphi^X(a)$$

for all $l \in \mathcal{L}$ and $a \in \mathbb{C}^n$.

Proof It is a direct verification. □

Definition 3.7 Let $X \in \mathcal{U}(n, \mathbb{C})$. Define the linear operator $U_X : F^2(\mathbb{C}^n) \to F^2(\mathbb{C}^n)$ by

$$U_X f(z) = f(X^*z), \quad \forall z \in \mathbb{C}^n. \tag{3.2}$$

Since $X^* = X^{-1} \in \mathcal{U}(n, \mathbb{C})$, the operator U_X is unitary on $F^2(\mathbb{C}^n)$ and $U_X^* = U_{X^*} = U_{X^{-1}}$.

Definition 3.8 Let $\varphi \in F^2(\mathbb{C}^n)$ and $X \in \mathcal{U}(n, \mathbb{C})$. Define φ_X on $\mathbb{C}^n$ by

$$\varphi_X(z) = \varphi(Xz), \quad \forall z \in \mathbb{C}^n. \tag{3.3}$$

From Eq. (3.2), we have that

$$\varphi_X \in F^2(\mathbb{C}^n) \text{ with } \|\varphi_X\|_{F^2} = \|\varphi\|_{F^2}.$$

Lemma 3.9 *Let $\varphi \in F^2(\mathbb{C}^n)$. Let $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. If $\varphi_{X^*} = U_X\varphi$, then whenever $H_\varphi^X \in \mathcal{B}(F^2(\mathbb{C}^n))$, we have*

$$H_\varphi^X = U_X^* H_{\varphi_{X^*}} U_X,$$

where $H_{\varphi_{X^*}}$ and H_φ^X are defined by the Eqs. (1.3) and (1.4) respectively.

Proof Given that $\varphi \in F^2(\mathbb{C}^n)$, $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Suppose H_φ^X is bounded. Then for all $f \in F^2(\mathbb{C}^n)$ and $z \in \mathbb{C}^n$,

$$\begin{aligned} U_X H_\varphi^X U_X^* f(z) &= H_\varphi^X U_X^* f(X^* z) \\ &= \int_{\mathbb{C}^n} (U_X^* f)(w) \varphi(X^* z + X^* \overline{Xw}) e^{\langle X^* z, w\rangle} d\lambda(w) \\ &= \int_{\mathbb{C}^n} f(Xw) \varphi(X^*(z + \overline{Xw})) e^{\langle z, Xw\rangle} d\lambda(w) \\ &= \int_{\mathbb{C}^n} f(Xw) \varphi_{X^*}(z + \overline{Xw}) e^{\langle z, Xw\rangle} d\lambda(w). \end{aligned}$$

Using the change of variable $w \to X^* w$ we get

$$\begin{aligned} U_X H_\varphi^X U_X^* f(z) &= \int_{\mathbb{C}^n} f(w) \varphi_{X^*}(z + \overline{w}) e^{\langle z, w\rangle} d\lambda(w) \\ &= H_{\varphi_{X^*}} f(z). \end{aligned}$$

Hence, we have $H_\varphi^X = U_X^* H_{\varphi_{X^*}} U_X$. □

If $\mathcal{L}$ is a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$, then as a consequence of Theorem 2.12 and Lemma 3.9, we have the following corollary from which we get that every $\mathcal{L}$-invariant operator can be respresented in the form 1.4 and vice-versa.

Corollary 3.10 *Let $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Then a bounded operator T on $F^2(\mathbb{C}^n)$ is $\mathcal{L}$-invariant if and only if there exists a unique $\varphi \in F^2(\mathbb{C}^n)$ such that $T = H_\varphi^X$, where H_φ^X is given by (1.4).*

Theorem 3.11 (Boundedness of H_φ^X) *Let $\varphi \in F^2(\mathbb{C}^n)$, $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Then the operator H_φ^X defined by the Eq. (1.4) is bounded on the Fock space $F^2(\mathbb{C}^n)$ if and only if there exists $m \in L^\infty(\mathbb{R}^n)$ such that*

