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On the Some Spectral Domains in the Spectrum Preserves

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Abstract

Let $\mathcal{B}(X)$ be the algebra of all bounded linear operators on an infinite dimensional complex Banach space X. For $A \in \mathcal{B}(X)$, $\sigma_1(A)$, $\sigma_2(A)$, and $\sigma_3(A)$ are the semi-Fredholm domain, the Fredholm domain, and the Weyl domain of A in the spectrum $\sigma(A)$, respectively. For an integer $k \geq 2$, let $(i_1, \dots, i_m)$ be a finite sequence with terms chosen from $\{1, \dots, k\}$ and assume that at least one of the terms in $(i_1, \dots, i_m)$ appears exactly once. The generalized product of k operators $A_1, \dots, A_k \in \mathcal{B}(X)$ is defined by

$$A_1 * A_2 * \cdots * A_k = A_{i_1} A_{i_2} \cdots A_{i_m},$$

and includes the usual product and the triple product. In this paper we characterize the form of the map ϕ from $\mathcal{B}(X)$ into itself which satisfies

$$\sigma_i(A_1 * \cdots * A_k) = \sigma_i(\phi(A_1) * \cdots * \phi(A_k)),$$

for all $A_1, \dots, A_k \in \mathcal{B}(X)$ and $i \in \{1, 2, 3\}$.

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1 Introduction

Throughout this paper, X denotes an infinite-dimensional complex or real Banach space and $\mathcal{B}(X)$ the algebra of all bounded linear operators acting on X. The dual space of X will be denoted by X^* and the adjoint of an operator $A \in \mathcal{B}(X)$ will be denoted by A^*. For n a positive integer, $\mathcal{F}_n(X)$ denotes the set of operators of rank n on X and $\mathcal{F}(X)$ the set of all finite rank operators. For a non zero vector $x \in X$ and a non zero linear functional f in the dual space X^*, $x \otimes f$ stands for the operator of rank one defined by $(x \otimes f)y := f(y)x, \quad (y \in X)$, and every operator of rank at most one on $\mathcal{B}(X)$ can be written in this form. By $\mathcal{N}_1(X)$ we denote the set of rank one nilpotent operators on X. It is clear that $x \otimes f \in \mathcal{N}_1(X)$ if and only if $f(x) = 0$. Let $A \in \mathcal{B}(X)$, the spectrum, the null space, and the range of A are denoted by $\sigma(A)$, $N(A)$, and $R(A)$ respectively. Recall that A is said to be lower semi-Fredholm if $codimR(A) < \infty$, and A is said to be upper semi-Fredholm if $dimN(A) < \infty$ and $R(A)$ is closed. The semi-Fredholm operator is lower semi-Fredholm or a upper semi-Fredholm operator. A is said to be Fredholm if it is both lower and upper semi-Fredholm, in this case we define the index of A, $indA$, by

$$indA := dimN(A) - codimR(A).$$

Obviously, any Fredholm operator has a finite index, if $indA = 0$ we say that A is Weyl operator. The semi-Fredholm domain, the Fredholm domain and the Weyl domain of the operator A are defined, respectively, by

$$\begin{aligned} \rho_{sf}(A) &:= \{\lambda \in \mathbb{C}/A - \lambda I \quad \text{is semi-Fredholm operator}\}, \\ \rho_f(A) &:= \{\lambda \in \mathbb{C}/A - \lambda I \quad \text{is Fredholm operator}\}, \\ \rho_w(A) &:= \{\lambda \in \mathbb{C}/A - \lambda I \quad \text{is Weyl operator}\}. \end{aligned}$$

It is clear that $\rho_w(A) \subset \rho_f(A) \subset \rho_{sf}(A)$. The semi-Fredholm domain, the Fredholm domain and the Weyl domain in the spectrum of the operator A are the subsets of the complex field defined, respectively, by

$$\begin{aligned} \sigma_1(A) &:= \{\lambda \in \sigma(A)/A - \lambda I \quad \text{is semi-Fredholm operator}\}, \\ \sigma_2(A) &:= \{\lambda \in \sigma(A)/A - \lambda I \quad \text{is Fredholm operator}\}, \\ \sigma_3(A) &:= \{\lambda \in \sigma(A)/A - \lambda I \quad \text{is Weyl operator}\}. \end{aligned}$$

Obviously, for all $A \in \mathcal{B}(X)$, $\sigma_1(A) = \sigma(A) \cap \rho_{sf}(A)$, $\sigma_2(A) = \sigma(A) \cap \rho_f(A)$ and $\sigma_3(A) = \sigma(A) \cap \rho_w(A)$. Moreover it is clear that $\sigma_3(A) \subset \sigma_2(A) \subset \sigma_1(A)$. For $x \in X$ and $f \in X^*$ if $f(x) \neq 0$, then

$$\sigma_i(\lambda x \otimes f) = \{f(\lambda x)\} = \lambda\{f(x)\} = \lambda\sigma_i(x \otimes f), \tag{1.1}$$

for all $\lambda \in \mathbb{C}$ and $i \in \{1, 2, 3\}$.

