Complex Analysis and Operator Theory (2023) 17:2
https://doi.org/10.1007/s11785-022-01300-0

Complex Analysis and Operator Theory

Geometric Invariants for a Class of Submodules of Analytic Hilbert Modules Via the Sheaf Model

Shibananda Biswas[1] · Gadadhar Misra[2,3] · Samrat Sen[4]

Received: 5 May 2022 / Accepted: 28 October 2022 / Published online: 19 November 2022

Abstract

Let $\Omega \subseteq \mathbb{C}^m$ be a bounded connected open set and $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be an analytic Hilbert module, i.e., the Hilbert space $\mathcal{H}$ possesses a reproducing kernel K, the polynomial ring $\mathbb{C}[z] \subseteq \mathcal{H}$ is dense and the point-wise multiplication induced by $p \in \mathbb{C}[z]$ is bounded on $\mathcal{H}$. We fix an ideal $\mathcal{I} \subseteq \mathbb{C}[z]$ generated by $p_1, \ldots, p_t$ and let $[\mathcal{I}]$ denote the completion of $\mathcal{I}$ in $\mathcal{H}$. The sheaf $\mathcal{S}^{\mathcal{H}}$ associated to analytic Hilbert module $\mathcal{H}$ is the sheaf $\mathcal{O}(\Omega)$ of holomorphic functions on Ω and hence is free. However, the subsheaf $\mathcal{S}^{[\mathcal{I}]}$ associated to $[\mathcal{I}]$ is coherent and not necessarily locally free. Building on the earlier work of Biswas, Misra and Putinar (Journal fr die reine und angewandte Mathematik (Crelles Journal) 662:165–204, 2012), we prescribe a hermitian structure for a coherent sheaf and use it to find tractable invariants. Moreover, we prove that if the zero set $V_{[\mathcal{I}]}$ is a submanifold of codimension t, then there is a unique local decomposition for the kernel $K_{[\mathcal{I}]}$ along the zero set that serves as a holomorphic frame for a vector bundle on $V_{[\mathcal{I}]}$. The complex geometric invariants of this vector bundle are also unitary invariants for the submodule $[\mathcal{I}] \subseteq \mathcal{H}$.

Dedicated to the memory of Jörg Eschmeier.

Communicated by Mihai Putinar.

The second-named author would like to acknowledge funding through the J C Bose National Fellowship and the MATRICS grant for his research.
A number of the results presented in this paper are from the PhD thesis of the third named author submitted to the Indian Institute of Science, Bangalore.

This article is part of the topical collection "Multivariable Operator Theory. The Jörg Eschmeier Memorial" edited by Raul Curto, Michael Hartz, Mihai Putinar and Ernst Albrecht.

✉ Gadadhar Misra
gm@isibang.ac.in

Extended author information available on the last page of the article

Keywords Hilbert module · Reproducing kernel function · Analytic Hilbert module · Submodule · Resolution · Holomorphic Hermitian vector bundle · Coherent sheaf · Linear space

Mathematics Subject Classification Primary: 47B13 · 47B32 · 47B35 · 32A10 · 32A36 · 32A38

1 Introduction

The notion of a Hilbert module over a function algebra was introduced by R. G. Douglas in the late eighties. Over the past couple of decades, problems of multi-variate operator theory have been discussed using the language of Hilbert modules. In this paper, we continue this tradition. Let us begin by setting up some conventions that will be in force throughout.

(1) $\mathbb{C}[z] := \mathbb{C}[z_1, \ldots, z_m]$ is the polynomial ring in m variables.
(2) $\Omega \subseteq \mathbb{C}^m$ is a bounded domain.
(3) $\mathcal{O}(\Omega)$ is the ring of holomorphic functions on Ω.
(4) $\mathcal{H}$ is a complex separable Hilbert space and $\mathcal{L}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$.

First, we recall some useful notions and terminology that we would be using through out this paper. For $1 \leq i \leq m$, let $T_i : \mathcal{H} \to \mathcal{H}$ be a commuting set of bounded linear operators on a Hilbert space $\mathcal{H}$ and let $\boldsymbol{T}$ denote the commuting tuple $(T_1, \ldots, T_m)$. For a polynomial $p \in \mathbb{C}[z]$, the map $p \to m_p := p(T)$ defines a homomorphism from $\mathbb{C}[z]$ to $\mathcal{L}(\mathcal{H})$. Thus the map $(p, h) \to p(T)h$, $h \in \mathcal{H}$, defines module multiplication on the Hilbert space $\mathcal{H}$ over the polynomial ring $\mathbb{C}[z]$.

The Hilbert space $\mathcal{H}$ equipped with this multiplication is said to be a *Hilbert module* over the polynomial ring $\mathbb{C}[z]$.

A closed subspace $\mathcal{H}_0 \subseteq \mathcal{H}$ is said to be a *submodule* of $\mathcal{H}$ if it is invariant under the module multiplication, i.e., $m_p f \in \mathcal{H}_0$ for all $f \in \mathcal{H}_0$. The *quotient module* $\mathcal{Q}$ is the quotient space $\mathcal{H}/\mathcal{H}_0$, which is the ortho-complement of $\mathcal{H}_0$ in $\mathcal{H}$. The module multiplication on this space is defined by compression of the multiplication on $\mathcal{H}$ to $\mathcal{Q}$, i.e., $\mathfrak{m}_p f = P_{\mathcal{H}_0^\perp}\big(m_p f\big)$, $f \in \mathcal{Q}$.

(The original definition of the Hilbert module required the module map to be continuous in both the variables, however, here we dispense with this requirement.)

Two Hilbert modules $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are said to be "unitarily" equivalent if there exists a unitary module map $\theta : \mathcal{H} \to \tilde{\mathcal{H}}$ intertwining the module maps, m and $\tilde{m}$, that is, $\tilde{m}_p \theta = \theta m_p$.

Definition 1.1 A Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ over the polynomial ring $\mathbb{C}[z]$ is said to be an *analytic Hilbert module* if it possesses a reproducing kernel K and the polynomial ring $\mathbb{C}[\underline{z}]$ is included in $\mathcal{H}$ and it is dense. In particular, $K(w, w) \neq 0$, $w \in \Omega$.

Fix an ideal $\mathcal{I} \subseteq \mathbb{C}[z]$ generated by $p_1, \ldots, p_t$ and let $[\mathcal{I}]$ denote the completion of $\mathcal{I}$ in $\mathcal{H}$. Thus

$$0 \longrightarrow [\mathcal{I}] \longrightarrow \mathcal{H} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

is a short exact sequence of Hilbert modules, where $\mathcal{Q}$ is the quotient module $[\mathcal{I}]^{\perp}$. The rigidity phenomenon for Hilbert modules is the question of determining when two submodules of the form $[\mathcal{I}]$, $[\mathcal{I}']$ of an analytic Hilbert module are equivalent. There is also the question of finding invariants for the sub and quotient modules $[\mathcal{I}]$, $[\mathcal{I}]^{\perp}$. In this paper, we mostly discuss the pair $(\mathcal{H}, [\mathcal{I}])$, where $\mathcal{H}$ is an *a*nalytic Hilbert modules.

For any submodule $\mathcal{M}$ of an analytic Hilbert module, we define the subsheaf $\mathcal{S}^{\mathcal{M}}$ of the sheaf $\mathcal{O}(\Omega)$ of holomorphic functions on Ω as follows:

$$\mathcal{S}^{\mathcal{M}}(U) := \left\{ \sum_{i=1}^{n} (f_i|_U) h_i : f_i \in \mathcal{M}, h_i \in \mathcal{O}(U), n \in \mathbb{N} \right\},$$

for any open $U \subset \Omega$. We note $\mathcal{S}^{\mathcal{H}}(U) = \mathcal{O}(U)$ and thus it is free and naturally gives rise to a holomorphic line bundle on Ω. However, in general, the sheaf corresponding to the sub-module $[\mathcal{I}]$ is not free, not even locally free, but only coherent.

As we have seen, the analytic Hilbert modules give rise to a holomorphic hermitian line bundle. Consequently, the methods of Cowen and Douglas developed for the class $B_1(\Omega)$ in [9] applies. The class $B_1(\Omega)$ consists of those commuting tuples of operators $\boldsymbol{T} := (T_1, \ldots, T_m)$ on a Hilbert space $\mathcal{H}$ possessing an open set of eigenvalues, say $\Omega \subset \mathbb{C}^m$, a holomorphic map $\gamma : \Omega \to \mathcal{H}$ with the property that $(T_i - w)\gamma(w) = 0$, $1 \leq i \leq m$, for each $w \in \Omega$ and finally $\gamma(w)$ modulo the constants is the only such vector. Clearly, the map $w \to \gamma(w)$ defines a trivial holomorphic line bundle $\mathcal{L}$ on Ω with a hermitian structure inherited from the Hilbert space $\mathcal{H}$, namely, $\|\gamma(w)\|$. Recall that the curvature of $\mathcal{L}$ is defined to be the $(1, 1)$ form:

$$\mathcal{K}_{\mathcal{L}}(z) := \sum_{i,j=1}^{n} \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \|\gamma(z)\|^2 dz_i \wedge d\bar{z}_j.$$

The Cowen–Douglas theorem says that the curvature is a complete invaraint for the commuting tuple of operators $\boldsymbol{T}$. Thus two analytic Hilbert modules are equivalent if and only if their curvatures are equal. However, there is a large class of Hilbert modules, such as $[\mathcal{I}]$, where the dimension of the joint kernel $\cap_{i=1}^{m} \ker(M_i - w_i)^*$ is not constant. For instance, let $H^2_{(0,0)}(\mathbb{D}^2)$ be the submodule of functions vanishing at $(0, 0)$ of the Hardy module $H^2(\mathbb{D}^2)$. In this case, $\dim \cap_{i=1}^{2} \ker(M_i - w_i)^* = 1$ if $(w_1, w_2) \neq (0, 0)$, while it is 2 if $(w_1, w_2) = (0, 0)$, cf. [12].

We study a class of Hilbert modules very similar to the ones introduced in [5], see also [3], and designated $\mathfrak{B}_1(\Omega)$ in that paper. Here, while we retain the same symbol, the definition instead of requiring $\dim \left(\mathcal{H} \Big/ \left(\sum_{i=1}^{m} (z_i - w_i)\mathcal{H} \right)\right) < \infty$, merely requires that $\dim \bigcap_{i=1}^{m} \ker(M_i - w_i)^* < \infty$, $w \in \Omega$. For what we do here, this is adequate, and more importantly, this condition is not very hard to verify as in Proposition 2.3.

Definition 1.2 The class $\mathfrak{B}_1(\Omega)$ consists of Hilbert modules $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ possessing a reproducing kernel K and such that $\dim \cap_{i=1}^{m} \ker(M_i - w_i)^* < \infty$, $w \in \Omega$.

All the analytic Hilbert modules $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ are in $\mathfrak{B}_1(\Omega)$, see Sect. 2. However, the reproducing kernel K of a Hilbert module in $\mathfrak{B}_1(\Omega)$ may vanish, that is, $K(w, w) = 0$

for w in some closed subset of Ω, unlike the case of the analytic Hilbert modules. Indeed the modules in this class are the ones where the dimension of the joint kernel $\cap_{i=1}^{m} \ker(M_i - w_i)^*$ of the module multiplication is not necessarily constant. Therefore the techniques from complex geometry developed in [9, 10] do not apply directly. The typical example that we will be considering is a pair $(\mathcal{H}, [\mathcal{I}])$, where $\mathcal{H}$ is an analytic Hilbert module and $[\mathcal{I}]$ is the completion of a polynomial ideal $\mathcal{I}$ in the norm topology of $\mathcal{H}$. If $\mathcal{I} =< p >$ is a principal ideal, then the kernel $K_{[\mathcal{I}]}$ factors: $K_{[\mathcal{I}]}(z, w) = p(z)\chi(z, w)\overline{p(w)}$. Thus $w \mapsto \chi(\cdot, w)$ is holomorphic frame defining a holomorphic hermitian line bundle which serves as an invariant for $[\mathcal{I}]$. However, to study modules of the form $[\mathcal{I}]$ without assuming that $\mathcal{I}$ is necessarily principal, we make crucial use of the decomposition theorem from [5], which is reproduced below.

