Complex Analysis and Operator Theory (2023) 17:71
https://doi.org/10.1007/s11785-023-01369-1

Complex Analysis and Operator Theory

Orthogonal Harmonic and Quaternionic Monogenic Functions in the Exterior of a Spheroid

J. Morais[1]

Received: 16 November 2022 / Accepted: 31 May 2023 / Published online: 29 June 2023
© The Author(s), under exclusive licence to Springer Nature Switzerland AG 2023

Abstract

We propose a single one-parameter family of orthogonal harmonic functions expressed in terms of spheroidal coordinates as independent variables to construct a common orthogonal basis for the L_2-Hilbert spaces of quaternionic monogenic functions in the space exterior of a spheroidal domain (either prolate or oblate). We give recurrence relations for the elements that constitute such a basis, which are particularly easy to handle from a computational point of view. Conversion formulas among the classes of harmonic and monogenic functions associated with a spheroid of arbitrary eccentricity to those related to the Euclidean ball are derived.

Keywords Quaternionic analysis · Associated Legendre functions of the first and the second kinds · Neumann's formula · Spheroidal harmonics · Spheroidal monogenics

1 Introduction

The theory of *monogenic* (or Fueter regular) functions of a vector variable in a domain in three-dimensional Euclidean space, taking values in the space of quaternions, has a wide range of applications. A significant part of the theory of monogenic functions has been built around the study of quaternionic counterparts of holomorphic functions of one complex variable, offering a refinement of classical harmonic analysis in three and four dimensions. Said sort depends on whether monogenic functions present certain peculiarities, such as continuity, differentiability or integrability, orthogonality with

---

Communicated by Roman Lavicka.

This article is part of the topical collection "Higher Dimensional Geometric Function Theory and Hypercomplex Analysis" edited by Irene Sabadini, Michael Shapiro and Daniele Struppa.

✉ J. Morais
joao.morais@itam.mx

[1] Department of Mathematics, ITAM, Río Hondo #1, Col. Progreso Tizapán, 01080 Mexico City, Mexico

respect to different inner products, and so on, throughout the domain of the variables. The study of the fundamental properties of monogenic functions is linked to harmonic functions through the Riesz and Moisil-Teodorescu systems of first-order constant coefficient partial differential equations generalizing the Cauchy–Riemann equations [5, 6, 10–13, 26–29].

The original impetus in initiating the investigation of orthogonal bases spanning the Hilbert spaces of square-integrable harmonic and monogenic functions defined in the interior of spheroidal (resp. prolate and oblate) domains of the form

$$\left\{\mathbf{x} \in \mathbb{R}^3 : \frac{x_0^2}{\cosh^2\alpha} + \frac{x_1^2 + x_2^2}{\sinh^2\alpha} = 1\right\}$$

and

$$\left\{\mathbf{x} \in \mathbb{R}^3 : \frac{x_0^2}{\sinh^2\alpha} + \frac{x_1^2 + x_2^2}{\cosh^2\alpha} = 1\right\}$$

for $\alpha > 0$ was proposed in [7], and [16, 17, 23]. These (confocal) domains become rounder as they degenerate with $\alpha \to \infty$ (since $\tanh\alpha \to 1$). In [8], the spheroidal harmonics were defined following [7], with a rescaling factor that permits including the Euclidean ball to limit both the prolate and oblate cases. We refer to [9] and [20] concerning further properties of spheroidal harmonic and monogenic polynomials. These works do not include the harmonics and monogenics vanishing at infinity, which are perhaps the more fascinating classes from the point of view of a physical application. Although such types of functions have not yet been used to any great extent in mathematical physics, from the perspective of the theory of the solution of Laplace's equation in three variables, it would be interesting to address the problem of constructing orthogonal sets of harmonic and monogenic functions defined in a region outside a spheroid, whose elements are parametrized by the shape of the corresponding spheroids. In this sense, we propose a single one-parameter family of orthogonal spheroidal harmonics to build a common orthogonal basis for the Hilbert spaces of square-integrable monogenic functions in the space exterior of a spheroidal domain of arbitrary eccentricity. To the best of our knowledge, these ideas seem to be new.

The outline of the paper is as follows. Section 3 employs the two kinds of associated Legendre functions to construct the basic external spheroidal harmonics that assume prescribed values on the boundary of the corresponding spheroids, combined into a single one-parameter family. These functions are shown to include the ordinary solid spherical, prolate, and oblate spheroidal harmonics as limiting cases. The orthogonality of the basic harmonics is taken with respect to two natural inner products, leading to the discussion of the proper external spheroidal harmonics. The main difficulties of this investigation will center on analyzing these functions. Conversion formulas that relate the coefficients of the expansions among the spheroidal and spherical harmonic systems are obtained. The basic external spheroidal monogenic functions are calculated in Sect. 4, and explicit formulas for their nonscalar parts are obtained in terms of the proper harmonics. We prove that these functions form a common orthogonal basis for the one-parameter family of L_2-Hilbert spaces of monogenic functions defined in a region outside a spheroid. Besides, we show the corresponding orthogonality of the spheroidal monogenics over the surface of the spheroids with respect to a suitable weight function.

2 Notation and Preliminaries

The majority of functions used in technical and applied mathematics originated from investigating practical problems. A relevant example is the *Ferrers' associated Legendre functions* of the first and the second kinds, $P_n^m(z)$ and $Q_n^m(z)$, of degree n and order m, for $z \in [-1, 1]$ and $z \in (1, \infty)$. When n and m are nonnegative integers, these functions are defined by (cf. [14, Ch. III])

$$P_n^m(t) = (-1)^m (1 - t^2)^{m/2} \frac{d^m P_n(t)}{dt^m}, \quad t \in [-1, 1],$$

and

$$Q_n^m(s) = (s^2 - 1)^{m/2} \frac{d^m Q_n(s)}{ds^m}, \quad |s| > 1,$$

where

$$P_n(t) = \frac{1}{2^n n!} \frac{d^n}{dt^n}(t^2 - 1)^n, \quad t \in \mathbb{R},$$

is the Legendre polynomial (or Legendre function of the first kind), and

$$Q_n(s) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(t)}{s - t} \, dt, \quad |s| > 1, \tag{1}$$

is the Legendre function of the second kind. The relation (1) is known as *Neumann's integral formula* [22, p. 24] (cf. [14, p. 63]).

We shall observe a slight difference between the variation of the index m in any of the previous definitions. Although the $P_n^m(t)$ are only defined for nonnegative integer values of m, which are equal to or less than n, the functions $Q_n^m(s)$ are defined for all nonnegative integer values of m.

In addition to the definitions introduced above, we further have from [14, p. 108]:

Lemma 1 *Let n and m be nonnegative integers, and let $|s| > 1$. Then*

$$Q_n^m(s) = \frac{(-1)^m (n+m)! (s^2 - 1)^{m/2}}{2^{n+1} (1/2)_{n+1} \, s^{n+m+1}} \, {}_2F_1\left(\frac{n+m+2}{2}, \frac{n+m+1}{2}; n + \frac{3}{2}; \frac{1}{s^2}\right). \tag{2}$$

Here ${}_2F_1$ is the usual notation for the classical Gaussian hypergeometric function and the Pochhammer symbol is $(a)_n = a(a+1)\cdots(a+n-1)$ with $(a)_0 = 1$ by convention.

The associated Legendre functions of the first and the second kinds are defined for negative integer order m by

$$\begin{cases} P_n^{-m}(t) = (-1)^m \dfrac{(n-m)!}{(n+m)!}\, P_n^m(t), \quad t \in [-1, 1], \\ Q_n^{-m}(s) = \dfrac{(n-m)!}{(n+m)!}\, Q_n^m(s), \quad |s| > 1, \end{cases} \tag{3}$$

where it is supposed that $n \geq m \geq 0$.

There is a classical formula that expresses the product of two associated Legendre functions of the second kind in terms of an associated Legendre function of the same type [3].

Proposition 1 *Let* $A_r^m = (1/2 + m)_r$. *Define*

$$\alpha_{n_1,n_2}^{m,r} = \frac{(n_1+m+r)!(n_2+m+r)!(n_1+n_2+2r+1)!}{r!(n_1+n_2+2m+2r+1)!(n_1+n_2+r+1)!} \times \left(\frac{2n_1+2n_2+2m+4r+3}{2n_1+2n_2+2m+2r+3}\right)\left(\frac{A_r^m A_{n_1+n_2+m+r+2}^0}{A_{n_1+r+1}^0 A_{n_2+r+1}^0}\right).$$

For nonnegative integers n_1, n_2, *and* m, *the following relation holds:*

$$(s^2-1)^{-m/2} Q_{n_1}^m(s) Q_{n_2}^m(s) = (-2)^m \sum_{r=0}^{\infty} \alpha_{n_1,n_2}^{m,r}\, Q_{n_1+n_2+m+2r+1}^m(s). \tag{4}$$

We need the following preliminary result.

Lemma 2 *For all nonnegative integers* n, m, $Q_n^m(s) Q_{n+2}^m(s) > 0$ *in* $|s| > 1$. *Further, let* $c > 1$ *be a fixed real constant. Then the following identity holds:*

$$\begin{aligned} \int_c^{\infty} & Q_n^m(s) Q_{n+2}^m(s)\, ds \\ &= (-1)^m\, 4^{n+m+3/2} \sum_{r=0}^{\infty}\sum_{k=0}^{\infty} \alpha_{n,n+2}^{m,r} \frac{4^r (2n+m+2r+2k+5)_{m-1}}{2(n+m+r+k+3/2)}\, c^{-2(m+r+k+1)} \\ &\quad \times \frac{(2n+m+2r+2k+4)!(2n+m+2r+k+3)!}{k!(4n+2m+4r+2k+7)!} \\ &\quad \times {}_2F_1\left(-(n+m+r+k+3/2), -m; -(n+m+r+k+1/2); c^2\right). \end{aligned} \tag{5}$$

Here $\alpha_{n_1,n_2}^{m,r}$ *has the same meaning as in Proposition* 1.

Proof The statements follow by combining (4) with the identity [24, p. 123]

$$Q_n^m(s) = (-1)^m (s^2-1)^{m/2} \sum_{k=0}^{\infty} \frac{2^n (n+2k+1)!(n+k)!(n+2k+2)_{m-1}}{k!(2n+2k+1)!\, s^{n+m+2k+1}}.$$

□

We now consider the recurrence formulas for the associated Legendre functions of the first and the second kinds, which will be used in the forthcoming sections [14, 30].

Proposition 2 1. *Let $n \geq 0$ and $0 \leq m \leq n$, and let $t \in [-1, 1]$. Then*

$$(1-t^2)(P_{n+1}^m)'(t) = (n+1+m)P_n^m(t) - (n+1)tP_{n+1}^m(t), \tag{6}$$

$$(n+1-m)P_{n+1}^m(t) = (2n+1)tP_n^m(t) - (n+m)P_{n-1}^m(t), \tag{7}$$

$$(t^2-1)(P_{n+1}^m)'(t) = (1-t^2)^{1/2}P_{n+1}^{m+1}(t) + mtP_{n+1}^m(t), \tag{8}$$

$$2mtP_{n+1}^m(t) = -(1-t^2)^{1/2}\Big(P_{n+1}^{m+1}(t) + (n+1+m)(n+2-m)P_{n+1}^{m-1}(t)\Big), \tag{9}$$

$$(1-t^2)^{1/2}P_n^{m+1}(t) = (n-m)tP_n^m(t) - (n+m)P_{n-1}^m(t), \tag{10}$$

$$2mP_{n+1}^m(t) = -(1-t^2)^{1/2}\Big(P_n^{m+1}(t) + (n+m)(n+m+1)P_n^{m-1}(t)\Big), \tag{11}$$

$$(1-t^2)^{1/2}P_{n+1}^m(t) = \frac{1}{2n+3}\Big(-P_{n+2}^{m+1}(t) + P_n^{m+1}(t)\Big). \tag{12}$$

2. *Let n and m be nonnegative integers, and let $|s| > 1$. Then*

$$(1-s^2)(Q_{n+1}^m)'(s) = (n+1+m)Q_n^m(s) - (n+1)sQ_{n+1}^m(s), \tag{13}$$

$$(n+1-m)Q_{n+1}^m(s) = (2n+1)sQ_n^m(s) - (n+m)Q_{n-1}^m(s), \tag{14}$$

$$(s^2-1)(Q_{n+1}^m)'(s) = (s^2-1)^{1/2}Q_{n+1}^{m+1}(s) + msQ_{n+1}^m(s), \tag{15}$$

$$2msQ_{n+1}^m(s) = (s^2-1)^{1/2}\Big(-Q_{n+1}^{m+1}(s) + (n+1+m)(n+2-m)Q_{n+1}^{m-1}(s)\Big), \tag{16}$$

$$(s^2-1)^{1/2}Q_n^{m+1}(s) = (n-m)sQ_n^m(s) - (n+m)Q_{n-1}^m(s), \tag{17}$$

$$2mQ_{n+1}^m(s) = (s^2-1)^{1/2}\Big(-Q_n^{m+1}(s) + (n+m)(n+m+1)Q_n^{m-1}(s)\Big), \tag{18}$$

$$(s^2-1)^{1/2}Q_{n+1}^m(s) = \frac{1}{2n+3}\Big(Q_{n+2}^{m+1}(s) - Q_n^{m+1}(s)\Big). \tag{19}$$

3 Solutions of Laplace's Equation in Spheroidal Coordinates

This section considers the problem of finding single one-parameter families of harmonic functions applicable to the space exterior of a spheroid, with particular emphasis on those orthogonal in the L_2-Hilbert space structure. This cannot be done with models where the Euclidean ball is only treated as a degenerate case [7, 14]. It requires a separate yet utterly analogous treatment for prolate and oblate spheroids. The construction of external harmonics becomes much more complicated than internal harmonics since they contain logarithmic functions.

3.1 Basic External Spheroidal Harmonics

The starting point of the present investigation is a result previously published by the author [21]. Consider the nested family domains, bounded by *coaxial spheroids* scaled so that the major axis is of length 2:

$$\Omega_\mu = \left\{ \mathbf{x} \in \mathbb{R}^3 \colon x_0^2 + \frac{x_1^2 + x_2^2}{1-\mu^2} = 1 \right\}. \tag{20}$$

(Confocal spheroids are often used, which differ by a change of scale depending on μ.) The parameter μ denotes the eccentricity of Ω_μ, which by convention is in the interval $(0, 1)$ (prolate spheroid) or in $i\mathbb{R}^+$ (oblate spheroid). The intermediate value $\mu = 0$ gives the Euclidean unit ball

$$\Omega_0 = \{\mathbf{x} \in \mathbb{R}^3 \colon |\mathbf{x}|^2 < 1\}.$$

Suppose for the moment that $\mu \in (0, 1)$. In this way, $\Omega_\mu^* := \mathbb{R}^3 \backslash \overline{\Omega}_\mu$ (where $\overline{\Omega}_\mu$ denotes the closure of Ω_μ) is parametrized using *prolate spheroidal coordinates* $(\eta, \vartheta, \varphi)$, corresponding to the family (20), which are related to Cartesian coordinates by

$$x_0 = \mu \cosh\eta \cos\vartheta, \quad x_1 = \mu \sinh\eta \sin\vartheta \cos\varphi, \quad x_2 = \mu \sinh\eta \sin\vartheta \sin\varphi, \tag{21}$$

with $\eta \in [\eta_\mu, \infty)$, $\vartheta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$, where the boundary value η_μ is given by

$$\mu \cosh\eta_\mu = 1. \tag{22}$$

By equations (21), direct computation shows that

$$|\mathbf{x}|^2 + \mu^2 = \mu^2(\cosh^2\eta + \cos^2\vartheta),$$

and also

$$\mu^2 (\cosh\eta \pm \cos\vartheta)^2 = (x_0 \pm \mu)^2 + x_1^2 + x_2^2.$$

Hence

$$\cosh\eta = \frac{\omega(\mu)}{2\mu}, \quad \cos\vartheta = \frac{2x_0}{\omega(\mu)},$$

where

$$\omega(\mu) := \left((x_0+\mu)^2 + x_1^2 + x_2^2\right)^{1/2} + \left((x_0-\mu)^2 + x_1^2 + x_2^2\right)^{1/2} \tag{23}$$

is positive. In the considerations to follow, we will often omit the argument of (23) and write ω instead of $\omega(\mu)$.

The oblate case is obtained by (21) via analytic continuation using $\widetilde{\eta} = \eta - i\pi/2$, thinking of $\mu \in i\mathbb{R}^+$ as being boundary values of the first quadrant in the complex plane. The following terms

$$\zeta(\mu, \mathbf{x}) = |\mathbf{x}|^2 + \mu^2 + 2x_0\mu, \quad \overline{\zeta}(\mu, \mathbf{x}) = |\mathbf{x}|^2 + \mu^2 - 2x_0\mu$$

inside the radicals in (23) are now complex conjugates, where

$$|\zeta(\mu, \mathbf{x})| = \left|\mu^2 - \left(x_0 + i\,(x_1^2 + x_2^2)^{1/2}\right)^2\right| \tag{24}$$

equals to the product of the distances from any point on the prescribed spheroids Ω_μ to the two foci $(\pm\mu, 0, 0)$. The function defined by (24) will play an important role in the forthcoming sections.