For a long time, there are many results concerning linear operators on a Banach space and others corresponding to inner-product spaces are realized(see, [3, 11]).

The study of maps on operator algebras preserving certain properties is a topic which attracts the attention of many authors (see, [1, 2, 4, 5, 7, 10, 13, 15]) and the references therein. In recent years a great activity has occurred in the question of relaxing the assumption of linearity or additivity (see, e.g. [1, 5, 10, 14, 15]). In [14], Shi. and Ji. proved that if X is a complex Banach space and if a surjective additive map $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ satisfies $\sigma_i(\phi(A)) = \sigma_i(A)$ for all $A \in \mathcal{B}(X)$ and $i \in \{1, 2, 3\}$, then one of the following forms is hold:

(i) $\phi(A) = TAT^{-1}$ for all $A \in \mathcal{B}(X)$, where T is an invertible linear operator in $\mathcal{B}(X)$.
(ii) $\phi(A) = TA^*T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X^* \to X$ is bounded invertible linear operator. In this case, X must be reflexive.

In [9] Hajighasmi and Hejazian, replaced the additivity of ϕ and the condition $\sigma_i(\phi(A)) = \sigma_i(A)$ for all $A \in \mathcal{B}(X)$, and $i \in \{1, 2, 3\}$, by $\sigma_i(\phi(A)\phi(B)) = \sigma_i(AB)$ for all $A, B \in \mathcal{B}(X)$ and $i \in \{1, 2, 3\}$. They showed that there exists a scalar $\lambda \in \mathbb{C}$ with $\lambda^2 = 1$ such that either

(i) $\phi(A) = \lambda TAT^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X \to X$ is an invertible linear operator in $\mathcal{B}(X)$.
(ii) $\phi(A) = \lambda TA^*T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X^* \to X$ is bounded invertible linear operator. In this case, X must be reflexive.

They showed also that if ϕ is a surjective map from $\mathcal{B}(X)$ into itself satisfying $\sigma_i(\phi(A)\phi(B)\phi(A)) = \sigma_i(ABA)$ for all $A, B \in \mathcal{B}(X)$ and $i \in \{1, 2, 3\}$, then there exists $\lambda \in \mathbb{C}$ with $\lambda^3 = 1$ such that either

(i) $\phi(A) = \lambda TAT^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X \to X$ is bijective linear operator in $\mathcal{B}(X)$.
(ii) $\phi(A) = \lambda TA^*T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X^* \to X$ is bijective linear operator. In this case, X must be reflexive.

In [6], Bouramdane and Ech-cheérif El Kettani described maps from $\mathcal{B}(\mathcal{H})$ into itself preserving some spectral domains in the spectrum of skew-products of operators, and established the following results.

Theorem 1.1 [6] *Let* $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ *be a map such that its range contains the operators of rank at most two. Then* ϕ *satisfies*

$$\sigma_i(AB^*) = \sigma_i(\phi(A)\phi(B)^*),$$

for all $A, B \in \mathcal{B}(\mathcal{H})$ *and* $i \in \{1, 2, 3\}$ *if and only if there exist two unitary operators* $U, V \in \mathcal{B}(\mathcal{H})$ *such that*

$$\phi(A) = UAV \ \ \textit{for all} \ \ A \in \mathcal{B}(\mathcal{H}).$$

Theorem 1.2 [6] *Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a map such that its range contains the operators of rank at most two. Then ϕ satisfies*

$$\sigma_i(AB^*A) = \sigma_i(\phi(A)\phi(B)^*\phi(A),$$

for all $A, B \in \mathcal{B}(\mathcal{H})$ and $i \in \{1, 2, 3\}$ if and only if there exist a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that

$$\phi(A) = UAU^* \text{ for all } A \in \mathcal{B}(\mathcal{H}).$$

Recall, for an integer $k \geq 2$, let $(i_1, \ldots, i_m)$ be a finite sequence with terms chosen from $\{1, \ldots, k\}$, such that $\{i_1, \ldots, i_m\} = \{1, \ldots, k\}$ and at least one of the terms in $(i_1, \ldots, i_m)$ appears exactly once. The generalized product of with m of k operators $A_1, \ldots, A_k \in \mathcal{B}(X)$ is defined by

$$A_1 * A_2 * \cdots * A_k = A_{i_1} A_{i_2} \cdots A_{i_m}.$$

Evidently, The generalized product includes the usual product and the triple product.

