Spectral Analysis Near Regular Point of Reducibility and Representations of Coxeter Groups

Michael I. Stessin[1]

Abstract

For a tuple of square matrices $A_1, \ldots, A_n$ the determinantal hypersurface is defined as

$$\sigma(A_1, \ldots, A_n)$$
$$= \left\{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} : det(x_1 A_1 + \cdots + x_n A_n) = 0 \right\}.$$

In this paper we develop a local spectral analysis near a regular point of reducibility of a determinantal hypersurface. We prove a rigidity type theorem for representations of Coxeter groups as an application.

Keywords Projective joint spectrum · Detarminantal manifold · Coxeter groups · Representations of coxeter groups

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1 Introduction and Statements of Results

Given n square $N \times N$ matrices, the determinant of their linear combination, $det(x_1 A_1 + \cdots + x_n A_n)$, (such linear combination is called *a pencil*) is a homoge-

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neous polynomial of degree N in variables $x_1, ..., x_n$. Zeros of this polynomial form an algebraic manifold in the projective space $\mathbb{CP}^{n-1}$. This manifold is called the **determinantal manifold** (or **determinantal hypersurface**) for the tuple $(A_1, \ldots, A_n)$. We use the following notation

$$\sigma(A_1, ..., A_n) = \left\{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} : \ det(x_1 A_1 + \cdots + x_n A_n) = 0 \right\}.$$

An infinite dimensional analog of determinantal manifold, called **projective joint spectrum** of a tuple of operators acting on a Hilbert space was introduced in [38]. For operators $A_1, \ldots, A_n$ acting on a Hilber space H it is defined by

$$\sigma(A_1, \ldots, A_n)$$
$$= \left\{ [x_1 : \cdots : x_n] \in \mathbb{CP}^{n-1} : \ x_1 A_1 + \cdots + x_n A_n \text{ is not invertible} \right\}.$$

To avoid trivial redundancies it is frequently assumed that at least one of the operators is invertible, and, thus, can be assumed to be the identity. In what follows we always assume that $A_{n+1} = I$. It was shown in [35] that a lot of information can be obtained from the part of the joint spectrum that lies in the chart $\{x_{n+1} \neq 0\}$ (and, therefore, we can take $x_{n+1} = -1$). We call this part the **proper joint spectrum** and denote it $\sigma_p(A_1, \ldots, A_n)$,

$$\sigma_p(A_1, \ldots, A_n) = \left\{ (x_1, ..., x_n) \in \mathbb{C}^n : \ x_1 A_1 + \cdots + x_n A_n - I \right.$$
$$\left. \text{is not invertible} \right\}.$$

When the dimension of H is finite the projective joint spectrum coincides with the corresponding determinantal manifold. This is why we use the same notation in both cases. In this paper we concentrate on the finite dimensional case, so we will use "determinantal manifold" and "projective joint spectrum" interchangiably. Of course, in the finite dimensional case

$$\sigma_p(A_1, \ldots, A_n) = \left\{ (x_1, , \ldots, , x_n) \in \mathbb{C}^n : det(x_1 A_1 + \cdots + x_n A_n - I) = 0 \right\}.$$

Matrix pencils have been under investigation for a long time. Ultimately this line of research led Frobenius to laying out the foundation of representation theory, cf [17–19].

One of the basic questions regarding determinantal manifolds was when a hypersurface in the projective space admits a determinantal representation. The number of publications in this area is very substantial. Without trying to give an exhaustive account of the results here we just refer a reader to [4, 9–14, 24, 25, 28, 29, 37] and references there.

In this paper we investigate joint spectra from a different angle: given that an algebraic hypersurface in $\mathbb{CP}^n$ has a determinantal representation, what does the geometry of the surface tell us about relations between operators in a representing pencil? This

line of research was paid much less attention until recently. The only early result we can mention is the one by Motzkin and Taussky [33]. The situation changed in the last decade when joint spectra have been intensely scrutinized exactly from this point of view (see [2, 3, 5, 8, 15, 16, 21–23, 30, 34–36, 38], and references there).

In particular, papers [34, 35] contain a local spectral analysis near the reciprocal of a spectral point of one of the operators when this point does not belong to the singular locus of the joint spectrum. Results of this analysis led to a spectral characterization of representations of non-special finite Coxeter groups in [8] and of Hadamard matrices of Fourier type in [34]. Non-singularity was essential in these cases. Here we concentrate on the local spectral analysis in a neighborhood of a point at which the joint spectrum is reducible, and, therefore, this point belongs to the singular locus of the joint spectrum. One of the difficulties of carrying out spectral analysis in this setting is that, in general, spectral projections related to operator families analytically depending on parameters (in our setting parameters are coordinates $(x_1, \ldots, x_n)$ of a point in the joint spectrum) might blow up when approaching a singular point. It turned out that if one of the matrices is diagonal, and the joint spectrum satisfies regularity conditions (a) and (b) introduced below, then the limit projections exist when approaching a singular point along a curve in a spectral component that is non-tangential to the singular locus.

Write

$$\sigma(A_1, ..., A_n, -I) = \left\{ R_1(x_1, .., x_n, x_{n+1})^{m_1} ... R_s(x_1, ..., x_n, x_{n+1})^{m_s}, = 0 \right\}, \quad (1.1)$$

where $R_1, ..R_s$ are irreducible polynomials.

Suppose that $\lambda \in \sigma(A_1)$. We introduce the regularity conditions at λ, which we mentioned above, separately for $\lambda \neq 0$ and $\lambda = 0$..
(1). $\lambda \neq 0$.

In this case

$$\sigma_p(A_1, ..., A_n) = \left\{ R_1(x_1, .., x_n, 1)^{m_1} ... R_s(x_1, ..., x_n, 1)^{m_s} = 0 \right\}. \quad (1.2)$$

Let us denoted by I_λ the set of indexes $1 \leq j \leq s$ corresponding to the components of the proper projective spectrum passing through $(1/\lambda, 0, ..., 0, -1)$:

$$I_\lambda = \left\{ 1 \leq j \leq s : \ R_j(1/\lambda, 0..., 0, 1) = 0 \right\}. \quad (1.3)$$

We call the following conditions **the regularity conditions at** λ:

(a) $\frac{\partial R_j}{\partial x_1}(1/\lambda, 0, ..., 0, 1) \neq 0, \ j \in I_\lambda$
(b) For every pair $i \neq j$, $i, j \in I_\lambda$ the vectors $\partial R_i(1/\lambda, 0, ..., 0, 1)$ and $\partial R_j(1/\lambda, 0, ..., 0, 1)$ are not proportional, where

$$\partial R_l = \left(\frac{\partial R_l}{\partial x_1}, \ldots, \frac{\partial R_l}{\partial x_n} \right).$$

(2). $\lambda = 0$.

In this case we pass to the chart $\{x_1 \neq 0\}$ (and we take $x_1 = 1$ here). Let us denote by $\tilde{\sigma}_{p,1}(A_1, ..., A_n)$ the part of the projective joint spectrum that lies in this chart:

$$\tilde{\sigma}_{p,1}(A_1, ..., A_n)$$
$$= \left\{ (x_2, ..., x_n, x_{n+1}) \in \mathbb{C}^n : A_1 + \left(\sum_{j=2}^{n} x_j A_j \right) - x_{n+1} I \text{ is not invertible} \right\}.$$

If $0 \in \sigma(A_1)$, then $0 \in \tilde{\sigma}_{p,1}(A_1, ..., A_n)$, and $\tilde{\sigma}_{p,1}(A_1, ..., A_n)$ is given by

$$\tilde{\sigma}_{p,1}(A_1, ..., A_n) = \{R_1(1, x_2, .., x_{n+1})^{m_1}...R_s(1, x_2, ..., x_{n+1})^{m_s} = 0\}. \tag{1.4}$$

Similarly, we denote by I_0 the set of indexes corresponding to the components of (1.4) passing through the origin

$$I_0 = \big\{ 1 \leq j \leq s : R_j(1, 0, ..., 0) = 0 \big\}. \tag{1.5}$$

The regularity conditions here are:

($\tilde{a}$) $\frac{\partial R_j}{\partial x_{n+1}}(0) \neq 0$, $j \in I_0$.

($\tilde{b}$) For every pair $i \neq j$, $i, j \in I_0$ the vectors ∂R_i and ∂R_j at the origin are not proportional, and here $\partial R_l = \big(\frac{\partial R_l}{\partial x_2}, \ldots, \frac{\partial R_l}{\partial x_{n+1}} \big)$.

In both cases $\lambda \neq 0$ and $\lambda = 0$ we write $\hat{x} = (x_2, ..., x_n)$.

Let $\mathcal{O}$ be a small neighborhood of $(1/\lambda, 0, ..., 0) \in \mathbb{C}^n$, if $\lambda \neq 0$, or of the origin in $\mathbb{C}^n$, if $\lambda = 0$. Write

$$\tilde{R}_j(x_1, ..., x_n) = R_j(x_1, ..., x_n, 1) \text{ if } \lambda \neq 0$$
$$\tilde{R}_j(x_2, ..., x_n, x_{n+1}) = R(1, x_2, ..., x_{n+1}) \text{ if } \lambda = 0.$$

The singular locus of the surface $\big\{ \prod_{j \in I_\lambda} \tilde{R}_j = 0 \big\} \cap \mathcal{O}$ is an algebraic manifold in $\mathcal{O}$ which we denote by $\mathcal{U}$. Of course, geometrically $\mathcal{U}$ coincides with singular locus of $\sigma_p(A_1, ..., A_n)$ or $\tilde{\sigma}_{p,1}(A_1, ..., A_n)$ in $\mathcal{O}$, but they have different multiplicities: the multiplicity of each regular point of $\big\{ \prod_{j \in I_\lambda} R_j = 0 \big\}$ is 1.

Conditions (a) and (b) ($\tilde{a}$ and $\tilde{b}$ respectively) imply that $\mathcal{U}$ has codimension greater than 1. Let us denote by $\tilde{\mathcal{U}}$ the orthogonal projection of $\mathcal{U}$ onto the hyperplain $\{x_1 = 0\}$ if $\lambda \neq 0$, and on the hyperplain $\{x_{n+1} = 0\}$, if $\lambda = 0$,

$$\tilde{\mathcal{U}} = \{(x_2, ..., x_n) : \exists (x_1, x_2, ..., x_n) \in \mathcal{U}\}, \ \lambda \neq 0,$$

and

$$\tilde{\mathcal{U}} = \{(x_2, ..., x_n) : \exists (x_2, ..., x_{n+1}) \in \mathcal{U}\}, \ \lambda = 0,$$

then $\tilde{\mathcal{U}}$ is a set of positive codimension in the plain $\{x_1 = 0\}$ for $\lambda \neq 0$, and in the plain $\{x_{n+1} = 0\}$ when $\lambda = 0$.