Hence, when $\mu \in i\mathbb{R}^+$, Ω^*_μ is parametrized by

$$x_0 = \frac{\mu}{i} \sinh \widetilde{\eta} \cos \vartheta, \quad x_1 = \frac{\mu}{i} \cosh \widetilde{\eta} \sin \vartheta \cos \varphi, \quad x_2 = \frac{\mu}{i} \cosh \widetilde{\eta} \sin \vartheta \sin \varphi,$$

where the coordinates range over $\widetilde{\eta} \in [\widetilde{\eta}_\mu, \infty)$ with $(\mu/i) \sinh \widetilde{\eta}_\mu = 1$, $\vartheta \in [0, \pi]$, and $\varphi \in [0, 2\pi)$.

The external harmonics to be employed in the sequel are defined as follows.

Definition 1 Let $n \geq 0$ and $0 \leq m \leq n$. For $\mathbf{x} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, the basic external spheroidal harmonics of degree $-(n+1)$ and order m are

$$U^{\pm}_{n,m}[\mu](\mathbf{x}) = U_{n,m}[\mu](\eta, \vartheta)\Phi^{\pm}_m(\varphi), \tag{25}$$

where $\Phi^+_m(\varphi) := \cos(m\varphi)$, $\Phi^-_m(\varphi) := \sin(m\varphi)$, and for $\mu \neq 0$,

$$U_{n,m}[\mu](\eta, \vartheta) = \frac{\beta_{n,m}}{\mu^{n+1}}\, P^m_n(\cos \vartheta)\, Q^m_n(\cosh \eta) \tag{26}$$

with

$$\beta_{n,m} = \frac{2^{n+1}(1/2)_{n+1}}{(n+m)!}. \tag{27}$$

The functions $U^-_{n,0}[\mu]$ vanish identically, as do all $U^{\pm}_{n,m}[\mu]$ for $m > n$. Therefore when we refer to the set $\{U^+_{n,m}[\mu],\ U^-_{n,m}[\mu]\}$, we always exclude the indices which apply to these trivial cases, even when we do not explicitly state $0 \leq m \leq n$ for the "$+$" case and $1 \leq m \leq n$ for the "$-$" case.

The basic harmonics (25), except for the constant factors $\beta_{n,m}$, and the rescaling of the $\mathbf{x}$ variable, are the functions defined in [14, Ch. X]. The motivation behind the choice for redefining these functions is explained in Proposition 3 below.

Table 1 Basic external spheroidal harmonic functions of degree $-(n+1) = -1, -2, -3$

n	m	$U^{\pm}_{n,m}[\mu] \quad (\tau = \frac{\omega + 2\mu}{\omega - 2\mu})$
0	0	$U^{+}_{0,0}[\mu] = \frac{\log(\tau)}{2\mu}$
1	0	$U^{+}_{1,0}[\mu] = \frac{3x_0\,(-4\mu + \omega\log(\tau))}{2\mu^3\omega}$
	1	$U^{+}_{1,1}[\mu] = \frac{3x_1(-4x_0^2+\omega^2)^{1/2}(-4\mu^2+\omega^2)^{1/2}\left(-4\mu\omega + (-4\mu^2+\omega^2)\log(\tau)\right)}{8\mu^3(x_1^2+x_2^2)^{1/2}(4\mu^2\omega-\omega^3)}$
		$U^{-}_{1,1}[\mu] = \frac{3x_2(-4x_0^2+\omega^2)^{1/2}(-4\mu^2+\omega^2)^{1/2}\left(-4\mu\omega + (-4\mu^2+\omega^2)\log(\tau)\right)}{8\mu^3(x_1^2+x_2^2)^{1/2}(4\mu^2\omega-\omega^3)}$
2	0	$U^{+}_{2,0}[\mu] = \frac{15(-12x_0^2+\omega^2)\left(12\mu\omega + (4\mu^2-3\omega^2)\log(\tau)\right)}{64\mu^5\omega^2}$
	1	$U^{+}_{2,1}[\mu] = \frac{15x_0x_1(-4x_0^2+\omega^2)^{1/2}\left(-32\mu^3+12\mu\omega^2-3\omega(-4\mu^2+\omega^2)\log(\tau)\right)}{8\mu^5\omega^2(x_1^2+x_2^2)^{1/2}(-4\mu^2+\omega^2)^{1/2}}$
		$U^{-}_{2,1}[\mu] = \frac{15x_0x_2(-4x_0^2+\omega^2)^{1/2}\left(-32\mu^3+12\mu\omega^2-3\omega(-4\mu^2+\omega^2)\log(\tau)\right)}{8\mu^5\omega^2(x_1^2+x_2^2)^{1/2}(-4\mu^2+\omega^2)^{1/2}}$
	2	$U^{+}_{2,2}[\mu] = -\frac{15(x_1^2-x_2^2)(-4x_0^2+\omega^2)\left(80\mu^3\omega-12\mu\omega^3+3(-4\mu^2+\omega^2)^2\log(\tau)\right)}{64\mu^5\omega^2(x_1^2+x_2^2)(4\mu^2-\omega^2)}$
		$U^{-}_{2,2}[\mu] = -\frac{15x_1x_2(-4x_0^2+\omega^2)\left(80\mu^3\omega-12\mu\omega^3+3(-4\mu^2+\omega^2)^2\log(\tau)\right)}{32\mu^5\omega^2(x_1^2+x_2^2)(4\mu^2-\omega^2)}$

Some examples of (25) in low degree are exhibited in Table 1.

The coefficients $\beta_{n,m}$ in the expression (26) is for the following.

Proposition 3 *For all* $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$, *the limit* $\lim_{\mu\to 0} U^{\pm}_{n,m}[\mu](\mathbf{x})$ *exists and is given by the external solid spherical harmonics*

$$U^{\pm}_{n,m}[0](\mathbf{x}) = \frac{1}{|\mathbf{x}|^{n+1}} P^m_n\left(\frac{x_0}{|\mathbf{x}|}\right)\Phi^{\pm}_m(\varphi), \tag{28}$$

where we employ spherical coordinates $x_0 = \rho\cos\theta$, $x_1 = \rho\sin\theta\cos\varphi$, *and* $x_2 = \rho\sin\theta\sin\varphi$.

Proof To prove this, we note that since the variable φ in (25) does not depend on the variable x_0, we examine the factors $P^m_n(2x_0/\omega)Q^m_n(\omega/(2\mu))$ in (26) with ω given by (23). Bearing in mind that

$$\left((x_0 \pm \mu)^2 + x_1^2 + x_2^2\right)^{1/2} = |\mathbf{x}| \pm \frac{x_0}{|\mathbf{x}|}\mu + O(\mu^2),$$

it follows that $\omega = 2|\mathbf{x}| + O(\mu^2)$ as $\mu \to 0$. Furthermore, we have once more from (23) that $2x_0/\omega = x_0/|\mathbf{x}| + O(\mu)$, so $P^m_n(2x_0/\omega) \to P^m_n(x_0/|\mathbf{x}|)$ as $\mu \to 0$. According to formula (2), direct calculation gives $\beta_{n,m}Q^m_n(s) \simeq 1/s^{n+1}$ as $s = \omega/2\mu$ tends to infinity, corresponding to $\mu \to 0$ for fixed $\mathbf{x}$. This establishes the statement. □

By the proposition just proved, it is observed that the external solid spherical harmonics (28) are embedded in the one-parameter family of basic external spheroidal harmonics. In contrast, in treatments such as [7, 14], the external harmonics degenerate as the eccentricity of the spheroid decreases.

It is clear that, unlike $U^{\pm}_{n,m}[0](\mathbf{x})$, the functions $U^{\pm}_{n,m}[\mu](\mathbf{x})$ are generally not homogeneous when $\mu \neq 0$.

3.2 Further Properties of the Basic Harmonics

We study the orthogonality of the external harmonics (25) with respect to two natural inner products.

Consider the Dirichlet inner product defined by

$$(f, g)_\mu = \iint_{\eta=\eta_\mu} f\,\frac{\partial g}{\partial \mathbf{n}}\,d\sigma, \quad (\Delta g = 0), \tag{29}$$

where $d\sigma$ denotes the area element on $\Gamma_\mu := \partial\Omega_\mu$, and

$$\mathbf{n} = \frac{1}{(1-\mu^2\cos^2\vartheta)^{1/2}}\left((1-\mu^2)^{1/2}\cos\vartheta + (\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi)\sin\vartheta\right) \tag{30}$$

is the unit outward normal vector to Γ_μ at the point $P^* := (\eta_\mu, \vartheta, \varphi)$, with $\cosh\eta_\mu = 1/\mu$ by (22).

Since the measure on the boundary is $d\sigma = (1-\mu^2)^{1/2}\sin\vartheta\, d\vartheta\, d\varphi$, we can compute the outward normal derivative $\partial/\partial\mathbf{n}$ of g at any point using (30) to obtain

$$(f,g)_\mu = \int_0^\pi \int_0^{2\pi} f(P^*)\,\frac{\partial g}{\partial \eta}(P^*)\,(1-\mu^2)^{1/2}\sin\vartheta\; d\varphi d\vartheta. \tag{31}$$

We will now show the orthogonality of the external harmonics (25) in the sense of the integral (29).

Proposition 4 *Let μ be fixed. For each $n \geq 0$, the collection*

$$\{U^+_{n,m}[\mu]\colon 0 \leq m \leq n\} \cup \{U^-_{n,m}[\mu]\colon 1 \leq m \leq n\} \tag{32}$$

is orthogonal in the sense of the Dirichlet integral (29) and their norms squared are equal to

$$\begin{aligned}\|U^{\pm}_{n,m}[\mu]\|_2^2 = \frac{(\beta_{n,m})^2}{\mu^{2n+1}} &\left(\frac{2\pi(1+\delta_{0,m})(n+m)!}{(2n+1)(n-m)!}\right)\\ &\times Q_n^m(1/\mu)\Big((1-\mu^2)^{1/2}\, Q_n^{m+1}(1/\mu) + m\, Q_n^m(1/\mu)\Big),\end{aligned}$$

where the coefficients $\beta_{n,m}$ have the same meaning as in (27). We use the symbol: $\delta_{m_1,m_2} = 0$ or 1, according as $m_1 \neq m_2$, or $m_1 = m_2$.

Proof For the sake of simplicity in the proof, we assume that $\mu \in (0,1)$ because the case $\mu \in i\mathbb{R}^+$ is similar. When $m_1 \neq m_2$, we have by the orthogonality of the set $\{\Phi^+_{m_1}, \Phi^-_{m_2} \mid m_1 \geq 0, m_2 \geq 1\}$ on $[0, 2\pi]$,

$$\begin{aligned}(U_{n_1,m_1}[\mu]\Phi^+_{m_1}(\varphi), U_{n_2,m_2}[\mu]\Phi^+_{m_2}(\varphi))_\mu &= 0,\\ (U_{n_1,m_1}[\mu]\Phi^-_{m_1}(\varphi), U_{n_2,m_2}[\mu]\Phi^-_{m_2}(\varphi))_\mu &= 0,\\ (U_{n_1,m_1}[\mu]\Phi^+_{m_1}(\varphi), U_{n_2,m_2}[\mu]\Phi^-_{m_2}(\varphi))_\mu &= 0,\\ (U_{n_1,m_1}[\mu]\Phi^-_{m_1}(\varphi), U_{n_2,m_2}[\mu]\Phi^+_{m_2}(\varphi))_\mu &= 0.\end{aligned}$$

According to (31), for $m_1 = m_2 = m$, a direct computation shows that

$$\begin{aligned}&(U_{n_1,m}[\mu]\Phi^+_m(\varphi), U_{n_2,m}[\mu]\Phi^+_m(\varphi))_\mu\\ &= \int_0^\pi \int_0^{2\pi} U_{n_1,m}[\mu](P^*)\,\frac{\partial U_{n_2,m}[\mu]}{\partial\eta}(P^*)\,(\Phi^+_m(\varphi))^2(1-\mu^2)^{1/2}\sin\vartheta\; d\varphi d\vartheta\\ &= \frac{\beta_{n_1,m}\,\beta_{n_2,m}}{\mu^{n_1+n_2+1}}\,\pi(1+\delta_{0,m})\, Q^m_{n_1}(1/\mu)\Big((1-\mu^2)^{1/2}\, Q^{m+1}_{n_2}(1/\mu) + m\, Q^m_{n_2}(1/\mu)\Big)\\ &\quad\times \int_0^\pi P^m_{n_1}(\cos\vartheta)P^m_{n_2}(\cos\vartheta)\sin\vartheta\; d\vartheta\\ &= \frac{(\beta_{n_1,m})^2}{\mu^{2n_1+1}}\left(\frac{2\pi(n_1+m)!}{(2n_1+1)(n_1-m)!}\right)(1+\delta_{0,m})\delta_{n_1,n_2}\end{aligned}$$

$$\times\, Q_{n_1}^m(1/\mu)\Big((1-\mu^2)^{1/2}\, Q_{n_1}^{m+1}(1/\mu) + m\, Q_{n_1}^m(1/\mu)\Big).$$

The same value is obtained when we replace $\Phi_m^+(\varphi)$ by $\Phi_m^-(\varphi)$ throughout, $m > 0$. Thus the statement is established. □

Next, we assert that the basic harmonics are not necessarily orthogonal in the closed subspaces $\mathrm{Har}_2(\Omega_\mu^*) = L_2(\Omega_\mu^*) \cap \mathrm{Har}(\Omega_\mu^*)$ of $L_2(\Omega_\mu^*)$ when $\mu \neq 0$ with respect to the ordinary L_2-inner product:

$$\langle f, g\rangle_{L_2(\Omega_\mu^*)} = \iiint_{\Omega_\mu^*} f(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}, \tag{33}$$

where $d\mathbf{x} = dx_0 dx_1 dx_2$.

Proposition 5 *The collection* (32) *does not form an orthogonal family of* $\mathrm{Har}_2(\Omega_\mu^*)$ *unless* $\mu = 0$.

Proof It is a simple matter to check that $U_{n,m}^{\pm}[\mu] \in L_2(\Omega_\mu^*)$. We assume again that $\mu \in (0, 1)$. Applying the coordinates (21), gives the infinitesimal volume element $d\mathbf{x} = dR d\varphi$, where

$$dR = \mu^3(\cosh^2\eta - \cos^2\vartheta)\sin\vartheta \sinh\eta\, d\vartheta d\eta. \tag{34}$$

It is clear that, when $m_1 \neq m_2$, we have

$$\langle U_{n_1,m_1}[\mu]\Phi_{m_1}^{\pm}(\varphi), U_{n_2,m_2}[\mu]\Phi_{m_2}^{\pm}(\varphi)\rangle_{L_2(\Omega_\mu^*)} = 0.$$

Let $m_1 = m_2 = m$. We compute

$$\begin{aligned}
&\langle U_{n_1,m}[\mu]\Phi_m^+(\varphi), U_{n_2,m}^+[\mu]\Phi_m^+(\varphi)\rangle_{L_2(\Omega_\mu^*)} \\
&= -\frac{\beta_{n_1,m}\,\beta_{n_2,m}}{\mu^{n_1+n_2-1}}(1+\delta_{0,m})\pi \times \Bigg(\int_0^\pi P_{n_1}^m(\cos\vartheta)P_{n_2}^m(\cos\vartheta)\sin\vartheta\, d\vartheta \\
&\quad\times \int_{\eta_\mu}^\infty Q_{n_1}^m(\cosh\eta)Q_{n_2}^m(\cosh\eta)\sinh\eta\cosh^2\eta\, d\eta \\
&\quad- \int_0^\pi P_{n_1}^m(\cos\vartheta)P_{n_2}^m(\cos\vartheta)\sin\vartheta\cos^2\vartheta\, d\vartheta \\
&\quad\times \int_{\eta_\mu}^\infty Q_{n_1}^m(\cosh\eta)Q_{n_2}^m(\cosh\eta)\sinh\eta\, d\eta\Bigg).
\end{aligned} \tag{35}$$

According to (7), we obtain

$$\begin{aligned}
&\int_0^\pi \cos^2\vartheta\, P_{n_1}^m(\cos\vartheta)P_{n_2}^m(\cos\vartheta)\sin\vartheta\, d\vartheta \\
&= \frac{(n_1+1-m)(n_2+1-m)}{(2n_1+1)(2n_2+1)}\int_0^\pi P_{n_1+1}^m(\cos\vartheta)P_{n_2+1}^m(\cos\vartheta)\sin\vartheta\, d\vartheta
\end{aligned}$$

$$+\frac{(n_1+1-m)(n_2+m)}{(2n_1+1)(2n_2+1)}\int_0^\pi P_{n_1+1}^m(\cos\vartheta)P_{n_2-1}^m(\cos\vartheta)\sin\vartheta\,d\vartheta$$
$$+\frac{(n_1+m)(n_2+1-m)}{(2n_1+1)(2n_2+1)}\int_0^\pi P_{n_1-1}^m(\cos\vartheta)P_{n_2+1}^m(\cos\vartheta)\sin\vartheta\,d\vartheta$$
$$+\frac{(n_1+m)(n_2+m)}{(2n_1+1)(2n_2+1)}\int_0^\pi P_{n_1-1}^m(\cos\vartheta)P_{n_2-1}^m(\cos\vartheta)\sin\vartheta\,d\vartheta.$$

By substituting this value into (35), it is found that the corresponding inner product is, in particular, distinct from zero when $n_2 - n_1 = \pm 2$. For the remainder of the proof, we consider when $n_2 - n_1 = 2$. The other case $n_2 - n_1 = -2$, can be treated analogously.