Recently, there has been a lot of activities on describing maps that preserve the generalized product of operators; see for example [1, 2, 4, 5, 15] and the references therein. In this paper, we propose to determine the form of mappings that preserve some spectral domains in the spectrum of generalized product of operators.

The aim of this paper is to characterize maps ϕ (with no additivity assumption) from $\mathcal{B}(X)$ into itself which satisfies

$$\sigma_i(A_1 * \cdots * A_k) = \sigma_i(\phi(A_1) * \cdots * \phi(A_k)),$$

for all $A_1, \ldots, A_k \in \mathcal{B}(X)$ and $i \in \{1, 2, 3\}$.

The main result is the following theorem.

Theorem 1.3 *Let $i \in \{1, 2, 3\}$, and let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map which satisfies the following condition:*

$$\sigma_i(\phi(A_1) * \cdots * \phi(A_k)) = \sigma_i(A_1 * \cdots * A_k) \text{ for all } A_1, \ldots, A_k \in \mathcal{B}(X).$$

Then there exists $\alpha \in \mathbb{C}$ with $\alpha^k = 1$ such that either

(i) $\phi(A) = \alpha TAT^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X \to X$ is a bounded invertible linear (or conjugate linear) operator in $\mathcal{B}(X)$.

(ii) $\phi(A) = \alpha TA^*T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X^* \to X$ is a bounded invertible linear (or conjugate linear) operator. In this case, X must be reflexive.

To establish our main results, we need the following well-known theorem. We recall that if $h : \mathbb{C} \to \mathbb{C}$ is a homomorphism, then an additive map $A : X \to X$ satisfying $A(\alpha x) = h(\alpha)A(x)$ $(x \in X, \alpha \in \mathbb{C})$ is called an h-quasilinear operator. If $h(\alpha) = \bar{\alpha}$, for $\alpha \in \mathbb{C}$, then A is said to be a conjugate linear operator.

Lemma 1.4 [12, Theoreme 3.3] *Let $\phi : \mathcal{F}(X) \to \mathcal{F}(X)$ be a bijective additive map preserving rank one operators in both directions. Then there exists a ring automorphism $h : \mathbb{C} \to \mathbb{C}$, and either there are h-quasilinear bijective mappings $A : X \to X$ and $B : X^* \to X^*$ such that:*

$$\phi(x \otimes f) = Ax \otimes Bf \ (x \in X, f \in X^*),$$

or there are h-quasilinear bijective mappings $C : X^* \to X$ and $D : X \to X^*$ such that:

$$\phi(x \otimes f) = Cf \otimes Dx \ (x \in X, f \in X^*).$$

Note that, if in Lemma 1.4 the map ϕ is linear, then h is the identity map on $\mathbb{C}$ and so the maps A, B, C and D are linear.

2 Preliminaries

In this section, we collect and introduce some lemmas that will be used in the sequel. We begin with the following lemma that will be used frequently in what follows.

Lemma 2.1 *For any $x \in X$ and $f \in X^*$, the following assertions hold:*

(i) $\sigma_i(x \otimes f) = \{f(x)\}$ *if and only if* $f(x) \neq 0$.
(ii) $\sigma_i(x \otimes f) = \emptyset$ *if and only if* $f(x) = 0$.

Proof [6, Lemme 2.3] □

In the sequel, for an operator $A \in \mathcal{B}(X)$ and $i \in \{1, 2, 3\}$, we use a useful notation defined by Bouramdane and Ech-cheérif El Kettani [6] by

$$\Lambda_i(A) := \begin{cases} \{0\}, & \text{if } \sigma_i(A) = \emptyset; \\ \sigma_i(A), & \text{Otherwise .} \end{cases}$$

Let ϕ be a map from $\mathcal{B}(X)$ into itself and r and s two nonegative integers such that $r + s \geq 1$ and $i \in \{1, 2, 3\}$. Then for all $A, B \in \mathcal{B}(X)$, we have

$$\sigma_i(\phi(A)^r\phi(B)\phi(A)^s) = \sigma_i(A^rBA^s) \Longleftrightarrow \Lambda_i(\phi(A)^r\phi(B)\phi(A)^s) = \Lambda_i(A^rBA^s). \tag{2.1}$$

We continue with the following identity principle.

