Complex Analysis and Operator Theory (2022) 16: 83
https://doi.org/10.1007/s11785-022-01257-0

Complex Analysis and Operator Theory

The Joint Spectrum for a Commuting Pair of Isometries in Certain Cases

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Received: 12 November 2021 / Accepted: 17 June 2022 / Published online: 18 July 2022

Abstract
We show that the joint spectrum of two commuting isometries can vary widely depending on various factors. It can range from being small (of measure zero or an analytic disc for example) to the full bidisc. En route, we discover a new model pair in the negative defect case and relate it to the modified bi-shift.

Mathematics Subject Classification Primary 47A13 · 47A45 · 47A65

1 Introduction

An isometry V is called *pure* if V^{*n} converges to 0 strongly as $n \to \infty$. This is equivalent to saying that V is the unilateral shift of multiplicity equal to the dimension of the range of the *defect operator* $I - VV^*$.

The famous Wold decomposition [17, 23] tells us that given an isometry V on a Hilbert space $\mathcal{H}$, the space $\mathcal{H}$ breaks uniquely into a direct sum $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$ of reducing subspaces such that $V|_{\mathcal{H}_0}$ is a unitary and $V|_{\mathcal{H}_0^{\perp}}$ is a pure isometry. This immediately implies that for a non-unitary isometry V (i.e., when the defect operator is

In fond memory of Jörg Eschmeier.

Communicated by Michael Hartz.

This article is part of the topical collection "Multivariable Operator Theory. The Jörg Eschmeier Memorial" edited by Raul Curto, Michael Hartz, Mihai Putinar and Ernst Albrecht.

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positive and not zero), the spectrum $\sigma(V)$ is the closed unit disc $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. The situation for a pair of commuting isometries is vastly different.

The topic of commuting isometries has been vigorously pursued in the last two decades, see [1, 2, 4–6, 11, 15, 16, 18, 20, 21] and the references therein. In [6] and [5], the novel idea of using graphs has led to a clear understanding of structures.

The *defect operator* $C(V_1, V_2)$ is introduced in [12] and [13], as

$$C(V_1, V_2) = I - V_1V_1^* - V_2V_2^* + V_1V_2V_2^*V_1^*.$$

In [13] and [16], the authors provide the characterization of (V_1, V_2), when the defect is positive, negative or zero. It is well known (see [11, 13]) that a pair has positive defect if and only if it is doubly commuting, and it has negative defect if and only if it is dual doubly commuting. In all the three cases, the defect is either a projection or negative of a projection. In general the defect is the difference of two projections; see [16].

In this paper we study the pairs of commuting isometries, whose defect is the difference of two mutually orthogonal projections. We characterize such pairs in Theorem 2.1 and we classify them in Table 1. We also provide the characterization for a few cases in Table 1; see Lemma 6.3 and Lemma 6.9. We rephrase the structure of (V_1, V_2) in each case, which appears in Table 1, in a unified approach using the Berger-Coburn-Lebow (BCL) Theorem. The joint spectrum is studied in detail for all the cases except the last one appearing in Table 1.

There is the related concept of the *fringe operators*:

$$F_1 : \ker V_1^* \to \ker V_1^* \quad \text{and} \quad F_2 : \ker V_2^* \to \ker V_2^*$$

defined by,

$$F_1(x) = P_{\ker V_1^*} V_2(x) \quad \text{and} \quad F_2(x) = P_{\ker V_2^*} V_1(x). \tag{1.1}$$

In various characterizations of Table 1, we shall point out the criteria in terms of the fringe operators for possible use in examples.

1.1 The Joint Spectrum

If (T_1, T_2) is a pair of commuting bounded operators on $\mathcal{H}$, then for defining (see [14, 22]) the *Taylor joint spectrum* $\sigma(T_1, T_2)$, one considers the *Koszul complex* $K(T_1, T_2)$:

$$0 \xrightarrow{\delta_0} \mathcal{H} \xrightarrow{\delta_1} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\delta_2} \mathcal{H} \xrightarrow{\delta_3} 0 \tag{1.2}$$

where $\delta_1(h) = (T_1h, T_2h)$ for $h \in \mathcal{H}$ and $\delta_2(h_1, h_2) = T_1h_2 - T_2h_1$ for $h_1, h_2 \in \mathcal{H}$. From the way the complex is constructed, ran $\delta_{n-1} \subseteq \ker \delta_n$. When ran $\delta_{n-1} = \ker \delta_n$ for all $n = 1, 2, 3$ we say that the *Koszul complex* $K(T_1, T_2)$ *is exact or the pair* (T_1, T_2) *is non-singular.* A pair $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ is said to be in the *joint spectrum* $\sigma(T_1, T_2)$ if the pair $(T_1 - \lambda_1 I, T_2 - \lambda_2 I)$ is singular. In the case of a singular pair, we say that the *non-singularity breaks at the stage n* if $\operatorname{ran} \delta_{n-1} \neq \ker \delta_n$.

Observe that the non-singularity breaks at stage 1 if and only if (λ_1, λ_2) is a joint eigenvalue for (T_1, T_2) and the non-singularity breaks at stage 3 if and only if the joint range of $(T_1 - \lambda_1 I, T_2 - \lambda_2 I)$ is not the whole space $\mathcal{H}$. If $(\overline{\lambda_1}, \overline{\lambda_2})$ is a joint eigenvalue of (T_1^*, T_2^*), then by the fact that $\operatorname{ran} T_1 + \operatorname{ran} T_2 = \mathcal{H}$ implies $\ker T_1^* \cap \ker T_2^* = \{0\}$, the non-singularity of the Koszul complex $K(T_1 - \lambda_1 I, T_2 - \lambda_2 I)$ breaks at stage 3. There are a few elementary results which we record as a lemma so that we can refer to it later.

Lemma 1.1 *Let $\mathcal{H}$ and $\mathcal{K}$ be two non-zero Hilbert spaces. Let (T_1, T_2) be a pair of commuting bounded operators on $\mathcal{H}$.*

(1) $\sigma(T_1, T_2) \subseteq \sigma(T_1) \times \sigma(T_2)$.
(2) If there is a non-trivial joint reducing subspace $\mathcal{H}_0$ for (T_1, T_2), i.e., if $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ and

$$T_i = \begin{array}{c} \quad \mathcal{H}_0 \quad \mathcal{H}_0^\perp \\ \begin{pmatrix} T_{i0} & 0 \\ 0 & T_{i1} \end{pmatrix} \end{array} \begin{array}{l} \mathcal{H}_0 \\ \mathcal{H}_0^\perp \end{array}$$

then

$$\sigma(T_1, T_2) = \sigma(T_{10}, T_{20}) \cup \sigma(T_{11}, T_{21}).$$

(3) $(z_1, z_2) \in \sigma(T_1, T_2)$ *if and only if* $(\overline{z_1}, \overline{z_2}) \in \sigma(T_1^*, T_2^*)$.
(4) $\sigma(I_{\mathcal{K}} \otimes T_1, I_{\mathcal{K}} \otimes T_2) = \sigma(T_1, T_2) = \sigma(T_1 \otimes I_{\mathcal{K}}, T_2 \otimes I_{\mathcal{K}})$.
(5) Let (S_1, S_2) be a pair of commuting bounded operators on a Hilbert space $\mathcal{K}$. If (T_1, T_2) is jointly unitarily equivalent to (S_1, S_2), then $\sigma(T_1, T_2) = \sigma(S_1, S_2)$.
(6) For any T in $\mathcal{B}(\mathcal{H})$ and S in $\mathcal{B}(\mathcal{K})$, the joint spectrum $\sigma(T \otimes I_{\mathcal{K}}, I_{\mathcal{H}} \otimes S)$ is the Cartesian product $\sigma(T) \times \sigma(S)$.

Thus, for commuting isometries V_1 and V_2, we have $\sigma(V_1, V_2) \subseteq \overline{\mathbb{D}^2}$. The joint spectrum of a pair of commuting unitary operators is contained in the torus $\mathbb{T}^2$, where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane.

This note uses the *fundamental pairs of isometries consisting of multiplication operators* to describe the structure of (V_1, V_2). These are fundamental in the sense that in each case the sign of the defect operator is dictated by the fundamental pair alone.

When the defect operator $C(V_1, V_2)$ of two commuting isometries is positive or negative, but not zero, then the whole space $\mathcal{H}$ breaks into a direct sum of reducing subspaces $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ in the style of Wold where the restriction of (V_1, V_2) on the $\mathcal{H}_0$ part is the fundamental pair and the restriction of (V_1, V_2) on the $\mathcal{H}_0^\perp$ part has defect zero. What the fundamental pair is depends on whether $C(V_1, V_2)$ is positive or negative. These are the contents of Theorem 4.11 and Theorem 5.10.

The fundamental pairs are such that in both cases (of $C(V_1, V_2)$ positive or negative), the joint spectrum of (V_1, V_2) is the whole closed bidisc $\overline{\mathbb{D}^2}$. These are done in Theorem

4.11 and Theorem 5.11. If the defect operator $C(V_1, V_2)$ is zero, the joint spectrum of (V_1, V_2) is contained in the topological boundary of the bidisc.

The structure theorem in the case ran $V_1 =$ ran V_2, shows that (V_1, V_2) is the direct sum of a prototypical pair (see Sect. 6.1.1) and a pair of commuting unitaries. The joint spectrum is computed. The joint spectrum of the prototypical pair of this case, is neither the closed bidisc nor contained inside the topological boundary of the bidisc.

In the case ran $V_2 \subsetneq$ ran V_1, the joint spectrum $\sigma(V_1, V_2) \subseteq \{(z_1, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\}$. The above inclusion is sharp; see Example 6.12, and it can be a strict inclusion; see Example 6.13. Note that $\{(z_1, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\}$ has measure non-zero and it is not equal to the closed bidisc.

In each case above except the case ran $V_2 \subsetneq$ ran V_1, we point out the stage of the Koszul complex where non-singularity is broken.

1.2 The Berger-Coburn-Lebow Theorem

For a Hilbert space $\mathcal{E}$, the Hardy space of $\mathcal{E}$-valued functions on the unit disc in the complex plane is

$$H^2_{\mathbb{D}}(\mathcal{E}) = \left\{ f : \mathbb{D} \to \mathcal{E} \mid f \text{ is analytic and } f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ with } \sum_{n=0}^{\infty} \|a_n\|^2_{\mathcal{E}} < \infty \right\}.$$

Here the a_n are from $\mathcal{E}$. This is a Hilbert space with the inner product

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle = \sum_{n=0}^{\infty} \langle a_n, b_n \rangle_{\mathcal{E}}$$

and is identifiable with $H^2_{\mathbb{D}} \otimes \mathcal{E}$ where $H^2_{\mathbb{D}}$ stands for the Hardy space of scalar-valued functions on $\mathbb{D}$. We shall use this identification throughout the paper, often without any further mention, and M_z denotes the multiplication operator by the coordinate function z on $H^2_{\mathbb{D}}$. For $\lambda \in \mathbb{D}$, let k_λ be the function in $H^2_{\mathbb{D}}$ given by

$$k_\lambda(z) = \sum_{n=0}^{\infty} z^n \overline{\lambda}^n = \frac{1}{1 - z\overline{\lambda}}.$$

It is well-known that the span of $\{k_\lambda : \lambda \in \mathbb{D}\}$ is dense in $H^2_{\mathbb{D}}$.

The space of $\mathcal{B}(\mathcal{E})$-valued bounded analytic functions on $\mathbb{D}$ will be denoted by $H^\infty_{\mathbb{D}}(\mathcal{B}(\mathcal{E}))$. Naturally, if $\varphi \in H^\infty_{\mathbb{D}}(\mathcal{B}(\mathcal{E}))$, then it induces a multiplication operator M_φ on $H^2_{\mathbb{D}}(\mathcal{E})$. One of the main tools for us is the Berger-Coburn-Lebow (BCL) theorem [3].

Theorem 1.2 *Let (V_1, V_2) be a commuting pair of isometries acting on $\mathcal{H}$. Then, up to unitary equivalence, the Hilbert space $\mathcal{H}$ breaks into a direct sum of reducing subspaces $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_u$ such that*

(1) There is a unique (up to unitary equivalence) triple $(\mathcal{E}, P, U)$ *where* $\mathcal{E}$ *is a Hilbert space,* P *is a projection on* $\mathcal{E}$ *and* U *is a unitary on* $\mathcal{E}$ *such that* $\mathcal{H}_p = H^2_{\mathbb{D}}(\mathcal{E})$*, the functions* φ_1 *and* φ_2 *defined on* $\mathbb{D}$ *by*

$$\varphi_1(z) = U^*(P^\perp + zP) \quad \textit{and} \quad \varphi_2(z) = (P + zP^\perp)U, \tag{1.3}$$

are commuting multipliers in $H^\infty_{\mathbb{D}}(\mathcal{B}(\mathcal{E}))$ *and* $(V_1|_{\mathcal{H}_p}, V_2|_{\mathcal{H}_p})$ *is equal to* $(M_{\varphi_1}, M_{\varphi_2})$*.*

(2) $V_1|_{\mathcal{H}_u}$ *and* $V_2|_{\mathcal{H}_u}$ *are commuting unitary operators.*

The result of the theorem above will be called the *BCL representation* of (V_1, V_2). Using Theorem 1.2 we can compute the defect operator (see [13]), because $M_{\varphi_1} = I_{H^2_{\mathbb{D}}} \otimes U^*P^\perp + M_z \otimes U^*P$ and $M_{\varphi_2} = I_{H^2_{\mathbb{D}}} \otimes PU + M_z \otimes P^\perp U$. Hence

$$C(M_{\varphi_1}, M_{\varphi_2}) = (I - M_z M_z^*) \otimes (U^*PU - P) = E_0 \otimes (U^*PU - P)$$

where E_0 is the *one dimensional projection onto the space of constant functions* in $H^2_{\mathbb{D}}$. Together with the fact that the defect operator of a pair of commuting unitary operators is zero, this means, in the decomposition $\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_u$ with $\mathcal{H}_p = H^2_{\mathbb{D}}(\mathcal{E})$,

$$C(V_1, V_2) = (E_0 \otimes (U^*PU - P)) \oplus 0. \tag{1.4}$$

Definition 1.3 (1) A *BCL triple* $(\mathcal{E}, P, U)$ is a Hilbert space $\mathcal{E}$, along with a projection P and a unitary U. It is said to be the *BCL triple for the pair of commuting isometries* (V_1, V_2) if $\mathcal{E}$, P and U are as in Theorem 1.2, part (1).

(2) Given a BCL triple $(\mathcal{E}, P, U)$, the functions $\varphi_1, \varphi_2 : \mathbb{D} \to \mathcal{B}(\mathcal{E})$ will always be as defined in (1.3).

(3) A pair of commuting isometries (V_1, V_2) is called pure if $\mathcal{H}_u = \{0\}$ in its BCL representation.

In fact, $\mathcal{H}_u$ is the unitary part of the product $V = V_1V_2$ in its Wold decomposition and $\mathcal{E} = \ker V^*$. Thus $\mathcal{H}_u = \{0\}$ if and only if V is pure.

The Berger-Coburn-Lebow theorem has an interesting consequence in the case when the part $\mathcal{H}_p$ is non-zero. We have $\varphi_1(z)\varphi_2(z) = z$ for every $z \in \mathbb{D}$ and hence $M_{\varphi_1}M_{\varphi_2} = M_z \otimes I_{\mathcal{E}}$. Hence, by the spectral mapping theorem for joint spectra (see for example [8]),

$$\{z_1z_2 : (z_1, z_2) \in \sigma(M_{\varphi_1}, M_{\varphi_2})\} = \sigma(M_z \otimes I_{\mathcal{E}}) = \overline{\mathbb{D}}.$$

Thus, if $\mathcal{H}_p \neq \{0\}$, then $\sigma(M_{\varphi_1}, M_{\varphi_2})$ cannot be contained in the torus $\mathbb{T}^2$. Hence by Lemma 1.1 part (2), $\sigma(V_1, V_2)$ cannot be contained in the torus $\mathbb{T}^2$.

Let $(\mathcal{E}, P, U)$ be a BCL triple. It is easy to see that if the Koszul complex $K(\varphi_1(z) - \lambda_1 I, \varphi_2(z) - \lambda_2 I)$ breaks at stage 3 for some $z \in \mathbb{D}$, then the Koszul complex $K(M_{\varphi_1} -$

$\lambda_1 I, M_{\varphi_2} - \lambda_2 I)$ breaks at stage 3. More generally, we show that, in all the cases under consideration in this note except the case ran $V_2 \subsetneq$ ran V_1, we have

$$\overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))} = \sigma(M_{\varphi_1}, M_{\varphi_2}). \tag{1.5}$$

See Theorem 3.8, Theorem 4.15, Theorem 5.13 and Theorem 6.8.

In the following, when we consider the pair (V_1, V_2) of commuting isometries, V denotes the product $V_1 V_2$, and $\{e_n : n \in \mathbb{Z}\}$ denotes the standard orthonormal basis of $l^2(\mathbb{Z})$.

We end this section with the comment that one of the strong points of the BCL theorem is that it models the V_i in terms of functions of one variable whereas V_i could, a priori, be dependent on two variables (multipliers on the Hardy space of the bidisc, for example). This strength will be greatly exploited in this note.