For $n_2 = n_1 + 2$, a straightforward computation shows that

$$\begin{aligned}&\langle U_{n_1,m}[\mu]\Phi_m^+(\varphi), U_{n_1+2,m}[\mu]\Phi_m^+(\varphi)\rangle_{L_2(\Omega_\mu^*)}\\&\quad = -\frac{2\pi(1+\delta_{0,m})(n_1+m)!}{(2n_1+1)(n_1-m)!}(\beta_{n_1,m})^2\,\mu^{-(2n_1+1)}I_{n_1,m}(\mu),\end{aligned}$$

where

$$I_{n,m}(\mu) := \int_{1/\mu}^\infty Q_n^m(s)Q_{n+2}^m(s)\,ds. \tag{36}$$

The same value is obtained when $\Phi_m^+(\varphi)$ is replaced by $\Phi_m^-(\varphi)$, $m > 0$. By Lemma 2, it follows that $\mu^{-(2n_1+1)}I_{n_1,m}(\mu) > 0$ for all $n_1, m = 0, 1, \ldots$ and fixed $\mu > 0$.

For the limiting case, when $\mu = 0$, we use Proposition 3 to show that $\lim_{\mu\to 0} I_{n_1,m}(\mu)/\mu^{2n_1+1} = 0$, for all $n_1 = 0, 1, \ldots$ $(m \geq 0)$. Thus, $\langle U_{n_1,m}^+[\mu], U_{n_1+2,m}^+[\mu]\rangle_{L_2(\Omega_\mu^*)}$ tends to zero as $\mu \to 0$. Similarly, we can prove that $\langle U_{n_1,m}^-[\mu], U_{n_1+2,m}^-[\mu]\rangle_{L_2(\Omega_\mu^*)} \to 0$ when μ tends to zero. This establishes the statement. □

The lack of orthogonality of the basic harmonics over the exterior of the prescribed spheroids in the usual L_2 sense means that defining suitable families of orthogonal external harmonics should be handled carefully. It is always possible to use an appropriate geometric weighting factor or apply an orthogonalization process to the prescribed harmonic functions, such as the Gram–Schmidt procedure that restores orthogonality. However, this orthogonalization process may be time-consuming and unstable. We preferably discuss a constructive approach discussed in [21] and show how it will be helpful not only from a function point of view but also for fast and stable computations.

3.3 Proper External Spheroidal Harmonics

We shall now be concerned with the following functions for the actual carrying out of the construction and given definition (25).

Definition 2 Let $U^{\pm}_{n,m}[\mu]$ have the same meaning as in Definition 1. Let $n \geq 0$ and $0 \leq m \leq n+1$. The proper external spheroidal harmonics of degree $-(n+3)$ and order m are

$$V^{\pm}_{n,m}[\mu](\mathbf{x}) = \frac{\partial}{\partial x_0} U^{\pm}_{n+1,m}[\mu](\mathbf{x}). \tag{37}$$

The proper harmonics $V^{\pm}_{n,m}[\mu]$ will play a crucial role in studying the basic external spheroidal monogenics in Sect. 4.

Following the notation already employed, we use $V^{\pm}_{n,m}[\mu] = V_{n,m}[\mu]\Phi^{\pm}_m$ when the factors $\Phi^{\pm}_m$ are not of interest.

It will be convenient before proceeding to investigate the algebraical forms of the *ansatz* functions $V_{n,m}[\mu]$. We will assume in the sequel that $\mu \in (0,1)$ because the case $\mu \in i\mathbb{R}^+$ is similar.

By differentiating (21),

$$\frac{\partial}{\partial x_0} = \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)} \left(\cos\vartheta \sinh\eta \frac{\partial}{\partial \eta} - \sin\vartheta \cosh\eta \frac{\partial}{\partial \vartheta} \right),$$

and combining (6)–(13) and (7)–(14) with the definition (37), we are thus led to the following remarkable representation:

$$\begin{aligned} V_{n,m}[\mu] = \frac{\beta_{n+1,m}\,(n+m+1)}{\mu^{n+3}(\cosh^2\eta - \cos^2\vartheta)} &\left(\cosh\eta\, P^m_n(\cos\vartheta) Q^m_{n+1}(\cosh\eta) \right. \\ &\left. - \cos\vartheta\, P^m_{n+1}(\cos\vartheta) Q^m_n(\cosh\eta) \right), \end{aligned} \tag{38}$$

where the coefficients $\beta_{n,m}$ have the same meaning as in (27).

As a consequence of (38) we notice that $V_{n,m}[\mu] = 0$ for $m > n+1$ since $P^m_n = 0$ for $m > n$.

According to (3) and (38), we deduce the following elementary identity.

Lemma 3 *For each $n \geq 0$ and $0 \leq m \leq n+1$,*

$$V_{n,-m}[\mu] = (-1)^m \frac{(n-m+1)!}{(n+m+1)!} V_{n,m}[\mu]. \tag{39}$$

Define

$$A_n[\mu] := \frac{P^n_n(\cos\vartheta) Q^n_{n-1}(\cosh\eta)}{\mu^{n+3}(\cosh^2\eta - \cos^2\vartheta)} \tag{40}$$

for $n > 0$ and fixed $\mu > 0$.

We have, for instance,

$$V_{n+1,n}[\mu] = \begin{cases} \frac{45}{4\mu^3} \cos\vartheta \left(\cosh\eta \, \log\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right) - 2\right) \\ \quad -\frac{15\cos\vartheta}{2\mu^3(\cosh^2\eta-\cos^2\vartheta)} & \text{if } n=0, \\ \frac{(2n+3)(2n+5)}{2\mu^2}\left(U_{n+1,n}[\mu] - \frac{2n+1}{2^{n-1}(n-1)!}\,\mu\cos\vartheta\, A_n[\mu]\right) & \text{if } n>0, \end{cases} \tag{41}$$

and

$$V_{n,n}[\mu] = \begin{cases} \frac{3}{2\mu^3}\log\left(\frac{\cosh\eta+1}{\cosh\eta-1}\right) - \frac{3\cosh\eta}{\mu^3(\cosh^2\eta-\cos^2\vartheta)} & \text{if } n=0, \\ \frac{(2n+1)(2n+3)}{\mu^2}\left(U_{n,n}[\mu] - \frac{\mu^2}{2^{n-1}(n-1)!}\cosh\eta\, A_n[\mu]\right) & \text{if } n>0. \end{cases} \tag{42}$$

It can further be seen that

$$V_{n,n+1}[\mu] = -\frac{2^{n+2}(1/2)_{n+2}}{(2n+1)!}\,\mu\cos\vartheta\, A_{n+1}[\mu]. \tag{43}$$

We shall return to these formulas in Sect. 3.5, where they will be used to prove the orthogonality of the proper harmonics (37) in the L_2-Hilbert space $\mathrm{Har}_2(\Omega^*_\mu)$.

Some examples of (37) in low degree are exhibited in Tables 2 and 3.

We now proceed to establish an elementary recurrence formula for the functions $V_{n,m}[\mu]$ to avoid the difficulties usually associated with manipulations such as formula (38).

Proposition 6 *For each $n \geq 2$, the following recurrence relation holds:*

$$\frac{\mu^2(n+1-m)(n-m)}{(2n+1)(2n+3)}\,V_{n,m}[\mu] = (n-m)U_{n,m}[\mu] + V_{n-2,m}[\mu]. \tag{44}$$

This uses the convention $V_{n-2,m}[\mu]=0$ when $m>n-1$.

Proof According to (7), (14), and (38), and bearing in mind that

$$\beta_{n+1,m} = \frac{(2n+1)(2n+3)}{(n+m+1)(n+m)}\,\beta_{n-1,m},$$

we have

$$V_{n,m}[\mu] = \frac{(2n+1)(2n+3)}{\mu^2(n+1-m)}\,U_{n,m}[\mu] \\ + \frac{\beta_{n-1,m}\,(2n+1)(2n+3)(n-1+m)}{\mu^{n+3}(\cosh^2\eta-\cos^2\vartheta)(n+1-m)(n-m)}$$

Table 2 Proper external spheroidal harmonic functions of degree $-(n+3) = -3, -4$

n	m	$V^{\pm}_{n,m}[\mu] \quad (\tau = \dfrac{\omega + 2\mu}{\omega - 2\mu})$
0	0	$V^{+}_{0,0}[\mu] = \dfrac{3}{2\mu^3}\left(\frac{4\mu\omega^3}{16x_0^2\mu^2 - \omega^4} + \log(\tau)\right)$
	1	$V^{+}_{0,1}[\mu] = -\dfrac{48x_0x_1(-4x_0^2+\omega^2)^{1/2}}{(x_1^2+x_2^2)^{1/2}(-4\mu^2+\omega^2)^{1/2}(-16x_0^2\mu^2+\omega^4)}$
		$V^{-}_{0,1}[\mu] = -\dfrac{48x_0x_2(-4x_0^2+\omega^2)^{1/2}}{(x_1^2+x_2^2)^{1/2}(-4\mu^2+\omega^2)^{1/2}(-16x_0^2\mu^2+\omega^4)}$
1	0	$V^{+}_{1,0}[\mu] = \dfrac{15x_0\left(4\mu(-48x_0^2\mu^2+4\mu^2\omega^2+3\omega^4) + (48x_0^2\mu^2\omega - 3\omega^5)\log(\tau)\right)}{64x_0^2\mu^7\omega - 4\mu^5\omega^5}$
	1	$V^{+}_{1,1}[\mu] = \dfrac{15x_1(-4x_0^2+\omega^2)^{1/2}}{8(x_1^2+x_2^2)^{1/2}\mu^5(-4\mu^2+\omega^2)^{1/2}(-16x_0^2\mu^2\omega+\omega^5)}$ $\left(-4(48x_0^2\mu^3\omega + 8\mu^3\omega^3 - 3\mu\omega^5) - 3(4\mu^2-\omega^2)(16x_0^2\mu^2-\omega^4)\log(\tau)\right)$
		$V^{-}_{1,1}[\mu] = \dfrac{15x_2(-4x_0^2+\omega^2)^{1/2}}{8(x_1^2+x_2^2)^{1/2}\mu^5(-4\mu^2+\omega^2)^{1/2}(-16x_0^2\mu^2\omega+\omega^5)}$ $\left(-4(48x_0^2\mu^3\omega + 8\mu^3\omega^3 - 3\mu\omega^5) - 3(4\mu^2-\omega^2)(16x_0^2\mu^2-\omega^4)\log(\tau)\right)$
	2	$V^{+}_{1,2}[\mu] = \dfrac{480x_0(x_1^2-x_2^2)(4x_0^2-\omega^2)}{(x_1^2+x_2^2)\omega(-4\mu^2+\omega^2)(-16x_0^2\mu^2+\omega^4)}$
		$V^{-}_{1,2}[\mu] = \dfrac{960x_0x_1x_2(4x_0^2-\omega^2)}{(x_1^2+x_2^2)\omega(-4\mu^2+\omega^2)(-16x_0^2\mu^2+\omega^4)}$

Table 3 Proper external spheroidal harmonic functions of degree $-(n+3)=-5$

n	m	$V^{\pm}_{n,m}[\mu] \quad (\tau = \frac{\omega+2\mu}{\omega-2\mu})$
2	0	$V^{+}_{2,0}[\mu] = \frac{35}{64\mu^7\omega^2(16x_0^2\mu^2-\omega^4)}\Big(-11520x_0^4\mu^3\omega + 64\mu^3\omega^5 - 60\mu\omega^7 + 240x_0^2(4\mu^3\omega^3+3\mu\omega^5) - 3(12\mu^2\omega^6 - 5\omega^8 + 320x_0^4(4\mu^4-3\mu^2\omega^2) + x_0^2(-192\mu^4\omega^2+60\omega^6))\log(\tau)\Big)$
	1	$V^{+}_{2,1}[\mu] = -\frac{105x_0x_1(-4x_0^2+\omega^2)^{1/2}}{16(x_1^2+x_2^2)^{1/2}\mu^7\omega^2(-4\mu^2+\omega^2)^{1/2}(16x_0^2\mu^2-\omega^4)}\Big(2560x_0^2\mu^5 - 64\mu^3(15x_0^2+2\mu^2)\omega^2 - 160\mu^3\omega^4 + 60\mu\omega^6 - 15\omega(-4\mu^2+\omega^2)(-16x_0^2\mu^2+\omega^4)\log(\tau)\Big)$
		$V^{-}_{2,1}[\mu] = -\frac{105x_0x_2(-4x_0^2+\omega^2)^{1/2}}{16(x_1^2+x_2^2)^{1/2}\mu^7\omega^2(-4\mu^2+\omega^2)^{1/2}(16x_0^2\mu^2-\omega^4)}\Big(2560x_0^2\mu^5 - 64\mu^3(15x_0^2+2\mu^2)\omega^2 - 160\mu^3\omega^4 + 60\mu\omega^6 - 15\omega(-4\mu^2+\omega^2)(-16x_0^2\mu^2+\omega^4)\log(\tau)\Big)$
	2	$V^{+}_{2,2}[\mu] = -\frac{105(x_1^2-x_2^2)(-4x_0^2+\omega^2)}{64(x_1^2+x_2^2)\mu^7\omega^2(4\mu^2-\omega^2)(16x_0^2\mu^2-\omega^4)}\Big(512\mu^5\omega^3 - 400\mu^3\omega^5 + 60\mu\omega^7 + 320x_0^2(20\mu^5\omega - 3\mu^3\omega^3) + 15(-4\mu^2+\omega^2)^2(16x_0^2\mu^2-\omega^4)\log(\tau)\Big)$
		$V^{-}_{2,2}[\mu] = -\frac{105x_1x_2(-4x_0^2+\omega^2)}{32(x_1^2+x_2^2)\mu^7\omega^2(4\mu^2-\omega^2)(16x_0^2\mu^2-\omega^4)}\Big(512\mu^5\omega^3 - 400\mu^3\omega^5 + 60\mu\omega^7 + 320x_0^2(20\mu^5\omega - 3\mu^3\omega^3) + 15(-4\mu^2+\omega^2)^2(16x_0^2\mu^2-\omega^4)\log(\tau)\Big)$
	3	$V^{+}_{2,3}[\mu] = -\frac{6720x_0(x_1^3-3x_1x_2^2)(-4x_0^2+\omega^2)^{3/2}}{(x_1^2+x_2^2)^{3/2}\omega^2(-4\mu^2+\omega^2)^{3/2}(-16x_0^2\mu^2+\omega^4)}$
		$V^{-}_{2,3}[\mu] = \frac{6720x_0x_2(-3x_1^2+x_2^2)(-4x_0^2+\omega^2)^{3/2}}{(x_1^2+x_2^2)^{3/2}\omega^2(-4\mu^2+\omega^2)^{3/2}(-16x_0^2\mu^2+\omega^4)}$

$$\times \left(\cosh\eta\, P^m_{n-2}(\cos\vartheta)\, Q^m_{n-1}(\cosh\eta) - \cos\vartheta\, P^m_{n-1}(\cos\vartheta)\, Q^m_{n-2}(\cosh\eta)\right).$$

The result now follows. □

The following "reverse Appell property", which follows from (44), involves the derivatives of the external solid spherical harmonics defined by (28) with respect to x_0:

$$\frac{\partial}{\partial x_0} U^{\pm}_{n+1,m}[0](\mathbf{x}) = -(n+2-m) U^{\pm}_{n+2,m}[0](\mathbf{x}). \tag{45}$$

The functions $V^{\pm}_{n,m}[\mu]$ are not so simply related to $U^{\pm}_{n,m}[\mu]$ for $\mu \neq 0$, as we show below.

