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Faster Convergence in the Free Central Limit Theorem

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Abstract
We show that there is a faster convergence in the free central limit theorem for measures of bounded support when we have vanishing free cumulants. We give estimates for the extremes of the support and the density of the converging measure. As a consequence, we obtain a more precise Berry–Esseen type estimate than previous results.

Keywords Free central limit theorem · Density estimate · Berry–Esseen theorem

1 Introduction

Free probability theory studies the distribution of self-adjoint operators that obey a new notion of independence called freeness. It is established now that it is a parallel theory with classical probability. One of its main results is the free central limit theorem. Given two probability measures μ and ν, the free convolution $\mu \boxplus \nu$ is defined as the distribution of $X + Y$ where X and Y are free self-adjoint operators of distributions μ and ν, respectively. The free central limit theorem states that if μ is a probability measure of zero mean and unit variance, then $D_{1/\sqrt{n}}\mu^{\boxplus n}$ converges in distribution to the semicircle distribution γ, where γ is given by the density $f_\gamma(t) = \sqrt{4-t^2}/2\pi$ for $t \in [-2, 2]$ and $D_c\mu$ stands for the dilation of a measure μ by a factor $c > 0$; this is $D_c\mu(B) = \mu(c^{-1}B)$ for all Borel sets $B \subset \mathbb{R}$.

Bercovici and Voiculescu [2] discovered that in the free central limit theorem the convergence is much stronger than in the classical case. They found that for a probability measure μ of bounded support the measure $D_{1/\sqrt{n}}\mu^{\boxplus n}$ becomes Lebesgue absolutely continuous, and its density $d\mu_n/dx$ converges uniformly to $d\gamma/dx$ on $\mathbb{R}$ as $n \to \infty$. Moreover, if $a_n = \inf(supp(D_{1/\sqrt{n}}\mu^{\boxplus n}))$ and $b_n =$

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$\sup(supp(D_{1/\sqrt{n}}\mu^{\boxplus n}))$, then $a_n \to -2$ and $b_n \to 2$. They named superconvergence to this type of strong convergence.

Many articles continued the study of the superconvergence phenomena. We next recall some of the main works. Wang [8] proved the superconvergence for unbounded measures. In this case, it is no longer true that the support becomes bounded in finite time. Now, for bounded measures, Arizmendi and Vargas [1] obtained the estimates $|a_n - (-2)| = O(n^{-1/2})$ and $|b_n - 2| = O(n^{-1/2})$ for the extremes $\{a_n, b_n\}$ of the support of $D_{1/\sqrt{n}}\mu^{\boxplus n}$. In a more general setting, Huang [5] studied the support of the measures in the free additive convolution semigroup $\{\mu^{\boxplus t} \mid t > 0\}$.

Chistyakov and Götze [3] gave an estimate of the density f_n of $D_{1/\sqrt{n}}\mu^{\boxplus n}$ for measures of finite fourth moment. They showed that $f_n(x + m_3(\mu)/\sqrt{n}) = v_n(x) + \rho_{n_1}(x) + \rho_{n_2}(x)$, where $v_n(x) = p_n(x) f_\gamma(e_n x)$, $p_n(x) \to 1$ and $e_n \to 1$ as $n \to \infty$, and the functions $|\rho_{n_1}(x)|$ and $|\rho_{n_2}(x)|$ converge to 0 as $n \to \infty$. As a consequence, they obtained that

$$\int_{\mathbb{R}} |f_n(x) - f_\gamma(x)| dx = 2|m_3(\mu)|/(\pi\sqrt{n}) + c(\mu)\theta((\epsilon_n/n)^{3/4} + 1/n), \tag{1}$$

where θ is a constant, $c(\mu)$ is a constant depending on μ, and ϵ_n is a sequence of positive numbers converging to 0. They also proved in [4] the free analogue of the Berry–Esseen theorem; if μ is a measure with $m_1(\mu) = 0$, $m_2(\mu) = 1$, and $\int |x|^3 d\mu < \infty$, then $\sup_{x\in\mathbb{R}} |D_{\frac{1}{\sqrt{n}}}\mu^{\boxplus n}(-\infty, x] - \gamma(-\infty, x]| = O(n^{-1/2})$.

