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Compact Complex *p*-Kähler Hyperbolic Manifolds

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Abstract

In this paper, we generalize tow new notions of hyperbolicity introduced recently in Marouani and Popovici (Balanced hyperbolic and divisorially hyperbolic compact complex manifolds) and we give some properties of this new notions.

Keywords Compact Käler manifolds · Holomorphic function · Metrics

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1 Introduction

Alessandrini and Andreatta [2] generalized the notion of Kähler manifold by introducing the notion of p-Kähler manifold for all $1 \leq p \leq n-1$ where n is the complex dimension of the manifold. As far as our work is concerned, we are basically interested in generalizing the notion of hyperbolic Kähler manifold in the sense of Gromov where we introduce a new notion: p-**Kähler hyperbolic.** We attempt to generalize this concept within the framework of p-Kähler manifolds by presenting examples of p-Kähler hyperbolic manifolds as well as displaying a more general version of (*Theorem 2.6.*) in [6]. In fact, we depict the nonexistence of holomorphic function of maximum rank which has **p-subexponential growth** for $\mathbb{C}^p$ to p-Kähler n-dimensional hyperbolic manifold. Next we prove the non-existence of certain L1 currents on the universal

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covering space of a p-Kähler hyperbolic manifold. Finally we prove vanishing theorem for the L2 harmonic spaces on the universal cover of a class of these manifolds.

2 Basic Definitions

A form α on a complex manifold (X, ω) is bounded with respect to the metric ω if the L^∞-norm of α is finite,

$$||\alpha||_{L^\infty(X)} := \sup_{x \in X} ||\alpha(x)||_\omega < \infty.$$

It called $\tilde{d}$-*bounded* if the lift $\tilde{\alpha}$ of α to the universal covering $\tilde{X} \to X$ is d-*bounded* over $(\tilde{X}, \pi^*(\omega))$ w.r.t $\tilde{\omega} = \pi^*(\omega)$ (see [5]). If X is compact, then every smooth form α is bounded and α is d-*bounded* if and only if it is d-exact. However, if X is non-compact, then an exact bounded form is not necessarily d-*bounded*.

Recall that a Hermitian metric on a complex manifold X identifies with a C^∞ positive definite $(1, 1)$-form ω on X. If we put $\dim_{\mathbb{C}} X = n \geq 2$, a Hermitian metric ω is said to be **Kähler** if ω is d-closed and it said to be **balanced** (see [Gau77] where these metrics were introduced under the name of semi-Kähler and [Mic83] where they were given this name) if $d\omega^{n-1} = 0$. Moreover, ω is said to be **degenerate balanced** (see [Pop15a] for the name) if ω^{n-1} is d-exact. Unlike in the Kähler setting, where no d-exact Hermitian metric ω can exist on a *compact* complex manifold, compact complex manifolds carrying degenerate balanced metrics do exist.

Definition 2.1 (*Grom91*) A compact Kähler complex n-dimensional manifold (X, ω) is called **Kähler hyperbolic** if ω is $\tilde{d}$-*bounded*.

Let X be a *compact* complex manifold. Recall that X is said to be *Brody hyperbolic* ([Bro78, Theorem 4.1]) if there are no entire curves in X, namely there exists no non-constant holomorphic map $f : \mathbb{C} \longrightarrow X$. Since X is compact, this is known to be equivalent to X being *Kobayashi hyperbolic* in the sense that the Kobayashi pseudo-distance on X is actually a distance. (See e.g. [Kob70].) In [MP21, Definition 2.5], Marouani and Popovici introduced the following 1-codimensional analogue of this:

Let $n \geq 2$ be an integer. An n-dimensional compact complex manifold X is said to be **divisorially hyperbolic** *if there exists no holomorphic map $f : \mathbb{C}^{n-1} \longrightarrow X$ such that f is non-degenerate at some point $x \in \mathbb{C}^{n-1}$ and has subexponential growth.*

What we mean by f having a *subexponential growth* is spelt out in Definition 2.3 of [MP21]. It refers to the growth of the volume of the ball $B_r \subset \mathbb{C}^{n-1}$ of radius $r > 0$ centred at $0 \in \mathbb{C}^{n-1}$, with respect to the degenerate metric $f^\star\omega$ on $\mathbb{C}^{n-1}$ that is the pullback under f of an arbitrary Hermitian metric ω on X, as r tends to $+\infty$.

In [MP21], the notion of degenerate balanced compact complex manifolds is generalized by introducing the following 1-codimensional analogue of Kähler hyperbolicity:

Definition 2.2 ([6]) An n-dimensional compact complex manifold X is said to be **balanced hyperbolic** if there exists a balanced metric ω on X such that ω^{n-1} is $\widetilde{d}$(bounded) with respect to ω.

Let E be a complex vector space of dimension n and let E^* be its dual. $\Omega \in \wedge^{p,p}(E^*)$ is called **strictly weakly positive** (see [3], and [2]) if and only if for all $\alpha_i \in E^*$ and for all $I = (i_1, ..., i_k)$ with $k + p = n$, $\Omega \wedge \sigma_k \alpha_I \wedge \overline{\alpha_I}$ is a strictly positive (n, n)-form for all linearly independent $(\alpha_{i_1}, ..., \alpha_{i_k})$ (i.e $\sigma_k \alpha_I \wedge \overline{\alpha_I} \neq 0$).