Let $j \in I_\lambda$. Since by condition a) when $\lambda \neq 0$, (by $\tilde{a}$), when $\lambda = 0$), $\frac{\partial R_j}{\partial x_1}(1/\lambda, 0, ..., 0) \neq 0$, $\left(\frac{\partial R_j}{\partial x_{n+1}}(0, ..., 0) \neq 0\right)$, by the implicit function theorem, in a small neighborhood of the origin in $\mathbb{C}^{n-1}$, $\mathcal{O}(\epsilon_j)$, the first coordinate x_1 (the $(n+1)$st coordinate x_{n+1}, respectively) can be expressed as an analytic function $x_{1,\lambda,j}(\hat{x})$ $(x_{n+1,0,j}(\hat{x})$ when $\lambda = 0)$, of $\hat{x} = (x_2, ..., x_n)$ satisfying

$$x_{1,\lambda,j}(0) = 1/\lambda, \ \tilde{R}_j(x_{1,\lambda,j}(\hat{x}), \hat{x}) = 0, \ \hat{x} \in \mathcal{O}(\epsilon_j), \ \text{if } \lambda \neq 0,$$
$$x_{n+1,0,j}(0) = 0, \ \tilde{R}_j(\hat{x}, x_{n+1,0,j}(\hat{x})) = 0, \ \hat{x} \in \mathcal{O}(\epsilon_j), \ \text{if } \lambda = 0. \quad (1.6)$$

Conditions a) and b) ($\tilde{a}$) and $\tilde{b}$)) imply that in $\mathcal{O}(\epsilon_j)$ we have $1 - x_{1,\lambda,j} = O(|\hat{x}|)$ (respectively $x_{n+1,0,j} = O(|\hat{x}|)$ for $\lambda = 0$), so that there are $d_j > 0$ such that

$$|1 - x_{1,\lambda,j}(\hat{x})| \leq d_j |\hat{x}|, \ \hat{x} \in \mathcal{O}(\epsilon_j), \ \lambda \neq 0$$
$$|x_{n+1,0,j}(\hat{x})| \leq d_j |\hat{x}|, \ \hat{x} \in \mathcal{O}(\epsilon_j), \ \lambda = 0. \quad (1.7)$$

Let $\lambda \in \sigma(A_1)$, $\lambda \neq 0$, $j \in I_\lambda$. Suppose that

$$y_{j,\lambda}(\hat{x}) = (x_{1,\lambda,j}(\hat{x}), \hat{x}) \quad (1.8)$$

is a regular spectral point which belongs to the jth component $\{\tilde{R}_j = 0\}$, and $\delta_j(x)$ is so small that no eigenvalue of $x_{1,\lambda,j}(\hat{x})A_1 + x_2 A_2 + \cdots + x_n A_n$ other than 1 is in the $\delta_j(x)$-neighborhood of 1, we denote by $P_{j,\lambda}(\hat{x})$ the projection

$$P_{j,\lambda}(\hat{x}) = \frac{1}{2\pi i} \int_{|w-1|=\delta_j(x)} \left(w - (x_{1,\lambda,j}(\hat{x})A_1 + \cdots + x_n A_n)\right)^{-1} dw. \quad (1.9)$$

If $\lambda = 0$ is in the spectrum of A_1, $x = (\hat{x}, x_{n+1,0,j}(\hat{x}))$ is in the jth component of $\tilde{\sigma}_{p,1}(A_1, ..., A_n)$, and $\delta_j(x)$ is small enough so that no non-zero eigenvalue of $A_1 + x_2 A_2 + , \ldots, + x_n A_n - x_{n+1,,0,j}(\hat{x})I)$ is in the $\delta_j(x)$-neighborhood of the origin, the corresponding projection $P_{j,0}(\hat{x})$ is given by

$$P_{j,0}(\hat{x}) = \frac{1}{2\pi i} \int_{|w|=\delta_j(x)} \left((w + x_{n+1,0,j}(\hat{x}))I - A_1 - \cdots - x_n A_n\right)^{-1} dw. \quad (1.10)$$

Our first result is

Theorem 1.11 *Let $(A_1, ..., A_n)$ be a matrix tuple, such that A_1 is a normal matrix. Suppose that $\lambda \in \sigma(A_1)$, and that $\sigma_p(A_1, \ldots, A_n)$, if $\lambda \neq 0$, or $\tilde{\sigma}_{p,1}(A_1, \ldots, A_n)$, if $\lambda = 0$, satisfies conditions (a) and (b) if $\lambda \neq 0$ and respctively conditions $\tilde{a}$) and $\tilde{b}$) if $\lambda = 0$. Fix $\hat{x}$ such that the line $\{t\hat{x} : t \in \mathbb{C}\}$ is not tangent to any component of $\tilde{\mathcal{U}}$ at the origin. Then for $j \in I_\lambda$ each projection $P_{j,\lambda}(t\hat{x})$ for $\lambda \neq 0$ and $P_{j,0}(t\hat{x})$ for $\lambda = 0$, can be extended to $t = 0$, so, it becomes an analytic function of t in some neighborhood $\mathcal{O}$ of the origin.*

We denote this limit projection by $\mathcal{P}_{j,\lambda}(\hat{x})$,

$$
\begin{aligned}
\mathcal{P}_{j,\lambda}(\hat{x}) &= \lim_{t \to 0} P_{j,\lambda}(t\hat{x}), \ \lambda \neq 0, \\
\mathcal{P}_{j,0}(\hat{x}) &= \lim_{t \to 0} P_{j,0}(t\hat{x}), \ \lambda = 0.
\end{aligned} \tag{1.12}
$$

Section 2 is devoted to the proof of this result.

We express certain pairwise relations between operators in the tuple in terms of these limit projections in Sect. 3. The main results here are the following Theorems formulated for the pair (A_1, A_2). Of course, when we have only 2 matrices in the tuple, that is when $n = 2$, the dependance on $\hat{x}$ is redundant, and we simply write $\mathcal{P}_{j,\lambda}$.

Let A_1 be a normal matrix and

$$
A_1 = \sum_{\lambda \in \sigma(A_1)} \lambda \mathscr{P}_\lambda \tag{1.13}
$$

be its spectral resolution. For $\lambda \in \sigma(A_1)$ write

$$
T_\lambda = \sum_{\mu \in \sigma(A_1), \mu \neq \lambda} \frac{\mathscr{P}_\mu}{\lambda - \mu}. \tag{1.14}
$$

Theorem 1.15 *Let A_1 and A_2 be $N \times N$ matrices with A_1 being normal. Suppose that the regularity conditions are satisfied at every spectral point of A_1. Further, suppose that $\lambda \in \sigma(A_1)$, and that the multiplicities m_j, $j \in I_\lambda$ are equal to 1. Then the following relations hold:*

$$
\mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} + x'_{1,\lambda,j}(0)\mathcal{P}_{j,\lambda} = 0, \ \lambda \neq 0, \tag{1.16}
$$

$$
\mathcal{P}_{j,0} A_2 \mathcal{P}_{j,0} - x'_{3,0,j}(0)\mathcal{P}_{j,0} = 0, \ \lambda = 0 \tag{1.17}
$$

$$
\mathcal{P}_{j,\lambda} A_2 T_\lambda A_2 \mathcal{P}_{j,\lambda} + \frac{x''_{1,\lambda,j}(0)}{2}\mathcal{P}_{j,\lambda} = 0, \ \lambda \neq 0, \tag{1.18}
$$

$$
\mathcal{P}_{j,0} A_2 T_0 A_2 \mathcal{P}_{j,0} - \frac{x''_{3,0,j}(0)}{2}\mathcal{P}_{j,0} = 0, \ \lambda = 0. \tag{1.19}
$$

Theorem 1.20 *Let $\lambda \in \sigma(A_1)$, $\lambda \neq 0$, and both pairs of matrices (A_1, A_2) and $(A_1, A_1 A_2)$ satisfy the conditions of Theorem 1.15. Then*

$$
\mathcal{P}_{j,\lambda} A_2^2 \mathcal{P}_{j,\lambda} = \frac{z''_{1,\lambda,j}(0) + 2\lambda\left(x'_{1,\lambda,j}(0)\right)^2 - \lambda^2 x''_{1,\lambda,j}(0)}{2\lambda} \mathcal{P}_{j,\lambda}, \tag{1.21}
$$

where $z_1 = z_{1,\lambda,j}(z_2)$ is the local representation of the jth component of $\sigma_p(A_1, A_1 A_2)$, $\{\tilde{R}_j(z_1, z_2) = 0\}$, near $(1/\lambda, 0))$.

Finally, in Sect. 4 we give an application of our technique to representations of Coxeter groups. Here we prove the following result which might be viewed as a rigidity type theorem for representations of Coxeter groups.

Theorem 1.22 *Let G be a Coxeter group with Coxeter generators $g_1, ..., g_n$, and let ρ be an m-dimensional linear representation of G. For every $2 \leq i \leq n$ denote by $\tilde{\rho}_i$ the representation of the Dihedral group D_{1i} generated by g_1 and g_i, which is induced by ρ. Assume that the following condition is satisfied:*

(∗) *For $2 \leq i \leq n$ no irreducibble representation of D_{1i} is included in the decomposition of $\tilde{\rho}_i$ with coefficient bigger than 1.*

Suppose that $A_1, ..., A_n$ are complex matrices in $M(N)$ such that A_1 is normal, $\| A_j \| = 1$, $j = 2, ..., n$, and conditions (a)–(b), $\tilde{a}$) - $\tilde{b}$) are satisfied for the pairs (A_1, A_j) and $(A_1, A_1 A_j)$, $j = 2, ..., n$ at every spectral point of A_1. Also suppose that

(I) $\sigma_p(A_1, ..., A_n) \supset \sigma_p(\rho(g_1), ..., \rho(g_n))$.
(II) $\exists \epsilon > 0$ *such that for every point* $\zeta_j^+ = (0, ..., \underbrace{1}_{J}, 0, ..., 0)$ *and* $\zeta_j^- =$

$(0, ..., \underbrace{-1}_{J}, 0, ..., 0)$, $j = 1, ..., n$ *we have*

$$\sigma_p\left(A_1, ..., A_n, A_1 A_2, ..., A_1 A_n\right) \cap \mathcal{O}_\epsilon(\zeta_j^\pm)$$
$$= \sigma_p\left(\rho(g_1), ..., \rho(g_n), \rho(g_1)\rho(g_2), ..., \rho(g_1)\rho(g_n)\right) \cap \mathcal{O}_\epsilon(\zeta_j^\pm).$$

Then

(1) *There exists an m-dimensional subspace L of $\mathbb{C}^N$ invariant under the action of each matrix A_j, $j = 1, ..., n$.*
(2) *Restrictions of A_j, $j = 1, ..., n$ to L are unitary, self-adjoint, and generate a representation, $\hat{\rho}$ of G, and*

$$\sigma_p\left(A_1\Big|_L, ..., A_n\Big|_L\right) = \sigma_p\left(\rho(g_1), ..., \rho(g_n)\right) \qquad (1.23)$$

.
(3) *If G is finite, non-special Coxeter group (that is either a Dihedral, or of types A,B, or D), then $\hat{\rho}$ is unitary equivalent to ρ.*

2 Limit Projections Along Components. Proof of Theorem 1.11

.
Observe that under conditions (a), (b) and ($\tilde{a}$), ($\tilde{b}$) we have:

- $M_\lambda = \sum_{j \in I_\lambda} m_j$, is the multiplicity of λ in $\sigma(A_1)$. Here m_j are exponents from (1.1).
- If $\lambda \neq 0$ and $\mathcal{O}$ is a small enough neighborhood of $(1/\lambda, 0 ..., 0)$, then for every $x = (x_1, ..., x_n)$ sufficiently close to $(1/\lambda, 0, ..., 0)$ and every $j \in I_\lambda$ the line through the origin and x has only one point of intersection with the hypersurface $\{\tilde{R}_j = 0\}$ which lies in $\mathcal{O}$. Similarly, if $\lambda = 0$ and $\mathcal{O}$ is a small enough neighborhood of the origin, every line parallel to the x_{n+1}-axis and passing through a point $\hat{x}$ close

to 0 has only one point of intersection with the hypersurface $\{\tilde{R}_j = 0\}$, $j \in I_0$ which lies in $\mathcal{O}$.