Theorem 1.3 (Theorem 1.5, [5]) *Let w_0 be a fixed but arbitrary point in Ω. Suppose $\mathcal{M}$ is in $\mathfrak{B}_1(\Omega)$ and let K be the reproducing kernel of $\mathcal{M}$. Assume that g_i^0, $1 \leq i \leq d$, is a minimal set of generators for the stalk $\mathcal{S}_{w_0}^{\mathcal{M}}$. Then*

(i) *there exists an open neighbourhood Ω_0 of w_0 such that*

$$K(\cdot, w) = \overline{g_1^0(w)}K^{(1)}(w) + \cdots + \overline{g_d^0(w)}K^{(d)}(w), \ w \in \Omega_0$$

for some choice of anti-holomorphic maps $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathcal{M}$,

(ii) *the vectors $K^{(i)}(w)$, $1 \leq i \leq d$, are linearly independent in $\mathcal{M}$ for w in some small neighbourhood of w_0,*

(iii) *the vectors $K^{(i)}(w_0)$, $1 \leq i \leq d$, are uniquely determined by the generators $g_1^0, \ldots, g_d^0$,*

(iv) *the linear span of the set of vectors $\{K^{(i)}(w_0) : 1 \leq i \leq d\}$ in $\mathcal{M}$ is independent of the choice of generators $g_1^0, \ldots, g_d^0$, and*

(v) *$M_p^* K^{(i)}(w_0) = \overline{p(w_0)}K^{(i)}(w_0)$, for $i = 1, \ldots, d$, where M_p denotes the module multiplication by the polynomial p.*

The decomposition of the kernel function given in Theorem 1.3 naturally gives rise to the hermitian structure on the "linear space" associated to the coherent sheaf $\mathcal{S}^{[\mathcal{I}]}$ described in Sect. 3. Moreover, it also enables us to define a holomorphic hermitian line bundle in a canonical manner leading to tractable invariants for the submodule $\mathcal{M}$.

On the other hand, part (iii) of the Theorem 1.3 gurantees only that $K^{(i)}$'s are uniquely determined at the point w_0. It is natural to ask what condition on an ideal $\mathcal{I}$ might ensure that these vectors are determined uniquely over the zero set $V_{[\mathcal{I}]}$. In Sect. 4, we prove that if $V_{[\mathcal{I}]}$ is a complex manifold codimension t and the ideal is generated by t polynomials, then the $K^{(i)}$'s are uniquely determined. This is part of Theorem 4.3 which improves Theorem 1.3 in this case. In Corollary 4.4, we have shown that the dimension formula holds for a larger class ideals than covered in Duan–Guo [14, Proposition 2.2 and Theorem 2.3]. In Sect. 5, Theorem 5.1, we show that the class of the vector bundle on the zero set $V_{[\mathcal{I}]}$ obtained from Theorem 4.3 is an invariant for the Hilbert module $[\mathcal{I}]$. We further show in Corollary 6.3, that in case of a homogeneous ideal, the curvatures are related via conjugation by a constant matrix with respect to two different minimal sets of generators.

2 Preliminaries

The relationship between analytic Hilbert modules and commuting tuples of operators in the Cowen-Douglas class is revealed by using the notion of a non-negative definite kernel K with the reproducing property discussed below. In particular, $w \to K(\cdot, w)$, $w \in \Omega$, serves as an anti-holomorphic frame for the eigenbundle for the operator (or, the commuting tuple of operators) in the Cowen-Douglas class.

Definition 2.1 The function $K : \Omega \times \Omega \to \mathbb{C}$ is said to be *non-negative definite kernel* if:

$$\left\langle \Big(\big(K(z_i, z_j)\big)\Big)x, x \right\rangle \geq 0, \ \{z_1, \ldots, z_n\} \subseteq \Omega, \ x \in \mathbb{C}^n, \ n \in \mathbb{N}.$$

In this paper, we assume the kernel function K to be holomorphic in first variable and anti-holomorphic in second. Let k_w be the holomorphic function defined by $k_w(z) := K(z, w)$.

Let $\mathcal{H}^0$ be the linear span of the vectors $\{k_w : w \in \Omega\}$. For any finite subset $\{z_1, \ldots, z_n\}$ of Ω and complex numbers $x_1, \ldots, x_n$, set

$$\Big\| \sum_{i=1}^{n} x_j k_{z_j} \Big\|^2 := \left\langle \Big(\big(K(z_i, z_j)\big)\Big)x, x \right\rangle,$$

where x is the vector whose i-th coordinate is x_i. Since K is assumed to be non-negative definite, this defines a semi-norm on the linear space $\mathcal{H}^0$.

Now, the sesquilinear form $K(z, w) = \langle K(\cdot, w), K(\cdot, z)\rangle$ is non-negative definite by assumption. However, for $f \in \mathcal{H}^0$, the Cauchy-Schwarz inequality gives

$$|f(w)|^2 = |\langle f, K(\cdot, w)\rangle|^2 \leq \|f\|^2 K(w, w), \ \ w \in \Omega.$$

It follows that if $\|f\| = 0$, then f is the zero vector in $\mathcal{H}^0$. Thus the semi-norm defined by K, as above, is indeed a norm on $\mathcal{H}^0$. The completion of $\mathcal{H}^0$ equipped with this norm is a Hilbert space, which consists of holomorphic functions on Ω (cf. [2, 17]). The function $k_w := K(\cdot, w)$, then has the reproducing property, namely,

$$\langle f, k_w\rangle = f(w), \ \ f \in \mathcal{H}, \ \ w \in \Omega.$$

Conversely, assume that the point evaluation $e_w : \mathcal{H} \to \mathbb{C}$, $w \in \Omega$, on a Hilbert space $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ is bounded, that is, $|f(w)| \leq C\|f\|$, $f \in \mathcal{H}$. Then $f(w) = \langle f, k_w\rangle$ for some vector $k_w \in \mathcal{H}$. It follows that $e_w^* = k_w$. Let $K(z, w) = e_z k_w = e_z e_w^*$. The function K is holomorphic in the first variable and anti-holomoprhic in the second. Also, $\overline{K(z, w)} = K(w, z)$. Finally, for any finite subset $\{z_1, \ldots, z_n\}$ of Ω, we have

$$0 \leq \Big\| \sum_{i=1}^{n} x_j k_{z_j} \Big\|^2 = \sum_{i,j=1}^{n} \bar{x}_i x_j K(z_i, z_j) = \left\langle \Big(\big(K(z_i, z_j)\big)\Big)x, x \right\rangle, \ \ x \in \mathbb{C}^n. \quad (2.1)$$

The non-negative definite function K is said to be the *reproducing kernel* of the Hilbert space $\mathcal{H}$.

There are several notions, namely, locally free modules [6], modules with sharp kernels [1], quasi-free modules [11], which are closely related to the notion of analytic Hilbert modules. In all of these variants, the definition ensures the existence of a holomorphic hermitian vector bundle corresponding to these Hilbert modules. The fundamental theorem of Cowen and Douglas (cf. [9]) then applies and says that the equivalence class of the Hilbert modules and those of the vector bundles determine each other. Finding tractable invariants for these remains a challenge.

It is easy to verify that $M_p^* k_w = \overline{p(w)} k_w$, or equivalently, k_w is in $\ker(M_p - p(w))^*$. If $\mathcal{H}$ is an analytic Hilbert module, then it follows that the $\dim \cap_{i=1}^m \ker(M_i - w_i)^* = 1$, $w \in \Omega$, where M_i is the operator of multiplication by the coordinate function z_i on $\mathcal{H}$. This is easily verified as follows. For any $f \in \cap_{i=1}^m \ker(M_i - w_i)^*$, $p \in \mathbb{C}[\underline{z}]$, we have

$$\langle f, p\rangle = \langle M_p^* f, 1\rangle = \langle \overline{p(w)} f, 1\rangle = \langle a k_w, p\rangle,$$

where $a = \langle f, 1\rangle$. Therefore, if $\mathcal{H}$ is an analytic Hilbert module, then the dimension of the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^*$ is 1 and is spanned by the vector k_w. Hence the map $\gamma : \Omega^* \to \mathcal{H}$, $\gamma(w) = k_{\bar{w}}$ is holomorphic for $w \in \Omega^* := \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$. Thus it defines a holomorphic hermitian line bundle $\mathcal{L}$ on Ω^*. If α is a non-vanishing holomorphic function defined on Ω^*, then $\alpha(w)\gamma(w)$ serves as a holomorphic frame for the line bundle $\mathcal{L}$ as well. The hermitian structures induced by these two holomorphic frames are $\|\gamma(w)\|^2$ and $|\alpha(w)|^2\|\gamma(w)\|^2$, respectively. These differ by the absolute square of a non-vanishing holomorphic function. However, the curvature $\mathcal{K}_{\mathcal{L}}$ defined relative to either one of these two frames is the same and therefore serves as an invariant for the holomorphic hermitian line bundle $\mathcal{L}$.

It is easy to see that $[\mathcal{I}]$ is in $\mathfrak{B}_1(\Omega)$ using the notion of *module tensor product*. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert modules over the polynomial ring $\mathbb{C}[\underline{z}]$. The Hilbert space tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of these two Hilbert modules has two natural module multiplications, namely, $m_p \otimes \mathrm{Id}(f_1 \otimes f_2) = m_p(f_1) \otimes f_2$ and $\mathrm{Id} \otimes m_p(f_1 \otimes f_2) = f_1 \otimes m_p(f_2)$. The module tensor product $\mathcal{H}_1 \otimes_{\mathbb{C}[\underline{z}]} \mathcal{H}_2$ is obtained by identifying the space on which these two multiplications coincide. Set $\mathcal{N}$ to be the closed subspace spanned by the set of vectors

$$\{m_p f_1 \otimes f_2 - f_1 \otimes m_p f_2 : f_1 \in \mathcal{H}_1, f_2 \in \mathcal{H}_2, p \in \mathbb{C}[\underline{z}]\} \subseteq \mathcal{H}_1 \otimes \mathcal{H}_2.$$

The subspace $\mathcal{N}$ is a submodule for both the left and the right multiplications: $m_p \otimes \mathrm{Id}$ and $\mathrm{Id} \otimes m_p$. On the quotient $\mathcal{N}^\perp = (\mathcal{H}_1 \otimes \mathcal{H}_2) \ominus \mathcal{N}$, these two module multiplications coincide (cf. [13]). The quotient Hilbert space $\mathcal{N}^\perp$ equipped with this multiplication is the module tensor product.

We consider the special case $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$, where $\mathbb{C}_w$ is $\mathbb{C}$ equipped with the module multiplication $m_p(\lambda) = p(w)\lambda$, $w \in \Omega$, $p \in \mathbb{C}[\underline{z}]$. In $\mathcal{H}$, let $\mathcal{J}(w)$ denote the joint kernel $\cap_{i=1}^m \ker(M_i - w_i)^* = \cap_{p \in \mathbb{C}[\underline{z}]} \ker(M_p - p(w))^*$. We have the following useful lemma.

Lemma 2.2 *Let $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be a Hilbert module. For any $w \in \Omega$, we have the equality*

$$\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = \Big(\mathcal{J}(w)\Big) \otimes \mathbb{C}.$$

Proof The proof consists of the following string of equalities:

$$\begin{aligned}
\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w &= \big(\mathcal{H} \otimes \mathbb{C}\big)/\big\{pf \otimes \lambda - f \otimes p(w)\lambda : f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\} \\
&= \big\{(p - p(w))f \otimes \lambda : f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\}^{\perp} \\
&= \big\{g \otimes \mu \in \mathcal{H} \otimes \mathbb{C} : \langle g, (p - p(w))f\rangle \mu\bar{\lambda} = 0, \ \ f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\} \\
&= \big\{g \otimes \mu : \langle M^*_{p-p(w)} g, f\rangle = 0, \ \ f \in \mathcal{H}, p \in \mathbb{C}[\underline{z}], \lambda \in \mathbb{C}\big\} \\
&= \Big(\mathcal{J}(w)\Big) \otimes \mathbb{C}.
\end{aligned}$$

These equalities are easily verified. □

Proposition 2.3 *The submodule $[\mathcal{I}]$ of an analytic Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ is in $\mathfrak{B}_1(\Omega)$.*

Proof We note that

$$\dim \bigcap_{i=1}^{m} \ker(M_i - w_i)^* = \dim \mathcal{J}(w) = \dim \big([\mathcal{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w\big).$$

Since the polynomial ring is Noetherian, it follows that $[\mathcal{I}]$ is finitely generated. Now, from [13, Lemma 5.11], it follows that the dim $\big([\mathcal{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w\big)$ is finite, completing the proof. □

Any module map $L : \mathcal{H} \to \widetilde{\mathcal{H}}$ must map the joint kernel $\mathcal{J}(w) \subseteq \mathcal{H}$ into the joint kernel $\widetilde{\mathcal{J}}(w) \subseteq \widetilde{\mathcal{H}}$. If the map L is assumed to be invertible then its restriction to the kernel $\mathcal{J}(w) \subseteq \mathcal{H}$ is evidently an isomorphism. Thus we have proved the following proposition.