Let X be a complex n-dimensional manifold. Recall that

(a) A smooth (p, p)-form $\alpha \in C^{\infty}_{p,p}(X, \mathbb{C})$ on X is said to be strictly weakly positive if for every point $x \in X$, $\alpha(x) \in \wedge^{p,p} T^*_x X$ is a strictly weakly positive (p, p)-form on the holomorphic tangent space $T_x X$ to X at x.
(b) Let p be an integer, $1 \leq p \leq n - 1$. X is said to be p-**Kähler** (see [2]) manifold if it has a closed strictly weakly positive (p, p)-form.

Obviously, $p = 1$ stands for the Kähler case. Moreover, the case $p = n - 1$ is well known. It corresponds to balanced manifolds.

3 Compact Complex *p*-Kähler Hyperbolic Manifolds

In this section, we introduce and discuss two hyperbolicity notions that generalise Kähler and balanced hyperbolicity, and divisorally hyperbolicity respectively.

The first notion that we introduce in this work combines Kähler hyperbolicity in the sense of Gromov and p-Kähler manifolds.

Definition 3.1 A compact complex n-dimensional manifold (X, ω) is called p-**Kähler hyperbolic** if it admits a strictly weakly positive (p, p)-form Ω, that is $\tilde{d}$*-bounded*.

A 1-Kähler hyperbolic manifold is simply a Kähler hyperbolic manifold. Also, an $(n - 1)$-Kähler hyperbolic is a balanced hyperbolic manifold (See [6]). It's an immediate consequence of the fact that a 1-Kähler and $(n - 1)$-Kähler manifold are Kähler and balanced respectively.

Recall that if (X, ω) a compact complex manifold of dimension n, then ω^p can never be d-exact for all $1 \leq p \leq n - 2$. In fact, if there exists a $p \in [1, n - 2]$ such that $\omega^p = d\Gamma$ for some smooth $(2p - 1)$-form Γ therefore, we get a Kähler metric ω since $d\omega^p = 0$ and $p < n - 1$ which implies that $\omega^n \in Im(d)$ what is impossible. But in the case where $p = n - 1$, such a manifold exists, that we called **degenerate balanced** manifold (See for example *Definition 2.9* in [6]). We can therefore ask the following question:

Question 3.2 *Is there a n-dimensional compact complex manifold X which admits a d-exact strictly weakly positive (p, p)-form?*

The immediate observation that provides the first class of examples of p-Kähler hyperbolic manifolds is the following

Lemma 3.3 *Let α and β be two d-closed forms on a compact complex manifold (X, ω), if α is a $\tilde{d}$*-bounded *form, then $\alpha \wedge \beta$ is a $\tilde{d}$*-bounded *form.*

Proof Let $\pi_X : \widetilde{X} \longrightarrow X$ be the universal cover of X. The $\widetilde{d}$(boundedness) assumption on α means that $\pi_X^\star \alpha = d\gamma$ on $\widetilde{X}$ for some smooth form γ on $\widetilde{X}$ that is bounded w.r.t.

the lift $\widetilde{\omega} = \pi_X^\star \omega$ of the Hermitian metric ω. Note that $d\gamma = \pi_X^\star \alpha$ and $\pi_X^\star \beta$ are trivially $\widetilde{\omega}$-bounded on $\widetilde{X}$ since α and β are ω-bounded on X thanks to X being *compact*.

We get: $\pi_X^\star \alpha \wedge \beta = d(\gamma \wedge \pi_X^\star(\beta))$ on $\widetilde{X}$, where both γ and $\pi_X^\star \beta$ are $\widetilde{\omega}$-bounded, hence so is $\gamma \wedge \pi_X^\star(\beta)$. □

We shall now point out an example of a *p-Kähler hyperbolic* manifold that is not Kähler hyperbolic in the sense of Gromov. We first recall the following result of Lucia Alessandrini.

Theorem 3.4 ([1, Theorem 5.5]) *Let X be a m-dimensional* **Kähler** *manifold, and let Y be a n-dimensional complex manifold which is s-Kähler for all $q \leq s < n$. Then $X \times Y$ is s-Kähler for all, $m + q \leq s < m + n$.*

Proposition 3.5 *Let X be a* **Kähler** *n-dimensional manifold and Y be a s-***Kähler hyperbolic** *manifold for all $q \leq s < n$. Then $X \times Y$ is s-Kähler hyperbolic for all, $m + q \leq s < m + n$.*

Proof Let (X, ω) be a compact complex Kähler manifold and Y be a p-Kähler hyperbolic manifold equipped with a strictly weakly positive (s, s)-forms Ω_s that are $\tilde{d}$-bounded.

Since X is Kähler, we can choose ω^s as a strictly weakly positive (s, s)-form on Y. Fix an index $j, m + q \leq j < m + n$, call $h = \max(0, j - n)$, and we consider the product strictly weakly positive (j, j)-form

$$\Theta_j = \sum_{h \leq k \leq m} i_1^\star \omega^k \wedge i_2^\star \Omega_{j-k}$$

on $X \times Y$, where $i_1 : X \times Y \to X$ and $i_2 : X \times Y \to Y$ are the projections on the two factors.