2 The Defect Operator

Recall that ([12] and [13]) the defect operator of a pair of commuting isometries (V_1, V_2) is defined as

$$C(V_1, V_2) = I - V_1 V_1^* - V_2 V_2^* + V_1 V_2 V_2^* V_1^*. \tag{2.1}$$

It is easy to see that (see [16]) the defect

$$C(V_1, V_2) = P_{\ker V_1^*} - P_{V_2(\ker V_1^*)} = P_{\ker V_2^*} - P_{V_1(\ker V_2^*)}, \tag{2.2}$$

and

$$\ker V_1^* \oplus V_1(\ker V_2^*) = \ker V_2^* \oplus V_2(\ker V_1^*) = \ker V_1^* V_2^*. \tag{2.3}$$

Note that the defect operator lives on $\ker V^*$ in the sense that the defect operator is zero on the orthogonal component of $\ker V^*$. Equation (2.2) shows that the defect is always a difference of two projections. Let

$$P_1 = P_{\ker V_1^*} \quad \text{and} \quad P_2 = P_{V_2(\ker V_1^*)}. \tag{2.4}$$

Define

$$\begin{aligned}
\mathcal{H}_1 &:= \operatorname{ran} P_1 \cap \ker P_2 = \ker V_1^* \cap \ker V_2^*,\\
\mathcal{H}_2 &:= \operatorname{ran} P_2 \cap \ker P_1 = V_1(\ker V_2^*) \cap V_2(\ker V_1^*),\\
\mathcal{H}_3 &:= \operatorname{ran} P_1 \cap \operatorname{ran} P_2 = \ker V_1^* \cap V_2(\ker V_1^*),\\
\mathcal{H}_4 &:= \ker P_1 \cap \ker P_2 = V_1(\ker V_2^*) \cap \ker V_2^*.
\end{aligned}$$

Notice that $\mathcal{H}_i \perp \mathcal{H}_j$ if $i \neq j$, and the $\mathcal{H}_i$ are reducing for P_1 and P_2 and

$$\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4 \subseteq \ker V^*. \tag{2.5}$$

Theorem 2.1 *The following are equivalent:*

(a) *The defect $C(V_1, V_2)$ is the difference of two mutually orthogonal projections.*
(b) $\ker V^* = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$.
(c) $\ker V_1^* = \mathcal{H}_1 \oplus \mathcal{H}_3$.
(d) $V_1(\ker V_2^*) = \mathcal{H}_2 \oplus \mathcal{H}_4$.
(e) *If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2),*

$$U^*(\operatorname{ran} P) = (U^*(\operatorname{ran} P) \cap \operatorname{ran} P) \oplus (U^*(\operatorname{ran} P) \cap \operatorname{ran} P^{\perp}). \tag{2.6}$$

Proof Suppose

$$C(V_1, V_2) = Q_1 - Q_2 \quad \text{with} \quad \operatorname{ran} Q_1 \perp \operatorname{ran} Q_2 \tag{2.7}$$

for some projections Q_1, Q_2 in $\ker V^*$. Notice that the pair (Q_1, Q_2) satisfying (2.7) is *unique* if it exists. Then for such a pair (V_1, V_2) we have

$$C(V_1, V_2) = \begin{pmatrix} I_{\operatorname{ran} Q_1} & 0 & 0 \\ 0 & -I_{\operatorname{ran} Q_2} & 0 \\ 0 & 0 & 0 \end{pmatrix} = P_1 - P_2. \tag{2.8}$$

For any two projections P and Q in $\mathcal{H}$ and $x \in \mathcal{H}$, $Px - Qx = x$ implies that $Px = x$ and $Qx = 0$. Using this fact and (2.8) we see that:

$$\mathcal{H}_1 = \operatorname{ran} Q_1 \quad \text{and} \quad \mathcal{H}_2 = \operatorname{ran} Q_2.$$

Note that $P_1 = P_2 = \begin{pmatrix} I_{\mathcal{K}} & 0 \\ 0 & 0_{\mathcal{L}} \end{pmatrix}$ on $\mathcal{K} \oplus \mathcal{L}$ for some $\mathcal{K}, \mathcal{L}$ that satisfy $\ker V^* = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{K} \oplus \mathcal{L}$. Hence, by (2.8) $\operatorname{ran} P_1 = \mathcal{H}_1 \oplus \mathcal{K}$ and $\operatorname{ran} P_2 = \mathcal{H}_2 \oplus \mathcal{K}$. Thus $\mathcal{K} = \operatorname{ran} P_1 \cap \operatorname{ran} P_2 = \mathcal{H}_3$. Similarly $\mathcal{L} = \ker P_1 \cap \ker P_2 = \mathcal{H}_4$. Therefore

$$\ker V^* = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4,$$

and in this decomposition

$$P_1 = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

This proves $(a) \implies (b)$, and $(b) \implies (a)$ is trivial.

Note the fact that, if $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ and $\mathcal{E}_4$ are any subspaces of $\mathcal{E}$ satisfying $\mathcal{E}_1 \oplus \mathcal{E}_2 = \mathcal{E} = \mathcal{E}_3 \oplus \mathcal{E}_4$, then $\mathcal{E}_1 = (\mathcal{E}_1 \cap \mathcal{E}_3) \oplus (\mathcal{E}_1 \cap \mathcal{E}_4)$ if and only if $\mathcal{E}_2 = (\mathcal{E}_2 \cap \mathcal{E}_3) \oplus (\mathcal{E}_2 \cap \mathcal{E}_4)$. Using the above fact the equivalence $(c) \iff (d)$ follows from (2.3).

The implication $(b) \implies (c)$ is clear. The implication $(c) \implies (b)$ follows from the equivalence of (c) and (d) and (2.3).

Table 1 Classification

$\ker V^* = \oplus_{i=1}^4 \mathcal{H}_i$ and the status of $\mathcal{H}_i$	The status of (V_1, V_2)	BCL characterization
$\mathcal{H}_i = 0$ for $i = 1, 2, 3, 4$	$C(V_1, V_2) = 0$ and both V_1 and V_2 are unitaries	$\mathcal{E} = 0$
$\mathcal{H}_i = 0$ for $i = 1, 2, 4$, $\mathcal{H}_3 \neq 0$	$C(V_1, V_2) = 0$ and V_2 is a unitary, V_1 is not a unitary	P=I
$\mathcal{H}_i = 0$ for $i = 1, 2, 3$, $\mathcal{H}_4 \neq 0$	$C(V_1, V_2) = 0$ and V_1 is a unitary, V_2 is not a unitary	P=0
$\mathcal{H}_i = 0$ for $i = 1, 2$, $\mathcal{H}_i \neq 0$ for $i = 3, 4$	$C(V_1, V_2) = 0$ and both V_1 and V_2 are not unitaries	P is non-trivial and U reduces ran P
$\mathcal{H}_1 = 0$, $\mathcal{H}_i \neq 0$ for $i = 2, 3, 4$	$C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$	$U(\text{ran } P^\perp) \subsetneq \text{ran } P^\perp$
$\mathcal{H}_2 = 0$, $\mathcal{H}_i \neq 0$ for $i = 1, 3, 4$	$C(V_1, V_2) \geq 0$ and $C(V_1, V_2) \neq 0$	$U(\text{ran } P) \subsetneq \text{ran } P$
$\mathcal{H}_i = 0$ for $i = 3, 4$, $\mathcal{H}_i \neq 0$ for $i = 1, 2$	ran V_1 = ran V_2 and V_i is not a unitary	P is non-trivial and $U(\text{ran } P) = \text{ran } P^\perp$
$\mathcal{H}_4 = 0$, $\mathcal{H}_i \neq 0$ for $i = 1, 2, 3$	ran $V_1 \subsetneq$ ran V_2 and V_2 is not unitary	$P \neq I$ and $U(\text{ran } P^\perp) \subsetneq \text{ran } P$
$\mathcal{H}_3 = 0$, $\mathcal{H}_i \neq 0$ for $i = 1, 2, 4$	ran $V_2 \subsetneq$ ran V_1 and V_1 is not unitary	$P \neq 0$ and $U(\text{ran } P) \subsetneq \text{ran } P^\perp$
$\mathcal{H}_i \neq 0$ for $i = 1, 2, 3, 4$	Unknown	—

Note that $\ker M_{\varphi_1}^* = \text{ran}(I - M_{\varphi_1} M_{\varphi_1}^*) = 1 \otimes \text{ran}(U^* P U) = 1 \otimes U^*(\text{ran } P)$, $\ker M_{\varphi_2}^* = 1 \otimes \text{ran } P^\perp$ and $M_{\varphi_2}(\ker M_{\varphi_1}^*) = 1 \otimes \text{ran } P$. Now the proof of $(a) \iff (e)$ follows from the equivalence $(a) \iff (c)$. □

Using the (e) part of the above theorem we easily get an example of a pair (V_1, V_2) whose defect is not a difference of two mutually orthogonal projections.

Example 2.2 Let $\mathcal{E} = \mathbb{C}^2$, $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The BCL triple $(\mathcal{E}, P, U)$ does not satisfy (2.6), hence the defect of the pair (V_1, V_2) corresponding to this BCL triple, is not the difference of two mutually orthogonal projections.

The Table 1 gives a neat classification. We leave the proofs to the reader for the first two columns. The contents of the third column will unfold as we progress. One can notice that certain cases are not mentioned in the table. That is because those cases cannot occur.

3 The Zero Defect Case

3.1 Structure

This subsection is mainly a rephrasing of known results. If one of the V_i's is a unitary, then it is trivial to check that the defect $C(V_1, V_2)$ is zero. The following example is the prototypical example of a pure pair of commuting isometries with defect zero; this example serves as a building block in the general structure, see Theorem 3.4.

Example 3.1 Let $\mathcal{L}$ be a non-zero Hilbert space and W be a unitary on $\mathcal{L}$. Consider the commuting pair of isometries $(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W)$ on $H^2_{\mathbb{D}} \otimes \mathcal{L}$. As $I \otimes W$ is a unitary, the defect $C(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W) = 0$. Also, $\sigma(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W) = \overline{\mathbb{D}} \times \sigma(W)$, by Lemma 1.1 part (6). Also see [7].

Now consider the unitary $\Lambda : H^2_{\mathbb{D}} \otimes \mathcal{L} \to H^2_{\mathbb{D}} \otimes \mathcal{L}$ given by

$$\Lambda\left(\sum_{m=0}^{\infty} a_m z^m\right) = \sum_{m=0}^{\infty} W^m(a_m) z^m. \tag{3.1}$$

Then, $\Lambda(M_z \otimes W^*)\Lambda^* = M_z \otimes I$ and $\Lambda(I \otimes W)\Lambda^* = I \otimes W$. This says that $(M_z \otimes W^*, I \otimes W)$ and $(M_z \otimes I, I \otimes W)$ are jointly unitarily equivalent. In particular, we have $C(M_z \otimes W^*, I \otimes W) = 0$ and

$$\sigma(M_z \otimes W^*, I \otimes W) = \overline{\mathbb{D}} \times \sigma(W) \tag{3.2}$$

for any unitary W.

We proceed towards the structure of an arbitrary pair (V_1, V_2) with $C(V_1, V_2) = 0$. A pair of commuting isometries (V_1, V_2) is called *doubly commuting* if V_1 commutes with V_2^*. The following lemma is proved in [13] and [16], it is relating positivity of the defect operator $C(V_1, V_2)$ with double commutativity of the pair (V_1, V_2). We give a short proof here.

Lemma 3.2 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) $C(V_1, V_2) \geq 0$.
(b) (V_1, V_2) *is doubly commuting.*
(c) *If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then* $U(\operatorname{ran} P) \subseteq \operatorname{ran} P$.

Proof Since commuting unitaries are always doubly commuting, it is enough to prove the equivalences when $(V_1, V_2) = (M_{\varphi_1}, M_{\varphi_2})$. By virtue of (1.4), we have $C(V_1, V_2) \geq 0$ if and only if $U^*PU \geq P$ which happens if and only if ran P is invariant under U. Now,

$$\begin{aligned} M_{\varphi_1} M_{\varphi_2}^* = M_{\varphi_2}^* M_{\varphi_1} &\text{ if and only if } (I - M_z M_z^*) \otimes (U^* P U^* P^{\perp}) = 0 \\ &\text{ if and only if } P^{\perp} U P = 0 \\ &\text{ if and only if } \operatorname{ran} P \text{ is invariant under } U. \end{aligned}$$

This completes the proof. □

We recall some characterization results from [13] and add a few new ones.

Lemma 3.3 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) $C(V_1, V_2) = 0$.
(b) $\ker V_2^*$ *is a reducing subspace for V_1 and $V_1|_{\ker V_2^*}$ is a unitary.*
(c) $\ker V_1^*$ *is a reducing subspace for V_2 and $V_2|_{\ker V_1^*}$ is a unitary.*
(d) The fringe operators F_1 and F_2 are unitaries.
(e) $\ker V_1^*$ *and* $\ker V_2^*$ *are orthogonal and their direct sum is* $\ker V^*$.
(f) $(\operatorname{ran} V_1 \ominus \operatorname{ran} V) \oplus (\operatorname{ran} V_2 \ominus \operatorname{ran} V) \oplus \operatorname{ran} V = \mathcal{H}$.
(g) If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then $\operatorname{ran} P$ *reduces U.*

Proof The equivalences of (a), (b), (c), (d) and (e) follows easily from (2.2) and (2.3).

$(e) \Rightarrow (f)$: Suppose (e) is true. We shall show that $\ker V_1^* = (\operatorname{ran} V_2 \ominus \operatorname{ran} V)$. Suppose $x \in \ker V_1^*$, which implies $x \in \operatorname{ran} V_2$ and $x \in (\operatorname{ran} V)^{\perp}$. So $x \in (\operatorname{ran} V_2 \ominus \operatorname{ran} V)$. If $x \in (\operatorname{ran} V_2 \ominus \operatorname{ran} V)$, then $x \in \operatorname{ran} V_2$ and $x \in \ker V^*$. So $x \in \ker V_1^*$. Similarly, $\ker V_2^* = \operatorname{ran} V_1 \ominus \operatorname{ran} V$. Hence (f) is true.

$(f) \Rightarrow (e)$: Suppose (f) is true. Since $\operatorname{ran} V \subseteq \operatorname{ran} V_1$, we have $\operatorname{ran} V_1 = (\operatorname{ran} V_1 \ominus \operatorname{ran} V) \oplus \operatorname{ran} V$. Therefore $\ker V_1^* = (\operatorname{ran} V_1)^{\perp} = (\operatorname{ran} V_2 \ominus \operatorname{ran} V)$. Similarly, $\ker V_2^* = (\operatorname{ran} V_1 \ominus \operatorname{ran} V)$. Hence $\ker V_1^* \oplus \ker V_2^* = \ker V^*$.

$(a) \Leftrightarrow (g)$: We use the formula $C(V_1, V_2) = (E_0 \otimes (U^*PU - P)) \oplus 0$ from (1.4). This gives

$$\begin{aligned} C(V_1, V_2) = 0 &\text{ if and only if } U^*PU - P = 0 \\ &\text{ if and only if } \operatorname{ran} P \text{ reduces } U. \end{aligned}$$

Thus, completes the proof. □

We now write the structure theorem given in Popovici [18, Sec. 4], which highlights the importance of Example 3.1. We give a proof for the completeness.

Theorem 3.4 *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$ with defect zero. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Let $\mathcal{E}_1 = \operatorname{ran} P$ and $\mathcal{E}_2 = \operatorname{ran} P^{\perp}$. Then $\mathcal{E}_1, \mathcal{E}_2$ are reducing subspaces for U, i.e.,*

$$\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2,\ U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} I_{\mathcal{E}_1} & 0 \\ 0 & 0 \end{pmatrix} \text{ in } \mathcal{B}(\mathcal{E}_1 \oplus \mathcal{E}_2)$$

for some unitaries U_1 and U_2 on $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively.

Also, $\mathcal{H} = (H^2_{\mathbb{D}} \otimes \mathcal{E}_1) \oplus (H^2_{\mathbb{D}} \otimes \mathcal{E}_2) \oplus \mathcal{K}$ where $\mathcal{K} = \bigcap_{n \geq 0} \operatorname{ran}(V_1V_2)^n$ and in this decomposition,

$$V_1 = \begin{pmatrix} M_z \otimes I_{\mathcal{E}_1} & 0 & 0 \\ 0 & I_{H^2_{\mathbb{D}}} \otimes U_2^* & 0 \\ 0 & 0 & W_1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} I_{H^2_{\mathbb{D}}} \otimes U_1 & 0 & 0 \\ 0 & M_z \otimes I_{\mathcal{E}_2} & 0 \\ 0 & 0 & W_2 \end{pmatrix},$$

up to unitarily equivalence, for some unitary U_i on $\mathcal{E}_i, i = 1, 2$ and commuting unitaries W_1, W_2 on $\mathcal{K}$.