"The component $\{\tilde{R}_j = 0\}$ of $\sigma_p(A_1, ..., A_n)$ ($\tilde{\sigma}_{p,1}(A_1, ..., A_n)$) has multiplicity m_j" means that for every regular point in this component that does not belong to any other component, the rank of the projection (1.9) is equal to m_j. As mentioned in the introduction, δ_j in (1.9) is so small that δ_j-neighborhood of 1 does not contain any eigenvalues of $x_{1j}(\hat{x})A_1 + \cdot + x_n A_n$ different from 1 (respectively δ_j-neighborhood of 0 does not contain non-trivial eigenvalues of $A_1 + x_2 A_2 + \cdot - x_{n+1} I$). The existence of such δ_j follows from conditions b) and $\tilde{b}$).

In general, even under the regularity conditions projection $P_{j,\lambda}(\hat{x})$ given by (1.9) might "blow up" as the point $\hat{x}$ approaches 0. A simple example is:

$$A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In this case $\hat{x} = x_2$, and the joint spectrum, $\sigma_p(A_1, A_2)$ is

$$\{x_1 + x_2 - 1 = 0\} \cup \{x_1 - x_2 - 1 = 0\},$$

so conditions (a) and (b) are satisfied at (1,0). When a point (x_1, x_2) approaches $(1, 0)$ along the components $\{x_1 + x_2 - 1 = 0\}$ and $\{x_1 - x_2 - 1 = 0\}$, the corresponding projections $P_{1,1}(x_2)$ and $P_{2,1}(x_2)$ are given by

$$P_{1,1}(x_2) = \begin{bmatrix} 1 & \frac{1-x_2}{2x_2} \\ 0 & 0 \end{bmatrix}, \quad P_{2,1}(x_2) = \begin{bmatrix} 0 & -\frac{1+x_2}{2x_2} \\ 0 & 1 \end{bmatrix}.$$

We see that the norm of each projection goes to ∞ as $x_2 \to 0$.

Theorem 1.11 claims that this does not happen in the case when A_1 is normal.

Proof of Theorem 1.11 1. $\lambda \neq 0$.
Since scaling preserves conditions (a) and (b), we may replace A_1 with A_1/λ and assume that $\lambda = 1$. To slightly simplify the notation, in this proof we will write $x_{1j}(\hat{x})$, $j \in I_1$ instead of $x_{1,1,j}(\hat{x})$ and $P_j(\hat{x})$ instead of $P_{j,1}(\hat{x})$.

Let $|\hat{x}| < \min\{\epsilon_j : j = 1, ..., r\}$, where ϵ_j are constants from (1.7), and suppose that $\mathcal{O} = \cap_j \mathcal{O}(\epsilon_j)$.

Denote by $\Gamma_{ij} = \{\hat{x} \in \mathcal{O} : x_{1i}(\hat{x}) = x_{1j}(\hat{x})\}$, $i \neq j$. Then $\mathcal{U} = \cup_{i,j=2}^k \Gamma_{ij}$, and $\tilde{\mathcal{U}} = \cup_{i,j=2}^k \tilde{\Gamma}_{ij}$, where $\tilde{\Gamma}_{ij}$ is the projection of Γ_{ij} onto $\{x_1 = 0\}$.

Condition b) implies that the tangent planes at the origin $\hat{x} = 0$ to the surfaces $\{x_1 = x_{1j}(\hat{x}), \ j \in I_1\}$ are pairwise different. These tangent planes are given by

$$x_1 - 1 = \sum_{l=2}^n \frac{\partial x_{1j}}{\partial x_l}(0)x_l,$$

so that the tangent plain in $\{x_1 = 0\}$ to $\tilde{\Gamma}_{ij}$ is given by

$$\left\{ \hat{x} : \sum_{l=2}^{n} \left(\frac{\partial x_{1j}}{\partial x_l}(0) - \frac{\partial x_{1j}}{\partial x_l}(0) \right) x_l = 0 \right\}. \tag{2.1}$$

If a point $\hat{x}$ does not belong to any of the plains given by (2.1), then there exists $a > 0$ and $\delta > 0$ such that for every $z \in \mathbb{C}$ with $|z| < \delta$ we have

$$|x_{1i}(z\hat{x}) - x_{1j}(z\hat{x})| \geq a|z|, \ i \neq j. \tag{2.2}$$

The constant a continuously depends on

$$\min_{i,j} \left\{ \left| \left| \sum_{k=2}^{n} \left(\frac{\partial x_{1j}}{\partial x_k}(0) - \frac{\partial x_{1j}}{\partial x_k}(0) \right) x_k \right| \right| \right\},$$

and, therefore, may be chosen such that (2.2) holds uniformly in a neighborhood of $\hat{x}$.

Recall that for each $j \in I_\lambda$ the point $y_{j,\lambda}(\hat{x})$ in the component $\{\tilde{R}_j = 0\}$ was defined by (1.8). In our case $\lambda = 1$, and instead of $y_{j,1}(\hat{x})$ we will write $y_j(\hat{x})$. For each $i \in I_1$ and $z \in \mathbb{C}$ close to 0 define $\tau_{ij}(z) \in \mathbb{C}$ by the condition that it is the closest to 1 root of the following equation in τ

$$\tilde{R}_i(\tau y_j(z\hat{x})) = 0. \tag{2.3}$$

As mentioned above, conditions (a) and (b) imply that every line in $\mathbb{C}^n$ passing through the origin and close to the x_1-axis has only one point of intersection with the hypersurface $\{\tilde{R}_i = 0\}$ that is close to $(1, 0, ..., 0)$. When this line is the line passing through the origin and $y_j(z\hat{x})$, this point determines $\tau_{ij}(z)$ in the following way. Write the homogeneous decomposition of $\tilde{R}_j$:

$$\tilde{R}_i(x) = \tilde{R}_i(x_1, \hat{x}) = \sum_{l=0}^{t_i} S_{il}(x),$$

where $S_{il}(x)$ is a homogeneous polynomial of degree l in $x_1, ..., x_n$ and t_i is the degree of $\tilde{R}_i$. We always assume that $\tilde{R}_0 = -1$. Then $\tau_{ij}(z)$ is the only root of the equation (in τ)

$$\sum_{l=0}^{t_i} \tau^l S_{il}(y_j(z\hat{x})) = 0$$

which is close to 1. Since this root has multiplicity 1, $\tau_{ij}(z)$ is an analytic functiion of z in a small neighborhood of the origin in $\mathbb{C}$, which we denote by U.

Of course, $\tau_{jj}(z) = 1$, $\tau_{ij}(z) \neq 1$ for $i \neq j$, $z \neq 0$, and, of courses, $\tilde{R}_i(\tau_{ij}(z)y_j(z\hat{x})) = 0$ for all $z \in U$. It is easy to see that (2.2) implies that there

exist positive constants c and C such that for all $j \neq i$

$$c|z| \leq |\tau_{ij}(z) - 1| \leq C|z|. \tag{2.4}$$

Using the notation $A(x) = x_1 A_1 + \cdots + x_n A_n$, it follows from the definition of the joint spectrum that the reciprocals $\mu_{ij}(z) = \frac{1}{\tau_{ij}(z)}$ are eigenvalues of $A(y_j(z\hat{x}))$. Now, (2.4) implies that a similar estimate (with different constants) holds for the eigenvalues μ_{ij}:

$$c_1|z| \leq |\mu_{ij}(z) - 1| \leq C_1|z|, \ \ j \neq i. \tag{2.5}$$

This shows that δ_j in (1.9) should be chosen smaller than $c_1|z|$ (of course, the integral (1.9) is the same for all $\delta_j < c_1|z|$). We call it $\delta_j(z)$.

Let $e_1, ..., e_N$ be an eigenbasis for A_1, where $e_1, ..., e_{M_1}$ are eigenvectors with eigenvalue 1, and $e_{M_1+1}, ..., e_N$ correspond to eigenvalues different from 1, so that in the basis $e_1, ..., e_N$ the matrix A_1 is diagonal with the first M_1 diagonal entries equal to 1 (recall, $M_1 = \sum_{j \in I_1} m_j$), and the others, which we denote by $\alpha_{M_1+1}, ..., \alpha_N$, being different from 1.

Write $A_2, ..., A_n$ in the basis $e_1, ..., e_N$:

$$A_j = \left[a_{lm}^j \right]_{l,m=1}^N .$$

For $t \in \mathbb{C}$ and $j \in I_1$ let $\mathcal{M}_j(t)$ be the $M_1 \times M_1$ block of the matrix

$$\left(t - \sum_{l=2}^n \frac{\partial x_{1j}}{\partial x_l} \bigg|_{(0,...,0)} x_l \right) I - \sum_{l=2}^n x_l A_l$$

formed by the first M_1 rows and the first M_1 columns. Here $\hat{x} = (x_2, ..., x_n)$ is the point that appeared in Theorem 1.11. Consider the following polynomials in t

$$\mathcal{S}_j(t) = det(\mathcal{M}_j(t)).$$

For each j this is a non-trivial polynomial of degree M_1 $\left(\mathcal{S}_j(t) \to \infty \text{ as } t \to \infty \right)$. Thus, there exists some positive number $b < c_1$ (the constant from (2.5)) such that these polynomials do not vanish in the punctured disk of radius b centered at the origin:

$$\mathcal{S}_j(t) \neq 0 \text{ for } t \in \mathbb{C}, \ 0 < |t| < b, \ j = 1, ..., k. \tag{2.6}$$

Choose some t_0 satisfying $0 < t_0 < b$.