Now, let μ be a probability measure of bounded support. The free cumulants $k_n(\mu)$ of μ, for $n = 1, 2, 3, ...$, are quantities (see below) that encode the free convolution and have analogue properties to the classical cumulants. In this note, we find that if μ has zero mean and unit variance, then the more vanishing free cumulants $k_n(\mu)$ for $n \geq 3$ there are, the faster the extremes of the support and the density of $D_{1/\sqrt{n}}\mu^{\boxplus n}$ converges. Next, we state our results and comment on their relationship with the above results.

Theorem 1 *Let μ be a probability measure such that $m_1(\mu) = 0$, $m_2(\mu) = 1$, and $supp(\mu) \subset [-K, K]$. Set $h = \min\{j \geq 3 : k_j(\mu) \neq 0\}$. Define $\mu_n = D_{1/\sqrt{n}}\mu^{\boxplus n}$. Let n be a large enough integer and denote by f_n the density of μ_n. Set $a_n = \inf(supp(\mu_n))$ and $b_n = \sup(supp(\mu_n))$. Then:*

(1) *if h is odd, then $a_n = -2 + \frac{k_h(\mu)}{n^{(h-2)/2}} + O(n^{-(h-1)/2})$ and $b_n = 2 + \frac{k_h(\mu)}{n^{(h-2)/2}} + O(n^{-(h-1)/2})$;*
(2) *if h is even, then $a_n = -2 - \frac{k_h(\mu)}{n^{(h-2)/2}} + O(n^{-(h-1)/2})$ and $b_n = 2 + \frac{k_h(\mu)}{n^{(h-2)/2}} + O(n^{-(h-1)/2})$.*

This theorem extends the result of Arizmendi and Vargas commented above. It implies that $|a_n - (-2)| = O(n^{-(h-2)/2})$ and $|b_n - 2| = O(n^{-(h-2)/2})$ for $h = \min\{j \geq 3 : k_j(\mu) \neq 0\}$. It also indicates that for n large enough, if h is odd and $k_h(\mu) > 0$, then $-2 < a_n < 2 < b_n$, and if $k_h(\mu) < 0$, then $a_n < -2 < b_n < 2$. But, if n is even and $k_h(\mu) > 0$, then $a_n < -2 < 2 < b_n$, and if $k_h(\mu) < 0$, then $-2 < a_n < b_n < 2$.

Our second result gives an estimate of the density of $D_{1/\sqrt{n}}\mu^{\boxplus n}$.

Theorem 2 *Under the assumptions of Theorem 1, we have*

$$|f_n(t) - f_\gamma(t)| \leq \begin{cases} \frac{2\delta_n}{\sqrt{4-t^2}} & \text{if } t \in [-2+2\delta_n, 2-2\delta_n], \\ 2\sqrt{\delta_n} & \text{if } t \in [-2+\delta_n, 2-\delta_n]^c \cap (a_n, b_n), \end{cases}$$

where $\delta_n = Cn^{-(h-2)/2}$ *and* C *is some constant that depends only on* μ.

As a consequence of this theorem and since $\int_2^2 \frac{1}{\sqrt{4-x^2}} < \infty$, we obtain a more concise estimate that the one in (1).

Corollary 1 *We have that*

$$\int_{\mathbb{R}} |f_n(x) - f_\gamma(x)| dx = O(n^{-(h-2)/2}).$$

In particular, this implies the following Berry–Esseen type estimate of the free central limit theorem for measures of bounded support:

$$\sup_{x\in\mathbb{R}} |\mu_n(-\infty, x] - \gamma(-\infty, x]| = O(n^{-(h-2)/2}).$$

We remark that in the classical case there is not improvement on the speed of convergence in the central limit theorem under the presence of vanishing cumulants, see [6].