Lemma 2.2 *Let $A, B \in \mathcal{B}(X)$, r and s two nonegative integers such that $r + s \geq 1$ and $i \in \{1, 2, 3\}$. Then the following statements are equivalent:*

(i) $A = B$.
(ii) $\sigma_i(R^rAR^s) = \sigma_i(R^rBR^s)$ *for all* $R \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$.
(iii) $\Lambda_i(R^rAR^s) = \Lambda_i(R^rBR^s)$ *for all* $R \in \mathcal{F}_1(X)$.

Proof The implication $i) \Longrightarrow ii)$ is evident.

Let $R = x \otimes f$ for $x \in X \backslash \{0\}$ and $f \in X^* \backslash \{0\}$ with $f(x) \neq 0$, then we have $R^r = f(x)^{r-1} x \otimes f$ and $R^s = f(x)^{s-1} x \otimes f$. Consequently $R^r A R^s = f(x)^{r+s-2} f(Ax) x \otimes f$ and $R^r B R^s = f(x)^{r+s-2} f(Bx) x \otimes f$, thus

$$\{f(x)^{r+s-1} f(Ax)\} = \sigma_i(R^r A R^s) = \sigma_i(R^r B R^s) = \{f(x)^{r+s-1} f(Bx)\}.$$

It follows that $f(x)^{r+s-1} f(Ax) = f(x)^{r+s-1} f(Bx)$, so we have $f(Ax) = f(Bx)$ for all $x \in X$ and $f \in X^*$, with $f(x) \neq 0$.

Now, if $f(x) = 0$, let $g \in X^*$ such that $g(x) \neq 0$. Then by the first case

$$(f+g)(Ax) = (f+g)(Bx) \quad \text{and} \quad g(Ax) = g(Bx).$$

Then

$$\begin{aligned} f(Ax) + g(Bx) &= f(Ax) + g(Ax) \\ &= (f+g)(Ax) \\ &= (f+g)(Bx) \\ &= f(Bx) + g(Bx). \end{aligned}$$

Thus $f(Ax) = f(Bx)$ in this case, it follows that $f(Ax) = f(Bx)$ for all $x \in X$ and $f \in X^*$. Thus $Ax = Bx$ for all $x \in X$. Which proves that $A = B$. □

The following lemma is a characterization of rank one operators in term of spectral domains in the spectrum.

Lemma 2.3 *Let $A \in \mathcal{B}(X)$, s and r are two nonegative integers, and $i \in \{1, 2, 3\}$. The following statements are equivalent*

(i) $A \in \mathcal{F}_1(X)$.
(ii) $\sigma_i(R^r A R^s)$ *contains at most one element for all* $R \in \mathcal{F}_2(X)$.
(iii) $\sigma_i(R^r A R^s)$ *contains at most one element for all* $R \in \mathcal{B}(X)$.

Proof [8, Lemme 2.4] □

We end this section with the following lemma which is a useful observation which allow us to prove that a map satisfying the equation (3.1) is additive.

Lemma 2.4 *Let $R \in \mathcal{F}_1(X)$ and $A, B \in \mathcal{B}(X)$, and $i \in \{1, 2, 3\}$. Then, The following statements hold:*

(i) $\Lambda_i((A+B)R) = \Lambda_i(AR) + \Lambda_i(BR)$.
(ii) $\Lambda_i(R(A+B)R) = \Lambda_i(RAR) + \Lambda_i(RBR)$.

Proof [6, Lemme 2.4] □

3 Main Result

Since all the necessary ingredients are collected in the preliminary section we will state and prove the promised main result. Let $A, B \in \mathcal{B}(X)$, set $A_{i_p} = B$ and $A_{i_j} = A$ for $j \neq p$ where i_p is the term which appears exactly once in $(i_1, \dots, i_m)$. Note that $A_1 * A_2 * \cdots * A_k = A^r B A^s$ for some nonegative integers r and s such that $r + s = m - 1$. Then, the Theorem 1.3 is a consequence of the following one.

Theorem 3.1 *Let r and s be two nonegative integers with $r + s \geq 1$, and $i \in \{1, 2, 3\}$. Let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map which satisfies the following condition:*

$$\sigma_i(\phi(A)^r \phi(B)\phi(A)^s) = \sigma_i(A^r B A^s) \text{ for all } A, B \in \mathcal{B}(X). \tag{3.1}$$

Then there exists $\alpha \in \mathbb{C}$ with $\alpha^{r+s+1} = 1$ such that either

(i) $\phi(T) = \alpha A T A^{-1}$ for all $T \in \mathcal{B}(X)$, where $A : X \to X$ is a bounded invertible linear (or conjugate linear) operator in $\mathcal{B}(X)$.
(ii) $\phi(T) = \alpha C T^* C^{-1}$ for all $T \in \mathcal{B}(X)$, where $C : X^* \to X$ is a bounded invertible linear (or conjugate linear) operator. In this case, X must be reflexive.