Next, we will show that there exists a positive constant $\mathcal{M}$ such that for $|w - 1| = t_0|z|$, where z is close to zero, the following norm estimate holds:

$$\left\| \left(wI - x_{1j}(z\hat{x})A_1 - \sum_{s=2}^n zx_s A_s \right)^{-1} \right\| \leq \frac{\mathcal{M}}{|z|}. \tag{2.7}$$

The norm here is understood as the norm of an operator acting on $\mathbb{C}^N$.

Let $\eta \in \mathbb{C}^N$, $\| \eta \| = 1$, and $\zeta \in \mathbb{C}^N$ satisfies

$$\left(wI - x_{1j}(z\hat{x})A_1 - \sum_{l=2}^{n} zx_l A_l \right) \zeta = \eta, \tag{2.8}$$

which is a system of N equations in variables $\zeta_1, ..., \zeta_N$, coordinates of ζ in the basis $e_1, ..., e_N$. Since $w - 1 = t_0 z e^{i\theta}$ and $1 - x_{1j}(z\hat{x}) = z\left(\sum_{s=2}^{n} \frac{\partial x_{1j}}{\partial x_s}|_{(0,...,0)} x_s \right) + O(|z|^2)$, the first M_1 equation of this system can be written as

$$z\left(\mathcal{M}_j(t_0 e^{i\theta}) + O(z)I_{M_1} \right)(^{M_1}\widehat{\zeta}) = (^{M_1}\widehat{\eta}) - z\mathcal{N} \, (\widehat{\zeta}^{N-M_1}), \tag{2.9}$$

where I_{M_1} is the $M_1 \times M_1$ identity matrix, $(^{M_1}\widehat{\zeta}) = (\zeta_1, ..., \zeta_{M_1})$ and $(^{M_1}\widehat{\eta}) = (\eta_1, ..., \eta_{M_1})$, $(\widehat{\zeta}^{N-M_1}) = (\zeta_{M_1+1}, ..., \zeta_N)$, and $\mathcal{N}$ is the $M_1 \times (N - M_1)$ part of the matrix $x_2 A_2 + \cdots + x_n A_n$ consisting of the entrees in the first M_1 rows and the last $N - M_1$ columns.

Consider (2.9) as a system of equations in $\zeta_1, ..., \zeta_{M_1}$. Then (2.6) implies that for sufficiently small z the main determinant of this system does not vanish (it is equal to $z^{M_1} \mathcal{S}_j(t_0 e^{i\theta}) + O(|z|^{M_1+1})$). Now, Cramer's rule implies that $\zeta_1, ..., \zeta_{M_1}$ are expressed in terms of $\zeta_{M_1+1}, ..., \zeta_N$ and $\eta_1, ..., \eta_{M_1}$ in the following way:

$$\zeta_i = \left[\frac{1}{z} \sum_{l=1}^{M_1} \frac{D_{il}(z)}{\mathcal{S}_j(t_0 e^{i\theta})} \eta_l \right] + \sum_{l=M+1}^{N} \frac{\psi_{il}(z)}{\mathcal{S}_j(t_0 e^{i\theta})} \zeta_l, \quad i = 1, ..., k_1. \tag{2.10}$$

where $D_{il}(z)$ and $\psi_{il}(z)$ are uniformly bounded analytic functions in a small punctured neighborhood of the origin (and, therefore, analytically extendable to the origin). Substitute these expressions of ζ_i, $i = 1, ..., M_1$ in the remaining $N - M_1$ equations of the system (2.8). We obtain a system in the following form:

$$\left[(1 + t_0 z e^{i\theta})I_{N-M_1} - x_{1j}(z\hat{x})\Lambda - \mathcal{L}(z) \right] (\widehat{\zeta}^{N-M_1})$$
$$= (\widehat{\eta}^{N-M_1}) - \mathcal{T}(z)(^{M_1}\widehat{\eta}), \tag{2.11}$$

where I_{N-M_1} is the $(N-M_1) \times (N-M_1)$ identity matrix, Λ is the $(N-M_1) \times (N-M_1)$ diagonal matrix with $\alpha_{M_1+1}, ..., \alpha_N$ on the main diagonal (they are coming from A_1), $\mathcal{L}(z)$ is an $(N - M_1) \times (N - M_1)$ matrix whose all entries are of the order of z, and $\mathcal{T}(z)$ is the following $(N - M_1) \times M_1$ matrix with entrees

$$(\mathcal{T}(z))_{im} = \frac{1}{\mathcal{S}_j(t_0 e^{i\theta})} \sum_{r=2}^{n} x_r \sum_{l=1}^{M_1} a_{il}^r D_{lm}(z),$$

that is the product of the lower left $(N - M_1) \times M_1$ block of the matrix $\sum_{r=2}^{n} x_r A_r$ and the matrix $D(z) = \left[\frac{D_{lm}(z)}{\mathcal{S}_j(t_0 e^{i\theta})} \right]$.

Since for z close to zero, $x_{1j}(z\hat{x})$ is close to one, and since $\alpha_{M+1}, ..., \alpha_N$ are not equal to one, it follows that the main determinant of (2.11) considered as a system in $\zeta_{M_1+1}, ..., \zeta_N$ stays away from 0 as $z \to 0$. Since for all $j = 1, ..., N$, $|\eta_J| \leq 1$, this implies that $\zeta_{M_1+1}, ..., \zeta_N$ stay bounded as $z \to 0$. In summary: there exists a constant $\tilde{\mathcal{M}}$ such that

$$|\zeta_j| \leq \frac{\tilde{\mathcal{M}}}{|z|} \ j = 1, ..., M_1, \ |\zeta_j| \leq \tilde{\mathcal{M}}, \ j = M_1 + 1, ..., N. \tag{2.12}$$

This yeilds

$$\| \ \zeta \ \| \leq \frac{\mathcal{M}_1}{|z|} \tag{2.13}$$

for some constant $\mathcal{M}_1$ independent of z and η, which proves (2.7).

Now, (2.7) implies

$$\left\| P_j(y(z\hat{x})) \right\| = \left\| \frac{1}{2\pi i} \int_{|w-1|=t_0|z|} \left(wI - x_{1j}(z\hat{x})A_1 - \sum_{l=2}^{n} zx_l A_l \right)^{-1} dw \right\|$$
$$\leq \frac{1}{2\pi} \left(\max_{|w-1|=t_0|z|} \left\| \left(wI - x_{1j}(z\hat{x})A_1 - \sum_{l=2}^{n} zx_l A_l \right)^{-1} \right\| \right) 2\pi t_0|z| \leq \mathcal{M}_1 t_0. \tag{2.14}$$

Since, the integral

$$\int_{|w-1|=t_0|z|} \left(wI - x_{1j}(z\hat{x})A_1 - \sum_{l=2}^{n} zx_l A_l \right)^{-1} dw$$

is an analytic function in z in a small punctured disk centered at the origin, Riemann's theorem implies that it is analytically extendable to $z = 0$. This completes the proof of Theorem 1.11 for $\lambda \neq 0$.

2. $\lambda = 0$.

The details of the proof in this case are very similar to those when $\lambda \neq 0$. The only differences are the following. If $\hat{x}$ is sufficiently close to 0, then close to 0 spectral points of the matrix $A_1 + x_2 A_2 + \cdots + x_n A_n - x_{n+1,j}(\hat{x})I$ are $x_{n+1,i}(\hat{x}) - x_{n+1,j}(\hat{x})$, $i \in I_0$ (here we again write $x_{n+1,j}$ instead of $x_{n+1,0,j}$). Now, condition $\tilde{b}$) implies that there exists $c > 0$ such that for sufficiently small z

$$|x_{n+1,i}(z\hat{x}) - x_{n+1,j}(z\hat{x})| > c|z|, \ i, j \in I_0, \ i \neq j.$$

Further, we choose a basis $e_1, \ldots, e_N$ which is an eigenbasis for A_1 so that eigenvectors $e_1, \ldots, e_{M_0}$ are 0-eigenvectors, and $e_{M_0+1}, \ldots, e_N$ are eigenvectors with non-trivial eigenvalues, and write $A_2, \ldots, A_n$ in this basis. The matrix $\mathcal{M}_j(t)$ in

this case is defined as $M_0 \times M_0$ block of the matrix

$$\left(t + \sum_{l+2}^{n} \frac{\partial x_{n+1,j}}{\partial x_l} x_l \right) I - \sum_{l=2}^{n} x_l A_l$$

of the first M_0 rows and the first M_0 columns. Again, there exists $b > 0$ such that the polynomial $\mathcal{S}_j(t) = det(\mathcal{M}_j(t))$ does not vanish in the punctured disk of radius b centered at the origin and we pick $0 < t_0 < b$. The remaining details of the proof are practically identical to those in the case $\lambda \neq 0$ (of course, the integral (1.9) in (2.14) is replaced with the integral (1.10)).

The proof is complete.

Remark 1 We would like to note that the above proof holds when A_1 is just a diagonal matrix, not necessarily normal.

Remark 2 In general, limit projections depend on the choice of $\hat{x}$.

Remark 3 It is easily seen that a similar proof holds when a point approaches $(1, 0, ..., 0)$ not along a line, but along a smooth curve $\gamma(z) \in \mathbb{C}^{n-1}$ which is not tangent to any component of $\hat{\mathcal{U}}$ at the origin.

3 Relations Between Component Projections $\mathcal{P}_{j,\lambda}$: Proofs of Theorems 1.15 and 1.20

Each projection $\mathscr{P}_\lambda$ in the spectral resolution (1.13) is, of course, represented by the integral

$$\mathscr{P}_\lambda = \frac{1}{2\pi i} \int_{\gamma_\lambda} (w - A_1)^{-1} dw,$$

where γ_λ is a contour which separates λ from the rest of the spectrum of A_1.

Proposition 3.1 *The limit projections $\mathcal{P}_{j,\lambda}$ from Theorem* 1.11 *satisfy the following relations:*

$$\mathcal{P}_{i,\lambda} \mathcal{P}_{j,\lambda} = 0 \text{ if } i \neq j, \ i, j \in I_\lambda. \tag{3.2}$$
$$\sum_{j \in I_\lambda} \mathcal{P}_{j,\lambda} = \mathscr{P}_\lambda \tag{3.3}$$

Proof The proofs for $\lambda \neq 0$ and $\lambda = 0$ are very similar, so we will give it for $\lambda \neq 0$.