Proof Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). By Lemma 3.3, we have $\mathcal{E}_1$ reduces U and in the decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, we have

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} I_{\mathcal{E}_1} & 0 \\ 0 & 0 \end{pmatrix}.$$

In this case, $\varphi_1(z) = \begin{pmatrix} zU_1^* & 0 \\ 0 & U_2^* \end{pmatrix}$ and $\varphi_2(z) = \begin{pmatrix} U_1 & 0 \\ 0 & zU_2 \end{pmatrix}$, for $z \in \mathbb{D}$. Therefore,

$$M_{\varphi_1} = \begin{pmatrix} M_z \otimes U_1^* & 0 \\ 0 & I_{H^2_{\mathbb{D}}} \otimes U_2^* \end{pmatrix}, \; M_{\varphi_2} = \begin{pmatrix} I_{H^2_{\mathbb{D}}} \otimes U_1 & 0 \\ 0 & M_z \otimes U_2 \end{pmatrix}.$$

Note that $(M_z \otimes U_1^*, I_{H^2_{\mathbb{D}}} \otimes U_1)$ and $(M_z \otimes I_{\mathcal{E}_1}, I_{H^2_{\mathbb{D}}} \otimes U_1)$ are jointly unitarily equivalent and $(I_{H^2_{\mathbb{D}}} \otimes U_2^*, M_z \otimes U_2)$ and $(I_{H^2_{\mathbb{D}}} \otimes U_2^*, M_z \otimes I_{\mathcal{E}_2})$ are jointly unitarily equivalent; see Example 3.1. This completes the proof, by Theorem 1.2. □

Remark 3.5 In the structure theorem (Theorem 3.4) one can write the U_1, U_2 and $\mathcal{E}_1, \mathcal{E}_2$ explicitly in terms of V_1 and V_2 as follows: $\mathcal{E}_1 = \ker V_1^*, \mathcal{E}_2 = \ker V_2^*$ and $U_1 = V_2|_{\ker V_1^*}, U_2 = V_1^*|_{\ker V_2^*}$. This is because of Lemma 3.3.

A comment is in order. Over several decades Marek Słociński, first by himself and then with his collaborators, has developed a complete structure theorem on commuting pairs of isometries; see [5] and the references therein. One of his early results is the following.

Theorem 3.6 (M. Słociński [21]) *Let (V_1, V_2) be a pair of doubly commuting isometries on a Hilbert space $\mathcal{H}$. Then there exists a unique decomposition*

$$\mathcal{H} = \mathcal{H}_{ss} \oplus \mathcal{H}_{su} \oplus \mathcal{H}_{us} \oplus \mathcal{H}_{uu},$$

where the subspace $\mathcal{H}_{ij}$ reduces both V_1 and V_2 for all $i, j \in \{s, u\}$. Moreover, V_1 on $\mathcal{H}_{ij}$ is a shift if $i = s$ and unitary if $i = u$ and V_2 is a shift if $j = s$ and unitary if $j = u$.

By Theorem 3.4, and the fact that if one of the V_i's is a unitary then the defect is zero, we have $C(V_1, V_2) = 0$ if and only if (V_1, V_2) is doubly commuting and $\mathcal{H}_{ss}$ in Theorem 3.6 is $\{0\}$.

3.2 Joint Spectrum

If (V_1, V_2) is pure and defect $C(V_1, V_2) = 0$, then by Theorem 3.4 and Remark 3.5,

$$\sigma(V_1, V_2) = \begin{cases} \overline{\mathbb{D}} \times \sigma(U_1) & \text{if } V_2 \text{ is unitary,} \\ \sigma(U_2^*) \times \overline{\mathbb{D}} & \text{if } V_1 \text{ is unitary,} \\ \overline{\mathbb{D}} \times \sigma(U_1) \cup \sigma(U_2^*) \times \overline{\mathbb{D}} & \text{if neither } V_1 \text{ nor } V_2 \text{ is a unitary,} \end{cases} \tag{3.3}$$

where $U_1 = V_2|_{\ker V_1^*}, U_2 = V_1^*|_{\ker V_2^*}$.

Lemma 3.7 *Let (V_1, V_2) be a pair of commuting isometries with defect zero and* $\ker V^* \neq \{0\}$. *Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Let $U_1 = U|_{\operatorname{ran} P}$ and $U_2 = U|_{\operatorname{ran} P^\perp}$. Then*

$$\sigma(\varphi_1(z), \varphi_2(z)) = \begin{cases} \{(z\overline{\lambda}, \lambda) : \lambda \in \sigma(U_1)\} & \text{if } V_2 \text{ is a unitary,} \\ \{(\overline{\mu}, z\mu) : \mu \in \sigma(U_2)\} & \text{if } V_1 \text{ is a unitary,} \\ \{(z\overline{\lambda}, \lambda), (\overline{\mu}, z\mu) : \lambda \in \sigma(U_1),\ \mu \in \sigma(U_2)\} & \text{otherwise.} \end{cases} \tag{3.4}$$

Here, for every point in the joint spectrum, the non-singularity breaks at stage 3.

Proof The proof in the case when neither V_1 nor V_2 is a unitary is done below in detail. The other cases follow similarly.

Letting $\mathcal{E}_1 = \operatorname{ran} P$ and $\mathcal{E}_2 = \operatorname{ran} P^\perp$, it is an easy check that both $\mathcal{E}_1$ and $\mathcal{E}_2$ are non-trivial. By Lemma 3.3, $\mathcal{E}_1$ reduces U. Hence in the decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, we have

$$\varphi_1(z) = \begin{pmatrix} zU_1^* & 0 \\ 0 & U_2^* \end{pmatrix} \quad \text{and} \quad \varphi_2(z) = \begin{pmatrix} U_1 & 0 \\ 0 & zU_2 \end{pmatrix},$$

for $z \in \mathbb{D}$. Then,

$$\sigma(\varphi_1(z)) = \sigma(zU_1^*) \cup \sigma(U_2^*) \quad \text{and} \quad \sigma(\varphi_2(z)) = \sigma(U_1) \cup \sigma(zU_2)$$

for all $z \in \mathbb{D}$. For $z \in \mathbb{D}$, we have

$$\varphi_1(z)\varphi_2(z) = zI_{\mathcal{E}_1 \oplus \mathcal{E}_2} = \varphi_2(z)\varphi_1(z).$$

Therefore, for all $z \neq 0$, by Lemma 1.1 part (1) and polynomial spectral mapping theorem, we get

$$\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \{(z\overline{\lambda}, \lambda), (\overline{\mu}, z\mu) : \lambda \in \sigma(U_1),\ \mu \in \sigma(U_2)\}.$$

For $z = 0$, it is easy to see that

$$\sigma(\varphi_1(0), \varphi_2(0)) \subseteq \{(0, \lambda), (\overline{\mu}, 0) : \lambda \in \sigma(U_1),\ \mu \in \sigma(U_2)\}.$$

To prove the other containments, let $\lambda \in \sigma(U_1)$. Then $U_1 - \lambda I$ is not onto, because U_1 is normal. Let $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathcal{E}_1 \oplus \mathcal{E}_2$. Then,

$$(\varphi_1(z) - z\overline{\lambda}I)k + (\varphi_2(z) - \lambda I)h = \begin{pmatrix} z(U_1^* - \overline{\lambda}I)k_1 + (U_1 - \lambda I)h_1 \\ (U_2^* - z\overline{\lambda}I)k_2 + (zU_2 - \lambda I)h_2 \end{pmatrix}.$$

Since U_1 is a normal operator, $\operatorname{ran}(U_1 - \lambda I) = \operatorname{ran}(U_1^* - \overline{\lambda} I)$ and hence the first component of the above spans only $\operatorname{ran}(U_1 - \lambda I)$. Hence we have

$$\operatorname{ran}(\varphi_1(z) - z\overline{\lambda} I) + \operatorname{ran}(\varphi_2(z) - \lambda I) \neq \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Therefore $(z\overline{\lambda}, \lambda) \in \sigma(\varphi_1(z), \varphi_2(z))$.

Similarly, if $\mu \in \sigma(U_2)$, we have

$$\operatorname{ran}(\varphi_1(z) - \overline{\mu} I) + \operatorname{ran}(\varphi_2(z) - z\mu I) \neq \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Therefore $(\overline{\mu}, z\mu) \in \sigma(\varphi_1(z), \varphi_2(z))$. So,

$$\sigma(\varphi_1(z), \varphi_2(z)) = \{(z\overline{\lambda}, \lambda), (\overline{\mu}, z\mu) : \lambda \in \sigma(U_1),\ \mu \in \sigma(U_2)\} \text{ for } z \in \mathbb{D}.$$

□

Theorem 3.8 *With the hypothesis of Lemma 3.7, we also have*

$$\sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))} = \begin{cases} \overline{\mathbb{D}} \times \sigma(U_1) & \text{if } V_2 \text{ is a unitary,} \\ \sigma(U_2^*) \times \overline{\mathbb{D}} & \text{if } V_1 \text{ is a unitary,} \\ \overline{\mathbb{D}} \times \sigma(U_1) \cup \sigma(U_2^*) \times \overline{\mathbb{D}} & \text{otherwise.} \end{cases}$$

and, for every point $(z_1, z_2) \in \begin{cases} \mathbb{D} \times \sigma(U_1) & \text{if } V_2 \text{ is a unitary,} \\ \sigma(U_2^*) \times \mathbb{D} & \text{if } V_1 \text{ is a unitary,} \\ \mathbb{D} \times \sigma(U_1) \cup \sigma(U_2^*) \times \mathbb{D} & \text{otherwise,} \end{cases}$ *the non-singularity of* $K(M_{\varphi_1} - z_1 I, M_{\varphi_2} - z_2 I)$ *breaks at stage 3.*

Proof As in Lemma 3.7, we prove only the case when neither V_1 nor V_2 is a unitary, other cases follow similarly.

We saw in the proof of Lemma 3.7 that for any point $z \in \mathbb{D}$, a pair of points $(z_1, z_2) \in \sigma(\varphi_1(z), \varphi_2(z))$ if and only if

$$\operatorname{ran}(\varphi_1(z) - z_1 I) + \operatorname{ran}(\varphi_2(z) - z_2 I) \neq \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Hence, if $(z_1, z_2) \in \sigma(\varphi_1(z), \varphi_2(z))$,

$$\operatorname{ran}(M_{\varphi_1} - z_1 I) + \operatorname{ran}(M_{\varphi_2} - z_2 I) \neq H^2(\mathcal{E}_1 \oplus \mathcal{E}_2). \tag{3.5}$$

Therefore, $(z_1, z_2) \in \sigma(M_{\varphi_1}, M_{\varphi_2})$, which implies that

$$\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \sigma(M_{\varphi_1}, M_{\varphi_2}).$$

Note that

$$\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z)) = \cup_{z\in\mathbb{D}}\{(z\overline{\lambda}, \lambda), (\overline{\mu}, z\mu) : \lambda \in \sigma(U_1),\ \mu \in \sigma(U_2)\}$$

$$= \mathbb{D} \times \sigma(U_1) \cup \sigma(U_2^*) \times \mathbb{D}. \tag{3.6}$$

Since

$$M_{\varphi_1} = \begin{pmatrix} M_z \otimes U_1^* & 0 \\ 0 & I_{H^2_{\mathbb{D}}} \otimes U_2^* \end{pmatrix} \quad \text{and} \quad M_{\varphi_2} = \begin{pmatrix} I_{H^2_{\mathbb{D}}} \otimes U_1 & 0 \\ 0 & M_z \otimes U_2 \end{pmatrix},$$

we have

$$\sigma(M_{\varphi_1}, M_{\varphi_2}) = \sigma(M_z \otimes U_1^*, I_{H^2_{\mathbb{D}}} \otimes U_1) \cup \sigma(I_{H^2_{\mathbb{D}}} \otimes U_2^*, M_z \otimes U_2).$$

By (3.2), we have

$$\sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\mathbb{D}} \times \sigma(U_1) \cup \sigma(U_2^*) \times \overline{\mathbb{D}}. \tag{3.7}$$

Therefore from (3.6) and (3.7) we have

$$\sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\mathbb{D}} \times \sigma(U_1) \cup \sigma(U_2^*) \times \overline{\mathbb{D}} = \overline{\cup_{z \in \mathbb{D}} \sigma(\varphi_1(z), \varphi_2(z))}.$$

The final thing to note is that for every point (z_1, z_2) in the set $\mathbb{D} \times \sigma(U_1) \cup \sigma(U_2^*) \times \mathbb{D}$, the non-singularity breaks at stage 3 and this is a direct consequence of (3.5). □

To conclude the section, we note that by Theorem 1.2, $\sigma(V_1, V_2) = \sigma(M_{\varphi_1}, M_{\varphi_2}) \cup \sigma(V_1|_{\mathcal{H}_u}, V_2|_{\mathcal{H}_u})$. Hence Theorem 3.8 tells that

$$\overline{\cup_{z \in \mathbb{D}} \sigma(\varphi_1(z), \varphi_2(z))} \subseteq \sigma(V_1, V_2). \tag{3.8}$$

The equality in (3.8) holds if and only if $\sigma(V_1|_{\mathcal{H}_u}, V_2|_{\mathcal{H}_u}) \subseteq \sigma(M_{\varphi_1}, M_{\varphi_2})$.

4 The Negative Defect Case

4.1 The Prototypical Example

When the defect operator is negative, a fundamental example plays an important role in much the same way the unilateral shift plays its role in the Wold decomposition of a single isometry. We shall first describe this example and then show how it is a part of every pair of commuting isometries with negative defect.

The *Hardy space of $\mathcal{E}$-valued functions on the bidisc* $\mathbb{D}^2$ is

$$H^2_{\mathbb{D}^2}(\mathcal{E}) = \left\{ f : \mathbb{D}^2 \to \mathcal{E} \mid f \text{ is analytic and } f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n \right.$$
$$\left. \text{with } \sum_{m,n=0}^{\infty} \|a_{m,n}\|^2_{\mathcal{E}} < \infty \right\}.$$

This is a Hilbert space with the inner product

$$\left\langle \sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n, \sum_{m,n=0}^{\infty} b_{m,n} z_1^m z_2^n \right\rangle = \sum_{m,n=0}^{\infty} \langle a_{m,n}, b_{m,n} \rangle_{\mathcal{E}}$$

and is identifiable with $H^2_{\mathbb{D}^2} \otimes \mathcal{E}$ where $H^2_{\mathbb{D}^2}$ stands for the Hardy space of scalar-valued functions on $\mathbb{D}^2$.

Let $U : H^2_{\mathbb{D}^2} \to H^2_{\mathbb{D}^2}$ be the unitary defined by

$$U(z_1^{m_1} z_2^{m_2}) = \begin{cases} z_1^{m_1+2} z_2^{m_2} & \text{if } m_1 \geq m_2, \\ z_1^{m_1+1} z_2^{m_2-1} & \text{if } m_1 + 1 = m_2, \\ z_1^{m_1} z_2^{m_2-2} & \text{if } m_1 + 2 \leq m_2. \end{cases} \tag{4.1}$$

on the orthonormal basis $\{z_1^{m_1} z_2^{m_2}\}_{m_1,m_2 \geq 0}$.

The pair of multipliers by the coordinate functions (M_{z_1}, M_{z_2}) forms a pair of doubly commuting isometries on $H^2_{\mathbb{D}^2}$. There is a natural isomorphism between the Hilbert spaces $H^2_{\mathbb{D}^2}$ and $H^2_{\mathbb{D}} \otimes H^2_{\mathbb{D}}$ wherein $z_1^{m_1} z_2^{m_2}$ is identified with $z_1^{m_1} \otimes z_2^{m_2}$. In this identification, the pair of coordinate multipliers (M_{z_1}, M_{z_2}) is identified with $(M_z \otimes I, I \otimes M_z)$.

Definition 4.1 The pair of bounded operators $\tau_1 := U^* M_{z_1}$ and $\tau_2 := M_{z_2} U$ on the Hardy space of the bidisc $H^2_{\mathbb{D}^2}$ will be called the fundamental isometric pair with negative defect.

The following lemma justifies the name except the word $fundamental$ which will be clear from Theorem 4.11.

Lemma 4.2 *The pair* (τ_1, τ_2) *is a pair of commuting isometries with defect negative and non-zero.*

Proof It is simple to check that the unitary U defined in (4.1) commutes with $M_{z_1 z_2}$, the operator of multiplication by the function $z_1 z_2$. That proves commutativity of τ_1 and τ_2. They are isometries because each is a product of an isometry and a unitary. Now, let

$$\mathcal{W}_1 = \ker(\tau_1^*) = \overline{\text{span}}\{z_2^2, z_2^3, z_2^4, \dots\} \quad \text{and} \quad \mathcal{W}_2 = \ker(\tau_2^*) = \overline{\text{span}}\{1, z_1, z_1^2, \dots\}.$$

Then, $\tau_2(\mathcal{W}_1) = \overline{\text{span}}\{z_2, z_2^2, z_2^3, \dots\}$. Thus, we have

$$C(\tau_1, \tau_2) = P_{\mathcal{W}_1} - P_{\tau_2(\mathcal{W}_1)} = -P_{\text{span}\{z_2\}} \leq 0.$$

That completes the proof. □

We shall prove the following lemma to compute the joint spectrum of (τ_1, τ_2).