Proof Assume that ϕ is a surjective map from $\mathcal{B}(X)$ into itself which satisfies the Eq. (3.1). We divide the proof into several steps.

Step 1. ϕ is injective.
Let $A, B \in \mathcal{B}(X)$ such that $\phi(A) = \phi(B)$. Then, by hypothesis, for every $T \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$ we have

$$\begin{aligned} \sigma_i(T^r A T^s) &= \sigma_i(\phi(T)^r \phi(A)\phi(T)^s) \\ &= \sigma_i(\phi(T)^r \phi(B)\phi(T)^s) \\ &= \sigma_i(T^r B T^s). \end{aligned}$$

It follows, by Lemma 2.2 that $A = B$, and so ϕ is injective. Thus ϕ is bijective since it is assumed to be surjective, moreover ϕ^{-1} satisfies the equation (3.1).

Step 2. ϕ preserves $\mathcal{F}_1(X)$ and $\mathcal{N}_1(X)$ in both directions.
Let R be a rank one operator, and $A \in \mathcal{B}(X)$ then, by Lemma 2.3, $\sigma_i(A^r R A^s)$ contains at most one element. It follows by hypothesis that $\sigma_i(\phi(A)^r \phi(R)\phi(A)^s)$ contains at most one element for every $A \in \mathcal{B}(X)$, again by the surjectivity of ϕ we deduce that $\phi(R)$ is a rank one operator. Since ϕ is bijective and ϕ^{-1} satisfies the equation (3.1), we conclude that ϕ preserves rank one operators in both directions.
Now, let $R \in \mathcal{F}_1(X)$, by hypothesis, we have $\sigma_i(R^{r+s+1}) = \sigma_i(\phi(R)^{r+s+1})$. If $R \in \mathcal{N}_1(X)$, then $R^{r+s+1} \in \mathcal{N}_1(X)$ and so $\sigma_i(\phi(R)^{r+s+1}) = \emptyset$. It follows that $\phi(R) \in \mathcal{N}_1(X)$. The inverse is similar.

Step 3. ϕ is linear and preserves $\mathcal{F}(X)$ in both directions.
Let us show first that ϕ is homogeneous. To do that consider $A \in \mathcal{B}(X)$ and $\lambda \in \mathbb{C}$. Note that $\sigma_i(\lambda R) = \lambda \sigma_i(R)$ for every $R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$, by the Eq. (1.1). Thus for

every $R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X)$, we have

$$\begin{aligned}\sigma_i(R^r \lambda A R^s) &= \lambda \sigma_i(R^r A R^s) \\ &= \lambda \sigma_i(\phi(R)^r \phi(A) \phi(R)^s) \\ &= \sigma_i(\phi(R)^r \lambda \phi(A) \phi(R)^s).\end{aligned}$$

And by hypothesis

$$\sigma_i(R^r \lambda A R^s) = \sigma_i(\phi(R)^r \phi(\lambda A) \phi(R)^s).$$

This implies that

$$\sigma_i(\phi(R)^r \phi(\lambda A) \phi(R)^s) = \sigma_i(\phi(R)^r \lambda \phi(A) \phi(R)^s) \quad \text{for all} \quad R \in \mathcal{F}_1(X) \backslash \mathcal{N}_1(X).$$

By appliying Lemma 2.2 and **Step 2.**, we conclude that

$$\phi(\lambda A) = \lambda \phi(A).$$

It remains to show that ϕ is additive. To do that let $A, B \in \mathcal{B}(X)$ and $R \in \mathcal{F}_1(X)$, then by appliying the Lemma 2.4, we have

$$\begin{aligned}\Lambda_i(\phi(R)^r \phi(A+B) \phi(R)^s) &= \Lambda_i(R^r (A+B) R^s) \\ &= \Lambda_i(R^r A R^s) + \Lambda_i(R^r B R^s) \\ &= \Lambda_i(\phi(R)^r \phi(A) \phi(R)^s) + \Lambda_i(\phi(R)^r \phi(B) \phi(R)^s) \\ &= \Lambda_i(\phi(R)^r (\phi(A) + \phi(B)) \phi(R)^s).\end{aligned}$$

So by (2.1)

$$\sigma_i(\phi(R)^r \phi(A+B) \phi(R)^s) = \sigma_i(\phi(R)^r (\phi(A) + \phi(B)) \phi(R)^s)$$

And by appliying the Lemma 2.2 and **Step 2.**, we get $\phi(A+B) = \phi(A) + \phi(B)$ for all $A, B \in \mathcal{B}(X)$, and ϕ is additive.