Theorem 7 *Let $n \geq 0$ and $0 \leq m \leq n+1$. The coefficients $v_{n,m,k}$ in the relation*

$$\begin{aligned} V^{\pm}_{n,m}[\mu] = &\sum_{0 \leq 2k \leq n-m-2} v_{n,m,k} \frac{U^{\pm}_{n-2k,m}[\mu]}{\mu^{2(k+1)}} \\ &+ \frac{(1/2)_{n+2}\, 2^{n-m}}{\mu^{n-m}(n+1-m)!} \begin{cases} \frac{1}{(1/2)_{m+2}}\, V^{\pm}_{m,m}[\mu] & \text{if } n-m \text{ is even,} \\ \frac{1}{(1/2)_{m+3}}\, \mu\, V^{\pm}_{m+1,m}[\mu] & \text{if } n-m \text{ is odd} \end{cases} \end{aligned} \tag{46}$$

are given by

$$v_{n,m,k} = \frac{4^{k+1}(n-m-2k)!(1/2)_{n+2}}{(n+1-m)!(1/2)_{n-2k}}. \tag{47}$$

Proof Suppose inductively that the formula holds when n is replaced by $n' < n$. Then

$$\begin{aligned} V^{\pm}_{n,m}[\mu] = &\frac{(2n+1)(2n+3)}{\mu^2(n+1-m)} U^{\pm}_{n,m}[\mu] \\ &+ \frac{(2n+1)(2n+3)}{(n+1-m)(n-m)} \sum_{0 \leq 2k \leq n-m-4} v_{n-2,m,k} \frac{U^{\pm}_{n-2(k+1),m}[\mu]}{\mu^{2(k+2)}} \\ &+ \frac{(1/2)_{n+2}\, 2^{n-m}}{\mu^{n-m}(n+1-m)!} \begin{cases} \frac{1}{(1/2)_{m+2}}\, V^{\pm}_{m,m}[\mu] & \text{if } n-m \text{ is even,} \\ \frac{1}{(1/2)_{m+3}}\, \mu\, V^{\pm}_{m+1,m}[\mu] & \text{if } n-m \text{ is odd.} \end{cases} \end{aligned}$$

Since, by (47),

$$v_{n,m,0} = \frac{(2n+1)(2n+3)}{n+1-m}, \quad v_{n,m,k+1} = \frac{(2n+1)(2n+3)}{(n+1-m)(n-m)} v_{n-2,m,k},$$

we find that the stated formula holds, completing the proof. □

3.4 Conversions Among Spheroidal and Solid Spherical Harmonics

In this section, we determine the coefficients α and $\widetilde{\alpha}$ of the following direct and inverse transformation formulas:

$$V^{\pm}_{n,m}[\mu] = \sum_{k} \alpha_{n+1,m,k}\, \mu^{2k} V^{\pm}_{n+2k,m}[0], \quad V^{\pm}_{n,m}[0] = \sum_{k} \widetilde{\alpha}_{n+1,m,k}\, \mu^{2k} V^{\pm}_{n+2k,m}[\mu].$$

Fix a value of μ. By referring to these expansions, we shall employ the constraints that the index m is not involved in the summations, and the values of the same evenness restrict the index k as a given n. It follows from symmetry considerations that the above relations will work for the "+" and "−" cases (cosines and sines) and, strikingly, for all values of μ.

For harmonic functions outside a prolate or an oblate spheroid, the transition from the expansion in external spheroidal harmonics to that in external solid spherical harmonics (and vice-versa) is worked out in [1]. Some of these formulas are discussed thoroughly in [2]. Two of these fundamental formulas, relevant to the sequel, are reproduced in our notation below (i.e., the factor (27) has been incorporated into (48) and (49)).

Proposition 8 *For $n \geq 0$, consider the rational constants*

$$\alpha_{n,m,k} = (-1)^k \frac{(n+2k-m)!(n+k)!(2n+1)!}{(n-m)!k!(2n+1+2k)!n!}, \tag{48}$$

$$\widetilde{\alpha}_{n,m,k} = \frac{(n+2k-m)!(2n+2k)!(n+2k)!}{(n-m)!k!(n+k)!(2n+4k)!} \tag{49}$$

for $0 \leq m \leq n$, and let $\alpha_{n,m,k} = \widetilde{\alpha}_{n,m,k} = 0$ otherwise. Then

$$U^{\pm}_{n,m}[\mu] = \sum_{k=0}^{\infty} \alpha_{n,m,k}\, \mu^{2k} U^{\pm}_{n+2k,m}[0],$$

$$U^{\pm}_{n,m}[0] = \sum_{k=0}^{\infty} \widetilde{\alpha}_{n,m,k}\, \mu^{2k} U^{\pm}_{n+2k,m}[\mu].$$

Since $\partial/\partial x_0$ is a linear operator, we automatically have the corresponding transformation formulas for the proper harmonics:

Corollary 1 *Let $n \geq 0$ and $0 \leq m \leq n+1$. Then*

$$V^{\pm}_{n,m}[\mu] = \sum_{k=0}^{\infty} \alpha_{n+1,m,k}\, \mu^{2k} V^{\pm}_{n+2k,m}[0],$$

$$V^{\pm}_{n,m}[0] = \sum_{k=0}^{\infty} \widetilde{\alpha}_{n+1,m,k}\, \mu^{2k} V^{\pm}_{n+2k,m}[\mu]. \tag{50}$$

Here $\alpha_{n,m,k}$ and $\widetilde{\alpha}_{n,m,k}$ have the same meaning as in Proposition 8.

3.5 Orthogonality Properties of the Proper Harmonics

We begin by formulating a technical proposition, which expresses an "individual" orthogonal property of the functions $V_{n,m}[\mu]$ over the interval $[0, \pi]$. We borrow from the techniques used in the earlier work [19] and extend those results for arbitrary μ (the method also relies on Neumann's formula (1) and identities (54) and (56) below. Hence, the proof is independent of the previous paper.)

Proposition 9 *Let $\mu \in [0, 1) \cup i\mathbb{R}^+$ be fixed. The following orthogonality relations hold for all $m = 0, 1, \ldots$ and each pair (n, k) such that $n, k \in \{m, m+1\}$,*

$$\int_0^{\pi} V_{n,m}[\mu] P_k^m(\cos\vartheta) \sin\vartheta \, d\vartheta = 0. \tag{51}$$

Proof Fix a value of μ. For the proof, let the left-hand sides of (51) be denoted by $C^m_{\varepsilon_1,\varepsilon_2}(\mu)$ with $n = m+\varepsilon_1$ and $k = m+\varepsilon_2$. We only use pairs in the set $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

We will assume that $\mu \in (0, 1)$ because the case $\mu \in i\mathbb{R}^+$ is similar (the intermediate case $\mu = 0$ is trivial, and it uses (45).) We have from (38) that

$$\begin{aligned} C^m_{(1,0)}(\mu) &= \int_0^{\pi} V_{m+1,m}[\mu] P_m^m(\cos\vartheta) \sin\vartheta \, d\vartheta \\ &= \frac{\beta_{m+2,m}\,(m+1)}{\mu^{m+4}} \int_0^{\pi} \Bigg((2m+3) P^m_{m+1}(\cos\vartheta) Q^m_{m+1}(\cosh\eta) \\ &\quad - 2m\, \frac{P^m_{m+1}(\cos\vartheta) Q^m_{m-1}(\cosh\eta)}{\cosh^2\eta - \cos^2\vartheta} \Bigg) P_m^m(\cos\vartheta) \sin\vartheta \, d\vartheta. \end{aligned}$$

Notice that the first term gives a zero-integral because of the orthogonality of the associated Legendre functions of the first kind over the interval $[0, \pi]$. The second term also has a vanishing integral because the underlying function is odd with respect to the variable $t = \cos\vartheta$. Similarly, it can be proved that $C^m_{(0,1)}(\mu) = 0$ for all $m = 0, 1, \ldots$.

We now consider the two remaining integrals, $C^m_{(0,0)}(\mu)$ and $C^m_{(1,1)}(\mu)$. For simplicity, we only sketch the proof for $C^m_{(0,0)}(\mu)$. The other integral can be derived straightforwardly. Let us begin by computing $C^0_{(0,0)}(\mu)$. As a consequence of Neumann's formula (1), we find

$$Q_0(\cosh\eta) = \frac{1}{2} \int_0^{\pi} \frac{\cosh\eta \sin\vartheta}{\cosh^2\eta - \cos^2\vartheta} \, d\vartheta. \tag{52}$$

Combining the explicit representation (42) for the $V_{0,0}[\mu]$ and (52), it follows that

$$\begin{aligned} C^0_{(0,0)}(\mu) &= \int_0^{\pi} V_{0,0}[\mu] P_0(\cos\vartheta) \sin\vartheta \, d\vartheta \\ &= \frac{3}{\mu^3} \log\left(\frac{\cosh\eta + 1}{\cosh\eta - 1}\right) - \frac{6}{\mu^3}\, Q_0(\cosh\eta) = 0. \end{aligned}$$

For an arbitrary $m > 1$, we have once again from (42) that

$$
\begin{aligned}
C^m_{(0,0)}(\mu) &= \int_0^\pi V_{m,m}[\mu]\, P^m_m(\cos\vartheta)\sin\vartheta\, d\vartheta \\
&= \frac{(2m+1)(2m+3)}{\mu^{m+3}} \int_0^\pi \Bigg(\beta_{m,m}\, P^m_m(\cos\vartheta)\, Q^m_m(\cosh\eta) \\
&\quad - \frac{\cosh\eta\, P^m_m(\cos\vartheta)\, Q^m_{m-1}(\cosh\eta)}{2^{m-1}(m-1)!(\cosh^2\eta - \cos^2\vartheta)}\Bigg) P^m_m(\cos\vartheta)\sin\vartheta\, d\vartheta \\
&= \frac{(2m+3)}{\mu^{m+3}} \Bigg(2^{m+2}(1/2)_{m+1}\, Q^m_m(\cosh\eta) \\
&\quad - \frac{(2m+1)}{2^{m-1}(m-1)!} \cosh\eta\, Q^m_{m-1}(\cosh\eta) \int_0^\pi \frac{(P^m_m(\cos\vartheta))^2}{\cosh^2\eta - \cos^2\vartheta} \sin\vartheta\, d\vartheta\Bigg).
\end{aligned}
\tag{53}
$$

Using the identity [14, p. 195]

$$
Q^m_n(s) = \frac{(-1)^m (n+m)!}{2^{n+1}\, n!} (s^2-1)^{-m/2} \int_{-1}^1 \frac{(1-t^2)^n}{(s-t)^{n-m+1}}\, dt, \tag{54}
$$

we can rewrite the integral in (53) as follows:

$$
\int_0^\pi \frac{(P^m_m(\cos\vartheta))^2}{\cosh^2\eta - \cos^2\vartheta} \sin\vartheta\, d\vartheta = (-1)^m \frac{(2m)!\,(\sinh\eta)^m}{2^{m-1}\, m!\cosh\eta}\, Q^m_m(\cosh\eta). \tag{55}
$$

Substituting this computation into $C^m_{(0,0)}$ and using the relation [31, Eq. (6.17)]

$$
Q^m_n(s) = (-1)^{n+1}\, 2^n n!\, (s^2-1)^{m/2} \frac{d^{m-n-1}}{ds^{m-n-1}} (s^2-1)^{-n-1} \quad (m > n) \tag{56}
$$

for $n = m-1$, we find $C^m_{(0,0)}(\mu) = 0$ for $m > 0$. It yields that $C^m_{(0,0)}(\mu) = 0$ for all $m = 0, 1, \ldots$. This completes the proof of the statements. □

The orthogonality of the proper harmonics (37) over the exterior of the prescribed spheroids Ω_μ is given in the following theorem:

Theorem 10 *Let μ be fixed. For each $n \geq 0$, the collection*

$$
\{V^+_{n,m}[\mu] \colon 0 \leq m \leq n+1\} \cup \{V^-_{n,m}[\mu] \colon 1 \leq m \leq n+1\} \tag{57}
$$

forms an orthogonal family in $\mathrm{Har}_2(\Omega^*_\mu)$ *with the norms*

$$
\| V^\pm_{n,m}[\mu] \|^2_2 = \frac{2\pi(1+\delta_{0,m})}{\mu^{2n+3}} \gamma_{n,m} I_{n,m}(\mu), \tag{58}
$$

where $I_{n,m}(\mu)$ has the same meaning as in (36) and

$$\gamma_{n,m} = \frac{2^{2n+3}(1/2)_{n+1}(1/2)_{n+2}\,(n+2-m)}{(n+m)!(n+1-m)!}. \tag{59}$$

Proof We will assume that $\mu \in (0,1)$, because the case $\mu \in i\mathbb{R}^+$ is similar. When $m_1 \neq m_2$, we have

$$\begin{aligned}
\langle V_{n_1,m_1}[\mu]\Phi^+_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^+_{m_2}(\varphi)\rangle_{L_2(\Omega^*_\mu)} &= 0,\\
\langle V_{n_1,m_1}[\mu]\Phi^-_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^-_{m_2}(\varphi)\rangle_{L_2(\Omega^*_\mu)} &= 0,\\
\langle V_{n_1,m_1}[\mu]\Phi^+_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^-_{m_2}(\varphi)\rangle_{L_2(\Omega^*_\mu)} &= 0,\\
\langle V_{n_1,m_1}[\mu]\Phi^-_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^+_{m_2}(\varphi)\rangle_{L_2(\Omega^*_\mu)} &= 0.
\end{aligned}$$

Using (33), we obtain

$$\begin{aligned}
&\langle V_{n_1,m}[\mu]\Phi^+_{m}(\varphi), V_{n_2,m}[\mu]\Phi^+_{m}(\varphi)\rangle_{L_2(\Omega^*_\mu)}\\
&\quad = (1+\delta_{0,m})\pi \int_{\eta_\mu}^{\infty}\int_0^{\pi} V_{n_1,m}[\mu]V_{n_2,m}[\mu]\, dR,
\end{aligned}$$

where dR has the same meaning as in (34).

Thus we need to study integrals of the form

$$\int_0^{\pi} V_{n_1,m}[\mu]\, V_{n_2,m}[\mu](\cosh^2\eta - \cos^2\vartheta)\sin\vartheta\, d\vartheta.$$

Without loss of generality, we assume that $n_1 > n_2$ and proceed to set $V_{n_1,m}[\mu]$ as (38) and $V_{n_2,m}[\mu]$ as (46). Since

$$\begin{aligned}
\int_0^{\pi} P^m_{n_1}(\cos\vartheta)P^m_{n_2-2k}(\cos\vartheta)\sin\vartheta\, d\vartheta &= 0,\\
\int_0^{\pi} P^m_{n_1+1}(\cos\vartheta)\cos\vartheta\, P^m_{n_2-2k}(\cos\vartheta)\sin\vartheta\, d\vartheta &= 0,
\end{aligned}$$

it follows that the remaining nonvanishing integrals are, respectively,

$$\int_0^{\pi} V_{m,m}[\mu]\, P^m_{n_1}(\cos\vartheta)\sin\vartheta\, d\vartheta,$$

according to $n_2 - m$ being even, or

$$\int_0^{\pi} V_{m+1,m}[\mu]\, P^m_{n_1}(\cos\vartheta)\sin\vartheta\, d\vartheta,$$

according to $n_2 - m$ being odd.

Furthermore, using (41) and (42), we are led toward the integrals of the form, as stated in Proposition 9. Hence, for $n_1 \neq n_2$,

$$\langle V^+_{n_1,m}[\mu], V^+_{n_2,m}[\mu]\rangle_{L_2(\Omega^*_\mu)} = 0,$$

and also

$$\langle V^-_{n_1,m}[\mu], V^-_{n_2,m}[\mu]\rangle_{L_2(\Omega^*_\mu)} = 0.$$

Using the orthogonality of the system $\Phi^\pm_m(\varphi)$ on $[0, 2\pi]$ again, we conclude that $\langle V^+_{n_1,m}[\mu], V^-_{n_2,m}[\mu]\rangle_{L_2(\Omega^*_\mu)} = 0$ when $n_1 \neq n_2$. This establishes the orthogonality statement.

For $n_1 = n_2 = n$ and $0 \leq m \leq n$, by (44), we find

$$\begin{aligned}
&\langle V_{n,m}[\mu]\Phi^+_m(\varphi), V_{n,m}[\mu]\Phi^+_m(\varphi)\rangle_{L_2(\Omega^*_\mu)} \\
&\quad= \frac{(2n+1)(2n+3)(n+m+1)}{\mu^{2n+3}(n+1-m)}\,\beta_{n,m}\,\beta_{n+1,m}\,(1+\delta_{0,m})\pi \\
&\qquad\times \int_{\eta_\mu}^{\infty}\int_0^{\pi} P^m_n(\cos\vartheta)Q^m_n(\cosh\eta)\big[\cosh\eta\,P^m_n(\cos\vartheta)Q^m_{n+1}(\cosh\eta) \\
&\qquad-\cos\vartheta\,P^m_{n+1}(\cos\vartheta)Q^m_n(\cosh\eta)\big]\sin\vartheta\,\sinh\eta\,d\vartheta\,d\eta.
\end{aligned}$$

Now, combining (7) and (14), it follows that

$$\begin{aligned}
&t\,R^m_n(t)R^m_{n+1}(t) \\
&\quad= \frac{1}{2n+3}\big((n+2-m)R^m_{n+2}(t) + (n+1+m)R^m_n(t)\big)\,R^m_n(t),
\end{aligned}$$

with $t = \cos\vartheta$ or $\cosh\eta$ and $R = P$ or Q.