Lemma 4.3 *For any $\lambda \in \mathbb{D}$, we have*

$$(\text{ran}(\tau_2 - \lambda I))^{\perp} = \left\{ (I - \bar{\lambda}\tau_2)^{-1} x : x \in \ker M_{z_2}^* \right\}. \tag{4.2}$$

Proof Using the Neumann series $(I - \bar{\lambda}A)^{-1} = \sum_{n \geq 0} \bar{\lambda}^n A^n$ for $\lambda \in \mathbb{D}$ and any contraction A, it is straightforward that the equality

$$(\text{ran}(A - \lambda I))^{\perp} = \{(I - \bar{\lambda}A)^{-1} x : x \in \ker A^*\}$$

is satisfied when A is an isometry. The proof is complete by noting that $\ker \tau_2^* = \ker U^* M_{z_2}^* = \ker M_{z_2}^*$. □

Recall the Koszul complex for a pair of commuting bounded operators (T_1, T_2) from (1.2):

$$0 \xrightarrow{\delta_0} \mathcal{H} \xrightarrow{\delta_1} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\delta_2} \mathcal{H} \xrightarrow{\delta_3} 0. \tag{4.3}$$

It is well-known that the most difficult stage to treat for the purpose of showing lack of exactness is the stage 2.

Proposition 4.4 *The fundamental isometric pair with negative defect has the full closed bidisc $\overline{\mathbb{D}^2}$ as its joint spectrum. Moreover, for every point in the open bidisc $\mathbb{D}^2$, the non-singularity breaks at stage 2.*

Proof Let $\lambda_1, \lambda_2 \in \mathbb{D}$. We shall find a non-zero function $h_2 \in (\text{ran}(\tau_2 - \lambda_2 I))^{\perp}$ such that

$$(\tau_1 - \lambda_1 I) h_2 \in \text{ran}(\tau_2 - \lambda_2 I). \tag{4.4}$$

This would imply that there exists $h_1 \in \mathcal{H} = H^2_{\mathbb{D}^2}$ such that

$$(\tau_1 - \lambda_1 I) h_2 = (\tau_2 - \lambda_2 I) h_1$$

producing a pair (h_1, h_2) in $\ker \delta_2$ which would not be in $\text{ran}\, \delta_1$.

To that end, we shall use the description of $(\text{ran}(\tau_2 - \lambda I))^{\perp}$ obtained in Lemma 4.3. Since any element from $\ker \tau_2^* = \ker M_{z_2}^*$ is of the form $\sum_{m=0}^{\infty} a_m z_1^m$ for a square summable sequence $\{a_m\}_{m \geq 0}$, it follows from Lemma 4.3 that

$$\begin{aligned} (\text{ran}(\tau_2 - \lambda_2 I))^{\perp} &= \left\{ (I - \overline{\lambda_2}\tau_2)^{-1} \sum_{m=0}^{\infty} a_m z_1^m : \sum_{m=0}^{\infty} |a_m|^2 < \infty \right\} \\ &= \left\{ \sum_{n=0}^{\infty} (\overline{\lambda_2} M_{z_2} U)^n \left(\sum_{m=0}^{\infty} a_m z_1^m \right) : \sum_{m=0}^{\infty} |a_m|^2 < \infty \right\} \\ &= \left\{ \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n : \sum_{m=0}^{\infty} |a_m|^2 < \infty \right\}. \end{aligned} \tag{4.5}$$

Our candidate for h_2 to satisfy (4.4) is

$$h_2 = \frac{1}{(1-\overline{\lambda_2} z_1^2 z_2)(1-\lambda_1 z_1)} = \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n \lambda_1^m z_1^{m+2n} z_2^n.$$

By (4.5), this function is in $(\operatorname{ran}(\tau_2 - \lambda_2 I))^\perp$. We shall verify below that $(\tau_1 - \lambda_1 I)h_2$ is in the closure of $\operatorname{ran}(\tau_2 - \lambda_2 I)$. Since $|\lambda_2| < 1$ and τ_2 is an isometry, $(\tau_2 - \lambda_2 I)$ is bounded below and hence its range is closed. That will complete the proof.

First note that

$$U \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n = \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n+2} z_2^n = M_{z_1}^2 \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \tag{4.6}$$

and

$$\begin{aligned}(M_{z_1}^* - \lambda_1 I)h_2 &= M_{z_1}^*(h_2) - \lambda_1 h_2 \\ &= \sum_{\substack{m,n=0 \\ (m,n)\neq(0,0)}}^{\infty} \overline{\lambda_2}^n \lambda_1^m z_1^{m+2n-1} z_2^n - \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n \lambda_1^{m+1} z_1^{m+2n} z_2^n \\ &= \sum_{m=1,n=0}^{\infty} \overline{\lambda_2}^n \lambda_1^m z_1^{m+2n-1} z_2^n + \sum_{n=1}^{\infty} \overline{\lambda_2}^n z_1^{2n-1} z_2^n - \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n \lambda_1^{m+1} z_1^{m+2n} z_2^n \\ &= \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n \lambda_1^{m+1} z_1^{m+2n} z_2^n + \sum_{n=1}^{\infty} \overline{\lambda_2}^n z_1^{2n-1} z_2^n - \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n \lambda_1^{m+1} z_1^{m+2n} z_2^n \\ &= \sum_{n=1}^{\infty} \overline{\lambda_2}^n z_1^{2n-1} z_2^n. \end{aligned} \tag{4.7}$$

We now compute the inner product between a typical element of $(\operatorname{ran}(\tau_2 - \lambda_2 I))^\perp$ and $(\tau_1 - \lambda_1 I)h_2$ by using the two equations above.

$$\begin{aligned}&\left\langle (\tau_1 - \lambda_1 I)h_2, \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle \\ &= \left\langle (U^* M_{z_1} - \lambda_1 I)h_2, \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle \\ &= \left\langle M_{z_1} h_2, U \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle - \lambda_1 \left\langle h_2, \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle \\ &= \left\langle M_{z_1} h_2, M_{z_1}^2 \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle - \lambda_1 \left\langle h_2, \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle \end{aligned}$$

$$= \left\langle (M_{z_1}^* - \lambda_1 I)h_2, \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle$$
$$= \left\langle \sum_{n=1}^{\infty} \overline{\lambda_2}^n z_1^{2n-1} z_2^n, \sum_{m,n=0}^{\infty} \overline{\lambda_2}^n a_m z_1^{m+2n} z_2^n \right\rangle = 0.$$

This shows that $(\tau_1 - \lambda_1 I)h_2 \in \overline{\text{ran}}(\tau_2 - \lambda_2 I) = \text{ran}(\tau_2 - \lambda_2 I)$ and hence completes the proof.

□

Note 4.5 *In Remark 5.4, we shall see that there is a joint invariant subspace $\mathcal{M}$ for (τ_1, τ_2) such that the defect operator of $(\tau_1|_{\mathcal{M}}, \tau_2|_{\mathcal{M}})$ is positive (and not zero).*

4.2 General Theory for the Negative Defect Case

Here we shall show that the fundamental example above is a typical example. This helps us to compute the joint spectrum of any commuting pair of isometries with negative defect.

In [11], Gaspar and Gaspar introduced the dual doubly commuting pairs. If $(\bar{V}_1, \bar{V}_2)$ is the *minimal unitary extension* to $\bar{\mathcal{H}}$ of (V_1, V_2) acting on $\mathcal{H}$, then the pair of commuting isometries $(\bar{V_1}^*|_{\bar{\mathcal{H}}\ominus\mathcal{H}}, \bar{V_2}^*|_{\bar{\mathcal{H}}\ominus\mathcal{H}})$ is called the *dual* of (V_1, V_2). If the dual is doubly commuting, then (V_1, V_2) is called a *dual doubly commuting* pair.

A pair (V_1, V_2) of commuting isometries is called a *bi-shift* (see [19]) if there is a wandering subspace $\mathcal{R}$ (i.e., $V_1^{p_1} V_2^{p_2}(\mathcal{R}) \perp V_1^{q_1} V_2^{q_2}(\mathcal{R})$ if $(p_1, p_2), (q_1, q_2) \in \mathbb{Z}_+^2$ and $(p_1, p_2) \neq (q_1, q_2)$) such that

$$\mathcal{H} = \bigoplus_{(n_1,n_2)\in\mathbb{Z}_+^2} V_1^{n_1} V_2^{n_2}(\mathcal{R}).$$

(V_1, V_2) is called a *modified bi-shift* if it is pure and its dual is a bi-shift.

Popovici used the concepts above greatly in his papers [19] and [20]. We are thankful to him for sending us his papers. First we shall give a characterizing lemma for this case; see also [13].

Lemma 4.6 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) $C(V_1, V_2) \leq 0$ *and* $C(V_1, V_2) \neq 0$.
(b) $V_2(\ker V_1^*) \supsetneq \ker V_1^*$.
(c) $V_1(\ker V_2^*) \supsetneq \ker V_2^*$.
(d) The adjoint of the fringe operators are isometries and not unitaries.
(e) (V_1, V_2) *is dual doubly commuting and* $C(V_1, V_2) \neq 0$.
(f) $\ker V_1^*$ *is orthogonal to* $\ker V_2^*$ *and* $\ker V_1^* \oplus \ker V_2^* \neq \ker V^*$.
(g) $C(V_1, V_2)$ *is the negative of a non-zero projection.*
(h) If $(\mathcal{E}, P, U)$ *is the BCL triple for* (V_1, V_2), *then* $U(\text{ran } P^\perp) \subsetneq \text{ran } P^\perp$.

Proof The equivalence $(e) \Leftrightarrow (h)$ is proved in [11]. All other proof are along the same lines as the proofs of various parts of Lemma 3.3. □

The geometrical structure and a model for dual doubly commuting isometries is known due to [11, 18]. Here we observe that the multiplication operators $M_{\varphi_i}, i = 1, 2$ associated to (V_1, V_2) as in Theorem 1.2, have some special forms in this case. This also helps us getting a model in the Hardy space of the bidisc. Some steps of the proof are used to obtain Theorem 4.15. The wandering space arguments used in the proof of the following theorem, appears in [18, Thm 4.3].

Let $\psi_1, \psi_2 : \mathbb{D} \to \mathcal{B}(l^2(\mathbb{Z}))$ be the multipliers associated with the BCL triple $(l^2(\mathbb{Z}), p_-, \omega)$ where p_- is the projection onto $\overline{\text{span}}\{e_n : n < 0\}$ and ω is the bilateral shift on $l^2(\mathbb{Z})$.

Theorem 4.7 *Let (V_1, V_2) be a pair of commuting isometries such that $C(V_1, V_2)$ is a non-zero negative operator. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Then, up to unitary equivalence*

$$\mathcal{E} = (l^2(\mathbb{Z}) \otimes \mathcal{L}) \oplus \mathcal{E}_2$$

for some non-trivial closed subspace $\mathcal{L}$ and a closed subspace $\mathcal{E}_2$ of $\mathcal{E}$. Moreover,

$$M_{\varphi_i} = \begin{array}{c} \\ \end{array} \begin{matrix} H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L} & H^2_{\mathbb{D}}(\mathcal{E}_2) & \\ \end{matrix} \atop \begin{pmatrix} M_{\psi_i} \otimes I_{\mathcal{L}} & 0 \\ 0 & M_{\varphi_i}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)} \end{pmatrix} \begin{matrix} H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L} \\ H^2_{\mathbb{D}}(\mathcal{E}_2) \end{matrix} \tag{4.8}$$

with $C(M_{\varphi_1}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}, M_{\varphi_2}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}) = 0$. In particular,

$$\sigma(M_{\psi_1}, M_{\psi_2}) = \sigma(M_{\psi_1} \otimes I_{\mathcal{L}}, M_{\psi_2} \otimes I_{\mathcal{L}}) \subseteq \sigma(M_{\varphi_1}, M_{\varphi_2}) \subseteq \sigma(V_1, V_2). \tag{4.9}$$

Proof Since $C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$, by Lemma 4.6, $U(\text{ran } P^{\perp}) \subsetneq \text{ran } P^{\perp}$. Consider $\mathcal{L} := (\text{ran } P^{\perp} \ominus U(\text{ran } P^{\perp})) \neq \{0\}$. Let $m, n \in \mathbb{Z}$ and $m > n$. Then for $x, y \in \mathcal{L}$ we have

$$\langle U^m x, U^n y\rangle = \langle U^{m-n} x, y\rangle = 0,$$

because $U^{m-n} x \in U(\text{ran } P^{\perp})$. Therefore $U^m(\mathcal{L}) \perp U^n(\mathcal{L})$ if $m, n \in \mathbb{Z}$ and $m \neq n$.

Set $\mathcal{E}_1 := \oplus_{n\in\mathbb{Z}} U^n(\mathcal{L})$. Clearly U reduces $\mathcal{E}_1$. Now $U(\text{ran } P^{\perp}) \subsetneq \text{ran } P^{\perp}$ implies:

$$\oplus_{n\geq 0}\, U^n(\mathcal{L}) \subseteq \text{ran } P^{\perp}. \tag{4.10}$$

For all $x \in \mathcal{L}$, $y \in \text{ran } P^{\perp}$ and $n < 0$, $\langle U^n x, y\rangle = \langle x, U^{-n} y\rangle = 0$ implies that

$$\oplus_{n<0}\, U^n(\mathcal{L}) \subseteq \text{ran } P. \tag{4.11}$$

It is clear from inclusions (4.10) and (4.11) that P also reduces $\mathcal{E}_1$. Now $U|_{\mathcal{E}_1} = W_{\mathcal{L}}$, the bilateral shift on $\mathcal{E}_1$ with the wandering subspace $\mathcal{L}$ and $P|_{\mathcal{E}_1} = P_{\mathcal{E}_1^-}$, the projection on $\mathcal{E}_1^- := \oplus_{n<0} U^n(\mathcal{L})$ in $\mathcal{E}_1$.

Let $\mathcal{E} = \ker V^* = \mathcal{E}_1 \oplus \mathcal{E}_2$. In this decomposition we have,

$$U = \begin{pmatrix} W_{\mathcal{L}} & 0 \\ 0 & U|_{\mathcal{E}_2} \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} P_{\mathcal{E}_1^-} & 0 \\ 0 & P|_{\mathcal{E}_2} \end{pmatrix}.$$

Let $\Theta : \mathcal{E}_1 \to l^2(\mathbb{Z}) \otimes \mathcal{L}$ be the unitary given by

$$\Theta(U^n x) = e_n \otimes x \text{ for } x \in \mathcal{L}, n \in \mathbb{Z}.$$

Then, with this identification (U, P) is jointly unitarily equivalent to

$$\left(\begin{pmatrix} \omega \otimes I_{\mathcal{L}} & 0 \\ 0 & U|_{\mathcal{E}_2} \end{pmatrix}, \begin{pmatrix} p_- \otimes I_{\mathcal{L}} & 0 \\ 0 & P|_{\mathcal{E}_2} \end{pmatrix} \right)$$

where ω is the bilateral shift on $l^2(\mathbb{Z})$ and p_- is the projection in $l^2(\mathbb{Z})$ onto $\overline{\text{span}}\{e_n : n < 0\}$. Therefore,

$$\begin{aligned} \varphi_1(z) &= \begin{pmatrix} \omega^* \otimes I_{\mathcal{L}} & 0 \\ 0 & U^*|_{\mathcal{E}_2} \end{pmatrix} \left[\begin{pmatrix} p_-^\perp \otimes I_{\mathcal{L}} & 0 \\ 0 & P^\perp|_{\mathcal{E}_2} \end{pmatrix} + z \begin{pmatrix} p_- \otimes I_{\mathcal{L}} & 0 \\ 0 & P|_{\mathcal{E}_2} \end{pmatrix} \right] \\ &= \begin{pmatrix} \omega^*(p_-^\perp + zp_-) \otimes I_{\mathcal{L}} & 0 \\ 0 & U^*(P^\perp + zP)|_{\mathcal{E}_2} \end{pmatrix} = \begin{pmatrix} \psi_1(z) \otimes I_{\mathcal{L}} & 0 \\ 0 & \varphi_1(z)|_{\mathcal{E}_2} \end{pmatrix}, \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \varphi_2(z) &= \left[\begin{pmatrix} p_- \otimes I_{\mathcal{L}} & 0 \\ 0 & P|_{\mathcal{E}_2} \end{pmatrix} + z \begin{pmatrix} p_-^\perp \otimes I_{\mathcal{L}} & 0 \\ 0 & P^\perp|_{\mathcal{E}_2} \end{pmatrix} \right] \begin{pmatrix} \omega \otimes I_{\mathcal{L}} & 0 \\ 0 & U|_{\mathcal{E}_2} \end{pmatrix} \\ &= \begin{pmatrix} (p_- + zp_-^\perp)\omega \otimes I_{\mathcal{L}} & 0 \\ 0 & (P + zP^\perp)U|_{\mathcal{E}_2} \end{pmatrix} = \begin{pmatrix} \psi_2(z) \otimes I_{\mathcal{L}} & 0 \\ 0 & \varphi_2(z)|_{\mathcal{E}_2} \end{pmatrix}, \end{aligned} \tag{4.13}$$

where $p_-^\perp$ denotes the projection $I_{l^2(\mathbb{Z})} - p_-$ in $l^2(\mathbb{Z})$. This in particular says that $\mathcal{E}_1$ reduces $\varphi_i(z)$ for $z \in \mathbb{D}$, $i = 1, 2$. Therefore $H^2_{\mathbb{D}}(\mathcal{E}_1)$ reduces M_{φ_i}, for $i = 1, 2$, and

$$M_{\varphi_i} = \begin{pmatrix} M_{\psi_i \otimes I_{\mathcal{L}}} & 0 \\ 0 & M_{\varphi_i|_{\mathcal{E}_2}} \end{pmatrix} = \begin{pmatrix} M_{\psi_i} \otimes I_{\mathcal{L}} & 0 \\ 0 & M_{\varphi_i}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)} \end{pmatrix}, \ i = 1, 2, \tag{4.14}$$

where

$$\psi_1(z) = \omega^*(p_-^\perp + zp_-), \ \psi_2(z) = (p_- + zp_-^\perp)\omega$$

and $\varphi_i|_{\mathcal{E}_2}(z) := \varphi_i(z)|_{\mathcal{E}_2} \in \mathcal{B}(\mathcal{E}_2)$, for $z \in \mathbb{D}$ and $i = 1, 2$.