Finally, since each $T \in \mathcal{F}(X)$ is a finite linear combination of rank one operators and since ϕ preserves $\mathcal{R}_1(X)$ in both directions, we have the result by linearity.

Step 4. ϕ takes the desired form.

It follows by has been already proved in the **Step 3.** and Lemma 1.4 that ϕ takes one of the following forms

(1) There exists a bounded invertible linear (or conjugate linear) operator $A : X \to X$ and $B : X^* \to X^*$ such that

$$\phi(x \otimes f) = Ax \otimes Bf, \quad \text{for all} \quad x \in X \quad \text{and} \quad f \in X^*.$$

(2) There exists a bounded invertible linear (or conjugate linear) $C : X^* \to X$ and $D : X \to X^*$ such that

$$\phi(x \otimes f) = Cf \otimes Dx, \quad \text{for all} \quad x \in X \quad \text{and} \quad f \in X^*.$$

Assume that ϕ has the first form. Let us show that $\phi(I) = \alpha I$ with $\alpha^{r+s+1} = 1$.

Since ϕ is surjective, there exist $S \in \mathcal{B}(X)$ such that $I = \phi(S)$, then $S = \lambda I$. Otherwise assume that there exist $x \in X$ such that Sx and x are linearly independent. Then there is $f \in X^*$ such that $f(Sx) = 0$ and $f(x) = 1$. Note that $Bf(Ax) \neq 0$. Then by Lemma 2.1 we have

$$\begin{aligned}
\emptyset &= \sigma_i(x \otimes f.S.x \otimes f) \\
&= \sigma_i((x \otimes f)^r.S.(x \otimes f)^s) \\
&= \sigma_i((Ax \otimes Bf)^{r+s}) \\
&= \{Bf(Ax)^{r+s}\} \\
&\neq \emptyset.
\end{aligned}$$

wich is a contradiction. Since $\phi(\lambda I) = I$, and by linearity of ϕ, we conclude that $\phi(I) = \alpha I$.

Let $f \in X^*$ and $x \in X$ such that $f(x) = 1$, then by Lemma 2.1 we obtain

$$\begin{aligned}
\{1\} &= \sigma_i(x \otimes f) \\
&= \sigma_i((x \otimes f)^r x \otimes f(x \otimes f)^s) \\
&= \sigma_i((Ax \otimes Bf)^r Ax \otimes Bf(Ax \otimes Bf)^s) \\
&= \sigma_i(Bf(Ax)^{r+s-2} Ax \otimes Bf.Ax \otimes Bf.Ax \otimes Bf) \\
&= \sigma_i(Bf(Ax)^{r+s} Ax \otimes Bf) \\
&= \left\{Bf(Ax)^{r+s+1}\right\}.
\end{aligned}$$

We conclude that

$$Bf(Ax)^{r+s+1} = 1. \tag{3.2}$$

On the other hand, we have

$$\begin{aligned}
\{1\} &= \left\{f(x)^{r+s}\right\} \\
&= \sigma_i((x \otimes f)^r I(x \otimes f)^s) \\
&= \sigma_i((Ax \otimes Bf)^r \alpha(Ax \otimes Bf)^s) \\
&= \sigma_i(\alpha Bf(Ax)^{r+s-1} Ax \otimes Bf) \\
&= \left\{\alpha Bf(Ax)^{r+s}\right\}.
\end{aligned}$$

Thus

$$\alpha Bf(Ax)^{r+s} = 1. \tag{3.3}$$

Otherwise

$$\begin{aligned}\{1\} &= \sigma_i(x \otimes f)\\ &= \sigma_i(I^r x \otimes f I^s)\\ &= \sigma_i(\alpha^{r+s} Ax \otimes Bf)\\ &= \left\{\alpha^{r+s} Bf(Ax)\right\}.\end{aligned}$$

So

$$\alpha^{r+s} Bf(Ax) = 1. \tag{3.4}$$

By combining of (3.2), (3.3) and (3.4), we get $\alpha^{r+s+1} = 1$.