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{C}^n} m(x) e^{-2(x - \frac{X(z+\overline{w})}{2})^2} dx \tag{3.4}$$

for all $z \in \mathbb{C}^n$. Also, we have

$$\|H_\varphi^X\|_{F^2 \to F} = \|m\|_{L^\infty(\mathbb{R}^n)}.$$

Proof From Theorems 2.13 and 3.9, we have $H_\varphi^X \in \mathcal{B}(F^2(\mathbb{C}^n))$ if and only if $H_{\varphi_{X^*}} \in \mathcal{B}(F^2(\mathbb{C}^n))$ if and only if there exists $m \in L^\infty(\mathbb{R}^n)$ such that

$$\varphi_{X^*}(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{C}^n} m(x) e^{-2(x-\frac{z}{2})^2} dx.$$

Since $\varphi_{X^*}(z) = \varphi(X^* z)$, we get

$$\varphi(z) = \left(\frac{2}{\pi}\right)^{\frac{n}{2}} \int_{\mathbb{C}^n} m(x) e^{-2(x-\frac{Xz}{2})^2} dx$$

for all $z \in \mathbb{C}^n$. Also,

$$\|H_\varphi^X\|_{F^2 \to F^2} = \|H_{\varphi_{X^*}}\|_{F^2 \to F^2} = \|m\|_{L^\infty(\mathbb{R}^n)}.$$

□

Lemma 3.12 *Let $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Let $a \in L^\infty(\mathbb{C}^n)$. Then we have*

$$T_a = U_X^* T_{a_{X^*}} U_X. \tag{3.5}$$

From Lemma 3.12, we have the following remark.

Remark 3.13 Let $\mathcal{L}$ be a Lagrangian plane and $a \in L^\infty(\mathbb{C}^n)$. Then the Toeplitz operator T_a is $\mathcal{L}$-invariant if and only if a is $\mathcal{L}$-invariant function.

Let $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ is such that $X\mathcal{L} = i\mathbb{R}^n$. Now we show that every Toeplitz operator T_a^X with an $\mathcal{L}$-invariant symbol $a \in L^\infty(\mathbb{C}^n)$ is of the form H_φ^X given by (1.4) for some $\varphi \in F^2(\mathbb{C}^n)$.

Theorem 3.14 *Let $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. For every Toeplitz operator T_a^X on $F^2(\mathbb{C}^n)$ with $\mathcal{L}$-invariant symobl $a \in L^\infty(\mathbb{C}^n)$, there exists $\varphi \in F^2(\mathbb{C}^n)$ such that $T_a^X = H_\varphi^X$, where*

$$\varphi(z) = \left(\frac{2}{\pi}\right)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} a_{X^*}(y) e^{-2(x-y)^2} dy \right) e^{-2(x-\frac{Xz}{2})^2} dx, \quad z \in \mathbb{C}^n.$$

Proof Let T_a^X be Toeplitz operator on $F^2(\mathbb{C}^n)$ with $\mathcal{L}$-invariant symbol. By Lemma 3.12, we have $U_X T_a^X U_X^* = T_{a_{X^*}}$ is Toeplitz operator with horizontal symbol a_{X^*}. Then Theorem 2.17 implies that $T_{a_{X^*}} = H_{\varphi_{X^*}}$, where

$$\varphi_{X^*}(z) = \left(\frac{2}{\pi}\right)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} a_{X^*}(y) e^{-2(x-y)^2} dy \right) e^{-2(x-\frac{z}{2})^2} dx.$$

Using Theorem 3.9, we get $U_X T_a^X U_X^* = U_X H_\varphi^X U_X^*$. Hence, $T_a^X = H_\varphi^X$, where

$$\begin{aligned}\varphi(z) &= \varphi(X^* X z)\\ &= \varphi_{X^*}(Xz)\\ &= \left(\frac{2}{\pi}\right)^n \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} a_{X^*}(y) e^{-2(x-y)^2} dy\right) e^{-2(x-\frac{Xz}{2})^2} dx.\end{aligned}$$

This proves the theorem. □

Let $\mathcal{L}$ be a Lagrangian plane, $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$ and $\mathcal{T}_{\mathcal{L}}(L^\infty)$ denote the C^*-algebra generated by Toeplitz operators with $\mathcal{L}$-invariant symbols. From Theorem 3.14, we observe that a bounded operator T on the Fock space $F^2(\mathbb{C}^n)$ belongs to $\mathcal{T}_{\mathcal{L}}(L^\infty)$ if and only if $U_X T U_X^* \in \mathcal{T}_{hor}(L^\infty)$. Hence, we have

$$\mathcal{T}_{\mathcal{L}}(L^\infty) = U_X^*(\mathcal{T}_{hor}(L^\infty))U_X = \left\{H_\varphi^X = U_X^* H_\varphi U_X : H_\varphi \in \mathcal{T}_{hor}\right\}. \tag{3.6}$$