It remains to prove that $C(M_{\varphi_1}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}, M_{\varphi_2}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}) = 0$. To that end, note that by Lemma 3.3, it is enough to show that $U(\text{ran } P|_{\mathcal{E}_2}) = \text{ran } P|_{\mathcal{E}_2}$. Notice from (4.10) and (4.11) that $\text{ran } P|_{\mathcal{E}_2} = \text{ran } P \ominus (\oplus_{n<0} U^n(\mathcal{L}))$. Hence

$$\begin{aligned} U(\text{ran } P|_{\mathcal{E}_2}) &= U(\text{ran } P \ominus (\oplus_{n<0} U^n(\mathcal{L}))) \\ &= U(\text{ran } P) \ominus (\oplus_{n<0} U^{n+1}(\mathcal{L})) \\ &= (U(\text{ran } P^{\perp}))^{\perp} \ominus (\mathcal{L} \oplus_{n<0} U^n(\mathcal{L})) \\ &= (\text{ran } P \oplus \mathcal{L}) \ominus (\mathcal{L} \oplus_{n<0} U^n(\mathcal{L})) \quad (\because \text{ran } P^{\perp} = \mathcal{L} \oplus U(\text{ran } P^{\perp})) \\ &= \text{ran } P \ominus (\oplus_{n<0} U^n(\mathcal{L})) = \text{ran } P|_{\mathcal{E}_2}. \end{aligned}$$

It remains only to prove (4.9), and it follows directly from (4.8) by Lemma 1.1. □

Lemma 4.8 *The pair (M_{ψ_1}, M_{ψ_2}) is jointly unitarily equivalent to (τ_1, τ_2). In particular,*

$$\sigma(M_{\psi_1}, M_{\psi_2}) = \overline{\mathbb{D}^2}$$

and for every point in $\mathbb{D}^2$, the non-singularity breaks at stage 2.

Proof Define the unitary $\Lambda : H^2_{\mathbb{D}^2} \to H^2_{\mathbb{D}}(l^2(\mathbb{Z}))$ by

$$\Lambda\left(\sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n\right) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{m+k,k} e_m + \sum_{m=1}^{\infty} a_{k,m+k} e_{-m}\right) z^k. \quad (4.15)$$

Then, $\Lambda \tau_i \Lambda^* = M_{\psi_i}$ for $i = 1, 2$. That is, (M_{ψ_1}, M_{ψ_2}) is jointly unitarily equivalent to (τ_1, τ_2). In particular,

$$\sigma(M_{\psi_1}, M_{\psi_2}) = \sigma(\tau_1, \tau_2) = \overline{\mathbb{D}^2} \quad (4.16)$$

and for every point in $\mathbb{D}^2$, the non-singularity breaks at stage 2 by Proposition 4.4. □

We shall use the following lemma proved in [11] and [9].

Lemma 4.9 *Let $(\mathcal{E}, P, U)$ be a BCL triple. Then the pair $(M_{\varphi_1}, M_{\varphi_2})$ has a non-trivial joint reducing subspace if and only if (P, U) has a non-trivial joint reducing subspace.*

Lemma 4.10 *The pair (τ_1, τ_2) does not have any non-trivial joint reducing subspace.*

Proof By Lemma 4.9, (τ_1, τ_2) has a non-trivial joint reducing subspace if and only if (p_-, ω) has a non-trivial joint reducing subspace in $l^2(\mathbb{Z})$. Let $\mathcal{E}_0 \neq \{0\}$ be a joint reducing subspace for (p_-, ω). Let $f = \sum_{n\in\mathbb{Z}} a_n e_n \in \mathcal{E}_0$, $a_{n_0} \neq 0$ for some $n_0 \in \mathbb{Z}$. Now

$$\omega^{*n_0}(f) = \sum_{n\in\mathbb{Z}} a_n e_{n-n_0}, \;\; p_-\omega^{*n_0}(f) = \sum_{n<n_0} a_n e_{n-n_0} \in \mathcal{E}_0.$$

Hence $\sum_{n\geq n_0} a_n e_{n-n_0} \in \mathcal{E}_0$ and

$$p_-\omega^*\left(\sum_{n\geq n_0} a_n e_{n-n_0}\right) = a_{n_0}e_{-1} \in \mathcal{E}_0.$$

Since $\mathcal{E}_0$ is reducing for ω and $e_{-1} \in \mathcal{E}_0$ we have $\mathcal{E}_0 = l^2(\mathbb{Z})$. □

The following theorem on the structure and the joint spectrum of commuting pair of isometries with negative defect follows directly from Theorem 1.2, Theorem 4.7, Lemma 1.1 and Lemma 4.8 and shows that when the defect of (V_1, V_2) is negative, then apart from a reduced part which has defect zero, (V_1, V_2) is the fundamental isometric pair with negative defect, albeit with a higher multiplicity.

Theorem 4.11 *Let (V_1, V_2) be a pair of commuting isometries such that $C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$. Then, there is a non-trivial subspace $\mathcal{L} \subsetneq \ker V^*$ such that, up to unitary equivalence, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$, where $\mathcal{H}_0 = H^2_{\mathbb{D}^2} \otimes \mathcal{L}$ and in this decomposition*

$$V_i = \begin{pmatrix} \tau_i \otimes I_{\mathcal{L}} & 0 \\ 0 & V_i|_{\mathcal{H}_0^\perp} \end{pmatrix}, i = 1, 2 \quad \textit{and} \quad C(V_1|_{\mathcal{H}_0^\perp}, V_2|_{\mathcal{H}_0^\perp}) = 0,$$

where the dimension of $\mathcal{L}$ is same as the dimension of the range of $C(V_1, V_2)$ and (τ_1, τ_2) is the fundamental isometric pair with negative defect. Moreover,

$$\sigma(V_1, V_2) = \overline{\mathbb{D}^2}$$

and for every point in $\mathbb{D}^2$, the non-singularity breaks at stage 2.

The pair (τ_1, τ_2) serves as a model in many ways.

Theorem 4.12 *A pair of commuting isometries (V_1, V_2) is a modified bi-shift if and only if (V_1, V_2) is jointly unitarily equivalent to $(\tau_1 \otimes I_{\mathcal{L}}, \tau_2 \otimes I_{\mathcal{L}})$ for some Hilbert space $\mathcal{L}$.*

Proof Let $(V_1, V_2) = (\tau_1 \otimes I_{\mathcal{L}}, \tau_2 \otimes I_{\mathcal{L}})$. Then

$$\ker V_1^* = \overline{\text{span}}\{z_2^2, z_2^3, \dots\} \otimes \mathcal{L}, \ \ker V_2^* = \overline{\text{span}}\{1, z_1, z_1^2, \dots\} \otimes \mathcal{L}.$$

Clearly $V_1^*|_{\ker V_2^*}$, $V_2^*|_{\ker V_1^*}$ and V_1V_2 are shifts. Therefore, by [20, Prop. 3.8] (V_1, V_2) is a modified bi-shift.

Suppose (V_1, V_2) is a modified bi-shift. If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then by [11, Thm. 2.4], $U(\text{ran } P^\perp) \subsetneq \text{ran } P^\perp$. Hence by Lemma 4.6, $C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$. Therefore, by Theorem 4.11, up to unitary equivalence, V_1 and V_2 are

$$V_1 = \begin{pmatrix} \tau_1 \otimes I_{\mathcal{L}} & 0 & 0 \\ 0 & M_z \otimes I_{\mathcal{E}_1} & 0 \\ 0 & 0 & I_{H^2_{\mathbb{D}}} \otimes U_2 \end{pmatrix}, V_2 = \begin{pmatrix} \tau_2 \otimes I_{\mathcal{L}} & 0 & 0 \\ 0 & I_{H^2_{\mathbb{D}}} \otimes U_1 & 0 \\ 0 & 0 & M_z \otimes I_{\mathcal{E}_2} \end{pmatrix}.$$

By [20, Prop. 3.8] we have $\mathcal{E}_1 = \mathcal{E}_2 = \{0\}$. Hence (V_1, V_2) is jointly unitarily equivalent to $(\tau_1 \otimes I_{\mathcal{L}}, \tau_2 \otimes I_{\mathcal{L}})$. □

Remark 4.13 $(l^2(\mathbb{Z}) \otimes \mathcal{L}, p_- \otimes I_{\mathcal{L}}, \omega \otimes I_{\mathcal{L}})$ is the BCL triple for a modified bi-shift (V_1, V_2), where $\dim \mathcal{L} = \dim(\operatorname{ran} C(V_1, V_2))$. (See also [11, Thm. 2.4]).

Some other descriptions of modified bi-shift are given in [11], by Gaspar and Gaspar. From the results in [11], it is clear that a dual doubly commuting pair is a direct sum of a zero defect part and a modified bi-shift. (caution: the W in [11] is U^* for us).

In the rest of this section we shall see the relation between the joint spectrum of a pair of commuting isometries (V_1, V_2) with defect negative and non-zero, and the joint spectrum of $(\varphi_1(z), \varphi_2(z))$, $z \in \mathbb{D}$, where φ_i's are the multipliers given in the BCL representation of (V_1, V_2). Indeed, we prove that

$$\sigma(V_1, V_2) = \sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\cup_{z \in \mathbb{D}} \sigma(\varphi_1(z), \varphi_2(z))} = \overline{\mathbb{D}^2}. \tag{4.17}$$

Lemma 4.14 *Let the operator valued functions ψ_1 and ψ_2 be as in Theorem 4.7. Then, For $z \in \mathbb{D}$, $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ is a joint eigenvalue for $(\psi_1(z), \psi_2(z))$ if $\lambda_1\lambda_2 = z$. Also*

$$\sigma(\psi_1(z), \psi_2(z)) = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\}, \tag{4.18}$$

and

$$\overline{\cup_{z \in \mathbb{D}} \sigma(\psi_1(z), \psi_2(z))} = \overline{\mathbb{D}^2} = \sigma(M_{\psi_1}, M_{\psi_2}). \tag{4.19}$$

Proof Let $z \in \mathbb{D} \setminus \{0\}$ and $\lambda_1, \lambda_2 \in \mathbb{D}$ be such that $\lambda_1\lambda_2 = z$. Consider

$$x_{\lambda_1,\lambda_2} = \sum_{n=0}^{\infty} \lambda_1^n e_n + \frac{1}{\lambda_1} \sum_{n=1}^{\infty} \lambda_2^{n-1} e_{-n}.$$

Note that

$$\psi_i(z) x_{\lambda_1,\lambda_2} = \lambda_i x_{\lambda_1,\lambda_2},\, i = 1, 2.$$

That is, (λ_1, λ_2) is a joint eigenvalue for $(\psi_1(z), \psi_2(z))$ with eigenvector x_{λ_1,λ_2}, where $z \neq 0$ and $\lambda_1, \lambda_2 \in \mathbb{D}$ are such that $\lambda_1\lambda_2 = z$. Therefore,

$$\sigma(\psi_1(z), \psi_2(z)) \supseteq \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\} \text{ for } z \neq 0.$$

As $\psi_1(z)$ and $\psi_2(z)$ are commuting contractions, $\sigma(\psi_1(z), \psi_2(z)) \subseteq \overline{\mathbb{D}^2}$. Since $\psi_1(z)\psi_2(z) = zI$, by spectral mapping theorem, we have the other inclusion. Hence

$$\sigma(\psi_1(z), \psi_2(z)) = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\} \text{ for } z \neq 0. \tag{4.20}$$

For the case $z = 0$, consider for $\lambda_1, \lambda_2 \in \mathbb{D} \setminus \{0\}$,

$$x_{\lambda_1,0} = \sum_{n=0}^{\infty} \lambda_1^n e_n + \frac{1}{\lambda_1} e_{-1},\ x_{0,\lambda_2} = \sum_{n=1}^{\infty} \lambda_2^{n-1} e_{-n} \quad \text{and} \quad x_{0,0} = e_{-1}.$$

Then $(\lambda_1, 0)$, $(0, \lambda_2)$ and $(0, 0)$ are joint eigenvalues for $(\psi_1(0), \psi_2(0))$ with the joint eigenvectors $x_{\lambda_1,0}, x_{0,\lambda_2}$ and $x_{0,0}$ respectively. Therefore, by the similar reasoning as in the case of $z \neq 0$, we get

$$\sigma(\psi_1(0), \psi_2(0)) = \overline{\mathbb{D}} \times \{0\} \cup \{0\} \times \overline{\mathbb{D}} = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = 0\}. \quad (4.21)$$

Hence, by (4.20), (4.21) and Lemma 4.8, we get (4.19).

Indeed, one can show that the joint eigen spaces are one dimensional, for all the joint eigenvalues of $(\psi_1(z), \psi_2(z))$, $z \in \mathbb{D}$. □

Theorem 4.15 *Let (V_1, V_2) be a pair of commuting isometries such that $C(V_1, V_2) \leq 0$ and $C(V_1, V_2) \neq 0$. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Then for $z \in \mathbb{D}$, $(\lambda_1, \lambda_2) \in \mathbb{D}^2$ is a joint eigenvalue for $(\varphi_1(z), \varphi_2(z))$ if $\lambda_1\lambda_2 = z$,*

$$\sigma(\varphi_1(z), \varphi_2(z)) = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\},$$

and

$$\sigma(V_1, V_2) = \sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\cup_{z\in\mathbb{D}} \sigma(\varphi_1(z), \varphi_2(z))} = \overline{\mathbb{D}^2}.$$

Moreover, the non-singularity breaks at stage 2.

Proof As $\varphi_1(z)$ and $\varphi_2(z)$ are commuting contractions, $\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \overline{\mathbb{D}^2}$. Since $\varphi_1(z)\varphi_2(z) = zI$, by spectral mapping theorem,

$$\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\}.$$

From (4.12) and (4.13) note that:

$$\varphi_i(z) = \begin{matrix} & \begin{matrix} l^2(\mathbb{Z}) \otimes \mathcal{L} & \mathcal{E}_2 \end{matrix} & \\ & \begin{pmatrix} \psi_i(z) \otimes I_{\mathcal{L}} & 0 \\ 0 & \varphi_i(z)|_{\mathcal{E}_2} \end{pmatrix} & \begin{matrix} l^2(\mathbb{Z}) \otimes \mathcal{L} \\ \mathcal{E}_2 \end{matrix} \end{matrix}, \ i = 1, 2.$$

Now the proof follows from Lemma 4.14, Lemma 1.1 and Lemma 4.8. □

5 The Positive Defect Case

By Lemma 3.2 (see also [13, 16]), $C(V_1, V_2) \geq 0$ if and only if (V_1, V_2) is doubly commuting. The structure of doubly commuting pair of isometries are well understood in the literature; see [11, 18, 21]. We rephrase and shine some of the existing results using the defect operator. We also study the joint spectrum in detail.

5.1 The Prototypical Example

Definition 5.1 The fundamental isometric pair of positive defect is the bi-shift (M_{z_1}, M_{z_2}), the pair of multiplication by the coordinate functions on $H^2_{\mathbb{D}^2}$.

It is folklore that the M_{z_i} are isometries and the defect is positive because of the following lemma. It will be clear from Theorem 5.10 why we call it $fundamental$.

Lemma 5.2 *The defect operator $C(M_{z_1}, M_{z_2})$ is the projection onto the one dimensional space of constant functions.*

Proof The proof is a straightforward computation. □

If (V_1, V_2) is a bi-shift on $\mathcal{H}$ with the wandering subspace $\mathcal{R}$, then $\Lambda : \mathcal{H} \to H^2_{\mathbb{D}^2} \otimes \mathcal{R}$ given by $\Lambda(V_1^{n_1} V_2^{n_2} x) = z_1^{n_1} z_2^{n_2} \otimes x$, $x \in \mathcal{R}$ is a unitary and $\Lambda V_i \Lambda^* = M_{z_i} \otimes I_{\mathcal{R}}$ for $i = 1, 2$. Also, by the above lemma $\dim(\operatorname{ran} C(V_1, V_2)) = \dim \mathcal{R}$. The following lemma is well known.