Therefore, when $0 \leq m \leq n$, we obtain

$$\begin{aligned}
&\langle V_{n,m}[\mu]\Phi^+_m(\varphi), V_{n,m}[\mu]\Phi^+_m(\varphi)\rangle_{L_2(\Omega^*_\mu)} \\
&\quad= \frac{(2n+1)(n+m+1)(n+2-m)}{\mu^{2n+3}(n+1-m)}\,\beta_{n,m}\,\beta_{n+1,m}\,(1+\delta_{0,m})\pi \\
&\qquad\times\bigg(\int_{\eta_\mu}^{\infty} Q^m_n(\cosh\eta)Q^m_{n+2}(\cosh\eta)\sinh\eta\,d\eta \\
&\qquad\times\int_0^{\pi}\big(P^m_n(\cos\vartheta)\big)^2\sin\vartheta\,d\vartheta \\
&\qquad-\int_{\eta_\mu}^{\infty}\big(Q^m_n(\cosh\eta)\big)^2\sinh\eta\,d\eta \\
&\qquad\times\int_0^{\pi} P^m_n(\cos\vartheta)P^m_{n+2}(\cos\vartheta)\sin\vartheta\,d\vartheta\bigg) \\
&\quad= \frac{(n+2-m)(n+m+1)!}{\mu^{2n+3}(n+1-m)!}\,\beta_{n,m}\,\beta_{n+1,m}\,2\pi(1+\delta_{0,m})\,I_{n,m}(\mu),
\end{aligned}$$

where $I_{n,m}(\mu)$ is defined by (36). We obtain the same value when $\Phi_m^+(\varphi)$ is replaced by $\Phi_m^-(\varphi)$ throughout, $0 < m \le n$.

When $m = n+1$, by use of (43), we have

$$\begin{aligned}
&\langle V_{n,n+1}[\mu]\Phi_{n+1}^+(\varphi), V_{n,n+1}[\mu]\Phi_{n+1}^+(\varphi)\rangle_{L_2(\Omega_\mu^*)} \\
&\quad = \frac{\pi\, 4^{n+2}\,((1/2)_{n+2})^2}{\mu^{2n+3}\,((2n+1)!)^2} \\
&\qquad \times \int_{\eta_\mu}^{\infty} \Big(Q_n^{n+1}(\cosh\eta)\Big)^2 \sinh\eta \left(\int_0^\pi \frac{\Big(P_{n+1}^{n+1}(\cos\vartheta)\Big)^2}{\cosh^2\eta - \cos^2\vartheta} \cos^2\vartheta \sin\vartheta\, d\vartheta\right) d\eta.
\end{aligned} \tag{60}$$

Using the identity (54), we find

$$\begin{aligned}
&\int_0^\pi \frac{\Big(P_{n+1}^{n+1}(\cos\vartheta)\Big)^2}{\cosh^2\eta - \cos^2\vartheta} \cos^2\vartheta \sin\vartheta\, d\vartheta \\
&\quad = (-1)^{n+1} \frac{2^{3n+4}\,((1/2)_{n+1})^2\,(n+1)!}{(2n+2)!} \\
&\qquad \times \left(Q_{n+1}^{n+1}(\cosh\eta) + \frac{1}{2n+3} \sinh\eta\, Q_{n+2}^{n+2}(\cosh\eta)\right) \frac{(\sinh\eta)^{n+1}}{\cosh\eta}.
\end{aligned}$$

Consequently, combining the relation (17) and identity (56), then (60) leads to

$$\| V_{n,n+1}^+[\mu]\|_2^2 = \frac{\pi\, 4^{n+3}(n+1)\,((1/2)_{n+2})^2}{\mu^{2n+3}(2n+3)!}\, I_{n,n+1}(\mu).$$

By the form of the $V_{n,m}^{\pm}[\mu]$, it follows that $\| V_{n,n+1}^+[\mu]\|_{L_2(\Omega_\mu^*)} = \| V_{n,n+1}^-[\mu]\|_{L_2(\Omega_\mu^*)}$. The proof is now completed. □

We now state and prove the orthogonality of the proper harmonics (37) over the surface of the prescribed spheroids with respect to a suitable weight function.

Theorem 11 *Let $|\zeta(\mu, \mathbf{x})|$ have the same meaning as in (24). For fixed μ, the collection (57) forms an orthogonal family over the surface of the spheroids Ω_μ in the sense of the scalar product*

$$\langle f, g\rangle_{L_2(\partial\Omega_\mu)} = \iint_{\eta=\eta_\mu} f(\mathbf{x}) g(\mathbf{x})\, |\zeta(\mu, \mathbf{x})|^{1/2}\, d\sigma. \tag{61}$$

Their norms squared are equal to

$$\| V_{n,m}^{\pm}[\mu]\|_2^2 = \frac{4^{n+2}\pi(1+\delta_{0,m})\sqrt{1-\mu^2}\,(2n+3)(1/2)_{n+1}(1/2)_{n+2}}{|\mu|^{2(n+2)}(n+m)!(n+1-m)!}\, Q_n^m(1/\mu)$$

$$\times \left(\frac{1}{\mu} \, Q^m_{n+1}(1/\mu) - \frac{(n+m+1)}{2n+3} \, Q^m_n(1/\mu) \right).$$

Proof For the proof, we consider the prolate case again. When $m_1 \neq m_2$, we have

$$\begin{aligned}
\langle V_{n_1,m_1}[\mu]\Phi^+_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^+_{m_2}(\varphi)\rangle_{L_2(\partial\Omega_\mu)} &= 0,\\
\langle V_{n_1,m_1}[\mu]\Phi^-_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^-_{m_2}(\varphi)\rangle_{L_2(\partial\Omega_\mu)} &= 0,\\
\langle V_{n_1,m_1}[\mu]\Phi^+_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^-_{m_2}(\varphi)\rangle_{L_2(\partial\Omega_\mu)} &= 0,\\
\langle V_{n_1,m_1}[\mu]\Phi^-_{m_1}(\varphi), V_{n_2,m_2}[\mu]\Phi^+_{m_2}(\varphi)\rangle_{L_2(\partial\Omega_\mu)} &= 0.
\end{aligned}$$

Let $P^* = (\eta_\mu, \vartheta, \varphi)$, i.e., with $\cosh \eta_\mu = 1/\mu$. For $m_1 = m_2 = m$, direct calculation leads readily to

$$\begin{aligned}
&\langle V_{n_1,m}[\mu]\Phi^+_m(\varphi), V_{n_2,m}[\mu]\Phi^+_m(\varphi)\rangle_{L_2(\partial\Omega_\mu)}\\
&\quad = \mu^3 \sinh \eta_\mu \,(1+\delta_{0,m})\pi \int_0^\pi V_{n_1,m}[\mu](P^*)V_{n_2,m}[\mu](P^*) \sin\vartheta \left|\sin(\vartheta - i\eta_\mu)\right|^2 d\vartheta,
\end{aligned}$$

where we have used that

$$\begin{aligned}
\left|\zeta(\mu, P^*)\right|^{1/2} &= \mu \left| 1 - (\cos\vartheta \cosh\eta_\mu + i \sin\vartheta \sinh\eta_\mu)^2 \right|^{1/2}\\
&= \mu \left|\sin(\vartheta - i\eta_\mu)\right|\\
&= \mu \Big(\cosh^2\eta_\mu - \cos^2\vartheta \Big)^{1/2}.
\end{aligned}$$

From this point on, the proof follows the argument used in Theorem 10 of assuming that $n_1 > n_2$, in association with the facts

$$\begin{aligned}
\int_0^\pi P^m_{n_1}(\cos\vartheta) P^m_{n_2-2k}(\cos\vartheta) \sin\vartheta \, d\vartheta &= 0,\\
\int_0^\pi P^m_{n_1+1}(\cos\vartheta) \cos\vartheta \, P^m_{n_2-2k}(\cos\vartheta) \sin\vartheta \, d\vartheta &= 0.
\end{aligned}$$

By (38), we then find in combination with Theorem 7 and Proposition 9 that

$$\begin{aligned}
&\langle V_{n_1,m}[\mu]\Phi^+_m(\varphi), V_{n_2,m}[\mu]\Phi^+_m(\varphi)\rangle_{L_2(\partial\Omega_\mu)}\\
&\quad = \frac{2\pi(1+\delta_{0,m})\,(n+m+1)!(n+2-m)}{\mu^{2n+3}(n+1-m)!} \, \beta_{n,m}\beta_{n+1,m}\,\delta_{n_1,n_2}\\
&\qquad \times \sinh\eta_\mu \, Q^m_n(1/\mu) Q^m_{n+2}(1/\mu), \qquad (62)
\end{aligned}$$

with exactly the same formula when $\Phi^+_m(\varphi)$ is replaced by $\Phi^-_m(\varphi)$, $m > 0$. The calculation of the norms of $V^\pm_{n,m}[\mu]$ comes from taking $n_1 = n_2$ in (62) and using relation (14). The statement follows. □

4 Spheroidal Monogenic Functions

In this section, we combine the results from the previous sections to construct a common orthogonal basis for the one-parameter family of monogenic L_2-Hilbert spaces applicable to the exterior of a spheroid of given eccentricity. We express the elements of such a basis in terms of the proper external harmonic functions (37). In an earlier paper [19], we treated the analogous problem for prolate spheroids. We borrow some of these techniques and fit many results into the present case. In particular, we can consider the prolate and oblate spheroidal monogenics simultaneously.

4.1 Notation

We are interested in a theory of functions from the space domain Ω^*_μ to $\mathbb{R}^3$. For this purpose, we consider the set

$$\mathbb{H} = \{\mathbf{x} = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3 : x_i \in \mathbb{R},\ i = 0, 1, 2, 3\}$$

of (real) quaternions, where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the quaternionic imaginary units obeying the multiplication rules $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. One usually writes $\overline{\mathbf{x}} = \mathrm{Sc}(\mathbf{x}) - \mathrm{Vec}(\mathbf{x})$ (here $\mathrm{Sc}(\mathbf{x}) = x_0$ and $\mathrm{Vec}(\mathbf{x}) = \mathbf{i}x_1 + \mathbf{j}x_2 + \mathbf{k}x_3$ denote the scalar and vector parts of $\mathbf{x}$) and $|\mathbf{x}| = (\mathbf{x}\overline{\mathbf{x}})^{1/2} = (\overline{\mathbf{x}}\mathbf{x})^{1/2} = \left(\sum_{i=0}^{3} x_i^2\right)^{1/2}$ for the conjugate and absolute value operations on $\mathbb{H}$ as in [10, 11, 18, 29]. We identify the Euclidean space $\mathbb{R}^3 = \{x = (x_0, x_1, x_2)\}$ with the real vector subspace of *reduced quaternions*

$$\{\mathbf{x} = x_0 + \mathbf{i}x_1 + \mathbf{j}x_2\} \subseteq \mathbb{H},$$

i.e., with vanishing $\mathbf{k}$-term. Although this subspace is not closed under quaternionic multiplication, it is possible to carry out a great deal of the analysis analogous to that of complex numbers [12, 13].

Consider the usual first-order differential generalized Cauchy–Riemann (or Fueter) operator

$$\overline{\partial} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}$$

and its conjugate

$$\partial = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2} \tag{63}$$

on functions $\mathbf{f}\colon \Omega \to \mathbb{R}^3$ on any domain $\Omega \subseteq \mathbb{R}^3$, with $\mathbf{f} = [\mathbf{f}]_0 + \mathbf{i}[\mathbf{f}]_1 + \mathbf{j}[\mathbf{f}]_2$ where $[\mathbf{f}]_i\colon \Omega \to \mathbb{R}$ $(i = 0, 1, 2)$. As in [5, 12, 13], $\mathbf{f}$ is called *monogenic* when $\overline{\partial}\mathbf{f} = 0$. While $\overline{\partial}$ does not generally commute with quaternionic functions, since we are considering $\mathbb{R}^3$-valued functions in this paper, the definition of monogenic function does not depend on whether one applies $\overline{\partial}$ from the left or the right since

$-\mathbf{k}(\overline{\partial}\mathbf{f})\mathbf{k} = \overline{\mathbf{f}\overline{\partial}}$. For domains $\Omega \subseteq \mathbb{R}^3$, we write $\mathcal{M}(\Omega) = \operatorname{Ker} \overline{\partial} \subseteq C^1(\Omega, \mathbb{R}^3)$ and $\mathcal{M}_2(\Omega) = \mathcal{M}(\Omega) \cap L_2(\Omega, \mathbb{R}^3)$, where $L_2(\Omega, \mathbb{R}^3)$ denotes the $\mathbb{R}$-linear space of all $\mathbb{R}^3$-valued functions $\mathbf{f}$, such that the real component functions $[\mathbf{f}]_i$ are in the usual $L_2(\Omega)$. Monogenic functions are harmonic, but not vice-versa.

4.2 Basic External Spheroidal Monogenics

In analogy to the definition (37) for harmonic functions, we define the required spheroidal monogenics to be employed for the space exterior of the prescribed spheroids (20) as follows.

Definition 3 Let $U^{\pm}_{n,m}[\mu]$ have the same meaning as in Definition 1. Let $n \geq -1$ and $0 \leq m \leq n+1$. The basic external spheroidal monogenics of degree $-(n+3)$ and order m are

$$\mathbf{X}^{\pm}_{n,m}[\mu] = \partial U^{\pm}_{n+1,m}[\mu]. \tag{64}$$

The statement that $\mathbf{X}^{\pm}_{n,m}[\mu]$ is monogenic is seen from the factorization of the Laplacian in $\mathbb{R}^3$ by $\Delta_3 = \overline{\partial}\partial$. We continue with the convention that $m \geq 1$ when the "-" sign appears in a superscript.

In the following theorem, we express the basic monogenics in terms of their quaternionic components. (The functions $V^{\pm}_{n,-1}[\mu]$ defined by (39) are involved in the representation (66) for zero-order monogenic functions.)

Theorem 12 *Let $V^{\pm}_{n,m}[\mu]$ have the same meaning as in (38). For all $n \geq 0$, the basic external spheroidal monogenics (64) are equal to*

$$\mathbf{X}^{+}_{-1,0}[\mu] = \frac{-\sinh\eta\cos\vartheta + (\mathbf{i}\cos\varphi + \mathbf{j}\sin\varphi)\cosh\eta\sin\vartheta}{\mu^2\sinh\eta\,(\cosh^2\eta - \cos^2\vartheta)}, \tag{65}$$

$$\begin{aligned}\mathbf{X}^{\pm}_{n,m}[\mu] = {} & V^{\pm}_{n,m}[\mu] + \frac{\mathbf{i}}{2}\left((n+2-m)V^{\pm}_{n,m-1}[\mu] - \frac{1}{n+1-m}\,V^{\pm}_{n,m+1}[\mu]\right) \\ & \mp\frac{\mathbf{j}}{2}\left((n+2-m)V^{\mp}_{n,m-1}[\mu] + \frac{1}{n+1-m}\,V^{\mp}_{n,m+1}[\mu]\right)\end{aligned} \tag{66}$$

for $0 \leq m \leq n$, and

$$\begin{aligned}\mathbf{X}^{\pm}_{n,n+1}[\mu] = {} & V^{\pm}_{n,n+1}[\mu] + \frac{\mathbf{i}}{2}\left(V^{\pm}_{n,n}[\mu] - \frac{\mu\cosh\eta}{2(2n+5)\cos\vartheta}\,V^{\pm}_{n+1,n+2}[\mu]\right) \\ & \mp\frac{\mathbf{j}}{2}\left(V^{\mp}_{n,n}[\mu] + \frac{\mu\cosh\eta}{2(2n+5)\cos\vartheta}\,V^{\mp}_{n+1,n+2}[\mu]\right).\end{aligned} \tag{67}$$

Further, they are functions in x_0, x_1, x_2.