Now, let us show that $\phi(x \otimes f) = \alpha Ax \otimes f A^{-1}$ for every $x \in X$ and $f \in X^*$. Let $x \in X$ and $f \in X^*$, we have

$$\begin{aligned}\{f(x)\} &= \sigma_i(x \otimes f)\\ &= \sigma_i(I^r x \otimes f I^s)\\ &= \sigma_i(\alpha^{r+s} Ax \otimes Bf)\\ &= \left\{\alpha^{r+s} Bf(Ax)\right\}.\end{aligned}$$

Then

$$\alpha^{r+s} Bf(Ax) = f(x).$$

Hence

$$Bf(Ax) = \alpha f(x). \tag{3.5}$$

Let us prove that A is continuous and $B = \alpha(A^*)^{-1}$. To do that consider $(x_n)_{n\geq 0}$ a sequence of vectors of X such that $\lim_{n\to+\infty} x_n = x \in X$ and $\lim_{n\to+\infty} Ax_n = y \in X$. We have

$$\begin{aligned}Bf(Ax) &= \lim_{n\to+\infty} Bf(Ax_n)\\ &= \lim_{n\to+\infty} \alpha f(x_n)\\ &= \alpha f(x) \quad \text{for all} \quad f \in X^*.\end{aligned}$$

Since B is bijective, we see that $Ax = y$ and the closed graph theorem tells us that A is continuous. It follows from (3.5) that $\alpha f(x) = A^* Bf(x)$, for all $x \in X$ and $f \in X^*$, wich implies that $A^* B = \alpha I_{X^*}$; as desired. Thus, for all $x \in X$ and $f \in X^*$, we have $\phi(x \otimes f) = \alpha Ax \otimes f A^{-1}$.

Now, let $T \in \mathcal{B}(X)$ and $R \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$, we have by hypothesis

$$\sigma_i(R^r T R^s) = \sigma_i(\phi(R)^r \phi(T) \phi(R)^s)$$

$$
\begin{aligned}
&= \sigma_i((\alpha ARA^{-1})^r \phi(T)(\alpha ARA^{-1})^s) \\
&= \sigma_i(\alpha^{r+s} R^r A^{-1}\phi(T) A R^s).
\end{aligned}
$$

This implies, by Lemma 2.2, that $T = \alpha^{r+s} A^{-1}\phi(T)A$, so $\phi(T) = \alpha ATA^{-1}$.

Now suppose that ϕ has the second form. Since ϕ is surjective, so there exist $S \in \mathcal{B}(X)$ such that $I = \phi(S)$. Then $S = \lambda I$. Otherwise assume there exist $x \in X$ such that Sx and x are linearly independent. Then there exist a linear form $f \in X^*$ such that $f(Sx) = 0$ and $f(x) = 1$. Note that $Dx(Cf) \neq 0$. Then by Lemma 2.1

$$
\begin{aligned}
\emptyset &= \sigma_i(x \otimes f.S.x \otimes f) \\
&= \sigma_i((x \otimes f)^r.S.(x \otimes f)^s) \\
&= \sigma_i((Cf \otimes Dx)^{r+s}) \\
&= \{Dx(Cf)^{r+s}\} \\
&\neq \emptyset,
\end{aligned}
$$

wich is a contradiction. Hence $\phi(\lambda I) = I$, and by linearity of ϕ, we conclude that $\phi(I) = \alpha I$.

Let $f \in X^*$ and $x \in X$ such that $f(x) = 1$, then by Lemma 2.1 we have

$$
\begin{aligned}
\{1\} &= \sigma_i(x \otimes f) \\
&= \sigma_i((x \otimes f)^r x \otimes f(x \otimes f)^s) \\
&= \sigma_i((Cf \otimes Dx)^r Cf \otimes Dx(Cf \otimes Dx)^s) \\
&= \sigma_i(Dx(Cf)^{r+s-2} Cf \otimes Dx.Cf \otimes Dx.Cf \otimes Dx) \\
&= \sigma_i(Dx(Cf)^{r+s} Cf \otimes Dx) \\
&= \left\{Dx(Cf)^{r+s+1}\right\}.
\end{aligned}
$$

We conclude that

$$
Dx(Cf)^{r+s+1} = 1, \tag{3.6}
$$

and,

$$
\begin{aligned}
\{1\} &= \left\{f(x)^{r+s}\right\} \\
&= \sigma_i((x \otimes f)^r I(x \otimes f)^s) \\
&= \sigma_i((Cf \otimes Dx)^r \alpha(Cf \otimes Dx)^s) \\
&= \sigma_i(\alpha Dx(Cf)^{r+s-1} Cf \otimes Dx) \\
&= \left\{\alpha Dx(Cf)^{r+s}\right\}.
\end{aligned}
$$