Fix $\lambda \in \sigma(A_1)$, $\lambda \neq 0$ and $j \in I_\lambda$. Let $\hat{x} \in \mathcal{O}$ - the neighborhood from the proof of Theorem 1.11. Equation (2.5) gives an estimate for $\mu_{ij}(\hat{x})$, $i \in I_\lambda$, the eigenvalues close to 1 of the operator $A(y_{j,\lambda}(\hat{x})) = x_{1,\lambda,j}(\hat{x}) A_1 + \cdots + x_n A_n$. Recall that $\mu_{ij}(\hat{x})$ are the reciprocals of the roots of Eq. (2.3) which in the proof of Theorem 1.11 were denoted by $\tau_{i,j}(1)$. In the proof of Theorem 1.11 we supresed the dependance of these

roots on λ, assuming that $\lambda = 1$, but here it is appropriate to call them $\tau_{i,j,\lambda}(z)$. Recall that $y_{j,\lambda}(\hat{x})$ was defined by (1.8). Write

$$\tau_{i,j,\lambda}(\hat{x}) = \tau_{i,j}(1), \ \ \tilde{y}_{i,j,\lambda}(\hat{x}) = \tau_{i,j,\lambda}(\hat{x})y_{j,\lambda}(\hat{x}).$$

Of course,

$$\tilde{y}_{j,j,\lambda}(\hat{x}) = y_{j,\lambda}(\hat{x}).$$

It was mentioned above that since $\tilde{R}_i$ is an irreducible polynomial, every regular point of $\{\tilde{R}_i = 0\}$ has multiplicity 1, so $\tau_{i,j,\lambda}(\hat{x})$ is a bounded analytic function of $\hat{x}$ in a punctured neighborhood of the origin, and, therefore, can be extended to the origin to become an analytic function in the whole neighborhood satisfying

$$\tau_{i,j,\lambda}(0) = 1.$$

It is also easy to compute that for $m = 2, \ldots, n$

$$\frac{\partial \tau_{i,j,\lambda}}{\partial x_m}(\hat{x}) = \frac{\tau_{i,j,\lambda}(\hat{x})\left(\frac{\partial x_{1,\lambda,i}}{\partial x_m}(\hat{x}) - \frac{\partial x_{1,\lambda,j}}{\partial x_m}(\hat{x})\right)}{x_{1,\lambda,j}(\hat{x}) - x_m\frac{\partial x_{1,\lambda,i}}{\partial x_m}(\hat{x})}, \tag{3.4}$$

and, therefore, (3.4)implies

$$\frac{\partial \tau_{j,\lambda,i}}{\partial x_m}(0) = \frac{\frac{\partial x_{1i}}{\partial x_m}(0) - \frac{\partial x_{1j}}{\partial x_m}(0)}{\lambda}. \tag{3.5}$$

Further, the projections $P_{i,\lambda}(\tilde{y}_{j,j,\lambda}(\hat{x}))$ and $P_{j,\lambda}(\tilde{y}_{i,j,\lambda}(\hat{x}))$ (defined by the integral (1.9)) satisfy

$$P_{i,\lambda}(\tilde{y}_{i,j,\lambda}(\hat{x}))P_{j,\lambda}(\tilde{y}_{j,j,\lambda}(\hat{x})) = 0, \ \ i \neq j.$$

Since $\tau_{i,j,\lambda}(\hat{x})\hat{x} \to 0$ as $\hat{x} \to 0$, Theorem 1.11 implies $P_{j,\lambda}(\tilde{y}_{j,j,\lambda}(\hat{x})) \to \mathcal{P}_{j,\lambda}$ and $P_{i,\lambda}(\tilde{y}_{i,j,\lambda}(\hat{x})) \to \mathcal{P}_{i,\lambda}$ as $\hat{x} \to 0$. This proves (3.2).

To prove (3.3) we fix a contour γ_λ that separates λ from the rest of the spectrum of A_1. Suppose that x is close to $(1/\lambda, 0, \ldots, 0)$. Consider the matrix

$$A(x) = x_1 A_1 + \cdots + x_n A_n.$$

Then $1 \in \sigma(A_1/\lambda)$ and $\mathcal{P}_\lambda$ is the projection on the 1-eigenspace of A_1/λ. Also, γ_λ/λ separates 1 from the rest of the spectrum of A_1/λ.

Of course, $A(x) \to A_1/\lambda$ as $\hat{x} \to 0$, and $x_1 \to 1/\lambda$. If $|x_1 - 1/\lambda| + |x_2| + \cdots + |x_n|$ is small enough, there are no spectral points of A_1/λ on γ_λ/λ, and, therefore,

$$\frac{1}{2\pi i}\int_{\gamma_\lambda/\lambda}(w - A(x))^{-1}dw \to \frac{1}{2\pi i}\int_{\gamma_\lambda/\lambda}\left(w - \frac{1}{\lambda}A_1\right)^{-1}dw = \mathcal{P}_\lambda,$$

as $x \to (1/\lambda, 0, \ldots, 0)$.

If $x_1 = x_{1,\lambda,j}(\hat{x})$, then the poles of $(w - A(x))^{-1}$ which lie inside γ_λ/λ are the reciprocals of $\tau_{i,j,\lambda}(\hat{x})$, $i \in I_\lambda$. Now, by the residue theorem

$$\frac{1}{2\pi i} \int_{\gamma_\lambda} (w - A(x))^{-1} dw = \sum_{i \in I_\lambda} P_{i,\lambda}(\tilde{y}_{i,j,\lambda}(\hat{x})) \to \sum_{i \backslash I_\lambda} P_{i,\lambda},$$

and we are done.

$\square$

We now concentrate on the case when $n = 2$. Of course, in this case $\hat{\mathcal{U}} = \{0\}$, and the limit projections are the same for all x_2 (which is $\hat{x}$ in this situation).

Proposition 3.6 *Let $n = 2$, $\lambda \in \sigma(A_1)$. Assume that conditions (a) and (b), or ã) and $\tilde{b}$) are satisfied at λ when $\lambda \neq 0$ or $\lambda = 0$ respectively. For $i, j \in I_\lambda$ we have*

$$\mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{i,\lambda} = 0, \ i \neq j \tag{3.7}$$

If there exists a neighborhood $\mathcal{O}$ of the origin, such that at every regular point of $\{\tilde{R}_j = 0\} \cap \mathcal{O}$, the range of $P_{j,\lambda}(x_2)$ consists of 1-eigenvectors for $A(y_{j,\lambda}((x_2)) = x_{1,\lambda,j}(x_2)A_1 + x_2 A_2$, if $\lambda \neq 0$, and , 0-eigenvectors of $A(y_{j,0}((x_2)) = A_1 + x_2 A_2 - x_{3,0,j}(x_2)I$ when $\lambda = 0$, or equivalently,

$$\left(A(y_{j,\lambda}(x_2)) - I\right)P_{j,\lambda}(x_2) = P_{j,\lambda}(x_2)\left(A(y_{j,\lambda}(x_2)) - I\right) = 0, \ \lambda \neq 0, \tag{3.8}$$
$$A(y_{j,0}((x_2))P_{j,0}(x_2) = P_{j,0}(x_2)A(y_{j,0}(x_2)) = 0, \ \lambda = 0, \tag{3.9}$$

then

$$\mathcal{P}'_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}_{j,\lambda} = \mathcal{P}_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}'_{j,\lambda}$$
$$= -\frac{x''_{1,\lambda,j}(0)}{2}\mathcal{P}_{j,\lambda}, \ \lambda \neq 0, \tag{3.10}$$
$$\mathcal{P}'_{j,0}\left(A_2 - x'_{3,.0,j}(0)I\right)\mathcal{P}_{j,0} = \mathcal{P}_{j,0}\left(A_2 - x'_{3,0,j}(0)I\right)$$
$$= \frac{x''_{3,0,j}(0)}{2}\mathcal{P}_{j,0}, \ \lambda = 0 \tag{3.11}$$
$$\mathcal{P}'_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}_{i,\lambda} = \mathcal{P}_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}'_{i,\lambda}$$
$$= 0, \ i \neq j, \ \lambda \neq 0, \tag{3.12}$$
$$\mathcal{P}'_{j,0}\left(A_2 - x'_{3,0,j}(0)I\right)\mathcal{P}_{i,0} = \mathcal{P}_{j,0}\left(A_2 - x'_{3,0,j}(0)I\right)\mathcal{P}'_{i,0}$$
$$= 0, \ i \neq j, \ \lambda = 0. \tag{3.13}$$

where $\mathcal{P}'_{j,\lambda} = \lim_{x_2 \to 0} \frac{dP_{j,\lambda}(x_2)}{dx_2}$, $j \in I_\lambda$.

Proof I. $\lambda \neq 0$.

We have for every x_2 close to 0:

$$P_{j,\lambda}(x_2)A(x_{1,\lambda,j}(x_2), x_2) = A(x_{1,\lambda,j}(x_2), x_2)P_{j,\lambda}(x_2). \qquad (3.14)$$

By Theorem 1.11 both sides of this equality are analytic functions of x_2 in a neighborhood of the origin. Differentiate with respect to x_2:

$$\frac{dP_{j,\lambda}(x_2)}{dx_2}A(x_{1,\lambda,j}(x_2), x_2) + P_{j,\lambda}(x_2)(x'_{1,\lambda,j}(x_2)A_1 + A_2)$$

$$= A(x_{1,\lambda,j}(x_2), x_2)\frac{dP_{j,\lambda}(x_2)}{dx_2} + (x'_{1,\lambda,j}(x_2)A_1 + A_2)P_{j,\lambda}(x_2).$$

Passing to the limit as $x_2 \to 0$ and using $\mathcal{P}_{j,\lambda}A_1 = A_1\mathcal{P}_{j,\lambda} = \lambda\mathcal{P}_{j,\lambda}$, we obtain

$$\frac{1}{\lambda}\mathcal{P}'_{j,\lambda}A_1 + \lambda x'_{1,\lambda,j}(0)\mathcal{P}_{j,\lambda} + \mathcal{P}_{j,\lambda}A_2 = \frac{1}{\lambda}A_1\mathcal{P}'_{j,\lambda} + \lambda x'_{1,\lambda,j}(0)\mathcal{P}_{j,\lambda} + A_2\mathcal{P}_{j,\lambda},$$

so that

$$\frac{1}{\lambda}\mathcal{P}'_{j,\lambda}A_1 + \mathcal{P}_{j,\lambda}A_2 = \frac{1}{\lambda}A_1\mathcal{P}'_{j,\lambda} + A_2\mathcal{P}_{j,\lambda}.$$

Multiply the last equality from the left by $\mathcal{P}_{i,\lambda}$ with $i \neq j$, $i \in I_\lambda$. Since $\mathcal{P}_{i,\lambda}A_1 = A_1\mathcal{P}_{i,\lambda} = \lambda\mathcal{P}_{i,\lambda}$, we obtain

$$\frac{1}{\lambda}\mathcal{P}_{i,\lambda}\mathcal{P}'_{j,\lambda}A_1 = \mathcal{P}_{i,\lambda}\mathcal{P}'_{j,\lambda} + \mathcal{P}_{i,\lambda}A_2\mathcal{P}_{j,\lambda}$$

yeilding

$$\mathcal{P}_{i,\lambda}\mathcal{P}'_{j,\lambda}\left(\frac{1}{\lambda}A_1 - I\right) = \mathcal{P}_{i,\lambda}A_2\mathcal{P}_{j,\lambda}.$$

Multiply the last equality by $\mathcal{P}_{j,\lambda}$ from the right. Since

$$\left(\frac{1}{\lambda}A_1 - I\right)\mathcal{P}_{j,\lambda} = 0,$$

and $\mathcal{P}_{j,\lambda}^2 = \mathcal{P}_{j,\lambda}$, we conclude that

$$\mathcal{P}_{i,\lambda}A_2\mathcal{P}_{j,\lambda} = 0,$$

and (3.7) is proved for $\lambda \neq 0$..