So we have a wedge product of a closed form $i_1^\star \omega^k$ with a $\tilde{d}$-bounded form $i_2^\star \Omega_{j-k}$, relying upon the previous Lemma, we get a $\tilde{d}$-bounded forms $\Theta_j = \sum_{h \leq k \leq m} i_1^\star \omega^k \wedge i_2^\star \Omega_{j-k}$ on $X \times Y$. □

Corollary 3.6 *Let (X, ω_X) be a n-dimensional compact Kähler manifold and (Y, ω_Y) be a Kähler hyperbolic manifold of* $\dim_{\mathbb{C}} = m$ *such that $m \geq n$, Then $X \times Y$ is p-Kähler hyperbolic for all $p > m$.*

Proof Let (X, ω_X) and (Y, ω_Y) be compact complex Kähler manifolds of respective dimensions n and m such that ω_Y be a $\tilde{d}$-bounded form, and let $\pi_X : \widetilde{X} \longrightarrow X$, $\pi_Y : \widetilde{Y} \longrightarrow Y$ be their respective universal covers.

Claim 3.7 *If ξ is $\tilde{d}$-bounded on a compact complex manifold, then ξ^p is $\tilde{d}$-bounded too.*

Proof of Claim. Let W be a compact complex manifold and $\pi : \widetilde{W} \longrightarrow W$ be the universal cover of W. The $\tilde{d}$(boundedness) assumption on ξ means that $\pi^\star \xi = d\beta$ for some smooth $(k-1)$-form β on $\widetilde{W}$ that is bounded. Note that $d\beta = \pi^\star \xi$ is trivially bounded on $\widetilde{W}$. We get: $\pi^\star \xi^p = d(\beta \wedge (d\beta)^{p-1})$ on $\widetilde{W}$, where both β and $d\beta$ are bounded, hence so is $\beta \wedge (d\beta)^{p-1}$. □

End of Proof of Corollary 3.6. The product metric

$$\omega = i_X^\star \omega_X + i_Y^\star \omega_Y$$

on $X \times Y$ is Kähler, where $i_X : X \times Y \to X$ and $i_Y : X \times Y \to Y$ are the projections on the two factors. The product map

$$\pi := \pi_X \times \pi_Y : \widetilde{X} \times \widetilde{Y} \longrightarrow X \times Y$$

is the universal cover of $X \times Y$. Let $p > \dim_{\mathbb{C}} Y = m$, we have

$$\begin{aligned} \pi^\star(\omega^p) &= (\pi^\star i_X^\star \omega_X + i_Y^\star \omega_Y)^p \\ &= \sum_{k=0}^{p} \binom{p}{k} \pi^\star (i_X^\star \omega_X)^k \wedge \pi^\star (i_Y^\star \omega_Y)^{p-k} \\ &= \sum_{k=0}^{p} \binom{p}{k} \tilde{i_X}^\star \pi_X^\star (\omega_X)^k \wedge \tilde{i_Y}^\star \pi_Y^\star (\omega_Y)^{p-k}. \end{aligned}$$

We obtain therefore a sum of wedge product of $\tilde{d}$-bounded form $\tilde{i_Y}^\star \pi_Y^\star (\omega_Y)^{p-k}$ with a d-closed form $\tilde{i_X}^\star \pi_X^\star (\omega_X)^k$, hence so a $\tilde{d}$-bounded form over $X \times Y$, thanks to *Lemma* 3.3. □

Proposition 3.8 *Let X be a p-Kähler hyperbolic manifold, and S be a submanifold of X, such that* $\dim_{\mathbb{C}} S > p$*, then S is a p-Kähler hyperbolic manifold too.*

Proof The natural inclusion $i : S \to X$ is a holomorphic immersion hence the pull-back commutes with the operators d, and preserves the different kinds of positivity. Let $\pi_X : \tilde{X} \to X$ and $\pi_S : \tilde{S} \to S$ be the universal coverings of X and S respectively. Since $\tilde{S}$ is simply connected, there exists a lifting map $\tilde{i} : \tilde{S} \to \tilde{X}$, such that the following diagram

$$\begin{array}{ccc} \tilde{S} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ S & \longrightarrow & X \end{array}$$

commutes. Since X is p-Kähler hyperbolic, then it admits a strictly weakly positive (p, p)-form Ω, such that $\pi_X^\star(\Omega) = d\Gamma$ for some bounded form Γ over $\tilde{X}$. Hence

$$\pi_S^\star(i^\star(\Omega)) = \tilde{i}^\star(\pi_X^\star(\Omega)) = \tilde{i}^\star(d\Gamma) = d(\tilde{i}^\star(\Gamma))$$

where $i^\star(\Omega)$ is a strictly weakly positive (p, p)-form on S, and $\tilde{i}^\star(\Gamma)$ is a bounded $(2p-1)$-form on $\tilde{S}$, since S and X are compact, π_S and π_X are local diffeomorphisms. □