Proposition 2.4 *Suppose $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ are two Hilbert module in $\mathcal{O}(\Omega)$, which are isomorphic via an invertible module map. Then $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ and $\widetilde{\mathcal{H}} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ are isomorphic for each $w \in \Omega$.*

Therefore, dim $\mathcal{J}(w)$ is clearly an invariant for the class of Hilbert modules in $\mathcal{O}(\Omega)$. For an analytic Hilbert module, this is a constant function.

Now we observe (as in [12]), for the Hardy module $H^2(\mathbb{D}^2)$, dim $\mathcal{J}(w)$ is identically 1 for all $w \in \mathbb{D}^2$ while for the submodule $H^2_{(0,0)}(\mathbb{D}^2)$, it equals 1 for $w \neq (0, 0)$ but is equal to 2 at $(0, 0)$. Thus $H^2(\mathbb{D}^2)$ and $H^2_{(0,0)}(\mathbb{D}^2)$ are not equivalent via any invertible module map.

The module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$ is said to be the *localization* of $\mathcal{H}$ at w and the set $\mathrm{Sp}(\mathcal{H}) := \big\{\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w : w \in \Omega\big\}$ is said to be the spectral sheaf. When $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ is an analytic Hilbert module, the spectral sheaf determines an anti-holomorphic line bundle via the frame $1 \otimes_{\mathbb{C}[\underline{z}]} 1_w$. The hermitian structure is

induced from $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w$. In general, however, the spectral sheaf is a direct sum of k copies of $\mathbb{C}_w$, where k is between 1 and t, which is the rank of $\mathcal{H}$, see below. In what follows, it will be convenient to use the notion of *locally free* module of rank n over $\Omega^* := \{w \in \mathbb{C}^m : \bar{w} \in \Omega\}$, where Ω is some open bounded subset of $\mathbb{C}^m$.

Definition 2.5 (Definition 1.4, [9]) Let $\mathcal{H}$ be a Hilbert module over $\mathbb{C}[\underline{z}]$. Let Ω be a bounded open connected subset of $\mathbb{C}^m$. We say $\mathcal{H}$ is locally free of rank n at w_0 in Ω^* if there exists a neighbourhood Ω_0^* of w_0 and holomorphic functions $\gamma_1, \gamma_2, \ldots, \gamma_n : \Omega_0^* \to \mathcal{H}$ such that the linear span of the set of n vectors $\{\gamma_1(w), \ldots, \gamma_n(w)\}$ is the module tensor product $\mathcal{H} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{\bar{w}}$. Following the terminology of [6], we say that a module $\mathcal{H}$ is *l*ocally free on Ω^* of rank n if it is locally free of rank n at every w in Ω^*.

Setting $V_{[\mathcal{I}]} := \big\{z \in \Omega : f(z) = 0,\, f \in [\mathcal{I}]\big\}$, in the examples $(\mathcal{H}, [\mathcal{I}])$, we have the following Lemma from [5, Lemma 1.3].

Lemma 2.6 *The submodule $[\mathcal{I}]$ of an analytic Hilbert module $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ is locally free on Ω^* of rank 1 if the ideal $\mathcal{I}$ is principal while if $p_1, \ldots p_t$, $t > 1$, is a minimal set of generators for $\mathcal{I}$, then $[\mathcal{I}]$ is locally free on $(\Omega \setminus V_{[\mathcal{I}]})^*$ of rank 1.*

Assume that there is finite set of generators for $[\mathcal{I}]$, moreover, let $\{p_1, \ldots, p_t\}$ be a minimal set of generators. A description of the spectral sheaf for a pair $(\mathcal{H}, [\mathcal{I}])$ is given below. For $w \in \Omega \setminus V_{[\mathcal{I}]}$, we have $[\mathcal{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = p_i \otimes_{\mathbb{C}[\underline{z}]} 1_w$, $1 \leq i \leq t$. However, note that

$$\begin{aligned} p_i \otimes_{\mathbb{C}[\underline{z}]} 1_w &= P_{\mathcal{J}(w)\otimes\mathbb{C}}(p_i \otimes 1) \\ &= \big(P_{\mathbb{C}[K_{[\mathcal{I}]}(\cdot,w)]} \otimes 1\big)(p_i \otimes 1) \\ &= \tfrac{p_i(w)}{K_{[\mathcal{I}]}(w,w)} K_{[\mathcal{I}]}(\cdot, w) \otimes 1. \end{aligned}$$

Here $\mathbb{C}[K_{[\mathcal{I}]}(\cdot, w)]$ denotes the one dimensional space spanned by the vector $K(\cdot, w)$. Thus the set of vectors $p_i \otimes_{\mathbb{C}[\underline{z}]} 1_w$ are linearly dependent and therefore $\dim[\mathcal{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = 1$ for $w \in \Omega \setminus V_{[\mathcal{I}]}$. Based on this observation and explicit computations in simple examples, it was conjectured in [12] that

$$\dim[\mathcal{I}] \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w = \begin{cases} 1 & \text{for } w \in \Omega \setminus V_{[\mathcal{I}]} \\ \text{codim of } V_{[\mathcal{I}]} & \text{for } w \in V_{[\mathcal{I}]} \end{cases}$$

This formula is shown to be false in general by means of several examples in the paper [14] by Duan and Guo. They show that the formula given above is valid if the ideal $\mathcal{I}$ has any one of the following properties:

(1) $\mathcal{I}$ is singly generated,
(2) $\mathcal{I}$ is a prime ideal in $\mathbb{C}[z_1, z_2]$
(3) $\mathcal{I}$ is a prime ideal in $\mathbb{C}[z_1, \ldots, z_m]$, $m > 2$ and w is a smooth point of $V_{[\mathcal{I}]}$.

3 Hermitian Structure

Let $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be an `analytic` Hilbert module over the polynomial ring $\mathbb{C}[z]$, i.e., it is a reproducing kernel Hilbert space such that the polynomial ring is densely contained in it. Let $\mathcal{M} := [\mathcal{I}]$ be the completion of an ideal $\mathcal{I} \subset \mathcal{H}$. Thus $\mathcal{M}$ is again a reproducing kernel Hilbert space. However, the reproducing kernel $K(\cdot, w)$ of the submodule $\mathcal{M}$ might be the zero vector for some $w \in \Omega$. Consequently, the dimension of the eigenspace $\cap_{i=1}^{m} \ker(M_i^* - \bar{w}_i)$ is no longer necessarily independent of $w \in \Omega$. For an analytic Hilbert module $\mathcal{H}$, the eigenspaces over $w \in \Omega$ defines a holomorphic hermitian vector bundle. As we have noted, any submodule $\mathcal{M}$ of the form $[\mathcal{I}]$ of an analytic Hilbert module $\mathcal{H}$ is in the class $\mathfrak{B}_1(\Omega)$. While $\mathcal{M}$ need not define a holomorphic vector bundle, it was shown in [5] that it defines a coherent sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$. We assume that $\mathcal{M}$ is a submodule of an analytic Hilbert module and with a slight abuse of notation, we let $\mathcal{S}$ denote the coherent sheaf $\mathcal{S}^{\mathcal{M}}(\Omega)$ defined by $\mathcal{M}$.

In what follows, we make the standing assumption that $\mathcal{M}$ is a submodule of an analytic Hilbert module $\mathcal{H}$ of the form $[\mathcal{I}]$.

The main goal here is to define a hermitian structure for the coherent sheaf $\mathcal{S}$ by taking advantage of its presentation via the short exact sequence: Since the sheaf $\mathcal{S}$ is a subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$, for a fixed $w_0 \in \Omega$, there is a open neighbourhood U_0 of w_0 and a natural number n such that $\mathcal{O}_{U_0}^n \to \mathcal{S}_{U_0} \to 0$ is exact. The Decomposition Theorem 1.3 produces linearly independent set of vectors $\{K^{(1)}(w), \ldots, K^{(n)}(w)\}$, $w \in U_0$, depending holomorphically on w. The size n of this set of vectors depends on w_0. Thus we have a Hermitian structure on the free module $\mathcal{O}_{U_0}^n$. We describe below how to transplant this Hermitian structure to the stalk of $\mathcal{S}$ at w_0.

For each $w \in \Omega$, let $\mathfrak{m}_w$ be the maximal ideal of $\mathcal{O}_w$ and let $\mathcal{Q}_w := \mathcal{O}_w/\mathfrak{m}_w$ be the quotient module. For any analytic sheaf $\mathcal{S}$, let $\overline{\mathcal{S}}_w$ be the module $\mathcal{S}_w \otimes_{\mathcal{O}_w} \mathcal{Q}_w = \mathcal{S}_w/\mathfrak{m}_w\mathcal{S}_w$. If $\mathcal{S}$ is coherent, then $\overline{\mathcal{S}}_w$ is a finite dimensional linear space.

If $\mathcal{O}_U^n \to \mathcal{S}_U \to 0$ is an exact sequence, then the induced map $\overline{\mathcal{O}}_w^n \to \overline{\mathcal{S}}_w \to 0$, $w \in U$, is also exact. If $\langle \cdot, \cdot \rangle_w$ is an inner product on $\overline{\mathcal{O}}_z^n$ ($\cong \mathbb{C}^n$), then we equip $\overline{\mathcal{S}}_w$ with the inner product induced by the onto map $\overline{\mathcal{O}}_w \to \overline{\mathcal{S}}_w \to 0$.

Definition 3.1 A hermitian structure on a coherent sheaf $\mathcal{S}$ is an assignment of inner products on each $\overline{\mathcal{S}}_w$, $w \in \Omega$.

If $\mathcal{S}$ is a coherent sheaf over Ω, then there exists an open neighbourhood U of a fixed but arbitrary $w_0 \in \Omega$ and a map φ such that $\mathcal{O}_U^n \xrightarrow{\varphi} \mathcal{S}_U \to 0$ is exact.

A hermitian structure on a coherent sheaf $\mathcal{S}$ is said to be smooth if for a fixed but arbitrary $w_0 \in \Omega$, there is a neighbourhood U of w_0 and a smooth inner product on $\mathcal{O}^n(U)$, that is, a smooth assignment of inner products on the locally free sheaf $\overline{\mathcal{O}}_w^n$, $w \in U$, such that the inner product on $\overline{\mathcal{S}}_w$ is the quotient inner product induced by the onto map $\overline{\mathcal{O}}_w \xrightarrow{\overline{\varphi}_w} \overline{\mathcal{S}}_w \to 0$.

Finally, a coherent sheaf equipped with a smooth hermitian structure, as above, is said to be a holomorphic hermitian sheaf.

For any fixed but arbitrary $w_0 \in \Omega$, let $\Omega_0 \subseteq \Omega$ be an open neighbourhood of w_0. Pick $h_1, \ldots, h_d$ in $\mathcal{O}(\Omega_0)$ and set $[h_1 g_1^0 + \cdots + h_d g_d^0]$ to be the equivalence class of

$h_1 g_1^0 + \cdots + h_d g_d^0$ in $\mathcal{S}_{w_0}/\mathfrak{m}_{w_0}\mathcal{S}_{w_0}$. The assignment

$$\left[h_1 g_1^0 + \cdots h_d g_d^0\right] \mapsto \left(h_1(w_0), \ldots, h_d(w_0)\right)$$

is well defined:

Suppose that $\sum_{i=1}^m h_i g_i^0 = \sum_{i=1}^m h_i' g_i^0$. Then $\sum_{i=1}^m (h_i - h_i') g_i^0$ belongs to $\mathfrak{m}_{w_0}\mathcal{S}(K)_{w_0}$ since $h_i(0) = h_i'(0)$, $i = 1, \ldots, m$. Therefore, $\left[\sum_{i=1}^m h_i g_i^0\right] = \left[\sum_{i=1}^m h_i' g_i^0\right]$.

This assignment provides a linear isomorphism between $\overline{\mathcal{S}}_{w_0}$ and $\mathbb{C}^d$.

For any fixed but arbitrary $w_0 \in \Omega$, there is an open neighbourhood $\Omega_0 \subseteq \Omega$ of w_0 such that $K(\cdot, w) = g_1^0(w) K_{w_0}^{(1)}(\cdot, w) + \cdots + g_d^0(w) K_{w_0}^{(d)}(\cdot, w)$, $w \in \Omega$, where $\{g_1^0, \ldots, g_d^0\}$ are a minimal set of generators for the stalk $\mathcal{S}_{w_0}$ at $w_0 \in \Omega_0$. Also, for any $w \in \Omega_0$, the vectors $\{K_{w_0}^{(1)}(\cdot, w), \ldots, K_{w_0}^{(d)}(\cdot, w)\} \subset \mathcal{M}$ are linearly independent.