If (3.8) holds, then

$$P_{j,\lambda}(x_2)A(x_{1,\lambda,j}(x_2), x_2) = P_{j,\lambda}(x_2) \tag{3.15}$$

$$A(x_{1,\lambda,j}(x_2), x_2)P_{j,\lambda}(x_2) = P_{j,\lambda}(x_2). \tag{3.16}$$

Differentiating twice (3.15) and (3.16) with respect to x_2 and passing to the limit as $x_2 \to 0$ yields

$$\tfrac{1}{\lambda}\mathcal{P}''_{j,\lambda}A_1 + 2\mathcal{P}'_{j,\lambda}(x'_{1,\lambda,j}(0)A_1 + A_2) + x''_{1,\lambda,j}(0)\mathcal{P}_{j,\lambda}A_1 = \mathcal{P}''_{j,\lambda},$$

$$x''_{1,\lambda,j}(0)A_1\mathcal{P}_{j,\lambda} + 2(x'_{1,\lambda,j}(0)A_1 + A_2)\mathcal{P}'_{j,\lambda} + \tfrac{1}{\lambda}A_1\mathcal{P}''_j = \mathcal{P}''_{j,\lambda},$$

where $\mathcal{P}''_{j,\lambda} = \lim_{x_2 \to 0}\frac{d^2 P_{j,\lambda}(x_2)}{dx_2^2}$. Moving the righthand side of the last two equations to the left gives

$$\mathcal{P}''_{j,\lambda}(\tfrac{1}{\lambda}A_1 - I) + 2\mathcal{P}'_{j,\lambda}(x'_{1,\lambda,j}(0)A_1 + A_2) + x''_{1,\lambda,j}(0)\mathcal{P}_{j,\lambda}A_1 = 0, \quad (3.17)$$

$$x''_{1,\lambda,j}(0)A_1\mathcal{P}_{j,\lambda} + 2(x'_{1,\lambda,jj}(0)A_1 + A_2)\mathcal{P}'_{j,\lambda} + (\tfrac{1}{\lambda}A_1 - I)\mathcal{P}''_{j,\lambda} = 0. \quad (3.18)$$

Multiply (3.17) from the right by $\mathcal{P}_{j,\lambda}$ and by $\mathcal{P}_{i,\lambda}$, $i \neq j$.
Again, since

$$\left(\tfrac{1}{\lambda}A_1 - I\right)\mathcal{P}_{l,\lambda} = \mathcal{P}_{l,\lambda}\left(\tfrac{1}{\lambda}A_1 - I\right) = 0, \ l \in I_\lambda,$$

$$\mathcal{P}_{l,\lambda}A_1 = A_1\mathcal{P}_{l,\lambda} = \lambda\mathcal{P}_{l,\lambda}, \ l \in I_\lambda$$

$$\mathcal{P}_{i,\lambda}\mathcal{P}_{j,\lambda} = \mathcal{P}_{j,\lambda}\mathcal{P}_{i,\lambda} = 0, \ i, j \in I_\lambda, \ i \neq j,$$

we obtain

$$\mathcal{P}'_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}_{j,\lambda} = -\tfrac{x''_{1,\lambda,j}(0)}{2}\mathcal{P}_{j,\lambda}$$

$$\mathcal{P}'_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}_{i,\lambda} = 0, \ i, j \in I_\lambda, \ i \neq j,$$

Similarly, multiplication of (3.18) from the left by $\mathcal{P}_{j,\lambda}$ and by $\mathcal{P}_{i,\lambda}$ results in

$$\mathcal{P}_{j,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}'_{j,\lambda} = -\tfrac{x''_{1,\lambda,j}(0)}{2}\mathcal{P}_{j,\lambda}$$

$$\mathcal{P}_{i,\lambda}\left(x'_{1,\lambda,j}(0)A_1 + A_2\right)\mathcal{P}'_{j,\lambda} = 0, \ i, j \in I_\lambda, \ i \neq j,$$

which finishes the proof for $\lambda \neq 0$.
 II $\lambda = 0$.

The proof in this case is similar and goes along the following lines.

$$\Big(A_1 + x_2 A_2 - x_{3,o,j}(x_2)I\Big)P_{j,0}(x_2) = P_{j,0}(x_2)\Big(A_1 + x_2 A_2 - x_{3,0,j}(x_2)I\Big),$$

$$\Big(A_2 - x'_{3,0,j}(0)I\Big)P_{j,0} + A_1\mathcal{P}'_{j,0} = \mathcal{P}'_{j,0}A_1 + \mathcal{P}_{j,0}\Big(A_2 - x'_{3,0,j}(0)I\Big).$$

Multiply the last relation from the right by $\mathcal{P}_{i,0}$ and use

$$\mathcal{P}_{j,0}\mathcal{P}_{i,0} = \mathcal{P}_{i,0}\mathcal{P}_{j,0} = 0, \ i \neq j, \ A_1\mathcal{P}_{l,0} = \mathcal{P}_{l,0}A_1 = 0, \ l \in I_0.$$

Since

$$P_{j,0}(x_2)P_{i,0}(x_2) \equiv 0, \ i \neq j, \ \mathcal{P}'_{j,0}\mathcal{P}_{i,0} = -\mathcal{P}_{j,0}\mathcal{P}'_{i,0},$$

we obtain

$$0 = -A_1\mathcal{P}_{j,0}\mathcal{P}'_{i,0} = A_1\mathcal{P}'_{j,0}\mathcal{P}_{i,0} = \mathcal{P}_{j,0}A_2\mathcal{P}_{i,0},$$

which is (3.7) for $\lambda = 0$.

Finally, under the assumption of (3.9), the proof of (3.11) and (3.13) is obtained the same way as the one of (3.10) and (3.12) by twice differentiating

$$P_{j,0}(x_2)\Big(A_1 + x_2 A_2\Big) = x_{3,,0,j}(x_2)P_{j,0}(x_2)$$

$$\Big(A_1 + x_2 A_2\Big)P_{j,0}(x_2) = x_{3,0,j}(x_2)P_{j,0}(x_2)$$

with respect to x_2, passing to the limit as $x_2 \to 0$, and multiplying by $\mathcal{P}_{j,0}$ and $\mathcal{P}_{i,0}$. We are done.   □

Obviously, (3.8) and (3.9) hold when every spectral component passing through $(1/\lambda, 0)$ has multiplicity 1.

Proof of Theorem 1.15 I. $\lambda \neq 0$

Fix $j \in I_\lambda$. In the results of the previous section we were interested only in the roots of (2.3)

$$\tilde{R}_i(\tau x_{1,\lambda,j}(x_2), \tau x_2) = 0. \tag{3.19}$$

which were close to 1. Now we will consider all the roots of this equation. Observe that if x_2 is small enough, all these roots are close to the ratios λ/μ where $\mu \in \sigma(A_1)$. Moreover, since condition b) holds at every spectral point of A_1, the multiplicity of each of these roots is 1 (when x_2 is close to 0). We denote them by $\tau_{j,\lambda,i,\mu}(x_2), \ 1 \leq i \leq s,$, where μ is defined by

$$\tau_{j,\lambda,i,\mu}(x_2) \to \frac{\lambda}{\mu}, \ \text{as } x_2 \to 0, \tag{3.20}$$

which, of course, implies $i \in I_\mu$. It was mentioned above that $\tau_{j,\lambda,i,\mu}(x_2)$ are analytic functions of x_2 in a neghborhood of the origin. Obviously, $\tau_{j,\lambda,j,\lambda}(x_2) = 1$.

Further, (1.9) defined projections $P_{j,\lambda}(x_2)$. We define

$$P_{j,\lambda,i,\mu}(x_2) = P_{i,\mu}(\tau_{j,\lambda,i,\mu}\, x_2). \tag{3.21}$$

It follows from Theorem 1.11 and (3.21) that $P_{j,\lambda,i,\mu}(x_2)$ are analytic operator-valued functions of x_2 in a neighborhood of the origin, and

$$\lim_{x_2\to 0} P_{j,\lambda,i,\mu}(x_2) = \mathcal{P}_{i,\mu}. \tag{3.22}$$

Also

$$\frac{dP_{j,\lambda,i,\mu}}{dx_2}(x_2) = \frac{dP_{i,\lambda}(\tau_{j,\lambda,i,\mu}\, x_2)}{dx_2}\left[\tau_{j,\lambda,i,\mu}(x_2) + x_2\frac{d\tau_{j,\lambda,i,\mu}(x_2)}{dx_2}\right],$$

so that (3.5) and (3.20) imply that for $\lambda = \mu$ we have

$$\frac{dP_{j,\lambda,i,\lambda}}{dx_2}(0) = \lim_{x_2\to 0}\frac{dP_{j,\lambda,i,\lambda}}{dx_2}(x_2) = \mathcal{P}'_{i,\lambda}(0). \tag{3.23}$$

Recall that projections (3.21) appeared in Proposition 3.1 for $\lambda = \mu$. We renamed them here, since we need these projections corresponding to all roots of (3.19) and want to indicate to which reciprocal of a point in $\sigma(A_1)$ the corresponding point of the proper joint spectrum converges. Note that $\tau_{j,\lambda,j,\lambda}(x_2) = 1$ implies $P_{j,\lambda,j,\lambda}(x_2) = P_{j,\lambda}(x_2)$.

Since for $i \in I_\lambda$ components $\{\tilde{R}_i(x_1, x_2) = 0\}$ have multiplicity 1 in the projective joint spectrum , the rank of $P_{j,\lambda,i,\mu}(x_2)$ is equal to 1 for every $i \in I_\lambda \cap I_\mu$. For other $i \notin I_\lambda$ the rank of $P_{j,\lambda,i,\mu}$ is equal to m_i - the multiplicity of the spectral component $\{\tilde{R}_i = 0\}$.