The proofs of our results rely on in some observations derived from the proof of the superconvergence in the free central limit theorem [2] and in a basic expansion of the $R-$transform.

Apart from this introduction, the sections of this paper are organized as follows. In Sect. 2, we present the preliminary material and set the framework to prove our results. In Sect. 3, we prove Theorems 1 and 2. In the last section, we present some examples that illustrate our estimates.

2 Preliminaries

2.1 The R-transform and the Semicircle Distribution

In this subsection, we introduce the Cauchy transform and derive from it the R-transform, which is the free analogue of the cumulant generating function. We also give some properties of the semicircle distribution related to these transforms. Throughout the paper, z denotes a complex number, and we write $z = x + iy$, where x and y are real numbers. By $\mathbb{C}^+$ and $\mathbb{C}^-$ we denote the open upper and lower complex half-planes, respectively.

The Cauchy transform of a probability measure of bounded support μ is defined as

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{t-z} d\mu(t) \quad \text{for } z \in \mathbb{C}\backslash supp(\mu).$$

The function G_μ is holomorphic in $\mathbb{C}\backslash supp(\mu)$. It maps $\mathbb{C}^+$ to $\mathbb{C}^+$ and $\mathbb{C}^-$ to $\mathbb{C}^-$. In the following proposition, we collect some facts of the Cauchy transform commented before Definition 1 in [2].

Proposition 1 *Let μ be a Lebesgue absolutely continuous probability distribution such that $supp(\mu) \subset [a, b]$. Set $f(x) = d\mu(x)$ and assume that f is analytical on (a, b). Then, if we define G^+_μ in $\mathbb{C}^+ \cup (a, b)$ by setting $G^+_\mu \cong G_\mu$ over $\mathbb{C}^+$ and $G^+_\mu(x) = \lim_{y\downarrow 0} G_n(x)$ for $x \in (a, b)$, then G^+_μ is continuous in $\mathbb{C}^+ \cup (a, b)$ and $1/\pi Im(G^+_\mu(x)) = f(x)$. On the other hand, if we define G^-_μ in $\mathbb{C}^- \cup (a, b)$ by setting $G^-_\mu \cong G_\mu$ over $\mathbb{C}^-$ and $G^-_\mu(x) = \lim_{y\uparrow 0} G_n(x)$ for $x \in (a, b)$, then G^-_μ is continuous in $\mathbb{C}^- \cup (a, b)$ and $-1/\pi Im(G^-_\mu(x)) = f(x)$.*

The next proposition, see [7], studies the inverse of the Cauchy transform.

Proposition 2 *Let μ be a probability measure such that $supp(\mu) \subset [-K, K]$. Then:*

(1) *G_μ is $1-1$ on $\{z : |z| > 4K\}$;*
(2) *$\{z : 0 < |z| < 1/(6K)\} \subset \{G_\mu(z) : |z| > 4K\}$;*
(3) *there is a function R_μ, analytical in $\{z : 0 < |z| < 1/(6K)\}$, such that $G_\mu(R_\mu(z) + 1/z) = z$ for $z \in \{z : 0 < |z| < 1/(6K)\}$;*
(4) *$R_\mu(z)$ admits the power series expansion $R_\mu(z) = \sum_{n=1}^{\infty} k_n z^{n-1}$ for $|z| < 1/(6K)$, where $\{k_n\}$ are some real numbers.*

The function R_μ is called the R-transform of μ. Now, the function $K_\mu(z) = R_\mu(z) + 1/z$ is the functional inverse of G_μ as indicated in the proposition. The values $\{k_n\}$, $n \in \mathbb{N}$, are called the free cumulants of μ. They behave like the classical cumulants. In particular, they satisfy $k_i(D_a\mu) = a^i k_i(\mu)$.