Proof As derivatives of functions, the basic monogenics are also functions in the variables x_0, x_1, x_2. To proceed with the proof, we write the operator (63) in prolate spheroidal coordinates (21):

$$
\begin{aligned}
\partial &= \frac{1}{\mu(\cosh^2\eta-\cos^2\vartheta)}\left(\cos\vartheta\sinh\eta\,\frac{\partial}{\partial\eta}-\sin\vartheta\cosh\eta\,\frac{\partial}{\partial\vartheta}\right)\\
&\quad-\frac{1}{\mu(\cosh^2\eta-\cos^2\vartheta)}\,(\mathbf{i}\cos\varphi+\mathbf{j}\sin\varphi)\left(\sin\vartheta\cosh\eta\,\frac{\partial}{\partial\eta}+\cos\vartheta\sinh\eta\,\frac{\partial}{\partial\vartheta}\right)\\
&\quad-\frac{1}{\mu\sin\vartheta\sinh\eta}\,(-\mathbf{i}\sin\varphi+\mathbf{j}\cos\varphi)\,\frac{\partial}{\partial\varphi}.
\end{aligned}
$$

The first line of the above expression applied to the functions $U^{\pm}_{n+1,m}[\mu]$ produces the scalar parts of $\mathbf{X}^{\pm}_{n,m}[\mu]$.

The expression (65) is trivial. Now, let $0\le m\le n$. For the nonscalar parts of $\mathbf{X}^{\pm}_{n,m}[\mu]$, we combine the identities (8)–(15) and (9)–(16) to find

$$
\begin{aligned}
&\frac{2\,\mu^{n+2}}{\beta_{n+1,m}\,\Phi^{\pm}_m}\left(\cos\vartheta\sinh\eta\,\frac{\partial}{\partial\vartheta}+\sin\vartheta\cosh\eta\,\frac{\partial}{\partial\eta}\right)U^{\pm}_{n+1,m}[\mu]\\
&\quad= (n+m+1)(n+2-m)\Big(-\cos\vartheta\sinh\eta\,P^{m-1}_{n+1}(\cos\vartheta)\,Q^{m}_{n+1}(\cosh\eta)\\
&\qquad+\sin\vartheta\cosh\eta\,P^{m}_{n+1}(\cos\vartheta)\,Q^{m-1}_{n+1}(\cosh\eta)\Big)\\
&\qquad+\cos\vartheta\sinh\eta\,P^{m+1}_{n+1}(\cos\vartheta)\,Q^{m}_{n+1}(\cosh\eta)\\
&\qquad+\sin\vartheta\cosh\eta\,P^{m}_{n+1}(\cos\vartheta)\,Q^{m+1}_{n+1}(\cosh\eta).
\end{aligned}
\tag{68}
$$

Next, we use the relations (7)–(14) and (12)–(18), obtaining

$$
\begin{aligned}
&\cos\vartheta\sinh\eta\,P^{m+1}_{n+1}(\cos\vartheta)\,Q^{m}_{n+1}(\cosh\eta)\\
&\quad+\sin\vartheta\cosh\eta\,P^{m}_{n+1}(\cos\vartheta)\,Q^{m+1}_{n+1}(\cosh\eta)\\
&\quad=\frac{\mu^{n+3}(\cosh^2\eta-\cos^2\vartheta)}{(n+1-m)(n+m+2)\beta_{n+1,m+1}}\,V_{n,m+1}[\mu].
\end{aligned}
$$

Furthermore, using (10) and its counterpart (17), we arrive at

$$
\begin{aligned}
-\frac{\mu^{n+3}(\cosh^2\eta-\cos^2\vartheta)}{\beta_{n+1,m-1}}\,V_{n,m-1}[\mu] &= \sin\vartheta\cosh\eta\,P^{m}_{n+1}(\cos\vartheta)\,Q^{m-1}_{n+1}(\cosh\eta)\\
&\quad-\cos\vartheta\sinh\eta\,P^{m-1}_{n+1}(\cos\vartheta)\,Q^{m}_{n+1}(\cosh\eta).
\end{aligned}
$$

With these calculations at hand, we have

$$
-\frac{1}{\mu(\cosh^2\eta-\cos^2\vartheta)}\left(\sin\vartheta\cosh\eta\frac{\partial}{\partial\eta}+\cos\vartheta\sinh\eta\frac{\partial}{\partial\vartheta}\right)U^{\pm}_{n+1,m}[\mu]
$$

$$= \frac{(n+2-m)}{2} V_{n,m-1}[\mu]\Phi_m^{\pm} - \frac{1}{2(n+1-m)} V_{n,m+1}[\mu]\Phi_m^{\pm}.$$

Similarly, according to (11)–(18) and (12)–(19), one can prove that

$$\begin{aligned}
&\frac{1}{\mu \sin\vartheta \sinh\eta} \frac{\partial}{\partial\varphi} U_{n+1,m}^{\pm}[\mu] \\
&= \mp \frac{m\,\beta_{n+1,m}}{\mu^{n+3}(\cosh^2\eta - \cos^2\vartheta)} \Phi_m^{\mp} \\
&\times \left(\frac{\sinh\eta\, P_{n+1}^m(\cos\vartheta) Q_{n+1}^m(\cosh\eta)}{\sin\vartheta} + \frac{\sin\vartheta\, P_{n+1}^m(\cos\vartheta) Q_{n+1}^m(\cosh\eta)}{\sinh\eta} \right) \\
&= \pm\frac{1}{2}\left(\frac{1}{n+1-m} V_{n,m+1}[\mu] + (n+2+m) V_{n,m-1}[\mu] \right) \Phi_m^{\mp}. \qquad (69)
\end{aligned}$$

Combining the above formulas, together with the relations

$$\Phi_m^{\pm}\cos\varphi \pm \Phi_m^{\mp}\sin\varphi = \Phi_{m-1}^{\pm}, \qquad (70)$$
$$-\Phi_m^{\pm}\cos\varphi \pm \Phi_m^{\mp}\sin\varphi = -\Phi_{m+1}^{\pm}, \qquad (71)$$
$$\Phi_m^{\pm}\sin\varphi \mp \Phi_m^{\mp}\cos\varphi = \mp\Phi_{m-1}^{\mp}, \qquad (72)$$
$$-\Phi_m^{\pm}\sin\varphi \mp \Phi_m^{\mp}\cos\varphi = \mp\Phi_{m+1}^{\mp}, \qquad (73)$$

one straightforwardly obtains the desired expressions for $(\partial/\partial x_1)U_{n+1,m}^{\pm}[\mu]$ and $(\partial/\partial x_2)U_{n+1,m}^{\pm}[\mu]$.

Now, we compute (67). By (68), we find

$$\begin{aligned}
&\frac{2\,\mu^{n+2}}{\beta_{n+1,n+1}\Phi_{n+1}^{\pm}} \left(\cos\vartheta \sinh\eta \frac{\partial}{\partial\vartheta} + \sin\vartheta \cosh\eta \frac{\partial}{\partial\eta} \right) U_{n+1,n+1}^{\pm}[\mu] \\
&= 2(n+1)\Big(-\cos\vartheta \sinh\eta\, P_{n+1}^n(\cos\vartheta) Q_{n+1}^{n+1}(\cosh\eta) \\
&\quad + \sin\vartheta \cosh\eta\, P_{n+1}^{n+1}(\cos\vartheta) Q_{n+1}^n(\cosh\eta) \Big) \\
&\quad + \sin\vartheta \cosh\eta\, P_{n+1}^{n+1}(\cos\vartheta) Q_{n+1}^{n+2}(\cosh\eta) \\
&= \mu^{n+3}(\cosh^2\eta - \cos^2\vartheta) \left(\frac{\mu(2n+2)!\cosh\eta}{2^{n+3}(1/2)_{n+3}\cos\vartheta} V_{n+1,n+2}[\mu] - \frac{2(n+1)}{\beta_{n+1,n}} V_{n,n}[\mu] \right).
\end{aligned}$$

On the other hand, we find from (69) that

$$\begin{aligned}
&\frac{1}{\mu \sin\vartheta \sinh\eta} \frac{\partial}{\partial\varphi} U_{n+1,n+1}^{\pm}[\mu] \\
&= \pm \frac{(n+1)\,\beta_{n+1,n+1}}{\beta_{n+1,n}} V_{n,n}[\mu]\Phi_{n+1}^{\mp}
\end{aligned}$$

$$\mp\frac{\beta_{n+1,n+1}}{2(2n+3)\mu^{n+3}(\cosh^2\eta-\cos^2\vartheta)}\,P_{n+2}^{n+2}(\cos\vartheta)Q_n^{n+2}(\cosh\eta)\Phi_{n+1}^{\mp}.$$

Using the expressions (70)–(73) for $m = n+1$, the required expression (67) follows. This establishes the proof of the theorem. □

To motivate the relevance of the expressions stated in Theorem 12, explicit recurrence rules between the basic external spheroidal monogenics are discussed below.

Proposition 13 *For each $n \geq 0$ and $0 \leq m \leq n$, the following recursive formula holds:*

$$\begin{aligned}\mathbf{X}_{n,m}^{\pm}[\mu] &= \frac{(n+2-m)}{2}\Big(\mathbf{X}_{n,m-1}^{\pm}[\mu]\,\mathbf{i}\mp\mathbf{X}_{n,m-1}^{\mp}[\mu]\,\mathbf{j}\Big)\\ &\quad-\frac{1}{2(n+1-m)}\Big(\mathbf{X}_{n,m+1}^{\pm}[\mu]\,\mathbf{i}\pm\mathbf{X}_{n,m+1}^{\mp}[\mu]\,\mathbf{j}\Big).\end{aligned} \tag{74}$$

For $m = 0$, we must take into account that

$$\mathbf{X}_{n,-1}^{\pm}[\mu] = \mp\frac{1}{(n+1)(n+2)}\,\mathbf{X}_{n,1}^{\pm}[\mu].$$

Proof The proof is an immediate consequence of Theorem 12 by direct inspection of the relations between the quaternionic components of the basic monogenics (66)–(67). □

From this, we easily deduce the further result:

Corollary 2 *For each $n \geq 0$ and $0 \leq m \leq n+1$, the following recursive formula holds:*

$$(n+2-m)\Big(\mathbf{X}_{n,m-1}^{+}[\mu]-\mathbf{X}_{n,m-1}^{-}[\mu]\mathbf{k}\Big)+\mathbf{X}_{n,m}^{+}[\mu]\mathbf{i}+\mathbf{X}_{n,m}^{-}[\mu]\mathbf{j}=0.$$

4.3 Conversions Among Spheroidal and Solid Spherical Monogenics

As a significant consequence of Theorem 12 and Proposition 3, together with (45), we obtain an explicit representation of the *external solid spherical monogenics* employing the external solid spherical harmonics.

Corollary 3 *For all $\mathbf{x} \in \mathbb{R}^3\backslash\{\mathbf{0}\}$, the limits $\lim_{\mu\to 0}\mathbf{X}_{-1,0}^{+}[\mu](\mathbf{x})$ and $\lim_{\mu\to 0}\mathbf{X}_{n,m}^{\pm}[\mu](\mathbf{x})$ exist and are given, respectively, by*

$$\begin{aligned}\mathbf{X}_{-1,0}^{+}[0](\mathbf{x}) &= -\frac{\overline{\mathbf{x}}}{|\mathbf{x}|^3},\\ \mathbf{X}_{n,m}^{\pm}[0](\mathbf{x}) &= -(n+2-m)U_{n+2,m}^{\pm}[0](\mathbf{x})\\ &\quad-\frac{\mathbf{i}}{2}\Big((n+2-m)(n+3-m)U_{n+2,m-1}^{\pm}[0](\mathbf{x})-U_{n+2,m+1}^{\pm}[0](\mathbf{x})\Big)\end{aligned}$$

$$\pm \frac{\mathbf{j}}{2}\bigg((n+2-m)(n+3-m)U^{\mp}_{n+2,m-1}[0](\mathbf{x}) + U^{\mp}_{n+2,m+1}[0](\mathbf{x})\bigg),$$

where the $U^{\pm}_{n,m}[0]$ have the same meaning as in (28).

This result shows that $\mathbf{X}^{+}_{-1,0}[\mu]$ leads to the Cauchy–Fueter kernel, except for the normalization factor $-1/4\pi$ when $\mu \to 0$. This observation is fundamental not only to ensure that the basic external spheroidal monogenics (64) are well-defined on the exterior of the prescribed spheroids Ω_μ but also it gives evidence of the completeness of these functions in the space $\mathcal{M}_2(\Omega^*_\mu)$ (see Theorem 18 below).

Given Corollary 1, it is natural to find the direct and inverse transformation formulas that permit passing from external solid spherical to spheroidal monogenics.

Theorem 14 *Let $n \geq 0$ and $0 \leq m \leq n+1$. Then*

$$\mathbf{X}^{\pm}_{n,m}[\mu] = \sum_{k=0}^{\infty} \alpha_{n+1,m,k}\, \mu^{2k}\, \mathbf{X}^{\pm}_{n+2k,m}[0], \tag{75}$$

$$\mathbf{X}^{\pm}_{n,m}[0] = \sum_{k=0}^{\infty} \widetilde{\alpha}_{n+1,m,k}\, \mu^{2k}\, \mathbf{X}^{\pm}_{n+2k,m}[\mu]. \tag{76}$$

Here $\alpha_{n,m,k}$ and $\widetilde{\alpha}_{n,m,k}$ have the same meaning as in Proposition 8.

Proof For simplicity in the proof, we only prove the direct transformation formula (75). The identity (76) can be established similarly. We fix n, m, μ, and the choice of sign $\pm$. According to (66), we want to show that $\mathbf{X}^{\pm}_{n,m}[\mu]$ is equal to

$$A = \sum_{k=0}^{\infty} \mu^{2k}\bigg(A_{0,k} + \frac{\mathbf{i}}{2}A_{1,k} \mp \frac{\mathbf{j}}{2}A_{2,k}\bigg),$$

where the quaternionic components are given by

$$\begin{aligned}
A_{0,k} &= \alpha_{n+1,m,k} V^{\pm}_{n+2k,m}[0],\\
A_{1,k} &= (n+2-m+2k)\alpha_{n+1,m,k} V^{\pm}_{n+2k,m-1}[0]\\
&\quad - \frac{1}{n+1-m+2k}\,\alpha_{n+1,m,k} V^{\pm}_{n+2k,m+1}[0],\\
A_{2,k} &= (n+2-m+2k)\alpha_{n+1,m,k} V^{\mp}_{n+2k,m-1}[0]\\
&\quad + \frac{1}{n+1-m+2k}\,\alpha_{n+1,m,k} V^{\mp}_{n+2k,m+1}[0].
\end{aligned}$$

According to (48), we obtain

$$\begin{aligned}
A_{1,k} &= (n+2-m)\alpha_{n+1,m-1,k} V^{\pm}_{n+2k,m-1}[0]\\
&\quad - \frac{1}{n+1-m}\,\alpha_{n+1,m+1,k} V^{\pm}_{n+2k,m+1}[0]
\end{aligned}$$

and

$$A_{2,k} = (n+2-m)\alpha_{n+1,m-1,k} V^{\mp}_{n+2k,m-1}[0] + \frac{1}{n+1-m} \alpha_{n+1,m+1,k} V^{\mp}_{n+2k,m+1}[0],$$

By Corollary 1, it follows that

$$\sum_{k=0}^{\infty} A_{0,k} = V^{\pm}_{n,m}[\mu],$$

$$\sum_{k=0}^{\infty} A_{1,k} = (n+2-m) V^{\pm}_{n,m-1}[\mu] + \frac{1}{n+1-m} V^{\pm}_{n,m+1}[\mu],$$

$$\sum_{k=0}^{\infty} A_{2,k} = (n+2-m) V^{\mp}_{n,m-1}[\mu] - \frac{1}{n+1-m} V^{\mp}_{n,m+1}[\mu].$$

This justifies the assertion $\mathbf{X}^{\pm}_{n,m}[\mu] = A$ according to (66). Similarly, we can show that (75) holds when $m = n+1$ in view of (67). The statement is established. □

4.4 Orthogonality Properties of the Basic Monogenics

In this section, we prove that the basic external spheroidal monogenics (64) are orthogonal over the exterior and surface of the prescribed spheroids Ω_{μ}.