Corollary 3.9 *If $X \times Y$ is a p-Kähler hyperbolic manifold, and $p <$* dim *X (respectively $p <$* dim *Y),then X(respectively Y) is a p-Kähler hyperbolic manifold.*

Remark 3.10 Recall that a (p, p)-form $\alpha \in \wedge^{p,p} V^*$ on a vector space V is strictly weakly positive if and only if its restriction $u_{|S}$ to every p-dimensional subspace $S \subset V$ is a positive volume form on S. (See e.g. [3, Chapter III]). Consequently, a smooth (p, p)-form $\alpha \in C^\infty_{p,p}(X, \mathbb{C})$ on a manifold X is strictly weakly positive if and only if, for every coordinate patch $U \subset X$ and every p-dimensional complex submanifold $Y \subset U$, its restriction $\alpha_{|Y}$ is a positive (i.e.>0) volume form on Y.

Let $\pi : \tilde{X} \to X$ be the universal covering of X we suppose that X is a p-Kähler compact manifold and let Ω be the (p, p)-strictly weakly positive form on X, if $\pi^\star(\Omega)$ be a d-exact form (need not necessarily be a $\tilde{d}$-bounded), then $\tilde{X}$ does not contain a compact p-dimensional submanifold S, in particular, we have the following:

Proposition 3.11 *If X is a p-Kähler hyperbolic manifold, then there is no holomorphic function $f : \mathbb{P}^p \to X$ such that $d_x f : T_x\mathbb{P}^p \longrightarrow T_{f(x)}X$ is of maximum rank at some point $x \in \mathbb{P}^p$.*

Proof Let $\pi : \tilde{X} \to X$ be the universal covering of X and let Ω be a (p, p)-form that is strictly weakly positive on X, such that $\pi^\star(\Omega) = d\Gamma$, for same smooth $(2p-1)$-form. Suppose that there exists a holomorphic function $f : \mathbb{P}^p \to X$ such that $d_x f : T_x\mathbb{P}^p \longrightarrow T_{f(x)}X$ is of maximum rank at some point $x \in \mathbb{P}^p$, then we get:

$$0 < \int_{\mathbb{P}^p} \pi^\star(\Omega) = \int_{\mathbb{P}^p} \pi^\star(d\Gamma) = \int_{\mathbb{P}^p} d\pi^\star(\Gamma) \stackrel{(a)}{=} \int_{\partial\mathbb{P}^p=\phi} \pi^\star(\Gamma) = 0,$$

where (a) is by Stokes theorem. Contradiction. □

Since we have not used the fact that X is p-Kähler hyperbolic, a question that arises.

Question 3.12 *Do we have the non-existence of non degenerate holomorphic map $f : \mathbb{C}^p \to X$?*

For $p = 1$, the answer is positive, it's quite simply Gromov's theorem. But in the case where $p = n - 1$, i.e. in the case where X is balanced hyperbolic, the answer is negative (See $[MPa21]$ example (IV)). However, in the case of balanced hyperbolic, if we add an additional condition we obtain a positive result similar to our question, which is the case if $2 \le p \le n-2$. Our additional condition on f is as follows:

Definition 3.13 Let X be a **compact** complex Hermitian manifold with $\dim_{\mathbb{C}} X = n$ equipped by a **strictly weakly positive** (p, p)-form Ω and let $f : \mathbb{C}^p \to X$ be a holomorphic map that is **non-degenerate** at some point $x \in \mathbb{C}^p$.

We say that f has **p-subexponential growth** if the following two conditions are satisfied:

(i) there exist constants $C_1 > 0$ and $r_0 > 0$ such that

$$\int_{S_t} |d\tau|_{f^\star\omega}\, d\sigma_{\omega, f, t} \le C_1 t \,\mathrm{Vol}_{\omega, f}(B_t), \quad t > r_0; \tag{1}$$

(ii)

$$\liminf_{b\to+\infty}\frac{F(b)}{b}=0, \tag{2}$$

where

$$F(b):=\int_0^b \mathrm{Vol}_{\Omega,\,f}(B_t)\,dt=\int_0^b\Bigg(\int_{B_t} f^\star\Omega\Bigg)\,dt,\quad b>0.$$

We also propose the following notion that generalises that of *Divisorial hyperbolicity*.

Definition 3.14 An n-dimensional compact complex manifold X is said to be p-**dimensional hyperbolic** if there is no holomorphic map that is **non-degenerate** at some point $x\in\mathbb{C}^p$ and has **p-subexponential growth** in the sense of Definition 3.13.

Our main theorem which generalizes *Theorem 2.6.* in [6] is the following:

Theorem 3.15 *Every* p-**Kähler hyperbolic** *compact complex manifold is* p-**dimensional hyperbolic**.

Proof Let (X,ω) be a compact complex manifold, with $\dim_{\mathbb{C}}X=n$, and Ω be a *p-Kähler hyperbolic* form on X. This means that, if $\pi_X:\widetilde{X}\longrightarrow X$ is the universal cover of X, we have

$$\pi_X^\star\Omega=d\Lambda\quad\text{on}\quad\widetilde{X},$$

where Λ is a smooth, $\widetilde{\omega}$-bounded $(2p-1)$-form on $\widetilde{X}$ and $\widetilde{\omega}=\pi_X^\star\omega$ is the lift of the metric ω to $\widetilde{X}$.