Next, we discuss some well known properties of the semicircle distribution γ. Its R-transform is $\mathcal{R}_\gamma(z) = z$, and $K_\gamma(z) = z + 1/z$. The Cauchy transform $G_\gamma(w)$ for $w \in \mathbb{C}\backslash[-2, 2]$ can be obtained by solving the equation $z + 1/z = w$, which gives $G_\gamma(w) = (w - \sqrt{w^2 - 4})/2$, where the branch of the square root is chosen so that $\sqrt{w^2 - 4}$ is holomorphic in $\mathbb{C} \backslash [-2, 2]$. The next estimation is useful for our purposes.

Proposition 3 *Let $\delta \in (0, 1/9)$. Suppose $0 < |y| < \delta$ and $|x-2| < 3\delta$ or $|x+2| < 3\delta$. Then $|Im(G_\gamma(z))| < 3\sqrt{\delta}$.*

Proof We have that $|Im(G_\gamma(z))| \le y/2 + \sqrt{|z^2 - 4|}/2$. Note that $|z^2 - 4| < |x^2 - 4| + y^2 + 2|xy|$ and $|x| < 7/3$. It follows that $|x^2 - 4| + y^2 + 2|xy| < (3\delta)(4 + 3\delta) + \delta^2 + 14\delta/3$. Therefore, $|Im(G_\gamma(z))| < 1/2(\delta + \sqrt{19\delta})$, and the desired result follows. □

Consider G^+_γ as given in Proposition 1. It is not so difficult to see that for $x \in (-2, 2)$ we have $|G^+_\gamma(x)| = 1$ and $K_\gamma(G^+_\gamma(x)) = x$. At last, we present a technical result on how behaves $G_\gamma \circ K_\gamma$ on $\mathbb{C}^+$.

Proposition 4 *Let $z \in \mathbb{C}^+$. Set $w = K_\gamma(z)$. Then:*

(1) *if $|z| > 1$, then $G_\gamma(w) = z$;*

(2) *if* $|z| < 1$, *then* $G_\gamma(w) = 1/z$;
(3) *if* $|z| = 1$, *then* $w \in (-2, 2)$ *and* $G^+_\gamma(w) = z$.

Proof Recall that $G_\gamma(w)$ is a solution of the equation $x + 1/x = w$. By definition $z + 1/z = w$, thus the solutions of that equation are $\{z, 1/z\}$. Note that $1/z \in \mathbb{C}^-$ and $Im(K_\gamma(z)) = y - |y|/|z|^2$. From these observations we conclude 1) and 2) as follows. If $|z| > 1$, then $w \in \mathbb{C}^+$, so $G_\gamma(w) \in \mathbb{C}^+$. Thus, $G_\gamma(w) = z$. Similarly, if $|z| < 1$, then $w \in \mathbb{C}^-$, so $G_\gamma(w) \in \mathbb{C}^-$. Hence, $G_\gamma(w) = 1/z$. Now, for the case $|z| = 1$, we clearly have $w \in (-2, 2)$ as $Re(K_\gamma(z)) = x + |x|/|z|^2$. Set $w_y = K_\gamma(z + iy)$. Then, from the continuity of G^+_γ and 1), it follows that $G^+_\gamma(w) = \lim_{y\downarrow 0} G_\gamma(w_y) = \lim_{y\downarrow 0}(z + iy) = z$. □

2.2 Free Convolution for Measures of Bounded Support

In this subsection, we first characterize the free convolution in terms of the R-transform. Then, we obtain some estimates of the transforms defined above for the converging measure in the free central limit theorem.