As a consequence of Theorem 3.14 and Eq. (3.6) we have the following:

Corollary 3.15 *Let $\mathcal{L}$ be a Lagrangian plane and $X \in \mathcal{U}(n, \mathbb{C})$ such that $X\mathcal{L} = i\mathbb{R}^n$. Then the C^*-algebra generated by Toeplitz operators with $\mathcal{L}$-invariant symbols is given by*

$$\mathcal{T}_{\mathcal{L}}(L^\infty) = \left\{H_\varphi^X : \varphi(z) = \left(\frac{2}{\pi}\right)^{n/2} \int_{\mathbb{R}^n} m(x) e^{-2(x-\frac{Xz}{2})^2} dx \text{ for some } m \in \mathcal{C}_{b,u}(\mathbb{R}^n)\right\}.$$

Acknowledgements The first author thanks the University Grant Commission (UGC), India for providing financial support. The authors thank the referee(s) for meticulously reading our manuscript and giving us several valuable suggestions. The authors also thank the handling editor for the help during the editorial process.

Author Contributions All authors equally contributed.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Bais, S.R., Venku Naidu, D.: Study of twisted Bargmann transform via Bargmann transform. Forum Math. **33**(6), 1659–1670 (2021)
2. Bargmann, V.: On a Hilbert space of analytic functions and an associated integral transform. Commun. Pure Appl. Math. **14**, 187–214 (1961)
3. de Gosson MA (2011) Symplectic methods in harmonic analysis and in mathematical physics. In: Pseudo-Differential Operators. Theory and Applications, 7, Birkhäuser/Springer Basel AG, Basel
4. Esmeral, K., Maximenko, E.A., Vasilevski, N.: C^*-algebra generated by angular Toeplitz operators on the weighted Bergman spaces over the upper half-plane. Integral Equ. Oper. Theory **83**(3), 413–428 (2015)
5. Esmeral, K., Vasilevski, N.: C*-algebra generated by horizontal Toeplitz operators on the Fock space. Bol. Soc. Mat. Mex. (3) **22**(2), 567–582 (2016)

6. Folland, G.B.: Harmonic Analysis in Phase Space. Annals of Mathematics Studies, 122. Princeton University Press, Princeton (1989)
7. Grudsky, S.M., Vasilevski, N.L.: Toeplitz operators on the Fock space: radial component effects. Integral Equ. Oper. Theory **44**(1), 10–37 (2002)
8. Grudsky, S., Quiroga-Barranco, R., Vasilevski, N.: Commutative C^*-algebras of Toeplitz operators and quantization on the unit disk. J. Funct. Anal. **234**(1), 1–44 (2006)
9. Grudsky, S., Vasilevski, N.: Bergman–Toeplitz operators: radial component influence. Integral Equ. Oper. Theory **40**(1), 16–33 (2001)
10. Herrera Yañez, C., Maximenko, E.A., Vasilevski, N.: Vertical Toeplitz operators on the upper half-plane and very slowly oscillating functions. Integral Equ. Oper. Theory **77**(2), 149–166 (2013)
11. Herrera Yañez, C., Hutník, O., Maximenko, E.A.: Vertical symbols, Toeplitz operators on weighted Bergman spaces over the upper half-plane and very slowly oscillating functions. C. R. Math. Acad. Sci. Paris **352**(2), 129–132 (2014)
12. Radjavi, H., Rosenthal, P.: Invariant Subspaces. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 77. Springer, New York (1973)
13. Vasilevski, N.L.: Commutative Algebras of Toeplitz Operators on the Bergman Space. Operator Theory: Advances and Applications, 185. Birkhäuser Verlag, Basel (2008)
14. Zhu, K.: Analysis on Fock Spaces. Graduate Texts in Mathematics, 263. Springer, New York (2012)
15. Zhu, K.: Towards a Dictionary for the Bargmann Transform. Handbook of Analytic Operator Theory, pp. 319–349. CRC Press/Chapman Hall Handbook in Mathematics Series, Boca Raton (2019)