Suppose there exists a holomorphic map $f:\mathbb{C}^p\longrightarrow X$, such that $d_xf:\mathbb{C}^p\longrightarrow T_{f(x)}X$ is of maximum rank at some point $x\in\mathbb{C}^p$.

There exists a lift $\widetilde{f}$ of f to $\widetilde{X}$, namely a holomorphic map $\tilde{f}:\mathbb{C}^p\longrightarrow\widetilde{X}$ such that $f=\pi_X\circ\tilde{f}$. The $(p,\,p)$-form $f^\star\Omega$ is ≥ 0 on $\mathbb{C}^p$ and may not be >0 on an analytic subset $\Sigma\subset\mathbb{C}^p$, since the notions of positivity coincide for p equal to the dimensional space. Thus, it can be written in the form (a) below for some $\mathbb{C}^\infty$ function $\mu:\mathbb{C}^p\longrightarrow[0,\,+\infty)$:

$$f^\star\Omega\overset{(a)}{=}\mu(z)\prod_{j=1}^{p}\frac{i}{2}dz_j\wedge d\bar{z}_j\overset{(b)}{=}\tilde{f}^\star(\pi_X^\star\Omega)\overset{(c)}{=}d(\tilde{f}^\star\Lambda)\qquad\text{on }\mathbb{C}^p,$$

where (b) follows from $f=\pi_X\circ\tilde{f}$.

We get also a $(p,\,p)$-form $f^\star\omega^p$ is ≥ 0 on $\mathbb{C}^p$ and may not be >0 on an analytic subset $\Sigma'\subset\mathbb{C}^p$. Thus, it can be written in the form (d) below for some $\mathbb{C}^\infty$ function $\mu':\mathbb{C}^p\longrightarrow[0,\,+\infty)$:

$$f^\star\omega^p\overset{(d)}{=}\mu'(z)\prod_{j=1}^{p}\frac{i}{2}dz_j\wedge d\bar{z}_j.$$

Hence, on $\mathbb{C}^p \setminus \{\Sigma \cup \Sigma'\}$ we get,

$$f^\star\Omega \overset{(a,d)}{=} \frac{\mu}{\mu'} f^\star\omega^p = h.f^\star\omega^p = (h^{\frac{1}{p}}.f^\star\omega)^p = d(\tilde{f}^\star\Lambda)$$

where $h = \frac{\mu}{\mu'}$ on $\mathbb{C}^p\setminus\{\Sigma \cup \Sigma'\}$.

For $z \in \mathbb{C}^p$, let $\tau(z) := |z|^2$ be its squared Euclidean norm. At every point $z \in \mathbb{C}^p\setminus\{\Sigma \cup \Sigma'\}$, we have:

$$\frac{d\tau}{|d\tau|_{h^{\frac{1}{p}}.f^\star\omega}} \wedge \star_{h^{\frac{1}{p}}.f^\star\omega}\left(\frac{d\tau}{|d\tau|_{h^{\frac{1}{p}}.f^\star\omega}}\right) = h\frac{f^\star\omega^p}{p!},$$

where $\star_{h^{\frac{1}{p}}.f^\star\omega}$ is the Hodge star operator induced by $h^{\frac{1}{p}}.f^\star\omega$. Thus, the $(2p-1)$-form

$$d\sigma_{\omega,\,f} := p!\star_{h^{\frac{1}{p}}.f^\star\omega}\left(\frac{d\tau}{|d\tau|_{h^{\frac{1}{p}}.f^\star\omega}}\right)$$

on $\mathbb{C}^p \setminus \{\Sigma \cup \Sigma'\}$ is the area measure induced by $h^{\frac{1}{p}}.f^\star\omega$ on the spheres of $\mathbb{C}^p$.

First we have the following □

Claim 3.16 *The* $(2p-1)$*-form* $\tilde{f}^\star\Lambda$ *is* $(f^\star\omega)$*-bounded on* $\mathbb{C}^p$.

Proof of Claim. For any tangent vectors $v_1, \ldots, v_{2p-1}$ in $\mathbb{C}^p$, we have:

$$\begin{aligned}|(\tilde{f}^\star\Lambda)(v_1,\ldots,v_{2p-1})|^2 &= |\Lambda(\tilde{f}_\star v_1,\ldots,\tilde{f}_\star v_{2p-1})|^2 \overset{(a)}{\leq} C\,|\tilde{f}_\star v_1|^2_{\widetilde{\omega}}\ldots|\tilde{f}_\star v_{2p-1}|^2_{\widetilde{\omega}}\\ &= C\,|v_1|^2_{\tilde{f}^\star\widetilde{\omega}}\ldots|v_{2p-1}|^2_{\tilde{f}^\star\widetilde{\omega}} \overset{(b)}{=} C\,|v_1|^2_{f^\star\omega}\ldots|v_{2p-1}|^2_{f^\star\omega},\end{aligned}$$

where $C > 0$ is a constant independent of the v_j's that exists such that inequality (a) holds thanks to the $\widetilde{\omega}$*-boundedness* of Λ on $\widetilde{X}$, while (b) follows from $\tilde{f}^\star\widetilde{\omega} = f^\star\omega$. □