Let μ_1 and μ_2 be probability measures of bounded support. Since the sum of two bounded operators is a bounded operator, then $\mu_1 \boxplus \mu_2$ has bounded support. Now, suppose that $supp(\mu_1) \subset [-r_1, r_1]$ and $supp(\mu_2) \subset [-r_2, r_2]$. Let $r = max\{r_1, r_2\}$. Then, the $R-$transform of $\mu_1 \boxplus \mu_2$ satisfies

$$\mathcal{R}_{\mu_1 \boxplus \mu_2}(z) = \mathcal{R}_{\mu_1}(z) + \mathcal{R}_{\mu_2}(z) \quad \text{for } z \in \{z \,:\, 0 < |z| < 1/(6r)\}, \tag{2}$$

where the free cumulants of $\mu_1 \boxplus \mu_2$ are given by

$$k_n(\mu \boxplus \nu) = k_n(\mu) + k_n(\nu) \quad \text{for } n \in \mathbb{N}. \tag{3}$$

Now, let μ be a probability measure such that $m_1(\mu) = 0$, $m_2(\mu) = 1$, and $supp(\mu) \subset [-r, r]$. In this case $k_1(\mu) = 0$ and $k_2(\mu) = 1$. Set $h = \min\{j \geq 3 \,:\, k_j(\mu) \neq 0\}$. Define $\mu_n = D_{1/\sqrt{n}}\mu^{\boxplus n}$. In the remainder of this subsection, we obtain some estimates of R_{μ_n} and K_{μ_n}. Note that $supp(D_{1/\sqrt{n}}\mu) \subset [-r/\sqrt{n}, r/\sqrt{n}]$. Hence, by Proposition 2 we have

$$R_{D_{1/\sqrt{n}}\mu}(z) = \frac{z}{n} + \frac{1}{z} + \sum_{j=h}^{\infty} \frac{k_j}{n^{j/2}} z^{j-1} \quad \text{for } |z| < \frac{\sqrt{n}}{6r}.$$

Equation (2) implies that $R_{\mu_n}(z) = nR_{D_{1/\sqrt{n}}\mu}(z)$ for $|z| < \frac{\sqrt{n}}{6r}$. Since $K_{\mu_n}(z) = R_{\mu_n}(z) + 1/z$, then

$$K_{\mu_n}(z) = z + \frac{1}{z} + \sum_{j=h}^{\infty} \frac{k_j}{n^{(j-2)/2}} z^{j-1} \quad \text{for } 0 < |z| < \frac{\sqrt{n}}{6r}. \tag{4}$$

Put $\xi(z) = \sum_{j=h}^{\infty} k_j n^{-(j-2)/2} z^{j-1}$ for $|z| < \frac{\sqrt{n}}{6r}$. Then for $|z| < m$ and large enough n and we have

$$|\xi(z)| = O(n^{-(h-2)/2}). \tag{5}$$

This can be seen from the properties of convergent power series and since $\xi(z)$ is analytical for $|z| < m$ because $\xi(z) = z^{(h-1)} n^{-(h-2)/2} \sum_{j=h}^{\infty} k_j (n^{-1/2} z)^{j-h}$ and $R_\mu(z) = z + z^{h-1} \sum_{j=h}^{\infty} k_j z^{j-h}$ is analytical for $|z| < 1/(6r)$.

Finally, we present a proposition with some properties of μ_n that are obtained in the proof of Theorem 3 in [2].

Proposition 5 *If $a_n = \inf(supp(\mu_n))$ and $b_n = \sup(supp(\mu_n))$, then:*

(1) *The derivative $dK_{\mu_n}(z)/dz$ has unique real zeros $x_1 < x_2$ in neighborhoods of -1 and 1. Moreover, $a_n = K_{\mu_n}(x_1)$ and $b_n = K_{\mu_n}(x_2)$;*
(2) *If G_n^+ is as in Proposition 1, then for $x \in (a_n, b_n)$ we have $G_n^+(x) \in \mathbb{C}^+$, $|G_n^+(x)| \approx 1$ and $K_{\mu_n}(G_n^+(x)) = x$.*

3 Proofs

3.1 Proof of Theorem 1

In order to estimate a_n and b_n, by Proposition 5 we need to find the unique zeros $x_1 < x_2$ of $dK_{\mu_n}(x)/dx$ that are near -1 and 1, and then we must estimate $a_n = K_{\mu_n}(x_1)$ and $b_n = K_{\mu_n}(x_2)$.