Now, we can assign an inner product on $\overline{\mathcal{S}}_{w_0}$ as follows: Let

$$H(w_0) = \left(\left(H_{ij}(w_0)\right)\right) := \left(\left(\left\langle K_{w_0}^{(i)}(\cdot, w_0), K_{w_0}^{(j)}(\cdot, w_0)\right\rangle\right)\right)$$

be the Grammian of the linearly independent vectors $\{K_{w_0}^{(1)}(\cdot, w_0), \ldots, K_{w_0}^{(d)}(\cdot, w_0)\}$. The matrix $H(w_0)$ is therefore positive definite and is invertible. We define a hermitian structure on $\overline{\mathcal{S}}_{w_0}$ using $H(w_0)$. Assertion (iv) of Theorem 1.3 shows that the hermitian structure at w_0 is independent of the choice of the generators.

We claim that the prescription of a hermitian structure on $\overline{\mathcal{S}}_w$, w in some open neighbourhood Ω_0 of $w_0 \in \Omega$ using $H(w)$, is a smooth hermitian structure on $\mathcal{S}$. We have a resolution $\mathcal{O}^d \to \mathcal{S} \to 0$. Let us define an inner product on $\overline{\mathcal{O}}_w^d$ using the hermitian structure $\mathbb{H}(w)^{-1}$, where $\mathbb{H}(w) = \left(\left(\left\langle K_{w_0}^{(i)}(\cdot, w), K_{w_0}^{(j)}(\cdot, w)\right\rangle\right)\right)$ and the set $\{K_{w_0}^{(1)}(\cdot, w), \ldots, k_{w_0}^{(d)}(\cdot, w)\}$, $w \in \Omega_0$, of linearly independent vectors are as in Theorem 1.3 except that we have added the subscript w_0 to emphasize the dependence of these vectors on the fixed but arbitrary point w_0.

We prove that the inner product induced by $\mathbb{H}$ on $\overline{\mathcal{S}}_{w_0}$, $w_0 \in \Omega_0$, coincides with the one we have already defined.

For any $w \in \Omega_0$, let $\{s_1^{(w)}, \ldots, s_e^{(w)}\}$ be a minimal set of generators of $\mathcal{S}_w$. We can then find linearly independent vectors $K_w^{(1)}(\cdot, u), \ldots, K_w^{(e)}(\cdot, u)$ that are anti-holomorphic in the variable u defined on some open neighbourhood of w such that $K(\cdot, u) = \sum_{j=1}^e s_j^{(w)}(u) K_w^{(j)}(\cdot, u)$. Since $\{g_1^{(0)}, \ldots, g_d^{(0)}\}$ is also a set of generators for $\mathcal{S}_w(K)$, not necessarily minimal, we must have a full rank matrix $G \in \mathcal{M}_{e\times d}(\mathcal{O}_w)$ such that

$$(s_1^{(w)}(u), \ldots, s_e^{(w)}(u)) G(u) = (g_1^{(0)}(u), \ldots, g_d^{(0)}(u)),$$

u in some small open neighbourhood of w.

Theorem 3.2 *The coherent sheaf $\mathcal{S}$ is a holomorphic hermitian sheaf, that is,*

$$\left(\overline{G(w)}\, \mathbb{H}(w)\, G(w)^\dagger\right) = \left(\left(\left\langle K_w^{(i)}(\cdot, w), K_w^{(j)}(\cdot, w)\right\rangle\right)\right)_{i,j=1}^e, \quad w \in \Omega_0,$$

where $G(w)^\dagger$ is the transpose of the matrix $G(w)$ and e depends on w.

Proof For $\alpha \in \mathcal{O}_w^d$, set $\beta := G\alpha$, and note that

$$\begin{aligned}\begin{pmatrix} g_1^{(0)} & \dots & g_d^{(0)} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} &= \begin{pmatrix} s_1^{(w)}(w) & \dots & s_e^{(w)}(w) \end{pmatrix} G(w) \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{pmatrix} \\ &= \begin{pmatrix} s_1^{(w)}(w) & \dots & s_e^{(w)}(w) \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix}.\end{aligned}$$

Suppose that $\overline{\mathcal{S}}_w$ is identified with $\mathbb{C}^e$ via the map: $\left[\sum_{i=1}^e \beta_i s_i^{(w)}\right] \mapsto \big(\beta_1(w), \dots, \beta_e(w)\big)$. Then the hermitian structure on $\overline{\mathcal{S}}_w$, by definition, is induced by the non-negative definite matrix $\Big(\Big(\big\langle K_w^{(i)}(\cdot, w), K_w^{(j)}(\cdot, w)\big\rangle\Big)\Big)_{i,j=1}^e$. The map $\mathbb{C}^d \cong \overline{\mathcal{O}}_w^d \xrightarrow{\overline{\varphi}_w} \overline{\mathcal{S}}_w \cong \mathbb{C}^e$ is then given by the formula:

$$(\alpha_1(w), \dots, \alpha_d(w)) \longrightarrow \big(\textstyle\sum_{i=1}^d \alpha_i g_i^{(0)}\big)_{|w} \longrightarrow \big(\textstyle\sum_{i=1}^d \beta_i s_i^{(w)}\big)_{|w} \longrightarrow (\beta_1(w), \dots, \beta_e(w)).$$

and $(\beta_1(w), \dots, \beta_e(w)) = (\alpha_1(w), \dots, \alpha_d(w))G(w)^\dagger$. Setting the hermitian structure on $\mathbb{C}^d$, induced by the set of linearly independent vectors $\{K_{w_0}^{(1)}(\cdot, w), \dots, k_{w_0}^{(d)}(\cdot, w)\}$ to be $\mathbb{H}(w) := \Big(\Big(\big\langle K_{w_0}^{(i)}(\cdot, w), K_{w_0}^{(j)}(\cdot, w)\big\rangle\Big)\Big)$, we see that the hermitian structure on $\mathbb{C}^e$ obtained via the map $\overline{\varphi}_w$ is of the form $\big(\overline{G(w)}\mathbb{H}(w)G^\dagger(w)\big)^{-1}$. Since

$$\begin{aligned} &s_1^{(w)}(w)K_w^{(1)}(\cdot, w) + \cdots + s_e^{(w)}(w)K_w^{(e)}(\cdot, w) = K(\cdot, w) \\ &= g_1^0(w)K_{w_0}^{(1)}(\cdot, w) + \cdots + g_d^0(w)K_{w_0}^{(d)}(\cdot, w),\end{aligned}$$

it follows that

$$\big(K_{w_0}^{(1)}(\cdot, w), \dots, K_{w_0}^{(d)}(\cdot, w)\big)\boldsymbol{g}^{(0)} = \big(K_{w_0}^{(1)}(\cdot, w), \dots, K_{w_0}^{(d)}(\cdot, w)\big)G(w)^\dagger \boldsymbol{s}^{(w)}.$$

By the uniqueness statement at w of the vectors $\{K_w^{(1)}(\cdot, u), \dots, K_w^{(e)}(\cdot, u)\}$ of Theorem 1.3, we conclude that

$$\big(K_w^{(1)}(\cdot, w), \dots, K_w^{(e)}(\cdot, w)\big) = \big(K_{w_0}^{(1)}(\cdot, w), \dots, K_{w_0}^{(d)}(\cdot, w)\big)G(w)^\dagger.$$

Taking inner product of the vectors on both sides of this equation verifies the assertion of the Theorem. □

For any fixed but arbitrary $w_0 \in \mathcal{Z}$, let d be the minimal set of generators $\overline{\mathcal{S}}_{w_0}$. Also, we have anti-holomorphic functions $K_{w_0}^{(i)}(\cdot, w)$, $1 \leq i \leq d$, defined on some open neighbourhood U_{w_0} of w_0. The real analytic function $H(w) = \big(\big(H_{ij}(w)\big)\big) := \Big(\Big(\big\langle K_{w_0}^{(i)}(\cdot, w), K_{w_0}^{(j)}(\cdot, w)\big\rangle\Big)\Big)$ defines a hermitian structure on the locally free sheaf $\mathcal{O}(U_{w_0})^d$. Set $\mathcal{P}_{w_0}$ to be the holomorphic hermitian vector bundle of rank d (d depends

on w_0) determined by the pair $(\mathcal{O}(U_{w_0})^d, H(w))$. Theorem 1.10 of [5] says that if two analytic sheaves $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are isomorphic (that is, there is a unitary module map between the modules $\mathcal{M}$ an $\tilde{M}$ they are associated with), then the holomorphic hermitian vector bundles $\mathcal{P}_{w_0}$ and $\tilde{\mathcal{P}}_{w_0}$ are equivalent (by shrinking U_{w_0} and $\tilde{U}_{w_0}$) via an isometric bundle map for each fixed but arbitrary $w_0 \in \mathcal{Z}$ on a common open neighbourhood U'_{w_0} of w_0. If w is outside the zero set, then there is an open neighbourhood U of w in which the dimension of the stalk $\overline{\mathcal{S}}_w$ is 1 and the eigenvectors $K(\cdot, w)$ define a holomorphic hermitian bundle $E(U)$ on the open set U, this is the Cowen-Douglas bundle of $\mathcal{M}$ on U. Moreover, if the two analytic sheaves $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are isomorphic, then these holomorphic hermitian bundles are (locally) equivalent.

Definition 3.3 If the vector bundles $\mathcal{P}_w$, $E(U)$ (determined by K) are equivalent to the corresponding vector bundles $\tilde{\mathcal{P}}_w$, $\tilde{E}(U)$ (determined by $\tilde{K}$) for all $w \in \mathcal{Z}$ and some open $U \subseteq \Omega \backslash \mathcal{Z}$, then we say that the hermitian sheaves $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are equivalent.

In particular, for each $w \in \mathcal{Z}$, the curvature of the determinant bundle $\det(\mathcal{P}_w)$ induced by the hermitian structure $\mathbb{H}(w)$ defined on U_w is an invariant for the analytic Hilbert module $\mathcal{M}$, or equivalently, that of the holomorphic hermitian sheaf $\mathcal{S}(K)$. Furthermore, if we assume that the zero variety $\mathcal{Z}$ is a complex manifold, then we can strengthen what we have just said, see Corollary 6.3. In this case, the dimension of $\overline{\mathcal{S}}_w$ is a constant, say $t > 1$, see Theorem 4.3. Also, for any $w \in \mathcal{Z}$, fixing a minimal set of generators $\{p_1, \ldots, p_t\}$, there exists a uniquely determined holomorphic frame $\{K^1, \ldots, K^t\}$ on some open neighbourhood Ω_w of w, again, see Theorem 4.3. Therefore, the locally-free sheaf $\mathcal{O}(\Omega_w)^t$ with the holomorphic frame $\{K^1, \ldots, K^t\}$ defines a smooth hermitian structure $\mathbb{H}$ for all w in $\Omega_w \cap \mathcal{Z}$ *simultaneously*. As before, the curvature of the determinant bundle induced by $\det(\mathcal{P})$ on the open set Ω_w is an invariant for the analytic Hilbert module $\mathcal{M}$. Summarizing this discussion, we list useful invariants for the holomorphic hermitian sheaf $\mathcal{S}$ associated to a reproducing kernel Hilbert space.

Theorem 3.4 *The curvature of the determinant bundles* $\det(\mathcal{P}_w)$, $w \in \mathcal{Z}$, *are invariants for the holomorphic hermitian sheaf* $\mathcal{S}$.

In what follows, we rework the (λ, μ) examples of [5, 4.1, p. 194] using the determinant bundle and arrive at the same conclusion.