End of Proof of Theorem 3.15 For any $r > 0$, let $\mathrm{Vol}_{\omega,\,f}(B_r)$ and $A_{\omega,\,f}(S_r)$ be the volume of the ball $B_r \subset \mathbb{C}^p$ w.r.t. the measure $h.f^\star\omega^p$ on $\mathbb{C}^p$, respectively the area of the sphere $S_r \subset \mathbb{C}^p$ w.r.t. $d\sigma_{\omega,\,f}$, we get:

- On the one hand, we have $d\tau = 2t\,dt$ and

$$\mathrm{Vol}_{\omega,\,f}(B_r) = \int_{B_r} h.f^\star\omega^p = \int_0^r\left(\int_{S_t} d\mu_{\omega,\,f,\,t}\right)dt = \int_{B_r} d\mu_{\omega,\,f,\,t} \wedge \frac{d\tau}{2t}, \qquad (3)$$

where $d\mu_{\omega,\,f,\,t}$ is the positive measure on S_t defined by

$$\frac{1}{2t}\,d\mu_{\omega,\,f,\,t} \wedge (d\tau)_{|S_t} = h.(f^\star\omega^p)_{|S_t}, \qquad t > 0.$$

This means that the measures $d\mu_{\omega,\,f,\,t}$ and $d\sigma_{\omega,\,f,\,t}$ on S_t are related by

$$\frac{1}{2t}\,d\mu_{\omega,\,f,\,t} = \frac{1}{|d\tau|_{f^\star\omega}}\,d\sigma_{\omega,\,f,\,t}, \quad t>0.$$

Now, the Hölder inequality yields:

$$\int_{S_t}\frac{1}{|d\tau|_{f^\star\omega}}\,d\sigma_{\omega,\,f,\,t} \geq \frac{A^2_{\omega,\,f}(S_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\,d\sigma_{\omega,\,f,\,t}}, \tag{4}$$

so together with (3) and (4) this leads to:

$$\begin{aligned}\mathrm{Vol}_{\omega,\,f}(B_r) &= \int_0^r\left(\int_{S_t}\frac{1}{2t}\,d\mu_{\omega,\,f,\,t}\right)d\tau = \int_0^r\left(\int_{S_t}\frac{1}{|d\tau|_{f^\star\omega}}\,d\sigma_{\omega,\,f,\,t}\right)d\tau \\ &\geq 2\int_0^r\frac{A^2_{\omega,\,f}(S_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\,d\sigma_{\omega,\,f,\,t}}\,t\,dt, \qquad r>0. \end{aligned} \tag{5}$$

On the other hand, for every $r>0$, we also have:

$$\begin{aligned}\mathrm{Vol}_{\omega,\,f}(B_r) &= \int_{B_r} h.f^\star\omega^p \overset{(d)}{=} \int_{B_r} d(\tilde{f}^\star\Lambda) \\ &= \int_{S_r}\tilde{f}^\star\Lambda \overset{(a)}{\leq} C\int_{S_r} d\sigma_{\omega,\,f} = C\,A_{\omega,\,f}(S_r),\end{aligned} \tag{6}$$

where $C>0$ is a constant that exists such that inequality (a) holds thanks to Claim 3.16.

Putting (5) and (6) together, we get for every $r>0$:

$$\begin{aligned}\mathrm{Vol}_{\omega,\,f}(B_r) &\geq \frac{2}{C^2}\int_0^r \mathrm{Vol}_{\omega,\,f}(B_t)\,\frac{t\,\mathrm{Vol}_{\omega,\,f}(B_t)}{\int_{S_t}|d\tau|_{f^\star\omega}\,d\sigma_{\omega,\,f,\,t}}\,dt \\ &\overset{(a)}{\geq} \frac{2}{C_1\,C^2}\int_{r_0}^r \mathrm{Vol}_{\omega,\,f}(B_t)\,dt \overset{(b)}{:=} C_2\,F(r),\end{aligned} \tag{7}$$

where (a) follows from the growth assumption (1) and (b) is the definition of a function $F:(r_0,\,+\infty)\longrightarrow(0,\,+\infty)$ with $C_2 := 2/(C_1\,C^2)$.

By taking the derivative of F, we get for every $r>r_0$:

$$F'(r) = \mathrm{Vol}_{\omega,\,f}(B_r) \geq C_2\,F(r),$$

where the last inequality is (7). This amounts to

$$\frac{d}{dt}\bigg(\log F(t)\bigg) \geq C_2, \quad t > r_0.$$

Integrating this over $t \in [a,\ b]$, with $r_0 < a < b$ arbitrary, we get:

$$\begin{aligned} -\log F(a) &\geq -\log F(b) + C_2\,(b-a) \\ &= b\bigg(\frac{-\log F(b)}{b} + C_2 + C_2\,\frac{(-a)}{b}\bigg), \qquad r_0 < a < b. \end{aligned} \tag{8}$$

Now, fix an arbitrary $a > r_0$ and let $b \to +\infty$. Thanks to the *subexponential growth* assumption (2) made on f, there exists a sequence of reals $b_j \to +\infty$ such that the right-hand side of inequality (8) for $b = b_j$ tends to $+\infty$ as $j \to +\infty$. This forces $F(a) = 0$ for every $a > r_0$, hence $\mathrm{Vol}_{\omega,\,f}(B_r) = 0$ for every $r > r_0$. This amounts to $f^\star\omega^p = 0$ on $\mathbb{C}^p$, in contradiction to the non-degeneracy assumption made on f at a point $x \in \mathbb{C}^p$. □

We end this section by proving the non-existence of certain L1 currents on the universal covering space of a p-Kähler hyperbolic manifold. We first need to recall the following lemma.