By the theory of Laurent series, we have

$$\frac{dK_{\mu_n}(z)}{dz} = 1 - \frac{1}{z^2} + \sum_{j=h}^{\infty} \frac{(j-1)k_j}{n^{(j-2)/2}} z^{j-2} \quad \text{for } r < |z| < \sqrt{n}/(6K),$$

for any $r > 0$. Since $1 - 1/z^2$ has zeros at 1 and -1, then from Rouche's theorem, we derive that there exist $\epsilon = O(n^{-(h-2)/2})$ and $\epsilon' = O(n^{-(h-2)/2})$ such that $-1+\epsilon$ and $1+\epsilon'$ are the roots of K_{μ_n}. Recall that

$$\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x} \quad \text{for } |x| < 1.$$

It follows that for n large enough

$$\begin{aligned} K_{\mu_n}(1+\epsilon') &= 1 + \epsilon' + \frac{1}{1+\epsilon'} + \sum_{j=h}^{\infty} \frac{k_j}{n^{(j-2)/2}} (1+\epsilon')^{j-1} \\ &= 1 + \epsilon' + 1 - \epsilon' + O((\epsilon')^2) + \frac{k_h(\mu)}{n^{(h-2)/2}} + O(n^{(h-1)/2}). \end{aligned}$$

Since $\epsilon' = O(n^{-(h-2)/2})$, then $b_n = 2 + k_h(\mu) n^{-(h-2)/2} + O(n^{-(h-1)/2})$.

On the other hand, we have that for n large enough

$$K_{\mu_n}(-1+\epsilon) = -1+\epsilon+\frac{1}{-1+\epsilon}+\sum_{j=h}^{\infty}\frac{k_j}{n^{(j-2)/2}}(-1+\epsilon)^{j-1}.$$

If n is odd, then

$$K_{\mu_n}(-1+\epsilon) = -1+\epsilon-1-\epsilon+O((\epsilon)^2)+\frac{k_h(\mu)}{n^{(h-2)/2}}+O(n^{-(h-1)/2}).$$

It follows that $a_n = -2+\frac{k_h(\mu)}{n^{(h-2)/2}}+O(n^{-(h-1)/2})$.

If n is even, then

$$K_{\mu_n}(-1+\epsilon) = -1+\epsilon-1-\epsilon+O((\epsilon)^2)-\frac{k_h(\mu)}{n^{(h-2)/2}}+O(n^{-(h-1)/2}).$$

In this case we have $a_n = -2-\frac{k_h(\mu)}{n^{(h-2)/2}}+O(n^{-(h-1)/2})$.

3.2 Proof of Theorem 2

By (5) and Theorem 1, there exists $C > 0$ such that for $\delta_n = Cn^{(h-2)/2}$ we have $|a_n+2| < \delta_n$, $|b_n-2| < \delta_n$, and $|\xi(z)| < \delta_n$ for $|z| < 3$.

Fix $x \in (a_n, b_n)$. By Proposition 5, $G_n^+(x) \in \mathbb{C}^+$ and $K_{\mu_n}(G_n^+(x)) = x$. Set $w = K_\gamma(G_n^+(x))$. Since $K_\gamma(G_n^+(x)) = K_{\mu_n}(G_n^+(x)) - \xi(G_n^+(x))$, then $|x-w| = |\xi(G_n^+(x))|$. Hence, $|x-w| < \delta_n$ as $|G_n^+(x)| \approx 1$.

First, suppose $|G_n^+(x)| > 1$. We have $G_\gamma(w) = G_n^+(x)$ according to Proposition 4. We claim that

$$|G_\gamma(w) - G_\gamma^+(x)| \le |w-x| \sup_{l^o(w,x)} |G_\gamma'(z)|, \tag{6}$$

where $l^o(a,b)$ denotes the straight line joining a with b without $\{a,b\}$. This follows by noticing that $G_\gamma(w) - G_\gamma(x+it) = \int_{l^o(w,(x+it))} G_\gamma'(z)dz$, and making $t \downarrow 0$.