We have

Theorem 15 *Let μ be fixed. For each $n \geq 0$, the collection*

$$\{\mathbf{X}^{+}_{-1,0}[\mu]\} \cup \{\mathbf{X}^{+}_{n,m}[\mu] \colon 0 \leq m \leq n+1\} \cup \{\mathbf{X}^{-}_{n,m}[\mu] \colon 1 \leq m \leq n+1\} \tag{77}$$

forms an orthogonal family in $\mathcal{M}_2(\Omega^*_{\mu})$ in the sense of the scalar inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L_2(\Omega^*_{\mu},\, \mathbb{R}^3)} = \mathrm{Sc} \iiint_{\Omega^*_{\mu}} \overline{\mathbf{f}}(\mathbf{x}) \mathbf{g}(\mathbf{x})\, d\mathbf{x}. \tag{78}$$

Their norms squared are equal to

$$\|\mathbf{X}^{+}_{-1,0}[\mu]\|_2^2 = \frac{4\pi}{\mu} \tanh^{-1}(\mu),$$

$$\|\mathbf{X}^{\pm}_{n,m}[\mu]\|_2^2 = \frac{\pi\, 4^{n+1} (1/2)_{n+1} (1/2)_{n+2}}{\mu^{2n+3}} \Bigg(\frac{2}{(n+1)^2 (n!)^2} \delta_{0,m} I_{n,1}(\mu) + \frac{1}{(n+1+m)!(n+1-m)!} \big(4(n+2-m)(n+1+m)$$

$$(1+\delta_{0,m})I_{n,m}(\mu) + ((n+1)^2-(m-1)^2)((n+1)^2-m^2)I_{n,m-1}(\mu) + I_{n,m+1}(\mu)\Big)$$

for $0 \leq m \leq n$, *and*

$$\|\mathbf{X}^{\pm}_{n,n+1}[\mu]\|_2^2 = \frac{2^{2n+5}\pi(n+1)}{\mu^{2n+3}}\Big(I_{n,n+1}(\mu) + (2n+1)(1+\delta_{0,n})I_{n,n}(\mu) + \frac{1}{8(n+1)}I_{n,n+2}(\mu)\Big),$$

where $I_{n,m}(\mu)$ *has the same meaning as in* (36).

Proof Combining (65) and (66), and applying the trigonometric identities

$$\cos\varphi\,\Phi^{\pm}_{m-1} \mp \sin\varphi\,\Phi^{\mp}_{m-1} = \Phi^{\pm}_{m},$$
$$\cos\varphi\,\Phi^{\pm}_{m+1} \pm \sin\varphi\,\Phi^{\mp}_{m+1} = \Phi^{\pm}_{m},$$

direct computation shows that

$$\begin{aligned}
&\mathrm{Sc}(\overline{\mathbf{X}^{+}_{-1,0}[\mu]}\,\mathbf{X}^{\pm}_{n,m}[\mu])\,\mu^3(\cosh^2\eta - \cos^2\vartheta)\sin\vartheta\sinh\eta \\
&= -\mu\Big(\cos\vartheta\sin\vartheta\sinh\eta\,V_{n,m}[\mu] - \frac{(n+2-m)}{2}\cosh\eta\sin^2\vartheta\,V_{n,m-1}[\mu] \\
&\quad + \frac{(n+2-m)}{2}\cosh\eta\sin^2\vartheta\,V_{n,m+1}[\mu]\Big)\Phi^{\pm}_{m}(\varphi).
\end{aligned}$$

According to the definition of the integral (78) and Proposition 9, the identities $\int_0^{2\pi}\Phi^{\pm}_m(\varphi)d\varphi = 0$ for $m > 0$ imply that

$$\begin{aligned}
&\langle \mathbf{X}^{+}_{-1,0}[\mu], \mathbf{X}^{\pm}_{n,m}[\mu]\rangle_{L_2(\Omega^*_\mu,\,\mathbb{R}^3)} \\
&= \iiint_{\Omega^*_\mu}\Big([\mathbf{X}^{+}_{-1,0}[\mu]]_0[\mathbf{X}^{\pm}_{n,m}[\mu]]_0 + [\mathbf{X}^{+}_{-1,0}[\mu]]_1[\mathbf{X}^{\pm}_{n,m}[\mu]]_1 \\
&\quad + [\mathbf{X}^{+}_{-1,0}[\mu]]_2[\mathbf{X}^{\pm}_{n,m}[\mu]]_2\Big)d\mathbf{x} \\
&= -2\pi\mu\Big(\int_{\eta_\mu}^{\infty}\sinh\eta\Big(\int_0^{\pi}V_{n,0}[\mu]P_1(\cos\vartheta)\sin\vartheta\,d\vartheta\Big)d\eta \\
&\quad + \frac{1}{n+1}\int_{\eta_\mu}^{\infty}\cosh\eta\Big(\int_0^{\pi}V_{n,1}[\mu]P_1^1(\cos\vartheta)\sin\vartheta\,d\vartheta\Big)d\eta\Big) \\
&= 0
\end{aligned}$$

for all $n \in \mathbb{N}_0$.

Likewise, by the same reasoning,

$$
\begin{aligned}
&\langle \mathbf{X}^{+}_{-1,0}[\mu], \mathbf{X}^{\pm}_{n,n+1}[\mu]\rangle_{L_2(\Omega^*_\mu,\, \mathbb{R}^3)} \\
&= -\mu\bigg(\int_{\eta_\mu}^{\infty}\int_0^{\pi}\bigg(\cos\vartheta\,\sin\vartheta\,\sinh\eta\; V_{n,n+1}[\mu] - \frac{1}{2}\sin^2\vartheta\,\cosh\eta\; V_{n,n}[\mu] \\
&\quad + \frac{\mu\sin^2\vartheta\,\cosh^2\eta}{4(2n+5)\cos\vartheta}\,V_{n+1,n+2}[\mu]\bigg)d\eta d\vartheta\bigg)\int_0^{2\pi}\Phi^{\pm}_{n+1}(\varphi)d\varphi \\
&= 0
\end{aligned}
$$

for all $n \in \mathbb{N}_0$.

Now, we compute $\langle \mathbf{X}^{\pm}_{n_1,m_1}[\mu], \mathbf{X}^{\pm}_{n_2,m_2}[\mu]\rangle_{L_2(\Omega^*_\mu,\, \mathbb{R}^3)}$. According to (66) and Theorem 10, we have

$$
\iiint_{\Omega^*_\mu}[\mathbf{X}^{\pm}_{n_1,m_1}[\mu]]_0[\mathbf{X}^{\pm}_{n_2,m_2}[\mu]]_0\,d\mathbf{x} = \|V^{\pm}_{n_1,m_1}[\mu]\|_2^2\,\delta_{n_1,n_2}\delta_{m_1,m_2}. \tag{79}
$$

Thus, to verify the orthogonality of the $\mathbf{X}^{\pm}_{n,m}[\mu]$, it suffices to show that the vector parts of the functions $\mathbf{X}^{\pm}_{n,m}[\mu]$ are orthogonal.

Expanding the integrands and applying the trigonometric identities

$$
\Phi^{\pm}_{m_1-1}\Phi^{\pm}_{m_2-1} + \Phi^{\mp}_{m_1-1}\Phi^{\mp}_{m_2-1} = \Phi^{+}_{m_1-m_2}, \tag{80}
$$

$$
\Phi^{\pm}_{m_1+1}\Phi^{\pm}_{m_2+1} + \Phi^{\mp}_{m_1+1}\Phi^{\mp}_{m_2+1} = \Phi^{+}_{m_1-m_2}, \tag{81}
$$

$$
-\Phi^{\pm}_{m_1-1}\Phi^{\pm}_{m_2+1} + \Phi^{\mp}_{m_1-1}\Phi^{\mp}_{m_2+1} = \mp\Phi^{+}_{m_1+m_2}, \tag{82}
$$

$$
-\Phi^{\pm}_{m_1+1}\Phi^{\pm}_{m_2-1} + \Phi^{\mp}_{m_1+1}\Phi^{\mp}_{m_2-1} = \mp\Phi^{+}_{m_1+m_2}, \tag{83}
$$

we find

$$
\begin{aligned}
&\iiint_{\Omega^*_\mu}\big([\mathbf{X}^{\pm}_{n_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{n_2,m_2}[\mu]]_1 + [\mathbf{X}^{\pm}_{n_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{n_2,m_2}[\mu]]_2\big)d\mathbf{x} \\
&= \frac{1}{4}\bigg(p_1p_2\iiint_{\Omega^*_\mu}V_{n_1,m_1-1}[\mu]V_{n_2,m_2-1}[\mu]\Phi^{+}_{m_1-m_2}\,d\mathbf{x} \\
&\quad \mp\frac{p_1}{p_2-1}\iiint_{\Omega^*_\mu}V_{n_1,m_1-1}[\mu]V_{n_2,m_2+1}[\mu]\Phi^{+}_{m_1+m_2}\,d\mathbf{x} \\
&\quad \mp\frac{p_2}{p_1-1}\iiint_{\Omega^*_\mu}V_{n_1,m_1+1}[\mu]V_{n_2,m_2-1}[\mu]\Phi^{+}_{m_1+m_2}\,d\mathbf{x} \\
&\quad + \frac{1}{(p_1-1)(p_2-1)}\iiint_{\Omega^*_\mu}V_{n_1,m_1+1}[\mu]V_{n_2,m_2+1}[\mu]\Phi^{+}_{m_1-m_2}\,d\mathbf{x}\bigg),
\end{aligned}
$$

where $p_i = n_i + 2 - m_i$ $(i = 1, 2)$.

We continue the calculations only for the prolate case. The following identities $\int_0^{2\pi} \Phi^+_{m_1 \pm m_2}(\varphi) d\varphi = 2\pi \delta_{m_1,m_2}$ for $m_1, m_2 > 0$ imply that

$$\iiint_{\Omega^*_\mu} \big([\mathbf{X}^\pm_{n_1,m_1}[\mu]]_1 [\mathbf{X}^\pm_{n_2,m_2}[\mu]]_1 + [\mathbf{X}^\pm_{n_1,m_1}[\mu]]_2 [\mathbf{X}^\pm_{n_2,m_2}[\mu]]_2\big) d\mathbf{x}$$
$$= \frac{\pi p_1 (n_2 + 2 - m_1)}{2} \delta_{m_1,m_2} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,m_1-1}[\mu] V_{n_2,m_1-1}[\mu]\, dR$$
$$- \frac{\pi (n_1 + 2)}{2(n_2 + 1)} \delta_{m_1,0} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,-1}[\mu] V_{n_2,1}[\mu]\, dR$$
$$- \frac{\pi (n_2 + 2)}{2(n_1 + 1)} \delta_{m_1,0} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,1}[\mu] V_{n_2,-1}[\mu]\, dR$$
$$+ \frac{\pi}{2(p_1 - 1)(n_2 + 1 - m_1)} \delta_{m_1,m_2} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,m_1+1}[\mu] V_{n_2,m_1+1}[\mu]\, dR,$$

where dR has the same meaning as in (34).

In consequence, using (39), we find

$$\iiint_{\Omega^*_\mu} \big([\mathbf{X}^\pm_{n_1,m_1}[\mu]]_1 [\mathbf{X}^\pm_{n_2,m_2}[\mu]]_1 + [\mathbf{X}^\pm_{n_1,m_1}[\mu]]_2 [\mathbf{X}^\pm_{n_2,m_2}[\mu]]_2\big) d\mathbf{x}$$
$$= \frac{\pi p_1 (n_2 + 2 - m_1)}{2} \delta_{m_1,m_2} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,m_1-1}[\mu] V_{n_2,m_1-1}[\mu]\, dR$$
$$+ \frac{\pi}{(n_1 + 1)(n_2 + 1)} \delta_{m_1,0} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,1}[\mu] V_{n_2,1}[\mu]\, dR$$
$$+ \frac{\pi}{2(p_1 - 1)(n_2 + 1 - m_1)} \delta_{m_1,m_2} \int_{\eta_\mu}^\infty \int_0^\pi V_{n_1,m_1+1}[\mu] V_{n_2,m_1+1}[\mu]\, dR.$$

Now, using again the orthogonality of Theorem 10, we are left with

$$\iiint_{\Omega^*_\mu} \big([\mathbf{X}^\pm_{n_1,m_1}[\mu]]_1 [\mathbf{X}^\pm_{n_2,m_2}[\mu]]_1 + [\mathbf{X}^\pm_{n_1,m_1}[\mu]]_2 [\mathbf{X}^\pm_{n_2,m_2}[\mu]]_2\big) d\mathbf{x}$$
$$= \frac{\pi}{2\mu^{2n_1+3}} \Bigg(p_1^2 \gamma_{n_1,m_1-1} I_{n_1,m_1-1}(\mu) + \frac{2}{(n_1+1)^2} \gamma_{n_1,1} I_{n_1,1}(\mu) \delta_{m_1,0}$$
$$+ \frac{1}{(p_1 - 1)^2} \gamma_{n_1,m_1+1} I_{n_1,m_1+1}(\mu) \Bigg) \delta_{n_1,n_2} \delta_{m_1,m_2}, \qquad (84)$$

with $I_{n,m}(\mu)$ defined by (36). Combining (79) and (84), we conclude that $\langle \mathbf{X}^+_{n_1,m_1}[\mu], \mathbf{X}^+_{n_2,m_2}[\mu] \rangle_{L_2(\Omega^*_\mu, \mathbb{R}^3)} = 0$ when $n_1 \neq n_2$ or $m_1 \neq m_2$. Similarly, $\langle \mathbf{X}^-_{n_1,m_1}[\mu], \mathbf{X}^-_{n_2,m_2}[\mu] \rangle_{L_2(\Omega^*_\mu, \mathbb{R}^3)} = 0$ when $n_1 \neq n_2$ or $m_1 \neq m_2$. Using the orthogonality of the system $\{\Phi^\pm_m\}$ on $[0, 2\pi]$ again, we conclude that $\langle \mathbf{X}^\pm_{n_1,m_1}[\mu], \mathbf{X}^\mp_{n_2,m_2}[\mu] \rangle_{L_2(\Omega^*_\mu, \mathbb{R}^3)} = 0$ when the indices do not coincide.

Now, applying the trigonometric identities

$$-\Phi^{\pm}_{m_1-1}\,\Phi^{\pm}_{n_2+2}+\Phi^{\mp}_{m_1-1}\,\Phi^{\mp}_{n_2+2}=\mp\Phi^{+}_{m_1+n_2+1},$$
$$\Phi^{\pm}_{m_1+1}\,\Phi^{\pm}_{n_2+2}+\Phi^{\mp}_{m_1+1}\,\Phi^{\mp}_{n_2+2}=\Phi^{+}_{m_1-n_2-1},$$

it follows from (43) and (67), in combination with Proposition 9, that

$$\langle \mathbf{X}^{\pm}_{n_1,m_1}[\mu],\mathbf{X}^{\pm}_{n_2,n_2+1}[\mu]\rangle_{L_2(\Omega^*_\mu,\,\mathbb{R}^3)}=0.$$

Similarly,

$$\langle \mathbf{X}^{\pm}_{n_1,m_1}[\mu],\mathbf{X}^{\mp}_{n_2,n_2+1}[\mu]\rangle_{L_2(\Omega^*_\mu,\,\mathbb{R}^3)}=0.$$

Furthermore, using the trigonometric identities (80)–(83) for $m_1=n_1+1$ and $m_2=n_2+1$, by (67) and (58), it follows that

$$\begin{aligned}&\langle \mathbf{X}^{\pm}_{n_1,n_1+1}[\mu],\mathbf{X}^{\pm}_{n_2,n_2+1}[\mu]\rangle_{L_2(\Omega^*_\mu,\,\mathbb{R}^3)}\\&\quad=\big(2\,\gamma_{n_1,n_1+1}I_{n_1,n_1+1}(\mu)+\gamma_{n_1,n_1}(1+\delta_{0,n_1})I_{n_1,n_1}(\mu)\big)\,\frac{\pi}{\mu^{2n_1+3}}\,\delta_{n_1,n_2}\\&\qquad+\frac{\mu^2\pi}{8(2n_1+5)(2n_2+5)}\,\delta_{n_1,n_2}\int_{\eta_\mu}^{\infty}\int_0^{\pi}\frac{V_{n_1+1,n_1+2}[\mu]V_{n_2+1,n_2+2}[\mu]\cosh^2\eta}{\cos^2\vartheta}\,dR.\end{aligned}\tag{85}$$

This establishes the orthogonality statement.

The calculation of the L_2-norms of $\mathbf{X}^{\pm}_{n,m}[\mu]$ for $0\le m\le n$ comes from taking $n_1=n_2$ and $m_1=m_2$ in (84) and adding expression (58). By the symmetric form of the $\mathbf{X}^{\pm}_{n,m}[\mu]$ in (66), it follows that $\|\mathbf{X}^{+}_{n,m}[\mu]\|_2=\|\mathbf{X}^{-}_{n,m}[\mu]\|_2$ when $m\neq 0$. Furthermore, taking $n_1=n_2$ in (85) and combining (43), (55), and (56), we find

$$\begin{aligned}&\|\mathbf{X}^{\pm}_{n,n+1}[\mu]\|_2^2\\&\quad=\big(2\,\gamma_{n,n+1}I_{n,n+1}(\mu)+\gamma_{n,n}(1+\delta_{0,n})I_{n,n}(\mu)\big)\,\frac{\pi}{\mu^{2n+3}}\\&\qquad+\frac{\pi\,4^{n+2}((1/2)_{n+3})^2}{(2n+5)^2(2n+3)!\,\mu^{2n+3}}\int_{\eta_\mu}^{\infty}\cosh\eta\,Q^{n+2}_{n+1}(\cosh\eta)\,Q^{n+2}_{n+2}(\cosh\eta)\sinh\eta\,d\eta.\end{aligned}$$

Using the relation $Q^{n+2}_{n}(\cosh\eta)=\sinh\eta\,Q^{n+2}_{n+1}(\cosh\eta)$, which follows from (56), the statement follows. The proof is now completed. □

The corresponding orthogonality of the basic spheroidal monogenics over the surface of the prescribed spheroids follows immediately from Theorems 11 and 15.