Example 3.5 For any pair $\lambda, \mu > 0$, let $\mathcal{H}^{(\lambda,\mu)}$ be the Hilbert module of wrigted Bergman spaces on the bidisc $\mathbb{D}^2$ determined by the kernel function $\frac{1}{(1-\bar{w}z)^\lambda}\frac{1}{(1-\bar{w}z)^\mu}$. Let $\mathcal{M} \subseteq \mathcal{H}^{(\lambda,\mu)}$ be the submodule of functions vanishing at $\{(0, 0)\}$. The hermitian metric for the sheaf $\mathcal{S}^{\mathcal{M}}$ in a neighbourhood of the point $\{(0, 0)\}$ was calculated in [5]. Taking the trace of the curvature of this metric from the bottom of page 29 in [5], we see that the coefficients of $dz_1 \wedge d\bar{z}_1$ and $dz_2 \wedge d\bar{z}_2$ of the curvature $\mathcal{K}^{(\lambda,\mu)}(z)_{|z=(0,0)}$ for the determinant bundle $\mathcal{P}_{(0,0)}$ are given by the formula: $\frac{\lambda+1}{2} + \frac{\lambda\mu^2}{(\lambda+\mu)^2}$, $\frac{\mu+1}{2} + \frac{\lambda^2\mu}{(\lambda+\mu)^2}$, respectively. Equating these coeffcients of $\mathcal{K}^{(\lambda,\mu)}(z)_{|z=(0,0)}$ and $\mathcal{K}^{(\lambda',\mu')}(z)_{|z=(0,0)}$, we

obtain the pair of equations:

$$\begin{aligned} \tfrac{\lambda+1}{2} + \tfrac{\lambda\mu^2}{(\lambda+\mu)^2} &= \tfrac{\lambda'+1}{2} + \tfrac{\lambda'\mu'^2}{(\lambda'+\mu')^2} \\ \tfrac{\mu+1}{2} + \tfrac{\lambda^2\mu}{(\lambda+\mu)^2} &= \tfrac{\mu'+1}{2} + \tfrac{\lambda'^2\mu'}{(\lambda'+\mu')^2} \end{aligned} \tag{3.1}$$

Set $k = \frac{\lambda}{\mu}$ and $k' = \frac{\lambda'}{\mu'}$. Dividing the first equation by the second in (3.1), we have that

$$\frac{k\left\{\frac{1}{2} + \frac{1}{(1+k)^2}\right\}}{\frac{1}{2} + \frac{k^2}{(1+k)^2}} = \frac{k'\left\{\frac{1}{2} + \frac{1}{(1+k')^2}\right\}}{\frac{1}{2} + \frac{k'^2}{(1+k')^2}} = \alpha \text{ (say)},$$

which simplifies to the equation $k(1+k)^2 + 2k = \alpha(1+k)^2 + 2\alpha k^2$. This shows k is a root of the equation

$$x^3 - (3\alpha - 2)x^2 - (2\alpha - 3)x - \alpha = 0. \tag{3.2}$$

We claim that this equation has only one positive root for every positive α and hence $k = k'$. From (3.1), it follows that $\lambda = \lambda'$ and $\mu = \mu'$. Any cubic equation can have either one real root and two complex roots or three real roots. In the first case, we are done as product of the roots is α. In case it has 3 real roots, since the product of the roots is positive, either there is one positive root and two negative roots or all the three roots are positive. In the first case, discarding the two negative roots, we are done again. We claim that the second case does not occur. If k_i, $i = 1, 2, 3$ are positive roots of the Eq. (3.2), then $\sum k_i = 3\alpha - 2$, $\sum_{i<j} k_i k_j = 3 - 2\alpha$ and $k_1 k_2 k_3 = \alpha$ and it follows that α lies in $[2/3, 3/2]$. However, the identity $\sum_{i\leq j} k_i k_j = 1/2\left\{\sum_{i<j}(k_i + k_j)^2\right\}$ gives that α must lie in $(1, 3/2]$. Now observing that $1/k$ is also a root of $x^3 - (3\beta - 2)x^2 - (2\beta - 3)x - \beta = 0$ with $\beta = 1/\alpha$, and using the same identity, we arrive at a contradiction that α can not be greater than 1.

4 Finding a Holomorphic Frame in an Open Subset of the Zero Set

Let Ω be a bounded domain in $\mathbb{C}^m$ and $\mathcal{M}$ be a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_m]$, in the class $\mathfrak{B}_1(\Omega)$. We construct a sheaf $\mathcal{S}^{\mathcal{M}}$ for the Hilbert module $\mathcal{M}$ as follows:

$$\mathcal{S}^{\mathcal{M}}(U) = \left\{ \sum_{i=1}^{n} (f_i|_U) g_i : f_i \in \mathcal{M}, g_i \in \mathcal{O}(U), n \in \mathbb{N} \right\}, \ U \text{ open in } \Omega$$

or equivalently,

$$\mathcal{S}^{\mathcal{M}}_w = \left\{ (f_1)_w \mathcal{O}_w + \cdots + (f_n)_w \mathcal{O}_w : f_1, \ldots, f_n \in \mathcal{M}, n \in \mathbb{N} \right\}, \ w \in \Omega.$$

Clearly, $\mathcal{S}^{\mathcal{M}}$ is a subsheaf of the sheaf of holomorphic functions $\mathcal{O}_\Omega$. From [5, Proposition 2.1], it follows that $\mathcal{S}^{\mathcal{M}}$ is coherent. In particular, for each fixed $w \in \Omega$, $\mathcal{S}^{\mathcal{M}}_w$ is generated by finitely many elements from $\mathcal{O}_w$.

Now, suppose that $\mathcal{H}$ is an analytic Hilbert module and $\mathcal{M}$ is the closure of a polynomial ideal $\mathcal{I}$ in $\mathcal{H}$ generated by $\{p_1, \ldots, p_t\}$. For each $w \in \Omega$, from [7, Lemma 2.3.2], it follows that $\{(p_1)_w, \ldots, (p_t)_w\}$ generates the stalk $\mathcal{S}^{\mathcal{M}}_w$. In the following lemma we provide a sufficient condition for the minimality of such a generating set. We let $Z(p_1, \ldots, p_t)$ denote the common zero set of the polynomials $p_1, \ldots, p_t$.

Lemma 4.1 *If $V(\mathcal{M}) := Z(p_1, \ldots, p_t) \cap \Omega$ is a submanifold of codimension t, then*

(a) *$\{p_1, \ldots, p_t\}$ is a minimal set of generators for $\mathcal{I}$ and*
(b) *for each $w \in V(\mathcal{M})$, $(p_1)_w, \ldots, (p_t)_w$ is a minimal generator of $\mathcal{S}^{\mathcal{M}}_w$.*

Proof Assume that $p_t = q_1 p_1 + \cdots + q_{t-1} p_{t-1}$, for $q_1, \ldots, q_{t-1} \in \mathbb{C}[z_1, \ldots, z_m]$. Then we have $V(\mathcal{M}) = Z(p_1, \ldots, p_{t-1}) \cap \Omega$. Since $p_1|_\Omega, \ldots, p_{t-1}|_\Omega$ are $t-1$ holomorphic functions from Ω to $\mathbb{C}$, applying [8, Section 3.5] we obtain that $\operatorname{codim}(V(\mathcal{M}))$ is at most $t-1$ which contradicts the hypothesis of the lemma. This proves part a).

To prove part b), assume that there exists a point $w_0 \in V(\mathcal{M})$ such that

$$(p_t)_{w_0} = (a_1)_{w_0}(p_1)_{w_0} + \cdots + (a_{t-1})_{w_0}(p_{t-1})_{w_0},$$

where $a_1, \ldots, a_{t-1}$ are holomorphic functions defined on some neighbourhood N_{w_0} of w_0 in Ω. Going to a smaller neighbourhood if necessary, we have $p_t = a_1 p_1 + \cdots + a_{t-1} p_{t-1}$ on N_{w_0}. As a result, $V(\mathcal{M}) \cap N_{w_0} = Z(p_1, \ldots, p_{t-1}) \cap N_{w_0}$. Now, since $p_1|_{N_{w_0}}, \ldots, p_{t-1}|_{N_{w_0}}$ are $t-1$ holomorphic functions on N_{w_0} and $Z(p_1|_{N_{w_0}}, \ldots, p_{t-1}|_{N_{w_0}}) = V(\mathcal{M}) \cap N_{w_0}$ is a submanifold, from [8, Section 3.5] it follows that $\operatorname{codim}(V(\mathcal{M}) \cap N_{w_0})$ is at most $t-1$. Again, this is a contradiction to the hypothesis saying $\operatorname{codim}(V(\mathcal{M}) \cap N_{w_0}) = \operatorname{codim}(V(\mathcal{M})) = t$. □

Remark 4.2 Suppose $Z(p_1, \ldots, p_t)$ is a complete intersection, that is, the tuple $(p_1, \ldots, p_t) : \Omega \to \mathbb{C}^t$ is a submersion at every point $w \in V(\mathcal{M})$. Then $V(\mathcal{M})$ is a submanifold of codimension t. In this case, we can give a more direct proof of Lemma 4.1 as follows.

If there exists a point $w_0 \in V(\mathcal{M})$ such that $p_t = a_1 p_1 + \cdots + a_{t-1} p_{t-1}$ on N_{w_0}, then, for $j = 1, \ldots, m$,

$$\begin{aligned}\frac{\partial p_t}{\partial w_j}(w_0) &= \frac{\partial (a_1 p_1)}{\partial w_j}(w_0) + \cdots + \frac{\partial (a_{t-1} p_{t-1})}{\partial w_j}(w_0) \\ &= a_1(w_0)\frac{\partial p_1}{\partial w_j}(w_0) + \cdots + a_{t-1}(w_0)\frac{\partial p_{t-1}}{\partial w_j}(w_0).\end{aligned}$$

As a result, the matrix

$$\begin{pmatrix} \frac{\partial p_1}{\partial w_1}(w_0) & \ldots & \frac{\partial p_1}{\partial w_t}(w_0) \\ \vdots & & \vdots \\ \frac{\partial p_t}{\partial w_1}(w_0) & \ldots & \frac{\partial p_t}{\partial w_t}(w_0) \end{pmatrix} = \begin{pmatrix} \frac{\partial p_1}{\partial w_1}(w_0) & \ldots & \frac{\partial p_1}{\partial w_t}(w_0) \\ \vdots & & \vdots \\ \frac{\partial p_{t-1}}{\partial w_1}(w_0) & \ldots & \frac{\partial p_{t-1}}{\partial w_t}(w_0) \\ \sum_{i=1}^{t-1} a_i(w_0)\frac{\partial p_i}{\partial w_1}(w_0) & \ldots & \sum_{i=1}^{t-1} a_i(w_0)\frac{\partial p_i}{\partial w_t}(w_0) \end{pmatrix}$$

has rank at most $t-1$. This contradicts the fact that $Z(p_1,\dots,p_t)$ is a complete intersection at w_0.

The following Theorem is a generalization of [5, Theorem 1.5] reproduced earlier in this paper as Theorem 1.3.

Theorem 4.3 *Let $\mathcal{H}\subseteq\mathcal{O}(\Omega)$ be an analytic Hilbert module for some bounded domain Ω in $\mathbb{C}^m$. Also, let $\mathcal{M}$ be a submodule of $\mathcal{H}$ of the form $[\mathcal{I}]$, that is, $\mathcal{M}$ is the completion of some polynomial ideal $\mathcal{I}\subseteq\mathcal{H}$ with generators $p_1,\dots,p_t$. Furthermore, assume that $V(\mathcal{M})$ is a submanifold of codimension t. Then there exist anti-holomorphic maps $F_1,\dots,F_t : V(\mathcal{M})\to\mathcal{M}$ such that we have the following.*

1. *For each $w\in V(\mathcal{M})$, there exists a neighbourhood Ω_w of w in Ω, anti-holomorphic maps $F^1_{\Omega_w},\dots,F^t_{\Omega_w}:\Omega_w\to\mathcal{M}$ with the properties listed below.*
   (a) *$F^j_{\Omega_w}(v)=F_j(v)$, for all $v\in V(\mathcal{M})\cap\Omega_w$, $j\in\{1,\dots,t\}$.*
   (b) *$k_u=\sum_{j=1}^t\overline{p_j(u)}F^j_{\Omega_w}(u)$, for all $u\in\Omega_w$, where $k_w:=K(\cdot,w)$, $w\in\Omega$, with K being the reproducing kernel of the submodule $\mathcal{M}$.*
   (c) *$\{F^1_{\Omega_w}(u),\dots,F^t_{\Omega_w}(u)\}$ is a linearly independent set for each $u\in\Omega_w$.*
2. *The set $\{F_1,\dots,F_t\}$ is uniquely determined by $\{p_1,\dots,p_t\}$, that is, if $G_1,\dots,G_t$ is another collection of anti-holomorphic maps from $V(\mathcal{M})$ to $\mathcal{M}$ satisfying* 1.a) *and* 1.b), *then $G_j=F_j$, $1\le j\le t$.*
3. *$M_p^*F_j(v)=\overline{p(v)}F_j(v)$, for all $j=1,\dots,t$, $v\in V(\mathcal{M})$, where M_p is the multiplication by the polynomial p.*
4. *For each $v\in V(\mathcal{M})$, the linear span of the set of vectors $\{F_1(v),\dots,F_t(v)\}$ in $\mathcal{M}$ is the joint kernel of $\mathcal{M}$ at v and hence is independent of the choice of generators $p_1,\dots,p_t$.*

Proof Pick an arbitrary point $w\in V(\mathcal{M})$. From Lemma 4.1, $(p_1)_w,\dots,(p_t)_w$ is a minimal set of generator of $\mathcal{S}^{\mathcal{M}}_w$. Consequently, there exists a neighbourhood Ω_w of w in Ω such that for all $u\in\Omega_w$, $K(\cdot,u)=\sum_{j=1}^t\overline{p_j(u)}F^j_{\Omega_w}(u)$, where $F^1_{\Omega_w},\dots,F^t_{\Omega_w}$ are anti-holomorphic maps from Ω_w to $\mathcal{M}$ satisfying conditions (ii) to (v) of Theorem 1.3.