Lemma 5.3 *The joint spectrum $\sigma(M_{z_1}, M_{z_2})$ is the whole bidisc $\overline{\mathbb{D}^2}$. Indeed, every point in the open bidisc is a joint eigenvalue for $(M^*_{z_1}, M^*_{z_2})$. In particular, for every $(w_1, w_2) \in \mathbb{D}^2$, the non-singularity of $K(M_{z_1} - w_1 I, M_{z_2} - w_2 I)$ is broken at the third stage.*

Remark 5.4 Recall the pair (τ_1, τ_2) defined in Sect. 4.1. One can observe that $\{\tau_1^m \tau_2^n(1) : m, n \geq 0\}$ is an orthonormal subset of $H^2_{\mathbb{D}^2}$. Let $\mathcal{M} = \overline{\operatorname{span}}\{\tau_1^m \tau_2^n(1) : m, n \geq 0\} \subset H^2_{\mathbb{D}^2}$. Clearly it is a joint invariant subspace (but not reducing) for (τ_1, τ_2). Identify $\mathcal{M}$ and $H^2_{\mathbb{D}^2}$ via $\tau_1^m \tau_2^n(1) \mapsto z_1^m z_2^n$. Clearly, the pair of isometries $(\tau_1|_{\mathcal{M}}, \tau_2|_{\mathcal{M}})$ gets identified with (M_{z_1}, M_{z_2}) on $H^2_{\mathbb{D}^2}$. Since $C(M_{z_1}, M_{z_2}) \geq 0$ and $C(M_{z_1}, M_{z_2}) \neq 0$, the same is true for the pair $(\tau_1|_{\mathcal{M}}, \tau_2|_{\mathcal{M}})$. Thus, a pair of commuting isometries with negative defect, when restricted to a joint invariant subspace, can have positive defect.

5.2 The General Case of the Positive Defect Operator

We start with a characterization available in [13] and [16].

Lemma 5.5 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) $C(V_1, V_2) \geq 0$ *and* $C(V_1, V_2) \neq 0$.
(b) $V_2(\ker V_1^*) \subsetneq \ker V_1^*$.
(c) $V_1(\ker V_2^*) \subsetneq \ker V_2^*$.
(d) The fringe operators are isometries and not unitaries.
(e) (V_1, V_2) *is doubly commuting and* $C(V_1, V_2) \neq 0$.
(f) $C(V_1, V_2)$ *is a non-zero projection.*
(g) If $(\mathcal{E}, P, U)$ *is the BCL triple for* (V_1, V_2)*, then* $U(\operatorname{ran} P) \subsetneq \operatorname{ran} P$.

Let $\eta_1, \eta_2 : \mathbb{D} \to \mathcal{B}(l^2(\mathbb{Z}))$ be the multipliers associated with the BCL triple $(l^2(\mathbb{Z}), p_{0+}, \omega)$ where p_{0+} is the projection onto $\overline{\text{span}}\{e_n : n \geq 0\}$ and ω is the bilateral shift on $l^2(\mathbb{Z})$. Now, we obtain a structure theorem which has its own independent interest and is applied later to obtain Theorem 5.13.

Theorem 5.6 *Let (V_1, V_2) be a pair of commuting isometries such that $C(V_1, V_2)$ is a non-zero positive operator. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Then, up to unitary equivalence*

$$\mathcal{E} = (l^2(\mathbb{Z}) \otimes \mathcal{L}) \oplus \mathcal{E}_2$$

for some non-trivial closed subspace $\mathcal{L}$ and a closed subspace $\mathcal{E}_2$ of $\mathcal{E}$. Moreover

$$M_{\varphi_i} = \begin{array}{c} \begin{array}{cc} H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L} & H^2_{\mathbb{D}}(\mathcal{E}_2) \end{array} \\ \begin{pmatrix} M_{\eta_i} \otimes I_{\mathcal{L}} & 0 \\ 0 & M_{\varphi_i}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)} \end{pmatrix} \end{array} \begin{array}{l} H^2_{\mathbb{D}}(l^2(\mathbb{Z})) \otimes \mathcal{L} \\ H^2_{\mathbb{D}}(\mathcal{E}_2) \end{array} \tag{5.1}$$

with $C(M_{\varphi_1}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}, M_{\varphi_2}|_{H^2_{\mathbb{D}}(\mathcal{E}_2)}) = 0$. In particular,

$$\sigma(M_{\eta_1}, M_{\eta_2}) = \sigma(M_{\eta_1} \otimes I_{\mathcal{L}}, M_{\eta_2} \otimes I_{\mathcal{L}}) \subseteq \sigma(M_{\varphi_1}, M_{\varphi_2}) \subseteq \sigma(V_1, V_2). \tag{5.2}$$

Proof We shall not give details of this proof because it follows the same line as the proof of Theorem 4.7. □

Lemma 5.7 *The pair (M_{η_1}, M_{η_2}) is jointly unitarily equivalent to (M_{z_1}, M_{z_2}). In particular, $\sigma(M_{\eta_1}, M_{\eta_2}) = \overline{\mathbb{D}^2}$.*

Proof Define the unitary $\Lambda : H^2_{\mathbb{D}^2} \to H^2_{\mathbb{D}}(l^2(\mathbb{Z}))$ by

$$\Lambda\left(\sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n\right) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{\infty} a_{m+k,k} e_{-(m+1)} + \sum_{m=1}^{\infty} a_{k,m+k} e_{m-1}\right) z^k. \tag{5.3}$$

Then, $\Lambda M_{z_i} \Lambda^* = M_{\eta_i}$ for $i = 1, 2$. That is, (M_{η_1}, M_{η_2}) is jointly unitarily equivalent to (M_{z_1}, M_{z_2}).

In particular, $\sigma(M_{\eta_1}, M_{\eta_2}) = \sigma(M_{z_1}, M_{z_2}) = \overline{\mathbb{D}^2}$, by Lemma 5.3. □

Remark 5.8 $(l^2(\mathbb{Z}) \otimes \mathcal{L}, p_{0+} \otimes I_{\mathcal{L}}, \omega \otimes I_{\mathcal{L}})$ is the BCL triple for the bi-shift with wandering subspace $\mathcal{L}$. (Compare this with the Remark 4.13).

Lemma 5.9 *The pair (M_{z_1}, M_{z_2}) does not have any non-trivial joint reducing subspace.*

The proof of the above lemma is similar to the proof of Lemma 4.10. Now the following structure theorem for $C(V_1, V_2) \geq 0$ follows from Theorem 1.2, Theorem 5.6, Lemma 5.7 and Lemma 5.2. It shows the role played by the bi-shift of a possibly higher multiplicity and is a rephrasing of some results in [11, 21] using the defect operator.

Theorem 5.10 *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$ with $C(V_1, V_2) \geq 0$ and $C(V_1, V_2) \neq 0$. Then, there is a non-trivial Hilbert space $\mathcal{L} \subsetneq \ker V^*$ such that, up to unitary equivalence, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$, where $\mathcal{H}_0 = H^2_{\mathbb{D}^2} \otimes \mathcal{L}$. In this decomposition*

$$V_i = \begin{pmatrix} M_{z_i} \otimes I_{\mathcal{L}} & 0 \\ 0 & V_{i0} \end{pmatrix}$$

and the defect operator $C(V_{10}, V_{20})$ is zero. Moreover, the dimension of $\mathcal{L}$ is the same as the dimension of the range of $C(V_1, V_2)$.

Theorem 5.11 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$ with $C(V_1, V_2) \geq 0$ and $C(V_1, V_2) \neq 0$. Then, $\sigma(V_1, V_2) = \overline{\mathbb{D}^2}$ and every point $(w_1, w_2) \in \mathbb{D}^2$ is a joint eigenvalue of (V_1^*, V_2^*). In particular, for every $(w_1, w_2) \in \mathbb{D}^2$, the non-singularity of $K(V_1 - w_1 I, V_2 - w_2 I)$ is broken at the third stage.*

Proof The proof follows from Theorem 5.11 and Lemma 5.3. □

The following lemma is useful to see the relation between the joint spectrum of a pair of commuting isometries (V_1, V_2) with defect positive and non-zero, and the joint spectra $\sigma(\varphi_1(z), \varphi_2(z))$, $z \in \mathbb{D}$, where φ_i's are the multipliers given in the BCL theorem.

Lemma 5.12 *Let the operator valued functions η_1 and η_2 be as in Theorem 5.6. Then, for $z \in \mathbb{D}$, $(\overline{\lambda_1}, \overline{\lambda_2}) \in \mathbb{D}^2$ is a joint eigenvalue for $(\eta_1(z)^*, \eta_2(z)^*)$ if $\lambda_1\lambda_2 = z$. Also,*

$$\sigma(\eta_1(z), \eta_2(z)) = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\}, \tag{5.4}$$

and

$$\overline{\cup_{z\in\mathbb{D}}\sigma(\eta_1(z), \eta_2(z))} = \overline{\mathbb{D}^2} = \sigma(M_{\eta_1}, M_{\eta_2}). \tag{5.5}$$

Proof Let $z \in \mathbb{D} \setminus \{0\}$ and $\lambda_1, \lambda_2 \in \mathbb{D}$ be such that $\lambda_1\lambda_2 = z$. Consider

$$x_{\lambda_1,\lambda_2} = \sum_{n=0}^{\infty} \overline{\lambda_2}^n e_n + \frac{1}{\overline{\lambda_2}} \sum_{n=1}^{\infty} \overline{\lambda_1}^{n-1} e_{-n}.$$

Note that

$$\eta_i(z)^* x_{\lambda_1,\lambda_2} = \overline{\lambda}_i x_{\lambda_1,\lambda_2}, i = 1, 2.$$

That is, $(\overline{\lambda_1}, \overline{\lambda_2})$ is a joint eigenvalue for $(\eta_1(z)^*, \eta_2(z)^*)$ with eigenvector x_{λ_1,λ_2}, where $z \neq 0$ and $\lambda_1, \lambda_2 \in \mathbb{D}$ are such that $\lambda_1\lambda_2 = z$. Therefore,

$$\sigma(\eta_1(z), \eta_2(z)) \supseteq \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\} \text{ for } z \neq 0.$$

As $\eta_1(z)$ and $\eta_2(z)$ are commuting contractions, we have $\sigma(\eta_1(z), \eta_2(z)) \subseteq \overline{\mathbb{D}^2}$. Since $\eta_1(z)\eta_2(z) = zI$, by spectral mapping theorem, we have the other inclusion. Hence

$$\sigma(\eta_1(z), \eta_2(z)) = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\} \text{ for } z \neq 0. \tag{5.6}$$

For the case $z = 0$, consider for $\lambda_1, \lambda_2 \in \mathbb{D} \setminus \{0\}$,

$$x_{\lambda_1,0} = \sum_{n=1}^{\infty} \overline{\lambda_1}^{n-1} e_{-n}, \; x_{0,\lambda_2} = \sum_{n=0}^{\infty} \overline{\lambda_2}^{n} e_n + \frac{1}{\overline{\lambda_2}} e_{-1} \text{ and } x_{0,0} = e_{-1}.$$

Then $(\overline{\lambda_1}, 0)$, $(0, \overline{\lambda_2})$ and $(0, 0)$ are joint eigenvalues for $(\eta_1(0)^*, \eta_2(0)^*)$ with the joint eigenvectors $x_{\lambda_1,0}, x_{0,\lambda_2}$ and $x_{0,0}$ respectively. Therefore, by the similar reasoning as in the case of $z \neq 0$, we get

$$\sigma(\eta_1(0), \eta_2(0)) = \overline{\mathbb{D}} \times \{0\} \cup \{0\} \times \overline{\mathbb{D}} = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = 0\}. \tag{5.7}$$

Hence, by (5.6), (5.7) and Lemma 5.7, we get (5.5).

Indeed, one can show that the joint eigenspaces are one dimensional, for all the joint eigenvalues of $(\eta_1(z)^*, \eta_2(z)^*)$, $z \in \mathbb{D}$. □

Theorem 5.13 *Let (V_1, V_2) be a pair of commuting isometries such that $C(V_1, V_2) \geq 0$ and $C(V_1, V_2) \neq 0$. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Then, for $z \in \mathbb{D}$, $(\overline{\lambda_1}, \overline{\lambda_2}) \in \mathbb{D}^2$ is a joint eigenvalue for $(\varphi_1(z)^*, \varphi_2(z)^*)$ if $\lambda_1\lambda_2 = z$, also*

$$\sigma(\varphi_1(z), \varphi_2(z)) = \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\}. \tag{5.8}$$

Moreover, every point in the open bidisc is a joint eigenvalue for $(M_{\varphi_1}^*, M_{\varphi_2}^*)$, and

$$\sigma(V_1, V_2) = \sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))} = \overline{\mathbb{D}^2}. \tag{5.9}$$

Proof Since $\varphi_1(z)$ and $\varphi_2(z)$ are commuting contractions, $\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \overline{\mathbb{D}^2}$. As $\varphi_1(z)\varphi_2(z) = zI$, by spectral mapping theorem,

$$\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \{(\lambda_1, \lambda_2) \in \overline{\mathbb{D}^2} : \lambda_1\lambda_2 = z\}.$$

In the same manner as in the proof of Theorem 4.7, we get

$$\varphi_i(z) = \begin{matrix} & \begin{matrix} l^2(\mathbb{Z}) \otimes \mathcal{L} & \mathcal{E}_2 \end{matrix} & \\ & \begin{pmatrix} \eta_i(z) \otimes I_{\mathcal{L}} & 0 \\ 0 & \varphi_i(z)|_{\mathcal{E}_2} \end{pmatrix} & \begin{matrix} l^2(\mathbb{Z}) \otimes \mathcal{L} \\ \mathcal{E}_2 \end{matrix} \end{matrix}, \; i = 1, 2,$$

where $\mathcal{L} = \operatorname{ran} P \ominus U(\operatorname{ran} P)$ and $\mathcal{E}_2 = \ker V^* \ominus (l^2(\mathbb{Z}) \otimes \mathcal{L})$. Now the proof follows from Lemma 5.12, Lemma 5.7 and Lemma 5.3. □

6 Towards the General Defect Operator

This section deals with the cases of ran V_1 = ran V_2 and ran $V_2 \subsetneq$ ran V_1. We provide the characterization and study the joint spectrum for both the cases. At the end of this section we give an example for the unknown case of $\mathcal{H}_i \neq 0$ for all $i = 1, 2, 3, 4$.

6.1 Range of V_1 Equal to the Range of V_2

In this case the defect operator is the difference of two mutually orthogonal projections whose ranges together span the kernel of V^*. The structure in this case, known from [4], is briefly recalled below because it is needed for deciphering the joint spectrum.

6.1.1 The Prototypical Family of Examples

Let $\mathcal{L}$ be a Hilbert space and W be a unitary on $\mathcal{L}$. Consider the pair of commuting isometries $(M_z \otimes I, M_z \otimes W)$ on $H^2_{\mathbb{D}} \otimes \mathcal{L}$. Clearly $\text{ran}(M_z \otimes I) = \text{ran}(M_z \otimes W)$.

Lemma 6.1 *Let $\mathcal{L}$ and W be as above. If $(V_1, V_2) = (M_z \otimes I, M_z \otimes W)$, then there are two projections P and Q such that*

(1) they are mutually orthogonal to each other,
(2) the dimensions of ranges of P and Q are same,
(3) the span of the ranges of P and Q is the kernel of V^* and
(4) the defect operator $C(V_1, V_2)$ on $H^2_{\mathbb{D}} \otimes \mathcal{L}$ is $P - Q$.

Proof In keeping with the notation E_0 for the projection onto the one dimensional subspace of constants in $H^2_{\mathbb{D}}$, let us denote by E_1 the projection onto the one dimensional space spanned by z. The kernel of V^* is

$$\text{ran}(I \otimes I_{\mathcal{L}} - M_z^2 (M_z^2)^* \otimes I_{\mathcal{L}}) = \text{ran}((E_0 + E_1) \otimes I_{\mathcal{L}}) = \overline{\text{span}}\{1, z\} \otimes \mathcal{L}$$

and the defect operator is

$$\begin{aligned} C(V_1, V_2) &= I \otimes I_{\mathcal{L}} - M_z M_z^* \otimes I_{\mathcal{L}} - M_z M_z^* \otimes I_{\mathcal{L}} + M_z^2 (M_z^2)^* \otimes I_{\mathcal{L}} \\ &= E_0 \otimes I_{\mathcal{L}} - M_z E_0 M_z^* \otimes I_{\mathcal{L}} \\ &= E_0 \otimes I_{\mathcal{L}} - E_1 \otimes I_{\mathcal{L}}. \end{aligned}$$

Thus, setting $P = E_0 \otimes I_{\mathcal{L}}$ and $Q = E_1 \otimes I_{\mathcal{L}}$, we are done. □

Lemma 6.2 *If $\mathcal{L} \neq \{0\}$, then the joint spectrum of $(M_z \otimes I_{\mathcal{L}}, M_z \otimes W)$ is*

$$\sigma(M_z \otimes I_{\mathcal{L}}, M_z \otimes W) = \{z(1, \alpha) : z \in \overline{\mathbb{D}}, \alpha \in \sigma(W)\}.$$

Proof This is a straightforward application of the polynomial spectral mapping theorem. Consider the polynomial $f(x_1, x_2) = (x_1, x_1 x_2)$. Then,

$$\sigma(M_z \otimes I_{\mathcal{L}}, M_z \otimes W) = f(\sigma(M_z \otimes I_{\mathcal{L}}, I_{H^2_{\mathbb{D}}} \otimes W))$$

$$= f(\sigma(M_z) \times \sigma(W))$$
$$= \{z(1, \alpha) : z \in \overline{\mathbb{D}}, \alpha \in \sigma(W)\}. \tag{6.1}$$

□

6.1.2 General Theory

Theorem 6.3 *Let* (V_1, V_2) *be a pair of commuting isometries on a Hilbert space* $\mathcal{H}$. *Then the following are equivalent:*

(a) $\text{ran } V_1 = \text{ran } V_2$.
(b) $V_1(\ker V_2^*) = V_2(\ker V_1^*)$.
(c) *The defect operator* $C(V_1, V_2)$ *is a difference of two mutually orthogonal projections* Q_1, Q_2 *with* $\text{ran } Q_1 \oplus \text{ran } Q_2 = \ker V^*$.
(d) *The fringe operators* F_1 *and* F_2 *are zero.*
(e) *If* $(\mathcal{E}, P, U)$ *is the BCL triple for* (V_1, V_2), *then* $U(\text{ran } P) = \text{ran } P^{\perp}$ *(or equivalently* $U(\text{ran } P^{\perp}) = \text{ran } P$*).*

Proof $(a) \Rightarrow (b)$ This is immediate from the equality

$$\ker V_1^* \oplus V_1(\ker V_2^*) = \ker V^* = \ker V_2^* \oplus V_2(\ker V_1^*)$$

after we note that $\text{ran } V_1 = \text{ran } V_2$ implies that $\ker V_1^* = \ker V_2^*$.