As it was mentioned above, for a rank 1 projection (3.8) holds, and, therefore, if Δ is small, we have

$$\Big(x_{1,\lambda,j}(x_2 + \Delta)A_1 + (x_2 + \Delta)A_2 - I\Big)P_{j,\lambda}(x_2 + \Delta) = 0.$$

We write this relation in the form

$$\frac{1}{2\pi i}\int_\gamma (w - 1)\Big(w - x_{1,\lambda,j}(x_2 + \Delta)A_1 - (x_2 + \Delta)A_2\Big)^{-1}dw = 0, \tag{3.24}$$

where, as above, γ is a circle centered at 1 which separates 1 from all other eigenvalues of $x_{1,\lambda,j}(x_2)A_1 + x_2 A_2$, and, therefore, the same is true for all $x_{1,\lambda,j}(x_2 + \Delta)A_1 + (x_2 + \Delta)A_2$ for all sufficiently small Δ.

Let us write (3.24) as

$$
0 = \frac{1}{2\pi i} \int_\gamma (w-1)\Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1}
$$
$$
\times \left[I - \left((x'_{1,\lambda,j}(x_2)A_1 + A_2)\Delta + \left(\sum_{k=2}^\infty \frac{x_{1,\lambda,j}^{(k)}(x_2)}{k!}\Delta^k \right) A_1 \right) \right.
$$
$$
\left. \times \Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1} \right]^{-1} dw
$$
$$
= \frac{1}{2\pi i} \int_\gamma (w-1)\Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1}
$$
$$
\times \sum_{l=0}^\infty \left[\left((x'_{1,\lambda,j}(x_2)A_1 + A_2)\Delta + \left(\sum_{k=2}^\infty \frac{x_{1,\lambda,j}^{(k)}(x_2)}{k!}\Delta^k \right) A_1 \right) \right.
$$
$$
\left. \times \Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1} \right]^l dw. \tag{3.25}
$$

The right hand side of the last expression is an analytic function of Δ in a small neighborhood of 0, so that all the derivatives at $\Delta = 0$ vanish. We will need only the first and the second ones. Here are the corresponding relations.

$$
\frac{1}{2\pi i} \int_\gamma (w-1)\Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1}(x'_{1,\lambda,j}(x_2)A_1 + A_2)
$$
$$
\times \Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1} dw = 0. \tag{3.26}
$$
$$
\frac{1}{2\pi i} \int_\gamma (w-1)\Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1}\left\{ \Big((x'_{1,\lambda,j}(x_2)A_1 + A_2) \right.
$$
$$
\times \Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1}(x'_{1,\lambda,j}(x_2)A_1 + A_2)\Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1} \Big)
$$
$$
\left. + \frac{x''_{1,\lambda,j}(x_2)}{2} A_1 \Big(w - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2\Big)^{-1} \right\} dw = 0. \tag{3.27}
$$

We will now express relations (3.26) and (3.27) in terms of projections $P_{j,\lambda,i,\mu}(x_2)$.

The spectrum of the matrix $x_{1,\lambda,j}(x_2)A_1 + x_2 A_2$ consists of complex numbers $\mu_{j,\lambda,i,\nu}(x_2)$ which are reciprocals of $\tau_{j,\lambda,i,\nu}(x_2)$, $\nu \in \sigma(A_1)$ and, possibly, 0. If x_2 is close to 0, the latter might occur only when $0 \in \sigma(A_1)$. If 0 is an eigenvalue of $x_{1,\lambda,j}(x_2)A_1 + x_2 A_2$, we include it in the formulas below as $\mu_{j,\lambda,i,0}(x_2)$.

As mentiioned above, each eigenvalue $\mu_{j,\lambda,i,\nu}(x_2)$, $i \in I_\lambda$, $\nu = \lambda$ is close to 1. These eigenvalues tend to 1 as $x_2 \to 0$ (and, of course, $\mu_{j,\lambda,j,\lambda}(x_2) = 1$ for all x_2). All other $\mu_{j,\lambda,i,\nu}(x_2)$ stay away from 1 as $x_2 \to 0$, that is for some positive constant a

$$
|\mu_{j,\lambda,i,\nu}(x_2) - 1| > a, \text{ if } i \in I_\lambda \text{ and } \nu \neq \lambda, \text{ or } i \notin I_\lambda. \tag{3.28}
$$

The Jordan decomposition of the matrix $A(x_2) = x_{1,\lambda,j}(x_2)A_1 + x_2A_2$ consists of Jordan blocks which have sizes between 1 and m_i corresponding to eigenvalue $\mu_{j,\lambda,i,\nu}$ (we consider an eigenvector which is not in a Jordan cell of dimension higher than 1 as a Jordan cell of size 1), so for w close to 1 we have

$$
\left(w - x_{1j}(x_2)A_1 - x_2A_2\right)^{-1}
$$
$$
= \frac{P_{j,\lambda,j,\lambda}(x_2)}{w-1} + \sum_{i\in I_\lambda, i\neq j} \frac{P_{j,\lambda,i,\lambda}(x_2)}{w - \mu_{j,\lambda,i,\lambda}(x_2)}
$$
$$
+ \sum_{\nu\in\sigma(A_1),\nu\neq\lambda} \sum_{i\in I_\nu} \sum_{r=0}^{m_i-1} \frac{\left(\mu_{j,\lambda,i,\nu}(x_2)I - x_{1,\lambda,j}(x_2)A_1 - x_2A_2\right)^r P_{j,\lambda,i,\nu}(x_2)}{\left(w - \mu_{j,\lambda,i,\nu}(x_2)\right)^{r+1}},
$$
$$
(3.29)
$$

The integrands of (3.26) and (3.27) are meromorphic functions in w in a neighborhood of $w = 1$. We use (3.29) to compute their residues at 1. A simple but tedious computation shows that

$$
Res\Big|_{w=1}\left[(w-1)\left(w - x_{1,\lambda,j}(x_2)A_1 - x_2A_2\right)^{-1}(x'_{1,\lambda,j}(x_2)A_1 + A_2)\right.
$$
$$
\left. \times\left(w - x_{1,\lambda,j}(x_2)A_1 - x_2A_2\right)^{-1}\right]
$$
$$
= P_{j,\lambda,j,\lambda}(x_2)(x'_{1,\lambda,j}(x_2)A_1 + A_2)\mathcal{P}_{j,\lambda,j,\lambda}(x_2) = 0 \qquad (3.30)
$$

Passing to the limit as $x_2 \to 0$ we obtain

$$
\mathcal{P}_{j,\lambda}(x'_{1,\lambda,j}(0)A_1 + A_2)\mathcal{P}_{j,\lambda} = 0, \qquad (3.31)
$$

which is the first relation claimed in Theorem 1.15.

To simplify the computation of the residue of (3.27) at $w = 1$ let us introduce the following operators

$$
\mathcal{R}(x_2) = x'_{1,\lambda,j}(x_2)A_1 + A_2,
$$
$$
\mathcal{S}_1(x_2) = P_{j,\lambda,j,\lambda}(x_2)\mathcal{R}(x_2)\left(\sum_{i\in I_\lambda, i\neq j} \frac{P_{j,\lambda,i,\lambda}(x_2)}{1 - \mu_{j,\lambda,i,\lambda}(x_2)}\right)\mathcal{R}(x_2)P_{j,\lambda,j,\lambda}(x_2),
$$
$$
\mathcal{S}_2(x_2) = P_{j,\lambda,j,\lambda}(x_2)\mathcal{R}(x_2)
$$
$$
\times\left(\sum_{\nu\in\sigma(A_1),\nu\neq\lambda} \sum_{i\in I_\nu} \sum_{r=0}^{m_i-1} \frac{\left(\mu_{j,\lambda,i,\nu}(x_2)I - x_{1,\lambda,j}(x_2)A_1 - x_2A_2\right)^r P_{j,\lambda,i,\nu}(x_2)}{\left(1 - \mu_{j,\lambda,i,\nu}(x_2)\right)^{r+1}}\right)
$$
$$
\times\mathcal{R}(x_2)P_{j,\lambda,j,\lambda}(x_2).
$$

A straightforward computation using (3.29) and (3.30) shows that (3.27) turns into

$$\mathscr{S}_1(x_2) + \mathscr{S}_2(x_2) + \frac{x''_{1,\lambda,j}(x_2)}{2} P_{j,\lambda,j,\lambda}(x_2) A_1 P_{j,\lambda,j,\lambda}(x_2) = 0. \qquad (3.32)$$

Our next step is finding the limits of each of terms of (3.32) as $x_2 \to 0$.

If $i \in I_\lambda$, $\mu_{j,\lambda,i,\lambda}(x_2) \to 1$ as $x_2 \to 0$, and condition (b) implies that $\mu'_{j,\lambda,i,\lambda}(0) \neq 0$, so we have

$$\mathscr{S}_1(x_2) = P_{j,\lambda,j,\lambda}(x_2)\mathscr{R}(x_2)\left(\sum_{i \in I_\lambda, i \neq j} \frac{P_{j,\lambda,i,\lambda}(x_2)}{1 - \mu_{j,\lambda,i,\lambda}(x_2)}\right)\mathscr{R}(x_2)P_{j,\lambda,j,\lambda}(x_2)$$

$$= P_{j,\lambda,j,\lambda}(x_2)\mathscr{R}(x_2)\left(\sum_{i \in I_\lambda, i \neq j} \frac{P_{j,\lambda,i,\lambda}(x_2)}{-\mu'_{j,\lambda,i,\lambda}(0)x_2 + O(|x_2|^2)}\right)\mathscr{R}(x_2)P_{j,\lambda,j,\lambda}(x_2)$$

$$= \left(\sum_{i \in I_\lambda, i \neq j} \frac{P_{j,\lambda,j,\lambda}(x_2)\mathscr{R}(x_2)P_{j,\lambda,i,\lambda}(x_2)}{x_2(-\mu'_{j,\lambda,i,\lambda}(0) + o(1))}\right)\mathscr{R}(x_2)P_{j,\lambda,j,\lambda}(x_2)$$

$$= \left(\sum_{i \in I_\lambda, i \neq j} \frac{P_{j,\lambda,j,\lambda}(x_2)\left(\mathscr{R}(0) + (x''_{1,\lambda,j}(0)x_2 + O(|x_2|^2))A_1\right)P_{j,\lambda,i,\lambda}(x_2)}{x_2(-\mu'_{j,\lambda,i,\lambda}(0) + o(1))}\right)$$

$$\times\mathscr{R}(x_2)P_{j,\lambda,j,\lambda}(x_2).$$

Write

$$\Phi(x_2) = P_{j,\lambda,j,\lambda}(x_2)\mathscr{R}(0)P_{j,\lambda,i,\lambda}(x_2).$$

Since $\mathcal{P}_{j,\lambda}A_1\mathcal{P}_{i,\lambda} = \lambda\mathcal{P}_{j,\lambda}\mathcal{P}_{i,\lambda} = 0$ for $i, j \in I_\lambda$, $i \neq j$, relation (3.7) in Proposition 3.1 implies

$$\Phi(0) = 0.$$

Also, (3.23) and (3.12) yield

$$\Phi'(0) = 0,$$

and, hence,

$$\Phi(x_2) = O(|x_2|^2), \text{ as } x_2 \to 0,$$

resulting in

$$\lim_{x_2 \to 0} \mathscr{S}_1(x_2) = 0. \qquad (3.33)$$

To evaluate the limit of $\mathscr{S}_2(x_2)$ we observe that since A_1 is normal, Theorem 1.11 amd (3.20) impliy that for $\lambda \neq \nu$, $i \in I_\nu$ and

$$((\mu_{j,\lambda,i,\nu}(x_2)I - x_{1,\lambda,j}(x_2)A_1 - x_2 A_2)^r P_{j,\lambda,i,\nu}(x_2) \to 0 \text{ for } r > 0 \text{ as } x_2 \to 0.$$