Take $w_1,w_2\in V(\mathcal{M})$ such that $\Omega_{w_1}\cap\Omega_{w_2}\cap V(\mathcal{M})$ is non-empty. Now, for each $u\in\Omega_{w_1}\cap\Omega_{w_2}$ we have

$$K(\cdot,u)=\sum_{j=1}^t\overline{p_j(u)}F^j_{\Omega_{w_1}}(u)\quad\text{and}\quad K(\cdot,u)=\sum_{j=1}^t\overline{p_j(u)}\tilde{F}^j_{\Omega_{w_2}}(u).$$

This implies

$$\sum_{j=1}^t\overline{p_j(u)}\big(F^j_{\Omega_{w_1}}(u)-\tilde{F}^j_{\Omega_{w_2}}(u)\big)=0.$$

For each $u\in\Omega_{w_1}\cap\Omega_{w_2}$, $1\le j\le t$, set $\alpha_j(u)=\overline{\big(F^j_{\Omega_{w_1}}(u)-\tilde{F}^j_{\Omega_{w_2}}(u)\big)}$ and note that $\sum_{j=1}^t p_j(u)\alpha_j(u)=0$. Now, fix an arbitrary $v\in\Omega_{w_1}\cap\Omega_{w_2}\cap V(\mathcal{M})$ and

assume that $\alpha_1(v) \neq 0$. This gives $\sum_{j=1}^{t}(p_j)_v(\alpha_j)_v = 0$ in $\mathcal{O}_{\mathbb{C}^m,v}$ and $(\alpha_1)_v$ is an unit in $\mathcal{O}_{\mathbb{C}^m,v}$. Consequently, $(p_1)_v = -\sum_{j=2}^{t}\big((\alpha_1)_v^{-1}(\alpha_j)_v\big)(p_j)_v$ which says that $\{(p_1)_v, \ldots, (p_t)_v\}$ is not a minimal set of generators of $\mathcal{S}_v^{\mathcal{M}}$ contradicting Lemma 4.1. Thus,

$$\alpha_j(v) = 0 \Leftrightarrow F^j_{\Omega_{w_1}}(v) = \tilde{F}^j_{\Omega_{w_2}}(v), \forall v \in \Omega_{w_1} \cap \Omega_{w_2} \cap V(\mathcal{M}), 1 \leq j \leq t. \quad (4.1)$$

Since $\{\Omega_w \cap V(\mathcal{M})\}_{w\in V(\mathcal{M})}$ is an open cover of $V(\mathcal{M})$, for each $j = 1, \ldots, t$, we define $F_j : V(\mathcal{M}) \to \mathcal{M}$ as follows:

$$F_j|_{\Omega_w \bigcap V(\mathcal{M})}(v) := F^j_{\Omega_w}(v), \forall v \in \Omega_w \cap V(\mathcal{M}).$$

From 4.1 it follows that for each $j = 1, \ldots, t$, F_j is a well-defined, anti-holomorphic map satisfying 1.a),1.b) and 1.c).

To prove 2., assume that for each $w \in V(\mathcal{M})$, there exist a neighbourhood N_w of w in Ω, anti-holomorphic maps $G^1_{N_w}, \ldots, G^t_{N_w} : N_w \to \mathcal{M}$ such that a) $G^j_{N_w}(v) = G_j(v)$, $\forall v \in N_w \cap V(\mathcal{M})$ and b) $K(\cdot, u) = \sum_{j=1}^{t}\overline{p_j(u)}G^j_{N_w}(u)$, $\forall u \in N_w$. This gives

$$\sum_{j=1}^{t}\overline{p_j(u)}\big(F^j_{\Omega_w}(u) - G^j_{N_w}(u)\big) = 0, \forall u \in \Omega_w \cap N_w.$$

Following similar arguments as given above, for each $v \in \Omega_w \cap N_w \cap V(\mathcal{M})$, we have $G_j(v) = G^j_{N_w}(v) = F^j_{\Omega_w}(v) = F_j(v)$. In particular, $F_j(w) = G_j(w)$, for all $w \in V(\mathcal{M})$.

The proof of Part 3 is straightforward from condition (v) of Theorem 1.3 and from the observation that for each $w \in V(\mathcal{M})$, $F_j(w) = F^j_{\Omega_w}(w)$. From the same observation we obtain that $\mathrm{Span}\{F_1(w), \ldots, F_t(w)\}$ is a subspace of $\cap_{p\in\mathbb{C}[\underline{z}]}\mathrm{Ker}(M_p^* - \overline{p(w)})$, for each $w \in V(\mathcal{M})$. Now, from 1.a) and 1.c) it follows that the dimension of this subspace is at least t. On the other hand, from [13, Lemma 5.11] it follows that $\dim(\mathcal{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w) = \dim\big(\cap_{p\in\mathbb{C}[\underline{z}]}\ker(M_p - p(w))^* \otimes \mathbb{C}\big) \leq t$. So, for each $w \in V(\mathcal{M})$, $\mathrm{Span}\{F_1(w), \ldots, F_t(w)\} = \cap_{p\in\mathbb{C}[\underline{z}]}\ker(M_p^* - \overline{p(w)})$ proving Part 4. □

Corollary 4.4 *Let $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be an analytic Hilbert module for some bounded domain Ω in $\mathbb{C}^m$. Also, let $\mathcal{M}$ be a submodule of $\mathcal{H}$ of the form $[\mathcal{I}]$ and $p_1, \ldots, p_t$ be the generators of $\mathcal{I}$. Assume that $V(\mathcal{M})$ is a submanifold of codimension t. Then*

$$\dim\left(\bigcap_{i=1}^{m}\ker(M_{z_j} - w_j)^*\right) = \begin{cases} 1 & \text{for } w \notin V(\mathcal{I}) \cap \Omega \\ \text{codimension of } V(\mathcal{I}) & \text{for } w \in V(\mathcal{I}) \cap \Omega. \end{cases}$$

Proof For each $w \in V(\mathcal{M}) = V(\mathcal{I}) \cap \Omega$, from the proof of Part 4., Theorem 4.3, we infer that

$$\dim\left(\bigcap_{i=1}^{m} \ker(M_{z_j} - w_j)^*\right) = \dim\left(\bigcap_{p \in \mathbb{C}[\underline{z}]} \ker(M_p^* - \overline{p(w)})\right) = t.$$

Moreover, clearly $\mathcal{M} \in \mathfrak{B}_1(\Omega)$. Therefore, from [3, Lemma 1.11], it follows that $\mathcal{M}$ is locally free on $(\Omega \setminus V(\mathcal{M}))^*$ with the result that $\dim\left(\cap_{i=1}^{m} \ker(M_{z_j} - w_j)^*\right) = 1$, for $w \notin V(\mathcal{M})$. □

Observe that we do not need $\mathcal{I}$ to be a prime ideal to prove Corollary 4.4. This enables us to consider examples that satisfy the conjecture of Douglas, Misra and Varughese [12] but does not follow from [14, Theorem 2.3]. We will discuss an explicit example below to demonstrate this. In what follows, we let $< \{p_1, \ldots, p_t\} >$ denote the ideal generated by the polynomials $p_1, \ldots, p_t$. Consider the ideal $\mathcal{I} = < \{z_1 z_2, z_1 - z_2\} >$ in $\mathbb{C}[z_1, \ldots, z_m]$ and define $\mathcal{M} = [\mathcal{I}]$ in $\mathcal{H} = H^2(\mathbb{D}^m)$, where $m \geq 2$. If $z \in V(\mathcal{I})$, then $z_1 z_2 = 0$ and $z_1 - z_2 = 0$. Thus, $V(\mathcal{I}) \subseteq \{z_1 = z_2 = 0\}$. Furthermore, observe that

$$\mathcal{I} \subseteq \langle \{z_1, z_1 - z_2\} \rangle = \langle \{z_1, z_2\} \rangle.$$

Note that for two ideals $\mathcal{I}_1, \mathcal{I}_2$ in $\mathbb{C}[z_1, \ldots, z_m]$, if $\mathcal{I}_1 \subseteq \mathcal{I}_2$, then $V(\mathcal{I}_2) \subseteq V(\mathcal{I}_1)$. As a result, we obtain $V(\mathcal{I}) = \{(z_1, \ldots, z_m) : z_1 = z_2 = 0\}$ which implies that $V(\mathcal{M}) = \{(z_1, \ldots, z_m) \in \mathbb{D}^m : z_1 = z_2 = 0\}$. So, by Corollary 4.4 we have

$$\dim(\mathcal{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_w) = \begin{cases} 1 & \text{for } w \notin V(\mathcal{I}) \cap \mathbb{D}^m, \\ 2 & \text{for } w \in V(\mathcal{I}) \cap \mathbb{D}^m. \end{cases}$$

Now, we claim that $\mathcal{I}$ is not prime. To see this, it is enough to show that neither z_1 nor z_2 belongs to $\mathcal{I}$. Assume $z_1 \in \mathcal{I}$. Then, there exist $p_1, p_2 \in \mathbb{C}[z_1, \ldots, z_m]$ such that

$$z_1 = p_1 z_1 z_2 + p_2(z_1 - z_2)$$

which implies

$$z_1(1 - p_2 - p_1 z_2) = -p_2 z_2.$$

This means z_1 divides $p_2 z_2$. But z_1 is a prime element of $\mathbb{C}[z_1, \ldots, z_m]$ and z_1 does not divide z_2. So, z_1 divides p_2 and we will write $p_2 = q z_1$, for some $q \in \mathbb{C}[z_1, \ldots, z_m]$. Thus, we have

$$z_1 = p_1 z_1 z_2 + q z_1(z_1 - z_2).$$

Finally, dividing both sides by z_1 we have

$$1 = p_1 z_2 + q(z_1 - z_2).$$

This is a contradiction because the right hand side vanishes at the origin whereas the left hand side does not. This proves $z_1 \notin \mathcal{I}$. Following similar arguments one can show that $z_2 \notin \mathcal{I}$.

5 The Vector Bundle Associated to the Joint Kernel and Its Curvature

From Theorem 4.3 we obtain a rank t, trivial, anti-holomorphic bundle $E_{\mathcal{M}}$ on $V(\mathcal{M}) = V(\mathcal{I}) \cap \Omega$ corresponding to the set $\{p_1, \ldots, p_t\}$ given by

$$E_{\mathcal{M}} = \bigsqcup_{w \in V(\mathcal{M})} < \{F_1(w), \ldots, F_t(w)\} > .$$

Since, for each $w \in V(\mathcal{M})$, $(E_{\mathcal{M}})_w$ is a subspace of $\mathcal{M}$, we can give a hermitian structure on $E_{\mathcal{M}}$ which is canonically induced by the inner product of $\mathcal{M}$. This observation leads to the following theorem.

Theorem 5.1 *Let Ω be a bounded domain in $\mathbb{C}^m$ and $\mathcal{I}$ be a polynomial ideal with generators $p_1, \ldots, p_t$. Also, let $\mathcal{H}, \mathcal{H}'$ be analytic Hilbert modules in $\mathcal{O}(\Omega)$ and $\mathcal{M}, \mathcal{M}'$ be the closure of $\mathcal{I}$ in $\mathcal{H}, \mathcal{H}'$, respectively, with the property that $codim V(\mathcal{M}) = codim V(\mathcal{M}') = t$. If the modules $\mathcal{M}, \mathcal{M}'$ are "unitarily" equivalent, then we have the following:*

(a) *$E_{\mathcal{M}}$ is equivalent to $E_{\mathcal{M}'}$, where $E_{\mathcal{M}}, E_{\mathcal{M}'}$ are two bundles on $V(\mathcal{M}) = V(\mathcal{M}') = V(\mathcal{I}) \cap \Omega$ obtained from Theorem 4.3 and the discussion above.*

(b) *If $F_{\mathcal{M}} := \{F_1, \ldots, F_t\}, F_{\mathcal{M}'} := \{F_1^{'}, \ldots, F_t^{'}\}$ are the global frames of $E_{\mathcal{M}}, E_{\mathcal{M}'}$ respectively that are obtained from applying Theorem 4.3 on $\mathcal{M}, \mathcal{M}'$ with respect to the set $\{p_1, \ldots, p_t\}$, then*

$$\mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}}) = \mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'}).$$

Here $\mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}}), \mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})$ are the curvature matrices of $E_{\mathcal{M}}, E_{\mathcal{M}'}$ with respect to the frames $F_{\mathcal{M}}, F_{\mathcal{M}'}$, respectively.