$(b) \Rightarrow (c)$ Set $P_1 = P_{\ker V_1^*}$ and $P_2 = P_{V_2(\ker V_1^*)}$. By (2.2), $C(V_1, V_2) = P_1 - P_2$. Moreover,

$$\begin{aligned} \text{ran } P_1 \oplus \text{ran } P_2 &= \ker V_1^* \oplus V_2(\ker V_1^*) \\ &= \ker V_1^* \oplus V_1(\ker V_2^*) \quad \text{(by (b))} \\ &= \ker V^*. \end{aligned}$$

$(c) \Rightarrow (d)$ It is immediate by noticing that

$$Q_1 = P_{\ker V_1^*} = P_{\ker V_2^*} \quad \text{and} \quad Q_2 = P_{V_2(\ker V_1^*)} = P_{V_1(\ker V_2^*)}$$

from the discussions in Sect. 2.

$(c) \Rightarrow (a)$ Immediate from the last line above.

$(d) \Rightarrow (c)$ $F_1 = 0$ and $F_2 = 0$ implies that $\ker V_1^* \perp V_2(\ker V_1^*)$ and $\ker V_2^* \perp V_1(\ker V_2^*)$. Now from (2.3), we see that $V_1(\ker V_2^*) = V_2(\ker V_1^*)$. Set $P_1 = P_{\ker V_1^*}$ and $P_2 = P_{V_2(\ker V_1^*)}$, to obtain (c).

Thus, we have shown the equivalences of (a), (b), (c) and (d).

$(a) \iff (e)$ If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then

$$\text{ran } V_1 = \text{ran } V_2$$

if and only if

$$\ker V_1^* = \ker V_2^*$$

if and only if

$$\ker M_{\varphi_1}^* = \ker M_{\varphi_2}^*$$

if and only if

$$\operatorname{ran} E_0 \otimes U^* P U = \operatorname{ran} E_0 \otimes P^\perp$$

if and only if

$$U^* P U = P^\perp.$$

□

The following result is a special case of Remark 4.2 in [4], which shows that the pair $(M_z \otimes I, M_z \otimes W)$, studied in Sect. 6.1.1, is the prototypical pair in this case. We give a proof which is different from the one in [4].

Theorem 6.4 *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$ satisfying any of the equivalent conditions in Theorem 6.3. Then, there exist Hilbert spaces $\mathcal{L}$ and $\mathcal{K}$ such that up to unitarily equivalence $\mathcal{H} = (H^2_{\mathbb{D}} \otimes \mathcal{L}) \oplus \mathcal{K}$ and in this decomposition,*

$$V_1 = \begin{pmatrix} M_z \otimes I_{\mathcal{L}} & 0 \\ 0 & W_1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} M_z \otimes W & 0 \\ 0 & W_2 \end{pmatrix},$$

for some unitary W on $\mathcal{L}$ and commuting unitaries W_1, W_2 on $\mathcal{K}$.

Proof Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Let $\mathcal{L} = \operatorname{ran} P^\perp$. By Theorem 6.3 (*e*), we have $U(\mathcal{L}) = \mathcal{L}^\perp$ and $U(\mathcal{L}^\perp) = \mathcal{L}$. Let $U_1 = U|_{\mathcal{L}} : \mathcal{L} \to \mathcal{L}^\perp$ and $U_2 = U|_{\mathcal{L}^\perp} : \mathcal{L}^\perp \to \mathcal{L}$. In the decomposition $\mathcal{E} = \mathcal{L}^\perp \oplus \mathcal{L}$, we have

$$U = \begin{pmatrix} 0 & U_1 \\ U_2 & 0 \end{pmatrix} \quad \text{and} \quad P = \begin{pmatrix} I_{\mathcal{L}^\perp} & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $\varphi_1(z) = \begin{pmatrix} 0 & U_2^* \\ zU_1^* & 0 \end{pmatrix}$ and $\varphi_2(z) = \begin{pmatrix} 0 & U_1 \\ zU_2 & 0 \end{pmatrix}$ for $z \in \mathbb{D}$, and

$$M_{\varphi_1} = \begin{pmatrix} 0 & I \otimes U_2^* \\ M_z \otimes U_1^* & 0 \end{pmatrix} \quad \text{and} \quad M_{\varphi_2} = \begin{pmatrix} 0 & I \otimes U_1 \\ M_z \otimes U_2 & 0 \end{pmatrix}.$$

Let us define the unitary

$$\Lambda : (H^2_{\mathbb{D}} \otimes \mathcal{L}^\perp) \oplus (H^2_{\mathbb{D}} \otimes \mathcal{L}) \to H^2_{\mathbb{D}} \otimes \mathcal{L}$$

given by,

$$\Lambda\left(\begin{pmatrix} \sum_{n=0}^\infty a_n z^n \\ \sum_{n=0}^\infty b_n z^n \end{pmatrix}\right) := \sum_{n \text{ is even}} (U_2 U_1)^{\frac{n}{2}} (b_{\frac{n}{2}}) z^n + \sum_{n \text{ is odd}} (U_2 U_1)^{\frac{n-1}{2}} U_2 (a_{\frac{n-1}{2}}) z^n,$$

$a_n \in \mathcal{L}^{\perp}$, $b_n \in \mathcal{L}$. One can see that $\Lambda M_{\varphi_1}\Lambda^* = M_z \otimes I_{\mathcal{L}}$ and $\Lambda M_{\varphi_2}\Lambda^* = M_z \otimes W$, where $W = U_2U_1$. This completes the proof by Theorem 1.2. □

Corollary 6.5 *Let (V_1, V_2) be a pair of commuting isometries satisfying any of the equivalent conditions in Theorem 6.3. If V_1V_2 is pure, then both V_1 and V_2 are pure.*

Since the BCL triple is unique (up to unitary equivalence), we shall use the following convenient choice of BCL triple due to A. Maji et al in [16] for a pair of commuting isometries.

Theorem 6.6 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then, $(\ker V^*, P, U_0)$ is the BCL triple for (V_1, V_2), where $P \in \mathcal{B}(\ker V^*)$ is the orthogonal projection onto $V_2(\ker V_1^*)$ and*

$$U_0 = \begin{pmatrix} V_2|_{\ker V_1^*} & 0 \\ 0 & V_1^*|_{V_1(\ker V_2^*)} \end{pmatrix} : \begin{matrix} \ker V_1^* \\ \oplus \\ V_1(\ker V_2^*) \end{matrix} \to \begin{matrix} V_2(\ker V_1^*) \\ \oplus \\ \ker V_2^* \end{matrix}$$

is a unitary operator on $\ker V^*$.

Remark 6.7 In Theorem 6.4, we can write the W and $\mathcal{L}$ explicitly in terms of V_1 and V_2 as follows:

By Theorem 6.6 and Theorem 6.3, $\mathcal{L} = \operatorname{ran} P^{\perp} = \ker V_1^*$, $U_1 : \ker V_1^* \to V_1(\ker V_1^*)$ given by $U_1 = V_2|_{\ker V_1^*}$ and $U_2 : V_1(\ker V_1^*) \to \ker V_1^*$ given by $U_2 = V_1^*|_{V_1(\ker V_1^*)}$. Therefore $W = V_1^*|_{V_1(\ker V_1^*)} V_2|_{\ker V_1^*}$.

6.1.3 Joint Spectrum

If (V_1, V_2) is pure and satisfying any of the equivalent conditions in Theorem 6.3, then by Theorem 6.4, Remark 6.7 and by Sect. 6.1, we have

$$\sigma(V_1, V_2) = \{z(1, e^{i\theta}) : z \in \overline{\mathbb{D}},\ e^{i\theta} \in \sigma(U_2U_1)\}, \tag{6.2}$$

where $U_1 = V_2|_{\ker V_1^*}$, $U_2 = V_1^*|_{V_1(\ker V_1^*)}$.

The final theorem of this section tells us the nature of elements in the joint spectrum and the relation between the joint spectrum of the commuting isometries satisfying any of the equivalent conditions in Theorem 6.3 and the joint spectra of the associated multipliers at every point of $\mathbb{D}$.

Theorem 6.8 *Let (V_1, V_2) be a pair of commuting isometries on $\mathcal{H}$ satisfying any of the equivalent conditions in Theorem 6.3 and $\ker V^* \neq \{0\}$. Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Let $U_1 : \operatorname{ran} P^{\perp} \to \operatorname{ran} P$ be the unitary given by $U_1(x) = U(x)$, $x \in \operatorname{ran} P^{\perp}$ and $U_2 : \operatorname{ran} P \to \operatorname{ran} P^{\perp}$ be the unitary given by $U_2(y) = U(y)$, $y \in \operatorname{ran} P$. Then*

- $\sigma(\varphi_1(z), \varphi_2(z)) = \{\pm\sqrt{z}(e^{-i\frac{\theta}{2}}, e^{i\frac{\theta}{2}}) : e^{i\theta} \in \sigma(U_1U_2)\}$. *Here for every point in the joint spectrum, the non-singularity breaks at stage 3.*

- $\sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))} = \{z(1, e^{i\theta}) : z \in \overline{\mathbb{D}},\ e^{i\theta} \in \sigma(U_1U_2)\}$. *Here, for every point (z_1, z_2) in the set $\{z(1, e^{i\theta}) : z \in \mathbb{D},\ e^{i\theta} \in \sigma(U_1U_2)\}$, the non-singularity in the Koszul complex $K(M_{\varphi_1} - z_1I, M_{\varphi_2} - z_2I)$ breaks at stage 3.*

Proof Let $\mathcal{E}_1 = \operatorname{ran} P$ and $\mathcal{E}_2 = \operatorname{ran} P^{\perp}$. We have by Theorem 6.3 that $U(\mathcal{E}_1) = \mathcal{E}_2$ and $U(\mathcal{E}_2) = \mathcal{E}_1$. Hence in the decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$, we have $\varphi_1(z) = \begin{pmatrix} 0 & U_2^* \\ zU_1^* & 0 \end{pmatrix}$ and $\varphi_2(z) = \begin{pmatrix} 0 & U_1 \\ zU_2 & 0 \end{pmatrix}$, for $z \in \mathbb{D}$, where U_1 and U_2 are as in the statement.

Now, we shall show that: for $z \in \mathbb{D}$,

$$\sigma(\varphi_1(z)) = \{w \in \mathbb{D} : w^2 = \overline{\alpha}z \text{ for some } \alpha \in \sigma(U_1U_2)\}, \tag{6.3}$$

$$\sigma(\varphi_2(z)) = \{w \in \mathbb{D} : w^2 = \alpha z \text{ for some } \alpha \in \sigma(U_1U_2)\}. \tag{6.4}$$

Clearly (6.3) and (6.4) hold for $z = 0$. So assume that $z \neq 0$.

Assertion 1 If λ is an eigenvalue of $\varphi_1(z)$ then $\overline{\frac{\lambda^2}{z}}$ is an eigenvalue of U_1U_2. In particular, $\lambda \in \mathbb{D}$.

Proof of Assertion 1. If λ is an eigenvalue of $\varphi_1(z)$, there exists a non-zero vector $h_1 \oplus h_2 \in \mathcal{E}_1 \oplus \mathcal{E}_2$ such that $\varphi_1(z)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \lambda \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$ which implies $U_2{}^*(h_2) = \lambda h_1$ and $\frac{z}{\lambda}U_1^*(h_1) = h_2$. Hence $(U_1U_2)^*h_1 = \frac{\lambda^2}{z}h_1$. Notice that $h_1 \neq 0$, otherwise $U_2{}^*(h_2) = \lambda h_1$ implies $h_2 = 0$. □

Assertion 2 For $\lambda \in \mathbb{C}$, $(\varphi_1(z) - \lambda I_{\mathcal{E}})$ is not onto if and only if $((U_1U_2)^* - \frac{\lambda^2}{z}I_{\mathcal{E}_1})$ is not onto. In particular, if $(\varphi_1(z) - \lambda I_{\mathcal{E}})$ is not onto then $\lambda \in \mathbb{D}$. Also

$$\operatorname{ran}(\varphi_1(z) - \lambda I) = \left\{ \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{E}_1 \oplus \mathcal{E}_2 : \lambda h_1 + U_2^*h_2 \in \operatorname{ran}((U_1U_2)^* - \frac{\lambda^2}{z}I) \right\}. \tag{6.5}$$

Proof of Assertion 2. Let $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \mathcal{E}_1 \oplus \mathcal{E}_2$. We have $(\varphi_1(z) - \lambda I)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -\lambda h_1 + U_2{}^*h_2 \\ zU_1{}^*h_1 - \lambda h_2 \end{pmatrix}$. Let $k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathcal{E}_1 \oplus \mathcal{E}_2$. Now $(\varphi_1(z) - \lambda I)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$ if and only if $-\lambda h_1 + U_2{}^*(h_2) = k_1$ and

$$((U_1U_2)^* - \frac{\lambda^2}{z}I_{\mathcal{E}_1})(h_1) = \frac{\lambda k_1 + U_2{}^*(k_2)}{z}.$$

This proves Assertion 2.

Using Assertions 1 and 2 and the fact that for any bounded normal operator T if $\alpha \in \sigma(T)$ then $T - \alpha I$ is not onto, we get (6.3) for $z \neq 0$. In a similar way, one can show (6.4) and for $z \neq 0$

$$\operatorname{ran}(\varphi_2(z) - \lambda I) = \left\{ \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathcal{E}_1 \oplus \mathcal{E}_2 : \lambda k_1 + U_1 k_2 \in \operatorname{ran}((U_1U_2) - \frac{\lambda^2}{z} I) \right\}. \tag{6.6}$$

For $z \in \mathbb{D}$, since

$$\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \sigma(\varphi_1(z)) \times \sigma(\varphi_2(z))$$

and

$$\varphi_1(z)\varphi_2(z) = zI_{\mathcal{E}_1 \oplus \mathcal{E}_2} = \varphi_2(z)\varphi_1(z),$$

by spectral mapping theorem we get,

$$\begin{aligned} &\sigma(\varphi_1(z), \varphi_2(z)) \\ &\subseteq \{(z_1, z_2) \in \mathbb{D}^2 : {z_1}^2 = \overline{\alpha} z, {z_2}^2 = \beta z, z_1 z_2 = z, \text{ for some } \alpha, \beta \in \sigma(U_1U_2)\}. \\ &= \{(z_1, z_2) \in \mathbb{D}^2 : {z_1}^2 = \overline{\alpha} z, {z_2}^2 = \alpha z, z_1 z_2 = z, \text{ for some } \alpha \in \sigma(U_1U_2)\}. \end{aligned}$$

Let $z \neq 0$ and $z_1, z_2 \in \mathbb{D}$ be such that ${z_1}^2 = \overline{\alpha} z$, ${z_2}^2 = \alpha z$ and $z_1 z_2 = z$ for some $\alpha \in \sigma(U_1U_2)$. We shall show that $\operatorname{ran}(\varphi_1(z) - z_1 I) + \operatorname{ran}(\varphi_2(z) - z_2 I) \neq \mathcal{E}_1 \oplus \mathcal{E}_2$. For this, let $y \notin \operatorname{ran}(U_1U_2 - \alpha I)$. Suppose $\begin{pmatrix} y \\ 0 \end{pmatrix} \in \operatorname{ran}(\varphi_1(z) - z_1 I) + \operatorname{ran}(\varphi_2(z) - z_2 I)$. Then there exists $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \in \operatorname{ran}(\varphi_1(z) - z_1 I)$ and $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \operatorname{ran}(\varphi_2(z) - z_2 I)$ such that $\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}$. Since $\operatorname{ran} T = \operatorname{ran} T^*$ for any bounded normal operator T, note that from (6.5) and (6.6), we have $z_1 h_1 + {U_2}^* h_2, z_2 k_1 + U_1 k_2 \in \operatorname{ran}(U_1U_2 - \alpha I)$. Since $y \notin \operatorname{ran}(U_1U_2 - \alpha I)$, we have $(z_1 U_1 - z_2 {U_2}^*)(k_2) \notin \operatorname{ran}(U_1U_2 - \alpha I)$. So $(U_1U_2 - \alpha I)^*(U_1(k_2)) \notin \operatorname{ran}(U_1U_2 - \alpha I)$. Which is a contradiction, as $\operatorname{ran}(U_1U_2 - \alpha I) = \operatorname{ran}(U_1U_2 - \alpha I)^*$. So $\operatorname{ran}(\varphi_1(z) - z_1 I) + \operatorname{ran}(\varphi_2(z) - z_2 I) \neq \mathcal{E}_1 \oplus \mathcal{E}_2$ for all $z \neq 0$, $z_1, z_2 \in \mathbb{D}$ such that ${z_1}^2 = \overline{\alpha} z$, ${z_2}^2 = \alpha z$ and $z_1 z_2 = z$ for some $\alpha \in \sigma(U_1U_2)$. Also, note that $\operatorname{ran}(\varphi_1(0)) + \operatorname{ran}(\varphi_2(0)) = \mathcal{E}_1 \oplus 0$. Hence, we have

$$\begin{aligned} &\sigma(\varphi_1(z), \varphi_2(z)) \\ &= \{(z_1, z_2) \in \mathbb{D}^2 : {z_1}^2 = \overline{\alpha} z, {z_2}^2 = \alpha z, z_1 z_2 = z, \text{ for some } \alpha \in \sigma(U_1U_2)\} \\ &= \{\pm\sqrt{z}(e^{-i\frac{\theta}{2}}, e^{i\frac{\theta}{2}}) : e^{i\theta} \in \sigma(U_1U_2)\} \end{aligned}$$

for $z \in \mathbb{D}$.