Lemma 3.17 ([5, Lemma 1.1.A.]) *Let $(X,\ g)$ be a* **complete** *Riemannian manifold of real dimension m. Let η be an L^1_g-form on X of degree $m-1$ such that $d\eta$ is again L^1_g. Then*

$$\int_X d\eta = 0.$$

Proposition 3.18 *Let $(X,\ \omega)$ be a p***-Kähler hyperbolic manifold** *and let $\pi : \widetilde{X} \longrightarrow X$ be the universal cover of X. There exists no non-zero d-closed positive $(n-p,\ n-p)$-current $\widetilde{T} \geq 0$ on $\widetilde{X}$ such that $\widetilde{T}$ is $L^1_{\widetilde{\omega}}$, where $\widetilde{\omega} := \pi^\star\omega$ is the lift of ω to $\widetilde{X}$.*

Proof Let $n = \dim_{\mathbb{C}} X$. The p-Kähler hyperbolic assumption on X means that there exist a strictly weakly positive $(p,\ p)$-form Ω on X such that $\pi^\star\Omega = d\widetilde{\Lambda}$ on $\widetilde{X}$ for some smooth $L^\infty_{\widetilde{\omega}}$-form $\widetilde{\Lambda}$ of degree $(2p-1)$ on $\widetilde{X}$.

If a current $\widetilde{T}$ as in the statement existed on $\widetilde{X}$, we would have

$$0 < \int\limits_{\widetilde{X}} \widetilde{T} \wedge \pi^\star\Omega = \int\limits_{\widetilde{X}} d(\widetilde{T} \wedge \widetilde{\Lambda}) = 0, \tag{9}$$

which is contradictory.

The last identity in (9) follows from Lemma 3.17 applied on the complete manifold $(\widetilde{X},\ \widetilde{\omega})$ to the $L^1_{\widetilde{\omega}}$-current $\eta := \widetilde{T} \wedge \widetilde{\Lambda}$ of degree $2n-1$ whose differential $d\eta = \widetilde{T} \wedge \pi^\star\Omega$ is again $L^1_{\widetilde{\omega}}$. That η is $L^1_{\widetilde{\omega}}$ follows from $\widetilde{T}$ being $L^1_{\widetilde{\omega}}$ (by hypothesis) and from $\widetilde{\Lambda}$ being $L^\infty_{\widetilde{\omega}}$, while $d\eta$ being $L^1_{\widetilde{\omega}}$ follows from $\widetilde{T}$ being $L^1_{\widetilde{\omega}}$ and from $\pi^\star\Omega$ being $L^\infty_{\widetilde{\omega}}$ (as a lift of the smooth, hence bounded, form Ω on the *compact* manifold X). □

4 Case of Compact Complex Kähler Manifolds

In this very short section we will be interested in a particular case of p-Kähler hyperbolic manifolds. We introduce the following notion which is stronger to be p-Kähler hyperbolic and we give some properties such as Theorem 4.4 which generalizes **Lefschetz vanishing theorem**.

Recall that in a compact complex n-dimensional manifold (X, ω), for all $1 \leq p < n-1$, if ω^p is d-closed then ω is a Kähler metric.

Definition 4.1 Let X be a compact complex manifold with $\dim_{\mathbb{C}} X = n$. A Hermitian metric ω on X is said to be **strictly** p**-Kähler hyperbolic** if ω^p is $\widetilde{d}$(bounded) with respect to ω.

The manifold X is said to be strictly p-Kähler hyperbolic if it carries a *strictly p-Kähler hyperbolic metric.*

Proposition 4.2 *Let (X, ω) be a compact complex* **strictly** p**-Kähler hyperbolic**, *where $1 \leq p \leq n-2$, then (X, ω) is* **strictly** s**-Kähler hyperbolic**, *for all $s \geq p$.*

Proof For all $s \geq p$ we have $\omega^s = \omega^p \wedge \omega^{s-p}$ as a consequence of *Lemma* 3.3 we get a $\tilde{d}$-bounded ω^s. □

Our main result in this section is to prove vanishing theorem for the L2 harmonic spaces on the universal cover of **strictly** p**-Kähler hyperbolic** manifold. Let first recall the following theorem

Theorem 4.3 (see e.g. [4, VIII,Theorem 3.2.]) *Let $(X,\ g)$ be a* **complete** *Riemannian manifold of real dimension m. Then:*