Proof From Theorem 4.3, it follows that for each $w \in V(\mathcal{I}) \cap \Omega$, $(E_{\mathcal{M}})_w = \cap_{i=1}^m \ker(M_{z_i} - w_i)^*$ and $(E_{\mathcal{M}'})_w = \cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^*$, where $M_{z_i}, M_{z_i}^{'}$ are pointwise multiplication by z_i on $\mathcal{M}, \mathcal{M}'$ respectively. If $L : \mathcal{M} \to \mathcal{M}'$ is an unitary module map, then, for any $g \in \cap_{i=1}^m \ker(M_{z_i} - w_i)^*$, $(M_{z_i}^{'})^*(Lg) = L(M_{z_i}^* g) = L(\bar{w}_i g) = \bar{w}_i(Lg)$. Thus,

$$L\Big(\cap_{i=1}^m \ker(M_{z_i} - w_i)^* \Big) \subseteq \cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^*.$$

Similarly, it can be shown that

$$L^{-1}\Big(\cap_{i=1}^m \ker(M_{z_i}^{'} - w_i)^* \Big) \subseteq \cap_{i=1}^m \ker(M_{z_i} - w_i)^*,$$

which is equivalent to

$$\cap_{i=1}^{m} \ker(M_{z_i}^{'} - w_i)^* \subseteq L\Big(\cap_{i=1}^{m} \ker(M_{z_i} - w_i)^* \Big).$$

Consequently, $L\big(\cap_{i=1}^{m} \ker(M_{z_i} - w_i)^* \big) = \cap_{i=1}^{m} \ker(M_{z_i}^{'} - w_i)^*$ and hence L is an isometric isomorphism between $(E_{\mathcal{M}})_w$ and $(E_{\mathcal{M}'})_w$ for all $w \in V(\mathcal{I}) \cap \Omega$. Moreover, there exists an anti-holomorphic map $A : V(\mathcal{I}) \cap \Omega \mapsto GL(t, \mathbb{C})$ such that for each w, the following matrix equality is true:

$$\big[LF_1(w) \cdots LF_t(w) \big] = \big[F_1^{'}(w) \cdots F_t^{'}(w) \big] A(w). \tag{5.1}$$

Thus, L induces a bundle isomorphism between $E_{\mathcal{M}}$ and $E_{\mathcal{M}'}$ which proves part (a).

Next, observe that part (b) follows trivially from part (a) when $t = 1$. This is because from part (a) we obtain the equality of the first Chern forms on $E_{\mathcal{M}}, E_{\mathcal{M}'}$ given by $c_1(E_{\mathcal{M}}, N_{\mathcal{M}}) = c_1(E_{\mathcal{M}'}, N_{\mathcal{M}'})$ which implies $\frac{i}{2\pi}\mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}}) = \frac{i}{2\pi}\mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'}) \Leftrightarrow \mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}}) = \mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})$. Here $N_{\mathcal{M}}, N_{\mathcal{M}'}$ are the hermitian metrics on $E_{\mathcal{M}}, E_{\mathcal{M}'}$ induced by the inner products of $\mathcal{M}, \mathcal{M}'$, respectively.

For the case where $t \geq 2$, note that $\mathcal{M}, \mathcal{M}' \in \mathfrak{B}_1(\Omega)$. As a result, following [3, Lemma 1.11] we obtain that the reproducing kernels of $\mathcal{M}, \mathcal{M}'$ are sharp on $\Omega \setminus V(\mathcal{I})$. Since L is unitary, there exists a non-vanishing, holomorphic function ϕ on $\Omega \setminus V(\mathcal{I})$ such that

$$LK(\cdot, w) = \overline{\phi(w)} K'(\cdot, w), \tag{5.2}$$

for all $w \in \Omega \setminus V(\mathcal{I})$ [10, Theorem 3.7], where K, K' are the reproducing kernels of $\mathcal{M}, \mathcal{M}'$, respectively. But $\operatorname{codim} V(\mathcal{M}) \geq 2$. So, by the Hartog's Extension Theorem [16, Page 198] ϕ can be uniquely extended to Ω as a holomorphic function. Since an analytic function is unambiguously determined from its definition on any open set, it follows that Eq. (5.2) is true for all $w \in \Omega$.

Now, fix an arbitrary point $w_0 \in V(\mathcal{I}) \cap \Omega$. Then, from condition b) of Theorem 4.3 we have

$$K(\cdot, u) = \sum_{j=1}^{t} \overline{p_j(u)} F^{j}_{\Omega_{w_0}}(u) \text{ and } K'(\cdot, u) = \sum_{j=1}^{t} \overline{p_j(u)} F'^{j}_{\Omega_{w_0}}(u), \tag{5.3}$$

for all $u \in \Omega_{w_0}$. Applying L to the first equality of the Eq. (5.3) we obtain that

$$\overline{\phi(u)} K'(\cdot, u) = \sum_{j=1}^{t} \overline{p_j(u)} LF^{j}_{\Omega_{w_0}}(u)$$

or, equivalently,

$$K'(\cdot, u) = \sum_{j=1}^{t} \overline{p_j(u)} \frac{LF^j_{\Omega_{w_0}}(u)}{\overline{\phi(u)}}.$$

Since, $v \mapsto \frac{LF_j(v)}{\phi(v)}$, $v \mapsto F_j'(v)$ are anti-holomorphic maps from $V(\mathcal{M}')$ to $\mathcal{M}'$ for each $j = 1, \ldots, t$, satisfying 1.a) and 1.b) of Theorem 4.3, by the condition 2 of the same theorem, we have $LF_j(v) = \overline{\phi(v)} F_j'(v)$, for all $v \in V(\mathcal{I}) \cap \Omega$. As a result, from Eq. (5.1) we have $A(v) = \overline{\phi(v)} I_{t\times t}$, for all $v \in V(\mathcal{I}) \cap \Omega$. Finally,

$$\begin{aligned}
\big(\mathcal{K}_{E_{\mathcal{M}}}(F_{\mathcal{M}})\big)(v) &= \partial\Big(\big(N_{\mathcal{M}}(F_{\mathcal{M}})\big)^{-1}\bar{\partial}\big(N_{\mathcal{M}}(F_{\mathcal{M}})\big)\Big)(v) \\
&= \partial\Big(\big(N_{\mathcal{M}'}(LF_{\mathcal{M}})\big)^{-1}\bar{\partial}\big(N_{\mathcal{M}'}(LF_{\mathcal{M}})\big)\Big)(v) \\
&= \big(\mathcal{K}_{E_{\mathcal{M}'}}(LF_{\mathcal{M}})\big)(v) \\
&= \big(\mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'}A)\big)(v) \\
&= A(v)^{-1} \cdot \big(\mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})\big)(v) \cdot A(v) \\
&= \big(\mathcal{K}_{E_{\mathcal{M}'}}(F_{\mathcal{M}'})\big)(v),
\end{aligned}$$

which proves part (b) of the Theorem. □

The following example demonstrates the use of the curvature invariant given in Theorem 5.1(b). We first consider the case of a principal ideal and follow it up with the case of ideals that are not principal.

Example 5.2 Let $\mathcal{M} = \{f \in \mathcal{H}^{\lambda,\mu}(\mathbb{D}^2) : f_{|z_1=0} = \partial_1 f_{|z_1=0} = \cdots = \partial_1^{p-1} f_{|z_1=0} = 0\}$. By [15, Lemma 4.14], $\mathcal{M}$ is the closure $[\mathcal{I}]$ of the ideal $\mathcal{I} =< z_1^p >$ in $\mathcal{H}^{\lambda,\mu}(\mathbb{D}^2)$. On the zero set $\{(0, w_2) : w_2 \in \mathbb{D}\}$, the frame is given by

$$F_1(z, (0, w_2)) = \frac{(\lambda)_p z_1^p}{p!(1 - z_2\bar{w}_2)^{\mu}},$$

and hence

$$\|F_1(z, (0, w_2))\|^2 = \frac{(\lambda)_p}{p!} \sum_{n\geq 0} \frac{(\mu)_n}{n!} |w_2|^{2n},$$

where $(\lambda)_p = \lambda(\lambda + 1) \ldots (\lambda + p - 1)$ is the Pochhammer symbol. The curvature at 0 relative to this frame is

$$\partial\bar{\partial} \log \|F_1(z, (0, w_2))\|^2_{\{w_2=0\}} = \frac{\mu(\lambda)_p}{p!}. \tag{5.4}$$

We note that $\mathcal{I}_k =< z_1^{k+p} >$ is contained in $\mathcal{I}$ for each $k \in \mathbb{N}$. If the completion $[\mathcal{I}]$ of the ideal $\mathcal{I}$ in $\mathcal{H}^{\lambda,\mu}(\mathbb{D}^2)$ is equivalent to the completion of $[\mathcal{I}]'$ of the ideal $[\mathcal{I}]$

taken in $\mathcal{H}^{\lambda',\mu'}(\mathbb{D}^2)$, then the completion $[\mathcal{I}_k]$ of $\mathcal{I}_k$ in $\mathcal{H}^{\lambda,\mu}(\mathbb{D}^2)$ is equivalent to the completion $[\mathcal{I}_k]'$ of $\mathcal{I}_k$ in $\mathcal{H}^{\lambda',\mu'}(\mathbb{D}^2)$. This follows from the fact that the kernel $K_{[\mathcal{I}]}$ and $K_{[\mathcal{I}]'}$ factor: $K_{[\mathcal{I}]}(z,w) = z_1^p \chi(z,w)\bar{w}_1^p$ and $K_{[\mathcal{I}]'}(z,w) = z_1^p \chi'(z,w)\bar{w}_1^p$. Thus if $[\mathcal{I}]$ and $[\mathcal{I}]'$ are equivalent, then there exists a non-zero holomorphic map φ on $\mathbb{D}^2$ that induces the unitary module map between $\mathcal{I}]$ and $[\mathcal{I}]'$. Following the same idea as in the the proof of [4, Lemma 4.1], $[\mathcal{I}_k]$ and $[\mathcal{I}_k]'$ are equivalent via the module map induced by φ. By considering the case $k = 1$, from Theorem 5.1 and the curvature computation in Eq. (5.4), it follows that $\mu(\lambda)_p = \mu'(\lambda')_p$ and $\mu(\lambda)_{p+1} = \mu'(\lambda')_{p+1}$. Thus, we have $\lambda = \lambda'$ and $\mu = \mu'$.

Consider the submodule $[\mathcal{I}]$ that is the completion of the ideal $\mathcal{I} =< z_1^{i_1}, \ldots, z_m^{i_m} >$ in $\mathcal{H}^{\boldsymbol{\lambda}}(\mathbb{D}^n)$ for some $m < n$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$. The reproducing kernel of $\mathcal{H}^{\boldsymbol{\lambda}}(\mathbb{D}^n)$ is $1/\prod_{i=1}^n (1 - z_i\bar{w}_i)^{\lambda_i}$. On the zero set $\{(0, , \ldots, 0, w_{m+1}, \ldots, w_n) \in \mathbb{C}^n : w_i \in \mathbb{D}, m+1 \leq i \leq n\}$, the frame is given by

$$F_k(z, (0, , \ldots, 0, w_{m+1}, \ldots, w_n)) = \frac{(\lambda_k)_{i_k} z_1^{i_k}}{i_k! \prod_{i=m+1}^n (1 - z_i\bar{w}_i)^{\lambda_i}},$$

and hence

$$\|F_k(z, (0, , \ldots, 0, w_{m+1}, \ldots, w_n))\|^2 = \frac{(\lambda_k)_{i_k}}{i_k! \prod_{i=m+1}^n (1 - |w_i|^2)^{\lambda_i}},$$

for $1 \leq k \leq m$. Since $\langle F_i, F_j \rangle = 0$ for $i \neq j$, the metric is a $m \times m$ diagonal matrix with diagonals $\|F_k\|^2$ as above. Thus, we just need to look at

$$\frac{\partial^2}{\partial w_i \partial \bar{w}_i} \log \|F_1(z, (0, , \ldots, 0, w_{m+1}, \ldots, w_n))\|^2$$

for $1 \leq k \leq m$ and $m+1 \leq i \leq n$ as the mixed derivatives are zero at the origin. So the curvature computation at the origin and calculation similar to the case of the principal ideal, we have the completion $[\mathcal{I}]$ of the ideal $\mathcal{I}$ in $\mathcal{H}^{\boldsymbol{\lambda}}(\mathbb{D}^n)$ and the completion $[\mathcal{I}]'$ in $H^{\boldsymbol{\lambda}'}(\mathbb{D}^n)$ are equivalent if and only if $\boldsymbol{\lambda} = \boldsymbol{\lambda}'$.