As we saw, for any point $(z_1, z_2) \in \sigma(\varphi_1(z), \varphi_2(z))$, $z \in \mathbb{D}$,

$$\operatorname{ran}(\varphi_1(z) - z_1 I) + \operatorname{ran}(\varphi_2(z) - z_2 I) \neq \mathcal{E}_1 \oplus \mathcal{E}_2.$$

Hence

$$\text{ran}(M_{\varphi_1} - z_1 I) + \text{ran}(M_{\varphi_2} - z_2 I) \neq H^2(\mathcal{E}_1 \oplus \mathcal{E}_2).$$

Therefore $(z_1, z_2) \in \sigma(M_{\varphi_1}, M_{\varphi_2})$, which implies $\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z)) \subseteq \sigma(M_{\varphi_1}, M_{\varphi_2})$. Notice that for every point (z_1, z_2) in $\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))$, the non-singularity of $K(M_{\varphi_1} - z_1 I, M_{\varphi_2} - z_2 I)$ breaks at stage 3.

Take $z \in \mathbb{D}$ and $e^{i\theta} \in \sigma(U_1U_2)$. Let $w = ze^{\frac{i\theta}{2}}, z_1 = we^{-\frac{i\theta}{2}}$ and $z_2 = we^{\frac{i\theta}{2}}$. So $(z_1, z_2) \in \sigma(\varphi_1(w^2), \varphi_2(w^2))$ and $(z_1, z_2) = z(1, e^{i\theta})$. Hence we have the equality:

$$\begin{aligned}
&\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z)) \\
&= \cup_{z\in\mathbb{D}}\{(z_1, z_2) \in \mathbb{D}^2 : {z_1}^2 = \overline{\alpha}z, {z_2}^2 = \alpha z, z_1z_2 = z, \text{ for some } \alpha \in \sigma(U_1U_2)\} \\
&= \{z(1, \alpha) : z \in \mathbb{D}, \alpha \in \sigma(U_1U_2)\}. \qquad (6.7)
\end{aligned}$$

Now as in the proof of Theorem 6.4, we have $(M_{\varphi_1}, M_{\varphi_2})$ is jointly unitarily equivalent to $(M_z \otimes I_{\mathcal{E}_2}, M_z \otimes U_2U_1)$, hence from (6.1) and (6.7), we have

$$\begin{aligned}
\sigma(M_{\varphi_1}, M_{\varphi_2}) = \sigma(M_z \otimes I_{\mathcal{E}_2}, M_z \otimes U_2U_1) &= \{z(1, \alpha) : z \in \overline{\mathbb{D}}, \alpha \in \sigma(U_1U_2)\} \\
&= \overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))}.
\end{aligned}$$

□

Note that $\sigma(V_1, V_2) = \sigma(M_{\varphi_1}, M_{\varphi_2}) \cup \sigma(V_1|_{\mathcal{H}_u}, V_2|_{\mathcal{H}_u})$, by Theorem 1.2. Hence by Theorem 6.8,

$$\sigma(M_{\varphi_1}, M_{\varphi_2}) = \overline{\cup_{z\in\mathbb{D}}\sigma(\varphi_1(z), \varphi_2(z))} \subseteq \sigma(V_1, V_2). \qquad (6.8)$$

The above inclusion is an equality if and only if $\sigma(V_1|_{\mathcal{H}_u}, V_2|_{\mathcal{H}_u}) \subseteq \sigma(M_{\varphi_1}, M_{\varphi_2})$.

6.2 Range of one Isometry is Strictly Contained in the Range of Other

If (V_1, V_3) is any pair of commuting isometries with V_3 is not unitary, set $V_2 = V_1V_3$. Then (V_1, V_2) satisfy $\text{ran } V_2 \subsetneq \text{ran } V_1$. Since the study of the case $\text{ran } V_1 \subsetneq \text{ran } V_2$ is equivalent to the case of $\text{ran } V_2 \subsetneq \text{ran } V_1$, we consider only the case $\text{ran } V_2 \subsetneq \text{ran } V_1$.

Lemma 6.9 *Let (V_1, V_2) be a pair of commuting isometries on a Hilbert space $\mathcal{H}$. Then the following are equivalent:*

(a) $\text{ran } V_2 \subsetneq \text{ran } V_1$.
(b) $V_2(\text{ker } V_1^*) \subsetneq V_1(\text{ker } V_2^*)$.
(c) The fringe operator $F_1 = 0$ and $F_2 \neq 0$.
(d) If $(\mathcal{E}, P, U)$ is the BCL triple for (V_1, V_2), then $U(\text{ran } P) \subsetneq \text{ran } P^\perp$ (or equivalently $U^*(\text{ran } P) \subsetneq \text{ran } P^\perp$).
(e) If $\mathcal{F}$ is the defect space of V_2 and if $(M_z \otimes I_{\mathcal{F}}) \oplus W_2$ is the Wold decomposition of V_2, then $V_1 = M_\varphi \oplus W_1$, with $\varphi(z) = W^*(Q^\perp + zQ)$ for some unitary W and projection $Q \neq I$ in $\mathcal{B}(\mathcal{F})$, and a unitary W_1 on $\mathcal{H} \ominus H^2_{\mathbb{D}}(\mathcal{F})$.

Proof $(a) \iff (b)$ and $(a) \iff (c)$ follows from (2.3).

$(a) \iff (d)$: Let $(\mathcal{E}, P, U)$ be the BCL triple for (V_1, V_2). Note that $\ker V_1^* \subsetneq \ker V_2^*$ if and only if $\ker M_{\phi_1}^* \subsetneq \ker M_{\phi_2}^*$ if and only if $U^*PU \le P^\perp$ and $U^*PU \neq P^\perp$ if and only if $U(\operatorname{ran} P) \subsetneq \operatorname{ran} P^\perp$, because $\ker M_{\phi_1}^* = 1 \otimes \operatorname{ran}(U^*PU)$ and $\ker M_{\phi_2}^* = 1 \otimes \operatorname{ran} P^\perp$.

$(a) \iff (e)$: By Douglas Lemma ([10]), we have $V_2 = V_1 V_3$ for some isometry (non-unitary) V_3. Notice that V_3 commutes with V_1. Let $(\mathcal{F}, Q, W)$ be the BCL triple for (V_1, V_3). Then, (see Theorem 1.2)

$$V_1 = M_\varphi \oplus W_1, \quad V_3 = M_\psi \oplus W_3 \text{ in } \mathcal{B}(H^2_{\mathbb{D}}(\mathcal{F}) \oplus (\mathcal{H} \ominus H^2_{\mathbb{D}}(\mathcal{F}))),$$

where $\phi(z) = W^*(Q^\perp + zQ)$ and $\psi(z) = (Q + zQ^\perp)W$ for all $z \in \mathbb{D}$. Now since $V_2 = V_1 V_3$ and $\varphi(z)\psi(z) = z$ for all $z \in \mathbb{D}$ we have

$$V_2 = M_z \otimes I_{\mathcal{F}} \oplus W_2,$$

where $W_2 = W_1 W_3$. This completes the proof of $(a) \implies (e)$, because V_3 is not a unitary implies that $Q \neq I$. $(e) \implies (a)$ is trivial. □

6.2.1 Joint Spectrum

Consider the pair

$$(V_1, V_2) = (U^k V^n, U^l V^m) \tag{6.9}$$

with V an isometry (not unitary) and U a unitary which commutes with V, for some non-negative integers n, m, l, k and $n < m$, as in [4, Sec. 4]. Since U is a unitary commuting with V, we have

$$\operatorname{ran} V_2 = \operatorname{ran} U^l V^m = \operatorname{ran} V^m \subsetneq \operatorname{ran} V^n = \operatorname{ran} U^k V^n = \operatorname{ran} V_1.$$

In [4], Burdak has given the model for such pairs, viz.,

$$V_1 = (M_z^n \otimes W^k) \oplus W_1, V_2 = (M_z^m \otimes W^l) \oplus W_2, \tag{6.10}$$

for some unitary W, W_1, W_2. We shall now compute the joint spectrum $\sigma(M_z^n \otimes W^k, M_z^m \otimes W^l)$ of the pure part, as an easy application of the polynomial spectral mapping theorem, viz., consider the polynomial $f : \mathbb{C}^2 \to \mathbb{C}^2$ given by,

$$f(z_1, z_2) = (z_1^n z_2^k, z_1^m z_2^l).$$

By the polynomial spectral mapping theorem,

$$\begin{aligned}\sigma(M_z^n \otimes W^k, M_z^m \otimes W^l) &= \sigma(f(M_z \otimes I, I \otimes W)) \\ &= f(\sigma(M_z \otimes I, I \otimes W)\end{aligned}$$

$$= f(\overline{\mathbb{D}} \times \sigma(W))$$
$$= \{(z^n\alpha^k, z^m\alpha^l) : z \in \overline{\mathbb{D}}, \alpha \in \sigma(W)\}.$$

Following example shows that the class considered in [4], namely, the class of pairs of the type given in (6.9), is only a subclass of this case.

Example 6.10 Let $\mathcal{H} = H^2_{\mathbb{D}^2}$ and $(V_1, V_2) = (M_{z_1}, M_{z_1 z_2})$. Then there is no unitary U commuting with both V_1 and V_2 such that $V_1^m = UV_2^n$ for any m, n. In particular, (V_1, V_2) is not of the form given in (6.9). But clearly $\operatorname{ran} V_2 \subsetneq \operatorname{ran} V_1$.

Lemma 6.11 *Let* (V_1, V_2) *be a pair of commuting isometries, with* $\operatorname{ran} V_2 \subsetneq \operatorname{ran} V_1$. *Then,*

$$\sigma(V_1, V_2) \subseteq \{(z_1, z_2) : |z_2| \leq |z_1|, z_1 \in \overline{\mathbb{D}}\}. \tag{6.11}$$

Proof By Douglas Lemma ([10]), we have $V_2 = V_1 V_3$ for some isometry V_3, which commutes with V_1. Consider the polynomial $p : \mathbb{C}^2 \to \mathbb{C}^2$ given by $p(z_1, z_2) = (z_1, z_1 z_2)$. By the spectral mapping theorem,

$$\begin{aligned}\sigma(V_1, V_2) = \sigma(p(V_1, V_3)) &= p(\sigma(V_1, V_3))\\ &\subseteq p(\overline{\mathbb{D}} \times \overline{\mathbb{D}}) = \{(z_1, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\}\\ &= \{(z_1, z_2) : |z_2| \leq |z_1|, z_1 \in \overline{\mathbb{D}}\}.\end{aligned}$$

□

The inclusion in (6.11) is sharp; see Example 6.12, and it can be a strict inclusion; see Example 6.13.

Example 6.12 Consider the pair $(M_{z_1}, M_{z_1 z_2})$ of commuting isometries in $H^2(\mathbb{D}^2)$. Notice that

$$\begin{aligned}\operatorname{ran}(M_{z_1 z_2}) &= \left\{ \sum_{m,n \geq 1}^{\infty} a_{m,n} z_1^m z_2^n : \sum_{m,n \geq 1}^{\infty} |a_{m,n}|^2 < \infty \right\}\\ &\subsetneq \left\{ \sum_{m \geq 1, n \geq 0}^{\infty} a_{m,n} z_1^m z_2^n : \sum_{m \geq 1, n \geq 0}^{\infty} |a_{m,n}|^2 < \infty \right\} = \operatorname{ran} M_{z_1}.\end{aligned}$$

Consider the polynomial $p : \mathbb{C}^2 \to \mathbb{C}^2$ given by $p(z_1, z_2) = (z_1, z_1 z_2)$. By the spectral mapping theorem, we have

$$\begin{aligned}\sigma(M_{z_1}, M_{z_1 z_2}) = \sigma(p(M_{z_1}, M_{z_2})) = p(\sigma(M_{z_1}, M_{z_2})) &= p(\overline{\mathbb{D}} \times \overline{\mathbb{D}})\\ &= \{(z_1, z_2) : |z_2| \leq |z_1|, z_1 \in \overline{\mathbb{D}}\}.\end{aligned}$$

The measure of the above spectrum is non-zero, indeed it is $\frac{\pi^2}{2}$.

Example 6.13 Let $0 \leq m < n$. Consider (M_z^m, M_z^n). Then,

$$\text{ran } M_z^n = \left\{ \sum_{k=n}^{\infty} a_k z^k : \sum_{k=n}^{\infty} |a_k|^2 < \infty \right\} \subsetneq \left\{ \sum_{k=m}^{\infty} a_k z^k : \sum_{k=m}^{\infty} |a_k|^2 < \infty \right\} = \text{ran } M_z^m.$$

By the spectral mapping theorem,

$$\sigma(M_z^m, M_z^n) = \{(z^m, z^n) : z \in \overline{\mathbb{D}}\} \subsetneq \{(z_1, z_1 z_2) : z_1, z_2 \in \overline{\mathbb{D}}\}.$$

Example 6.14 Let $\mathcal{H} = H^2_{\mathbb{D}^2} \oplus H^2_{\mathbb{D}^2}$, and $V_i = M_{z_i} \oplus \tau_i$ for $i = 1, 2$. Then (V_1, V_2) is a pair of commuting isometries with defect $E_0 \oplus -P_{\text{span}\{z_2\}}$. Clearly the defect is a difference of two mutually orthogonal projections and (V_1, V_2) lies in the unknown case given in Table 1.

The following example gives an irreducible pair of commuting isometries lying in the unknown case given in Table 1.

Example 6.15 Consider the pure pair (V_1, V_2) with the BCL triple $(l^2(\mathbb{Z}), p_{01}, \omega)$ where ω is the bilateral shift and p_{01} is the projection in $l^2(\mathbb{Z})$ onto $\text{span}\{e_0, e_1\}$. Then

$$C(V_1, V_2) = E_0 \otimes (\omega^* p_{01} \omega - p_{01}) = E_0 \otimes p_{\text{span}\{e_{-1}\}} - E_0 \otimes p_{\text{span}\{e_1\}}, \tag{6.12}$$

where E_0 is the one dimensional projection onto the space of constant functions in $H^2_{\mathbb{D}}$. The pair (V_1, V_2) is irreducible. To show this, we shall show that the pair (p_{01}, ω) is irreducible; see [9, 11]. Suppose $\mathcal{E}_0 \neq \{0\}$ is a reducing subspace for (p_{01}, ω). Let $\sum_{n \in \mathbb{Z}} a_n e_n \in \mathcal{E}_0$ and $a_{n_0} \neq 0$ for some $n_0 \in \mathbb{Z}$. Consider

$$\omega^{*n_0} \sum_{n \in \mathbb{Z}} a_n e_n = \sum_{n \in \mathbb{Z}} a_n e_{n-n_0} \in \mathcal{E}_0.$$

This implies that $p_{01} \sum_{n \in \mathbb{Z}} a_n e_{n-n_0} = a_{n_0} e_0 + a_{n_0+1} e_1 \in \mathcal{E}_0$. Hence $\omega^*(a_{n_0} e_0 + a_{n_0+1} e_1) = a_{n_0} e_{-1} + a_{n_0+1} e_0 \in \mathcal{E}_0$. So $a_{n_0} e_{-1} \in \mathcal{E}_0$. Thus $e_{-1} \in \mathcal{E}_0$, shows that $\mathcal{E}_0 = l^2(\mathbb{Z})$.

The pair (V_1, V_2) lies in the unknown case of $\mathcal{H}_i \neq 0$ for all $i = 1, 2, 3, 4$ mentioned in Table 1. To see this first note that (6.12) shows that $C(V_1, V_2)$ is a difference of two mutually orthogonal projections and one can see that, it is not in any other case of the Table 1, using the characterization in terms of BCL given for those cases.

Acknowledgements The authors are thankful to the referee for a careful reading and pertinent comments. The first author is supported by the J C Bose Fellowship JCB/2021/000041 of SERB. The third author gratefully acknowledges support from the Indian Institute of Science through the Institute of Eminence post-doctoral fellowship and from the University Grants Commission (Grant no. MA/20-21/0047), India through a D. S. Kothari post-doctoral fellowship. We are thankful to Dan Popovici for bringing a good amount of existing literature to our notice.

Data Availability Statement Data sharing is not applicable to this article as no data sets were generated or analysed during the current study.

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