Now, we have $G_\gamma'(z) = 1/2 - z/(2\sqrt{z^2-4})$ for $z \in \mathbb{C}\backslash[-2,2]$. It follows that $|G_\gamma'(z)| < 1/2 + 3/(2\sqrt{|z^2-4|}) < 3/\sqrt{4-x^2}$ for $x \in (-2,2)$ and $y \in (0,1)$. It is not so hard to see that for $\theta \in [-2+2\delta, 2-2\delta]$

$$\sup_{t\in([\theta-\delta,\theta+\delta])} 1/\sqrt{4-t^2} < 2/\sqrt{4-\theta^2}.$$

From (6) it follows that $|G_\gamma(w) - G_\gamma^+(x)| < 6\delta_n/\sqrt{4-x^2}$ for $x \in [-2+2\delta_n, 2-2\delta_n]$. Therefore, Proposition 1 implies that

$$|f_n(x) - f_\gamma(x)| = (1/\pi)|Im(G_\gamma(w) - G_\gamma^+(x))| < \frac{2\delta_n}{\sqrt{4-x^2}},$$

for $x \in [-2+2\delta_n, 2-2\delta_n]$.

Now, consider $x \in [-2+2\delta_n, 2-2\delta_n]^c \cap [a_n, b_n]$. We have that $|x-2| < 2\delta_n$ or $|x+2| < 2\delta_n$. Recall that $|f_n(x)| = (1/\pi)Im(G_\gamma(w))$. Since $|w-x| < \delta_n$ and $w \in \mathbb{C}^+$, then $0 < Im(w) < \delta_n$ and $|Re(w)-2| < 3\delta_n$ or $|Re(w)+2| < 3\delta_n$. Hence, by Proposition 3, $Im(G_\gamma(w)) < 3\delta_n$, so $|f_n(x)| < 2\sqrt{\delta_n}$.

Next, suppose $|G_n^+(x)| < 1$. Proposition 4 implies that $G_\gamma(w) = 1/G_n^+(x)$. By the same argument as before, we have

$$|G_\gamma(w) - G_\gamma^-(x)| \leq |w-x| \sup_{l(w,x)} |G'_\gamma(z)|,$$

and

$$|G_\gamma(w) - G_\gamma^-(x)| < \frac{6\delta_n}{\sqrt{4-x^2}}, \tag{7}$$

for $x \in [-2+2\delta_n, 2-2\delta_n]$. Now, Proposition 5 says that $K_{\mu_n}(G_n^+(x)) = x$, so

$$x = G_n(x) + \frac{1}{G_n(x)} + \sum_{j=h}^{\infty} \frac{k_j}{n^{(j-2)/2}} (G_n(x))^{j-1}.$$

Therefore, $Im(G_n^+(x)) = -Im(1/G_n^+(x)) + Im(\xi(G_n^+(x)))$. We conclude that

$$|Im(G_n^+(x)) + Im(1/G_n^+(x))| \leq |\xi(G_n^+(x))|. \tag{8}$$

By Proposition 1, $|f_n(x) - f_\gamma(x)| = (1/\pi)(|Im(G_n^+(x)) + Im(G_\gamma^-(x))|)$. Note that

$$|G_n^+(x) + G_\gamma^-(x)| < |G_n^+(x) + 1/G_n^+(x)| + |-(1/G_n^+(x)) + G_\gamma^-(x)|.$$

Hence, we derive from (7) and (8) that $|f_n(x) - f_\gamma(x)| < (1/\pi)(\delta_n + \frac{6\delta_n}{\sqrt{4-x^2}})$ for $x \in [-2+2\delta_n, 2-2\delta_n]$. We conclude that $|f_n(x) - f_\gamma(x)| < \frac{2\delta_n}{\sqrt{4-x^2}}$ for $x \in [-2+2\delta_n, 2-2\delta_n]$.