(a) *The space $\mathcal{D}_\bullet(X,\ \mathbb{C})$ of compactly supported C^∞ forms of any degree (indicated by a $\bullet$) on X is* **dense** *in the domains Dom d, Dom $d^\star$ and in Dom $d \cap$ Dom $d^\star$ for the respective graph norms:*

$$u \mapsto ||u|| + ||du||, \quad u \mapsto ||u|| + ||d^\star u||, \quad u \mapsto ||u|| + ||du|| + ||d^\star u||.$$

(b) *The extension $d^\star$ of the formal adjoint of d to the L^2-space coincides with the Hilbert space adjoint of the extension of d.*

(c) *The d-Laplacian $\Delta = \Delta_g := dd^\star + d^\star d$ has the following property:*

$$\langle\langle \Delta u,\ u\rangle\rangle = ||du||^2 + ||d^\star u||^2$$

for every form $u \in$ Dom Δ. In particular, Dom $\Delta \subset$ Dom $d \cap$ Dom $d^\star$ and $\ker \Delta = \ker d \cap \ker d^\star$.

(d) *There are L^2*-**orthogonal decompositions** *in every degree (indicated by a $\bullet$):*

$$L^2_\bullet(X,\ \mathbb{C}) = \mathcal{H}^\bullet_\Delta(X,\ \mathbb{C}) \oplus \overline{Im\, d} \oplus \overline{Im\, d^\star}$$
$$\ker d = \mathcal{H}^\bullet_\Delta(X,\ \mathbb{C}) \oplus \overline{Im\, d} \quad \text{and} \quad \ker d^\star = \mathcal{H}^\bullet_\Delta(X,\ \mathbb{C}) \oplus \overline{Im\, d^\star}, \qquad (10)$$

where $\mathcal{H}^{\bullet}_{\Delta}(X, \mathbb{C}) := \{u \in L^2_{\bullet}(X, \mathbb{C}) \mid \Delta u = 0\}$ *is the space of* Δ-**harmonic** L^2*-forms, while*

$$Im\, d := L^2_{\bullet}(X, \mathbb{C}) \cap d(L^2_{\bullet -1}(X, \mathbb{C})) \quad and$$
$$Im\, d^{\star} := L^2_{\bullet}(X, \mathbb{C}) \cap d^{\star}(L^2_{\bullet +1}(X, \mathbb{C})).$$

This standard result saying that some further key facts in the Hodge Theory of compact Riemannian manifolds remain valid on *complete* such manifolds X when the differential operators involved (e.g. d, $d^{\star}$, Δ) are considered as *closed* and *densely defined* unbounded operators on the spaces $L^2_k(X, \mathbb{C})$ of L^2-forms of degree k on X.

Theorem 4.4 *Let* X *be a compact complex strictly* p*-Kähler hyperbolic manifold with* $dim_{\mathbb{C}} X = n$. *Let* $\pi : \widetilde{X} \longrightarrow X$ *be the universal cover of* X *and* $\widetilde{\omega} := \pi^{\star}\omega$ *the lift to* $\widetilde{X}$ *of a strictly* p*-Kähler hyperbolic metric* ω *on* X.

For all $s \leq n - p$, *there are no non-zero* $\Delta_{\widetilde{\omega}}$*-harmonic* $L^2_{\widetilde{\omega}}$*-forms of degrees* s *and* $2n - s$ *on* $\widetilde{X}$:

$$\mathcal{H}^{s}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = 0 \quad and \quad \mathcal{H}^{2n-s}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) = 0,$$

where $\Delta_{\widetilde{\omega}} := dd^{\star}_{\widetilde{\omega}} + d^{\star}_{\widetilde{\omega}}d$ *is the* d*-Laplacian induced by the metric* $\widetilde{\omega}$.

Proof Since $\pi^{\star}(\omega^p) = \widetilde{\omega}^p = d\Gamma$ for a smooth $(2p-1)$-bounded form Γ, then for all d-closed $L^2_{\widetilde{\omega}}$-form ξ, $\widetilde{\omega}^p \wedge \xi = d(\Gamma \wedge \xi) = d\Psi$, where Ψ is $L^2_{\widetilde{\omega}}$ since Γ is bounded. In the other hand, if ξ is harmonic, we get $\Delta_{\widetilde{\omega}}(\widetilde{\omega}^p \wedge \xi) = \widetilde{\omega}^p \wedge \Delta_{\widetilde{\omega}}(\xi) = 0$, so that $\widetilde{\omega}^p \wedge \xi \in \mathcal{H}^{\bullet}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) \cap \overline{\mathrm{Im}\, d} = 0$, where the last equality follows from (10). The Lefschetz theorem said that the map

$$\widetilde{\omega}^p \wedge \bullet : \mathcal{H}^{s}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C}) \to \mathcal{H}^{2p+s}_{\Delta_{\widetilde{\omega}}}(\widetilde{X}, \mathbb{C})$$

is injective for $2p + 2s \leq 2n = \dim_{\mathbb{R}}(\widetilde{X})$, since $s \leq n - p$, this implies that $\xi = 0$ □

Corollary 4.5 (Lefschetz vanishing theorem) *If* $p = 1$, *(i.e.* X *is Kähler hyperbolic) then* $\mathcal{H}^s = 0$, *unless* $s = n$.

Author contributions All authors reviewed the manuscript.

Declarations

Conflict of interest The authors declare that they have no competing interests.

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