Hence

$$\alpha Dx(Cf)^{r+s} = 1. \tag{3.7}$$

Otherwise

$$\begin{aligned}\{1\} &= \sigma_i(x \otimes f)\\ &= \sigma_i(I^r x \otimes f I^s)\\ &= \sigma_i(\alpha^{r+s} Cf \otimes Dx)\\ &= \left\{\alpha^{r+s} Dx(Cf)\right\}.\end{aligned}$$

Hence

$$\alpha^{r+s} Dx(Cf) = 1. \tag{3.8}$$

By combining (3.6), (3.7) and (3.8), we conclude that $\alpha^{r+s+1} = 1.$

Now, choose $f \in X^*$, we have

$$\begin{aligned}\{f(x)\} &= \sigma_i(x \otimes f)\\ &= \sigma_i((\phi(I)^r \phi(x \otimes f)\phi(I)^s)\\ &= \sigma_i(\alpha^{r+s} Cf \otimes Dx)\\ &= \left\{\alpha^{r+s} Dx(Cf)\right\}.\end{aligned}$$

Hence $Dx(Cf) = \dfrac{1}{\alpha^{r+s}} f(x)$. Since $\alpha^{r+s+1} = 1$, thus

$$Dx(Cf) = \alpha f(x). \tag{3.9}$$

Consequently, by similar reasoning as the first case and by using the closed graph theorem, separately for each of C and D, we conclude that these operators are bounded. Therfore, both $C^*\ X^* \rightarrow X^{**}$ and $D^*\ X^{**} \rightarrow X^*$ are invertible. Moreover, if β is the canonical embedding of X in X^{**}. Then by (3.9) we have $D^* \circ \beta \circ C = \alpha I_{X^*}$ which implies that $\beta \circ C = \alpha (D^*)^{-1}$. Since both C and $(D^*)^{-1}$ are surjective, β is also surjective and hence X is reflexive. Identifying X with X^{**}, so if $f(x) = 1$ then $D^*C = \alpha I_{X^*}$. Similarly, it follows from (3.9) that $C^*D = \alpha I$ and so $D = \alpha(C^*)^{-1} = \alpha(C^{-1})^*$. So $\phi(R) = \alpha C R^* C^{-1}$ for all $R \in \mathcal{F}_1(X)$. Since the spectrum of a bounded linear operator coincides with the spectrum of its adjoint, for arbitrary $T \in \mathcal{B}(X)$, we obtain that

$$\begin{aligned}\sigma_i(\phi(R)^r \phi(T)\phi(R)^s) &= \sigma_i(R^r T R^s)\\ &= \sigma_i(\alpha^{r+s+1}(R^*)^r T^*(R^*)^s)\\ &= \sigma_i(\alpha C R^* C^{-1} \alpha C R^* C^{-1} ... \alpha C T^* C^{-1} ... \alpha C R^* C^{-1})\\ &= \sigma_i(\phi(R)^r \alpha C T^* C^{-1} \phi(R)^s),\end{aligned}$$

For all $R \in \mathcal{F}_1(X)\backslash\mathcal{N}_1(X)$. By Lemma 2.2 and **step 2.**, we get $\phi(T) = \alpha C T^* C^{-1}$.

By taking $r = 0$ and $s = 1$ (resp. $r = s = 1$)the [9, Theoreme 2.3](resp. [9, Theoreme 3.5]) becomes corollary of Theoreme 3.1.

Corollary 3.2 *Let $i \in \{1, 2, 3\}$ and let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map satisfying*

$$\sigma_i(AB) = \sigma_i(\phi(A)\phi(B)) \quad \textit{for all} \quad A, B \in \mathcal{B}(X).$$

Then there exists a scalar α with $\alpha^2 = 1$ such that either

(1) *$\phi(A) = \alpha T A T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X \to X$ is bijective linear operator in $\mathcal{B}(X)$.*
(2) *$\phi(A) = \alpha T A^* T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X^* \to X$ is bijective linear operator. In this case, X must be reflexive.*

Corollary 3.3 *Let $i \in \{1, 2, 3\}$ and let $\phi : \mathcal{B}(X) \to \mathcal{B}(X)$ be a surjective map satisfying*

$$\sigma_i(ABA) = \sigma_i(\phi(A)\phi(B)\phi(A)) \quad \textit{for all} \quad A, B \in \mathcal{B}(X).$$

Then there exists a scalar α with $\alpha^3 = 1$ such that either

(1) *$\phi(A) = \alpha T A T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X \to X$ is bijective linear operator in $\mathcal{B}(X)$.*
(2) *$\phi(A) = \alpha T A^* T^{-1}$ for all $A \in \mathcal{B}(X)$, where $T : X^* \to X$ is bijective linear operator. In this case, X must be reflexive.*

□

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Declarations

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