Now, consider $x \in [-2+2\delta_n, 2-2\delta_n]^c \cap [a_n, b_n]$. Using the same ideas as above and Proposition 3, we obtain that $Im(G_\gamma(w)) < 3\sqrt{\delta_n}$. Since $|f_n(x)| = (1/\pi)|Im(G_n^+(x))|$ and $G_\gamma(w) = 1/G_n^+(x)$, then, by (8) we have $|f_n(x)| < (1/\pi)(\delta_n + 3\sqrt{\delta_n})$, which is less than $2\sqrt{\delta_n}$ for δ_n relatively small.

Finally, suppose $|G_n^+(x)| = 1$. Then, by Proposition 4, $w \in (-2, 2)$ and $G_\gamma^+(w) = G_n^+(x)$. From Proposition 1 we get that $(1/\pi)|Im(G_\gamma^+(w) - G_\gamma^+(x))| = |f_\gamma(w) - f_\gamma(x)|$. Note that

$$|f_\gamma(w) - f_\gamma(x)| \leq (1/\pi)|w-x| \sup_{l^o(w,x)} |t/\sqrt{4-t^2}|,$$

so by arguments as above, $|f_\gamma(w) - f_\gamma(x)| \leq 2\delta_n/\sqrt{4-x^2}$ for $x \in [-2+2\delta_n, 2-2\delta_n]$. Therefore, $|f_n(x) - f_\gamma(x)| \leq 2\delta_n/\sqrt{4-x^2}$ for $x \in [-2+2\delta_n, 2-2\delta_n]$.

Now, suppose that $x \in [-2+2\delta_n, 2-2\delta_n]^c \cap [a_n, b_n]$. Hence, $|x-2| < 2\delta_n$ or $|x+2| < 2\delta_n$. Recall that $f_n(x) = f_\gamma(w) = (1/\pi)\sqrt{4-w^2}$. Since $|w-x| < \delta_n$, then $w-2 < 3\delta_n$ or $w+2 < 3\delta_n$. Thus, $|f_n(x)| < 2\sqrt{\delta_n}$.

4 Examples

The following example is from Proposition 2.5 in [4]. Recall that for measures of zero mean and unit variance we have $k_3(\mu) = m_3(\mu)$ and $k_4(\mu) = m_4(\mu) - 2$. Let $\mu = q\delta_x + p\delta_y$, where $0 < p < 1$, $p+q = 1$, $x = -\sqrt{p/q}$, and $y = \sqrt{q/p}$. Define $\mu_n = D_{1/\sqrt{n}}\mu^{\boxplus n}$. We can easily verify that $m_3(\mu) = (q-p)/\sqrt{pq}$. Set $\alpha = m_3(\mu)$. Then

$$\frac{d\mu_n}{dx} = \frac{\sqrt{(x-x_1)(x-x_2)}}{1 - x(x/n - \alpha/\sqrt{n})} \quad \text{for } x_1 < x < x_2,$$

where $x_1 = \alpha/\sqrt{n} - 2\sqrt{1-1/n}$ and $x_2 = \alpha/\sqrt{n} + 2\sqrt{1-1/n}$. Since $\sqrt{1+c/n} = 1 + c/(2n) + O(n^{-2})$, then $x_1 = -2 + k_3(\mu)/\sqrt{n} + O(n^{-1})$ and $x_2 = 2 + k_3(\mu)/\sqrt{n} + O(n^{-1})$. Now, when $p = q$, we have $m_3(\mu) = 0$ and $m_4(\mu) = 1$, so $k_4(\mu) = -1$ By the previous formula, we have $x_1 = -2\sqrt{1-1/n} = -2 + 1/n + O(n^{-2})$ and $x_2 = 2\sqrt{1-1/n} = 2 - 1/n + O(n^{-2})$. Hence, $x_1 = -2 - k_4(\mu)/n + O(1/n^2)$ and $x_2 = 2 + k_4(\mu)/n + O(1/n^2)$. Therefore, these estimates agree with Theorem 1.

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Declarations

Conflict of interest The authors declare no conflict of interest.

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