6 On the Relationship of Curvature with the Generators

Two different sets of generators for an ideal $\mathcal{I}$ give rise to two distinct holomorphic frames for the holomorphic hermitian vector bundle associated to the completion $[\mathcal{I}]$ in the Hilbert module $\mathcal{H}$. The curvature computed relative to these frames are equivalenrt via an invertible map which is explicitly computed below in the case when Ω is a bounded domain containing the zero vector in $\mathbb{C}^m$ and $\{p_1, \ldots, p_t\}, \{q_1, \ldots, q_t\}$ are two sets of generators of the polynomial ideal $\mathcal{I}$ consisting of homogeneous polynomials of the same degree. Let $\mathcal{H}$ be an analytic Hilbert module in $\mathcal{O}(\Omega)$ and $\mathcal{M}$ be the closure of $\mathcal{I}$ in $\mathcal{H}$ with the property that $\text{codim} V(\mathcal{M}) = t$. Then, by [3, Lemma

4.7] we obtain that

$$\left\{p_1(\bar{D})K(\cdot,w)|_{w=0},\ldots,p_t(\bar{D})K(\cdot,w)|_{w=0}\right\} \text{ and}$$
$$\left\{q_1(\bar{D})K(\cdot,w)|_{w=0},\ldots,q_t(\bar{D})K(\cdot,w)|_{w=0}\right\}$$

are two bases of $\ker D_{M^*}$. As a result, from [4, Lemma 2.1] and [4, Proposition 2.10] it follows that there exists a constant invertible matrix $A = (a_{ij})_{i,j=1}^t$ such that,

$$q_j = \sum_{i=1}^{t} a_{ij} p_i,\ 1 \le j \le t. \tag{6.1}$$

Applying Theorem 4.3 separately with the choice of $\{p_1,\ldots,p_t\}$ and $\{q_1,\ldots,q_t\}$, we obtain the sets $\{F_1^p,\ldots,F_t^p\}$, $\{F_1^q,\ldots,F_t^q\}$, respectively which consist of the anti-holomorphic maps from $V(\mathcal{M})$ to $\mathcal{M}$ satisfying conditions 1) to 4) of the theorem. Now, we claim the following.

Lemma 6.1 *For each $w \in V(\mathcal{M})$, $\left[F_1^p(w)\ \ldots\ F_t^p(w)\right] = \left[F_1^q(w)\ \ldots\ F_t^q(w)\right]A^*$.*

Proof Fix an arbitrary point $w_0 \in V(\mathcal{M})$. Then from condition 1 of Theorem 4.3 there exist a neighbourhood Ω_{w_0} of w_0 in Ω, anti-holomorphic maps $F^k_{\Omega_{w_0},p}$, $F^k_{\Omega_{w_0},q} : V(\mathcal{M}) \to \mathcal{M}$, $k = 1,\ldots,t$ such that

(a) $F^k_{\Omega_{w_0},p}(v) = F_k^p(v)$, $F^k_{\Omega_{w_0},q}(v) = F_k^q(v)$, for all $v \in V(\mathcal{M}) \cap \Omega_{w_0}$, $k = 1,\ldots,t$ and

(b) $K(\cdot,u) = \sum_{i=1}^t \overline{p_i(u)} F^i_{\Omega_{w_0},p}(u)$, $K(\cdot,u) = \sum_{j=1}^t \overline{q_j(u)} F^j_{\Omega_{w_0},q}(u)$, for all $u \in \Omega_{w_0}$.

Now, applying Eq. (6.1) to the second equality of b) we obtain that

$$K(\cdot,u) = \sum_{i=1}^{t} \overline{p_i(u)}\Big(\sum_{j=1}^{t} \bar{a}_{ij} F^j_{\Omega_{w_0},q}(u)\Big),$$

for all $u \in \Omega_{w_0}$. Finally, observe that, for each $i \in \{1,\ldots,t\}$, $v \mapsto \sum_{j=1}^t \bar{a}_{ij} F_j^q(v)$ is an anti-holomorphic map from $V(\mathcal{M}) \to \mathcal{M}$ satisfying conditions 1.a) and 1.b) of Theorem 4.3 with respect to the set $\{p_1,\ldots,p_t\}$. So, from condition 2 of Theorem 4.3, it follows that, for each $i = 1,\ldots,t$, $v \in V(\mathcal{M})$,

$$F_i^p(v) = \sum_{j=1}^{t} \bar{a}_{ij} F_j^q(v),$$

proving the lemma. □

Note that each of the sets $\{F_1^p,\ldots,F_t^p\}$ and $\{F_1^q,\ldots,F_t^q\}$ canonically induces an anti-holomorphic frame of $E_{\mathcal{M}}$ on $V(\mathcal{M})$. If we denote the frames as $F^p_{\mathcal{M}}$ and $F^q_{\mathcal{M}}$ respectively, then we have

$$\mathcal{K}_{E_{\mathcal{M}}}(F^p_{\mathcal{M}}) = \mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}}A^*) = (A^*)^{-1} \cdot \mathcal{K}_{E_{\mathcal{M}}}(F^q_{\mathcal{M}}) \cdot A^*,$$

where $\mathcal{K}_{E_\mathcal{M}}(F^p_\mathcal{M}), \mathcal{K}_{E_\mathcal{M}}(F^q_\mathcal{M})$ are the curvature matrices of the bundle $E_\mathcal{M}$ with respect to the frames $F^p_\mathcal{M}$, $F^q_\mathcal{M}$, respectively. Thus, we have proved the following.

Proposition 6.2 *Let Ω be a bounded domain in $\mathbb{C}^m$ and $\{p_1, \ldots, p_t\}, \{q_1, \ldots, q_t\}$ be two generators of the polynomial ideal $\mathcal{I}$ consisting of homogeneous polynomials of same degree. Also, let $\mathcal{H} \subseteq \mathcal{O}(\Omega)$ be an analytic Hilbert module and $\mathcal{M}$ be the closure of $\mathcal{I}$ in $\mathcal{H}$ with the property that $codimV(\mathcal{M}) = t$. Furthermore, assume that $F^p_\mathcal{M}$, $F^q_\mathcal{M}$ are the global frames of $E_\mathcal{M}$ obtained by applying Theorem 4.3 on $\mathcal{M}$ with respect to the generators mentioned above. Then there exists a constant invertible matrix A such that*

$$\mathcal{K}_{E_\mathcal{M}}(F^p_\mathcal{M}) = (A^*)^{-1} \cdot \mathcal{K}_{E_\mathcal{M}}(F^q_\mathcal{M}) \cdot A^*.$$

Corollary 6.3 *Let $\Omega, \mathcal{I}$ be as above, $\mathcal{H}, \mathcal{H}' \subseteq \mathcal{O}(\Omega)$ be two analytic Hilbert modules and $\mathcal{M}, \mathcal{M}'$ be the closure of $\mathcal{I}$ in $\mathcal{H}, \mathcal{H}'$ respectively with $codimV(\mathcal{M}) = codimV(\mathcal{M}') = t$. Suppose that $F^p_\mathcal{M} := \{F^p_1, \ldots, F^p_t\}$, $F^q_{\mathcal{M}'} := \{F^{'q}_1, \ldots, F^{'q}_t\}$ are the global frames of $E_\mathcal{M}$, $E_{\mathcal{M}'}$ corresponding to the generators $\{p_1, \ldots, p_t\}, \{q_1, \ldots, q_t\}$, respectively. If the modules $\mathcal{M}$ and $\mathcal{M}'$ are "unitarily" equivalent, then there exists a constant invertible matrix A such that*

$$\mathcal{K}_{E_\mathcal{M}}(F^p_\mathcal{M}) = (A^*)^{-1} \cdot \mathcal{K}_{E_{\mathcal{M}'}}(F^q_{\mathcal{M}'}) \cdot A^*,$$

where $\mathcal{K}_{E_\mathcal{M}}(F^p_\mathcal{M}), \mathcal{K}_{E_{\mathcal{M}'}}(F^q_{\mathcal{M}'})$ are the curvature matrices of $E_\mathcal{M}$, $E_{\mathcal{M}'}$ with respect to the frames $F^p_\mathcal{M}$, $F^q_{\mathcal{M}'}$, respectively.

Proof If we apply Theorem 4.3 on $\mathcal{M}$ with respect to the generator $\{q_1, \ldots, q_t\}$, we will obtain a collection of anti-holomorphic maps $\{F^q_1, \ldots, F^q_t\}$ from $V(\mathcal{M})$ to $\mathcal{M}$. This set canonically induces a global frame of $E_\mathcal{M}$. Let us denote the frame by $F^q_\mathcal{M}$. Then, by Proposition 6.2 it follows that there exists a constant invertible matrix A such that

$$\mathcal{K}_{E_\mathcal{M}}(F^p_\mathcal{M}) = (A^*)^{-1} \cdot \mathcal{K}_{E_\mathcal{M}}(F^q_\mathcal{M}) \cdot A^*.$$

Finally, from Theorem 5.1 we obtain that

$$\mathcal{K}_{E_\mathcal{M}}(F^q_\mathcal{M}) = \mathcal{K}_{E_{\mathcal{M}'}}(F^q_{\mathcal{M}'})$$

which proves the corollary. □

Data availability Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

References

1. Agrawal, O.P., Salinas, N.: Sharp kernels and canonical subspaces. Am. J. Math. **110**(1), 23–47 (1988)

2. Aronszajn, N.: Theory of reproducing kernels. Trans. Am. Math. Soc. **68**(3), 337–404 (1950)
3. Biswas, S.: Geometric invariants for a class of semi-Fredholm Hilbert Modules, Ph.D. thesis, Indian Statistical Institute of Science (2010)
4. Biswas, S., Misra, G.: Resolution of singularities for a class of Hilbert modules. Indiana Univ. Math. J. **61**, 1019–1050 (2012)
5. Biswas, S., Misra, G., Putinar, M.: Unitary invariants for Hilbert modules of finite rank. Journal fr die reine und angewandte Mathematik (Crelles Journal) **662**, 165–204 (2012)
6. Chen, X.M., Douglas, R.G.: Localization of Hilbert modules. Mich. Math. J. **39**(3), 443–454 (1992)
7. Chen, X.M., Guo, K.: Analytic Hilbert Modules. Chapman and Hall/CRC, London (2003)
8. Chirka, E.M.: Complex Analytic Sets. Kluwer, Dordrecht (1989)
9. Cowen, M.J., Douglas, R.G.: Complex geometry and operator theory. Acta Math. **141**(3–4), 187–261 (1978)
10. Curto, R.E., Salinas, N.: Generalized Bergman kernels and the Cowen–Douglas theory. Am. J. Math. **106**, 447–488 (1984)
11. Douglas, R.G., Misra, G.: On quasi-free Hilbert modules. N. Y. J. Math. **11**, 547–561 (2005)
12. Douglas, R.G., Misra, G., Varughese, C.: Some geometric invariants from resolutions of Hilbert modules. In: Borichev, A.A., Nikolski, N.K. (eds.) Systems, Approximation, Singular Integral Operators, and Related Topics Operator Theory: Advances and Applications, vol. 129, pp. 241–270. Birkhäuser, Basel (2001)
13. Douglas, R.G., Paulsen, V.I.: Hilbert Modules Over Function Algebras. Longman Sc & Tech, New York (1989)
14. Duan, Y., Guo, K.: Dimension formula for localization of Hilbert modules. J. Oper. Theory **62**, 439–452 (2009)
15. Ghara, S., Misra, G.: Decomposition of the tensor product of two Hilbert modules. In: Operator Theory, Operator Algebras and Their Interactions with Geometry and Topology-Ronald G. Douglas Memorial, Operator Theory: Advances and Applications, vol. 278, pp. 221–265
16. Lojasiewicz, S.: Introduction to Complex Analytic Geometry. Birkhäuser, Basel (2013)
17. Paulsen, V.I., Raghupathi, M.: An Introduction to the Theory of Reproducing Kernel Hilbert Spaces. Cambridge Studies in Advanced Mathematics, vol. 152. Cambridge University Press, Cambridge (2016)

Authors and Affiliations

Shibananda Biswas[1] · Gadadhar Misra[2,3] · Samrat Sen[4]

Shibananda Biswas
shibananda@iiserkol.ac.in

Samrat Sen
samratsen@alum.iisc.ac.in

1 Department of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Nadia, Mohanpur, West Bengal 741246, India

2 Statistics and Mathematics Unit, Indian Statistical Institute, Bangalore 560059, India

3 Department of Mathematics, Indian Institute of Technology, Gandhinagar 382055, India

4 4L B.G. Bye Lane, Naktala, Kolkata 700047, India