Theorem 16 *Let $|\zeta(\mu,\mathbf{x})|$ have the same meaning as in (24). For fixed μ, the collection (77) forms an orthogonal family over the surface of the spheroids Ω_μ in the sense of* *the scalar product*

$$\langle \mathbf{f}, \mathbf{g} \rangle_{L_2(\partial\Omega_\mu, \mathbb{R}^3)} = \mathrm{Sc} \iint_{\eta=\eta_\mu} \overline{\mathbf{f}}(\mathbf{x})\mathbf{g}(\mathbf{x})\, |\zeta(\mu, \mathbf{x})|^{1/2}\, d\sigma. \tag{86}$$

4.5 An Orthogonal Basis of External Spheroidal Monogenics

The results of the preceding sections enable us to prove the following theorem about the approximation of a smooth function $\mathbf{f}$ in $\mathcal{M}_2(\Omega_\mu^*)$ expanded as a linear combination of basic external spheroidal monogenics. Although the proof of this fact follows the same argument as in [19], since it is necessary to employ the definitions (25) and (64), we include the proof for the completeness of the presentation.

Theorem 17 *Suppose* $\mathbf{f} \in \mathcal{M}_2(\Omega_\mu^*) \cap C^1(\Gamma_\mu, \mathbb{R}^3)$ *and let* μ *be fixed. The monogenic Fourier series expansion of* $\mathbf{f}$ *with respect to the collection* (77)*,*

$$\sum_{n=-1}^{\infty} \sum_{m=0}^{n+1} \frac{1}{\|\mathbf{X}_{n,m}^{\pm}[\mu]\|_2} \left(a_{n,m}^{+}[\mu]\, \mathbf{X}_{n,m}^{+}[\mu] + a_{n,m}^{-}[\mu]\, \mathbf{X}_{n,m}^{-}[\mu] \right) \tag{87}$$

converges to $\mathbf{f}$ *in the* L_2*-sense, where the (real) Fourier coefficients* $a_{k,m}^{\pm}[\mu]$ *are given by*

$$a_{k,m}^{\pm}[\mu] = \frac{1}{\|\mathbf{X}_{k,m}^{\pm}[\mu]\|_2} \langle \mathbf{f}, \mathbf{X}_{k,m}^{\pm}[\mu] \rangle_{L_2(\Omega_\mu^*, \mathbb{R}^3)}.$$

Proof Suppose that $\mathbf{f} \in \mathcal{M}_2(\Omega_\mu^*)$. Hence, there exists a real-valued harmonic function h in Ω_μ^* such that $(1/2)\partial h = \mathbf{f}$. Moreover, the restriction of h on Γ_μ is a twice continuously differentiable function. Now, let $g(\vartheta, \varphi)$ be a function defined on the unit sphere, which is related to the value of $Tr_{\Gamma_\mu} h$ on the prescribed spheroid by $g(\vartheta, \varphi) = Tr_{\Gamma_\mu} h(\vartheta, \varphi) = h(\eta_\mu, \vartheta, \varphi)$, with $\cosh \eta_\mu = 1/\mu$ by (22). (The trace operator Tr_{Γ_μ} describes just the restriction onto the boundary Γ_μ.) Since $g(\vartheta, \varphi)$ is a twice continuously differentiable function on the unit sphere, it can be expressed employing a series of surface spherical harmonics,

$$g(\vartheta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} P_n^m(\cos\vartheta)\big(\alpha_{n,m}^{+} \Phi_m^{+}(\varphi) + \alpha_{n,m}^{-} \Phi_m^{-}(\varphi)\big) = Tr_{\Gamma_\mu} h(\vartheta, \varphi). \tag{88}$$

It was shown in [15, 25] that the above expansion is absolutely and uniformly convergent with respect to $(\vartheta, \varphi) \in \Gamma_0$.

For simplicity, we again assume that $\mu \in (0, 1)$. Extending (88) to Ω_μ^* leads to a series expansion of h in terms of the basic external spheroidal harmonics (25):

$$h = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{\beta_{n,m}}{\mu^{n+1}} \frac{Q_n^m(\cosh\eta)}{Q_n^m(1/\mu)}\, P_n^m(\cos\vartheta)\big(\alpha_{n,m}^{+} \Phi_m^{+}(\varphi) + \alpha_{n,m}^{-} \Phi_m^{-}(\varphi)\big). \tag{89}$$

Using the results of [14, pp.417–421], we find $\left|Q_n^m(\cosh\eta)/Q_n^m(1/\mu)\right| < 1$ for all $n, m = 0, 1, \ldots$ and fixed $\mu > 0$. Under the previous circumstances, it then follows that the series (89) converges uniformly and absolutely to h in Ω_μ^*.

Moreover, since h is harmonic in Ω_μ^* and twice continuously differentiable on the boundary Γ_μ, it yields the absolute and uniform convergence of its first derivatives in $\Omega_\mu^* \cup \Gamma_\mu$. In particular, the corresponding series expansion for the derivatives, namely

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha_{n,m}^{+}[\mu]}{Q_n^m(1/\mu)}\,\mathbf{X}_{n-1,m}^{+}[\mu] + \frac{\alpha_{n,m}^{-}[\mu]}{Q_n^m(1/\mu)}\,\mathbf{X}_{n-1,m}^{-}[\mu]\right) \tag{90}$$

converges uniformly and absolutely to $\mathbf{f} = (1/2)\partial h$ in $\Omega_\mu^* \cup \Gamma_\mu$. This further implies the L_2-convergence in every subset $\Omega_\mu^* \cap B_r$, where B_r is a ball with some radius $r > 0$, which contains $\overline{\Omega}_\mu$. More precisely, denote by $\mathbf{S}_N$ the finite sum of the first N-summands in the series (90). For any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that

$$\sup_{\mathbf{x}\in\Omega_\mu^*\cup\Gamma_\mu} |\mathbf{S}_{N(\epsilon)}(\mathbf{x}) - \mathbf{f}(\mathbf{x})| < \epsilon.$$

We have then

$$\|\mathbf{S}_{N(\epsilon)} - \mathbf{f}\|^2_{L_2(\Omega_\mu^*\cap B_r,\mathbb{R}^3)} < \epsilon^2\, vol(\Omega_\mu^* \cap B_r).$$

For the exterior domain of B_r, we use another estimation. Let $0 < r_1 < r$ such that $\Omega_\mu \subset B_{r_1}$. By the Cauchy integral formula for quaternionic monogenic functions [11, pp. 87-88], for all $\mathbf{y} \in \mathbb{R}^3 \backslash B_r$, one finds

$$\begin{aligned}
|\mathbf{S}_{N(\epsilon)}(\mathbf{y}) - \mathbf{f}(\mathbf{y})| &\le \frac{1}{4\pi}\iint_{\partial B_{r_1}} \left|\mathbf{X}_{-1,0}^{+}[0](\mathbf{y}-\mathbf{z})\right| |\mathbf{S}_{N(\epsilon)}(\mathbf{z}) - \mathbf{f}(\mathbf{z})| d\sigma(\mathbf{z})\\
&< \frac{\epsilon}{4\pi}\iint_{\partial B_{r_1}} \frac{1}{|\mathbf{y}|^2 - |\mathbf{z}|^2}\, d\sigma(\mathbf{z})\\
&< \epsilon\,\frac{r_1^2}{|\mathbf{y}|^2 - r_1^2}.
\end{aligned}$$

Now, the L_2-norm of the difference between $\mathbf{S}_N$ and $\mathbf{f}$ can be approximated by

$$\begin{aligned}
\|\mathbf{S}_{N(\epsilon)} - \mathbf{f}\|^2_{L_2(\mathbb{R}^3\backslash B_r,\mathbb{R}^3)} &< (r_1^2\epsilon)^2 \int_r^\infty \int_0^\pi \int_0^{2\pi} \frac{\rho^2 \sin\theta}{(\rho^2 - r_1^2)^2}\, d\varphi d\theta d\rho\\
&< 4\pi (r_1^2\epsilon)^2 \int_r^\infty \frac{\rho^2}{(\rho^2 - r_1^2)^2}\, d\rho\\
&< 4\pi (r_1^2\epsilon)^2 \left(\frac{r}{2(r^2 - r_1^2)} - \frac{1}{4r_1}\log\frac{r - r_1}{r + r_1}\right).
\end{aligned}$$

To sum up, for an arbitrary small $\epsilon > 0$, we can find a natural number $N(\epsilon)$ such that $\|\mathbf{S}_{N(\epsilon)} - \mathbf{f}\|_2 < \epsilon$. Thus, the series expansion (90) converges to $\mathbf{f}$ in the whole domain Ω_μ^* in the sense of the L_2-norm, which is the desired result. The theorem follows. □

From this theorem, we deduce the further result:

Corollary 4 *Suppose* $\mathbf{f} \in \mathcal{M}_2(\Omega_\mu^*)$ *and let* μ *be fixed. Then the restriction of* $\mathbf{f}$ *in* Ω_μ^* *can be represented by its Fourier series expansion of the form* (87). *Further, this series expansion converges to* $\mathbf{f}$ *in the* L_2*-sense.*

Corollary 5 *Any function in the collection*

$$\{\mathbf{X}_{-1,0}^{+}[0]\} \cup \{\mathbf{X}_{n,m}^{+}[0] \colon 0 \leq m \leq n+1\} \cup \{\mathbf{X}_{n,m}^{-}[0] \colon 1 \leq m \leq n+1\} \qquad (91)$$

can be represented by its Fourier series expansion of the form (87).

To conclude, the general result is that

Theorem 18 *For fixed* μ*, the collection* (77) *forms an orthogonal basis of* $\mathcal{M}_2(\Omega_\mu^*)$.

Proof This result is proved by first approximating $\mathbf{f}$ in $\mathcal{M}_2(\Omega_\mu^*)$ from the set (91) (which, in fact, forms an orthogonal basis of $\mathcal{M}_2(\Omega_0^*)$, see [19]), and then by the collection (77) of basic external spheroidal monogenics. □

This theorem is the generalization of that of [19], which corresponds to the case of prolate spheroids.

5 Concluding Remarks

We introduced a single one-parameter family of basic external spheroidal harmonics that assume prescribed values on the boundary of the family of coaxial spheroidal domains Ω_μ. The basic harmonics are functions in x_0, x_1, x_2 which were normalized so that the limiting case $\mu \to 0$ gives the classical external solid spherical harmonics. The nonorthogonality of the basic harmonics in the L_2-Hilbert space structure led to the discussion of the proper external spheroidal harmonics. Underlying our manipulations is a set of conversion formulas that relate the coefficients of the expansions among the spheroidal and spherical harmonic systems. The basic external spheroidal monogenic functions were calculated explicitly, and formulas for their nonscalar parts were obtained in terms of the proper harmonics. Ordinary methods based on decomposing a function space into subspaces of homogeneous functions fail to prove the completeness of a monogenic function system constructed through the basic external spheroidal monogenics because of the appearance of logarithmic functions. The technique we employed to build an orthogonal basis for the one-parameter family of monogenic L_2-Hilbert spaces in the space exterior of Ω_μ was based on the harmonic extension of a function defined on the boundary of the spheroid Ω_μ to the exterior domain Ω_μ^*.

We will extend our results in a future paper to external spheroidal monogenics taking values in the space of full quaternions (with a right $\mathbb{R}$-linear structure), using the theoretical basis presented in [21].

Acknowledgements The author acknowledges financial support by the Asociación Mexicana de Cultura, A. C. The author would like to express his sincere thanks to Raybel A. García A. for his help in creating the tables in Sect. 3.

Author Contribution I state that the manuscript is not published elsewhere, accepted for publication, or under editorial review for publication elsewhere.

Declarations

Conflict of interest The authors declare no competing interests.

References

1. Antonov, V., Baranov, A.: Relation between the expansions of an external potential in spherical functions and spheroidal harmonics. Tech. Phys. **47**(3), 361–363 (2002)
2. Arnaoudov, I., Venkov, G.: Relations between spheroidal and spherical harmonics. Comptes rendus de l'Académie bulgare des sciences: sciences mathématiques et naturelles **7**, 971–978 (2010)
3. Bailey, W.N.: On the product of two Legendre polynomials. Math. Proc. Camb. Philos. Soc. **29**, 173–177 (1933)
4. Byerly, W.E.: An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics, pp. 251–258. Dover, New York (1959)
5. Brackx, F., Delanghe, R., Sommen, F.: Clifford Analysis. Pitman Advanced Publishing Program (1982)
6. Fueter, R.: Functions of a hyper complex variable. Lecture notes written and supplemented by E. Math. Inst. Univ. Zürich, Fall Semester, Bareiss (1949)
7. Garabedian, P.: Orthogonal harmonic polynomials. Pac. J. Math. **3**(3), 585–603 (1953)
8. García Ancona, R., Morais, J., Michael Porter, R.: Contragenic functions on spheroidal domains. Math. Methods Appl. Sci. **31**(7), 2575–2589 (2018)
9. García Ancona, R., Morais, J., Michael Porter, R.: Relations among spheroidal and spherical harmonics. Appl. Math. Comput. **384** (2020)
10. Gürlebeck, K., Sprössig, W.: Quaternionic Analysis and Elliptic Boundary Value Problems. Akademie Verlag, Berlin (1989)
11. Gürlebeck, K., Sprössig, W.: Quaternionic and Clifford Calculus for Physicists and Engineers. Wiley, Chichester (1997)
12. Gürlebeck, K., Habetha, W., Sprößig, W.: Holomorphic Functions in the Plane and n-Dimensional Space. Birkhäuser Verlag, Berlin (2008)
13. Gürlebeck, K., Habetha, W., Sprößig, W.: Application of Holomorphic Functions in Two and Higher Dimensions. Birkhäuser Verlag, Berlin (2016)
14. Hobson, E.: The Theory of Spherical and Ellipsoidal Harmonics. Cambridge (1931)
15. Kalf, H.: On the expansion of a function in terms of spherical harmonics in arbitrary dimensions. Bull. Belg. Math. Soc. Simon Stevin **2**, 361–380 (1995)
16. Morais, J.: A complete orthogonal system of spheroidal monogenics. J. Numer. Anal. Ind. Appl. Math. **6**, 105–119 (2011)
17. Morais, J.: An orthogonal system of monogenic polynomials over prolate spheroids in $\mathbb{R}^3$. Math. Comput. Model. **57**(3–4), 425–434 (2013)
18. Morais, J., Georgiev, S., Sprößig, W.: Real Quaternionic Calculus Handbook. Birkhäuser, Basel (2014)
19. Morais, J., Nguyen, M.H., Kou, K.I.: On 3D orthogonal prolate spheroidal monogenics. Math. Methods Appl. Sci. **39**(4), 635–648 (2016)
20. Morais, J.: Some comments on an orthogonal family of monogenic functions on spheroidal domains. In: Bernstein, S. (ed.) Topics in Clifford Analysis. Trends in Mathematics, pp. 251–266. Birkhäuser, Cham (2019)
21. Morais, J.: A quaternionic version theory related to spheroidal functions. Habilitation thesis, Freiberg University (2021)
22. Neumann, F.: Beiträge zur Theorie der Kugelfunktionen, Leipzig. Teubner, B.G (1878)

23. Nguyen, H.M., Gürlebeck, K., Morais, J., Bock, S.: On orthogonal monogenics in oblate spheroidal domains and recurrence formulae. Integral Transf. Spec. Funct. **25**(7), 513–527 (2014)
24. MacRobert, T.M.: Spherical Harmonics: An Elementary Treatise on Harmonic Functions with Applications. Methuen, London (1927)
25. Ragozin, D.: Uniform convergence of spherical harmonic expansions. Math. Ann. **195**, 87–94 (1972)
26. Riesz, M.: Clifford Numbers and Spinors. Institute for Physical Science, vol. 54. Kluwer Academic Publishers, Dorrecht (1993)
27. Stein, E.M., Weiß, G.: On the theory of harmonic functions of several variables, Part I: The theory of H^p spaces. Acta Math. **103**(1–2), 25–62 (1960)
28. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
29. Sudbery, A.: Quaternionic analysis. Math. Proc. Camb. Philos. Soc. **85**, 199–225 (1979)
30. Suschowk, D.: Formulae for 25 of the associated Legendre functions of the second kind. Math. Tables Other Aids Comput. **13**(68), 303–305 (1959)
31. Szmytkowski, R.: On the derivative of the associated Legendre function of the first kind of integer degree with respect to its order (with applications to the construction of the associated Legendre function of the second kind of integer degree and order). J. Math. Chem. **46**, 231–260 (2009)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor (e.g. a society or other partner) holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.