Since by (3.28) we have $|1 - \mu_{j,\lambda,i,\lambda}(x_2)| > a > 0$, and, since by Theorem 1.11 all projections $P_{j,\lambda,i,\nu}(x_2)$ are bounded, all terms in the expression of $\mathscr{S}_2(x_2)$ with $r > 0$ tend to 0 as $x_2 \to 0$. This yields

$$\lim_{x_2 \to 0} \mathscr{S}_2(x_2) = \mathcal{P}_{j,\lambda} A_2 \left(\sum_{\nu \in \sigma(A_1), \nu \neq \lambda} \sum_{i \in I_\nu} \frac{\mathcal{P}_{i,\nu}}{1 - \mu_{j,\lambda,i,\nu}(0)} \right) A_2 \mathcal{P}_{j,\lambda}. \qquad (3.34)$$

Equations (3.3) in Proposition 3.1, (3.20), and (3.34) now give

$$\lim_{x_2 \to 0} \mathscr{S}_2(x_2) = \mathcal{P}_{j,\lambda} A_2 \left(\sum_{\nu \in \sigma(A_1), \nu \neq \lambda} \frac{\lambda \mathscr{P}_\nu}{\lambda - \nu} \right) A_2 \mathcal{P}_{j,\lambda} = \lambda \mathcal{P}_{j,\lambda} A_2 T_\lambda A_2 \mathcal{P}_{j,\lambda}.$$
$$(3.35)$$

Finally, passing to the limit in (3.32) as $x_2 \to 0$, (3.33), and (3.35) along with $\mathcal{P}_{j,\lambda} A_1 = A_1 \mathcal{P}_{j,\lambda} = \lambda \mathcal{P}_{j,\lambda}$, $\mathcal{P}_{j,\lambda}^2 = \mathcal{P}_{j,\lambda}$, result in

$$\mathcal{P}_{j,\lambda} A_2 T_\lambda A_2 \mathcal{P}_{j,\lambda} + \frac{x_{1,\lambda,j}''(0)}{2} \mathcal{P}_{j,\lambda} = 0,$$

which finishes the proof of Theorem 1.15 for $\lambda \neq 0$.

II. $\lambda = 0$

In this case (1.10) gives the expression for $P_{j,0}(x_2)$.

The spectrum of $A_1 + x_2 A_2$ consists of the roots of

$$R_i(1, x_2, z) = \tilde{R}_j(x_2, z) = 0.$$

If x_2 is close to 0, we denote these roots by $x_{3,\nu,i}(x_2)$, where $\nu \in \sigma(A_1)$ is the eigenvalue of A_1 to which $x_{3,\nu,i}(x_2)$ converges as $x_2 \to 0$, and $i \in I_\nu$. Each $x_{3,\nu,i}$ has multiplicity m_i, and, in particular, this multiplicity is equal to 1 for $i \in I_0$.

Fix $j \in I_0$ and x_2 close to 0. Similar to what we had in the case $\lambda \neq 0$, here

$$\left(A_1 + (x_2 + \Delta)A_2 - x_{3,0,j}(x_2 + \Delta)I \right) P_{j,0}(x_2) = 0,$$

if Δ is small enough. An analog of (3.24) in this setting is

$$\frac{1}{2\pi i} \int_\gamma w \Big((w + x_{3,0,j}(x_2 + \Delta))I - A_1 - (x_2 + \Delta)A_2 \Big)^{-1} dw = 0, \qquad (3.36)$$

and, as above, γ separates 0 from the rest of the spectrum of $A_1 + x_2 A_2 - x_{3,0,j}(x_2)I$. Condition $\tilde{b}$) implies that there is $a > 0$ independent of x_2 such that γ can be taken to be circle centered at the origin of radius $a|x_2|$.

Following a similar line of presentation as for $\lambda \neq 0$, we write (3.36) as a power series in Δ:

$$
\begin{aligned}
0 &= \frac{1}{2\pi i} \int_\gamma w \Big((w + x_{3,0,j}(x_2)I - A_1 - x_2 A_2 \Big)^{-1} \\
&\quad \times \left[I - \left\{ (A_2 - x'_{3,0,j}(x_2)I)\Delta - \left(\sum_{k=2}^\infty \frac{x_{3,0,j}^{(k)}(x_2)}{k!} \Delta^k \right) I \right\} \right. \\
&\quad \times \left. \Big((w + x_{3,0,j}(x_2)I - A_1 - x_2 A_2 \Big)^{-1} \right]^{-1} dw \\
&= \frac{1}{2\pi i} \int_\gamma w \Big((w + x_{3,0,j}(x_2)I - A_1 - x_2 A_2 \Big)^{-1} \\
&\quad \times \sum_{l=0}^\infty \left[\left\{ (A_2 - x'_{3,0,j}(x_2)I)\Delta - \left(\sum_{k=2}^\infty \frac{x_{3,0,j}^{(k)}(x_2)}{k!} \Delta^k \right) I \right\} \right. \\
&\quad \times \left. \Big((w + x_{3,0,j}(x_2)I - A_1 - x_2 A_2 \Big)^{-1} \right]^l dw. \tag{3.37}
\end{aligned}
$$

and equate the coefficients for powers of Δ to 0. Again, we are interested in the the coefficients for Δ and Δ^2.

In this case the inverse is given by:

$$
\begin{aligned}
&\Big((w + x_{3,0,j}(x_2)I - A_1 - x_2 A_2 \Big)^{-1} \\
&= \frac{P_{j,0}(x_2)}{w} + \sum_{i \in I_0, i \neq j} \frac{P_{i,0}(x_2)}{w + x_{3,0,j}(x_2) - x_{3,0,i}(x_2)} \\
&\quad + \sum_{v \in \sigma(A_1), v \neq 0} \sum_{i \in I_v} \sum_{l=0}^{m_i - 1} \frac{(A_1 + x_2 A_2 - x_{3,v,i}(x_2)I)^l P_{i,0}(x_2)}{\big(w + x_{3,0,j}(x_2) - x_{3,v,i}(x_2) \big)^{l+1}}. \tag{3.38}
\end{aligned}
$$

We substitute (3.38) into (3.37) and evaluate the residues of the coefficients for Δ and Δ^2. The details of this evaluation are similar to the ones in the case $\lambda \neq 0$ and are omitted.

The proof of Theorem 1.15 is complete.

Our next result shows that under the assumptions of Theorem 1.15 the limit component projections coming out of the joint spectrum $\sigma_p(A_1, A_1 A_2)$ coincide with $\mathcal{P}_{j,\lambda}$.

In a similar way as in our previous consideration, it follows from the implicit function theorem that if $\sigma(A_1, A_1 A_2)$ satisfies conditions (a) and (b), then for each spectral component of $\sigma_p(A_1, A_1 A_2)$ passing through $(1/\lambda, 0)$ the first coordinate is expressed as an analytic function of the second one. To distinguish from the previous

case when we considered the pair (A_1, A_2), we denote the coordinates of a spectral point in $\sigma_p(A_1, A_1 A_2)$ by (z_1, z_2), so that the in the jth component passing through $(1/\lambda, 0)$, $z_1 = z_{1,\lambda,j}(z_2)$. Similarly, we denote by $Q_{j,\lambda}(z_2)$ the projection

$$Q_{j,\lambda}(z_2) = \frac{1}{2\pi i} \int_{|w-1|=\tilde{\delta}_j} \left(w - z_{1,\lambda,j}(z_2) A_1 - z_2 A_1 A_2 \right)^{-1} dw,$$

where the contours $\{w : |w - 1| = \tilde{\delta}_j\}$ separates 1 from the rest of the spectrum of $\left(z_{1,\lambda,j}(z_2) A_1 + z_2 A_1 A_2 \right)$, as it was in (1.9). Since A_1 is normal, Theorem 1.11 implies that there are limits $\mathcal{Q}_{j,\lambda}$ of $Q_{j,\lambda}(x_2)$ as $x_2 \to 0$.

Lemma 3.39 *Suppose that the pairs of matrices (A_1, A_2) and $(A_1, A_1 A_2)$ satisfy the conditions of Theorem 1.15, and let $\lambda \in \sigma(A_1)$, $\lambda \neq 0$. Then $\mathcal{P}_{j,\lambda} = \mathcal{Q}_{j,\lambda}$.*

Proof First we observe that relation (3.31) (the first relation in Theorem 1.15) implies that for $j \in I_\lambda$ the compression of A_2 to the range of $\mathcal{P}_{j,\lambda}$ is $-x'_{1,\lambda,j}(0) I_{\mathcal{R}(\mathcal{P}_{j,\lambda})}$, where $I_{\mathcal{R}(\mathcal{P}_{j,\lambda})}$ is the identity matrix on the range of $\mathcal{P}_{j,\lambda}$. This implies that $(-x'_{1,\lambda,j}(0))$, $j \in I_\lambda$ are the eigenvalues of the compression of A_2 to the λ-eigenspace of A_1, which by (3.3) is the sum of the ranges of $\mathcal{P}_{j,\lambda}$. Each of these ranges is of dimension 1, and by condition b) all numbers $(x'_{1,\lambda,j}(0))$ are different. Let e_j, $j \in I_\lambda$ be an eigenvector for $\mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda}$. Then these eigenvectors form a basis of the λ-eigenspace of A_1.

Similarly, applying Theorem 1.15 to the pair $(A_1, A_1 A_2)$ we obtain that the compression of $A_1 A_2$ to the λ-eigenspace of A_1 has $(-z'_{1,\lambda,j}(0))$ as eigenvalues of multiplicity one each.

Propositions 3.1 and 3.6 imply

$$\mathcal{P}_\lambda A_1 A_2 \mathcal{P}_\lambda = \left(\sum_{j=1}^{k_\lambda} \mathcal{P}_{j,\lambda} \right) A_1 A_2 \left(\sum_{j=1}^{k_\lambda} \mathcal{P}_{j,\lambda} \right) = A_1 \sum_{j=1}^{k_\lambda} \mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda},$$

so that for $j \in I_\lambda$

$$\mathcal{P}_\lambda A_1 A_2 \mathcal{P}_\lambda e_j = A_1 \mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} e_j = -x'_{1,\lambda,j}(0) A_1 e_j = -\lambda x'_{1,\lambda,j}(0) e_j.$$

This shows that $-\lambda x'_{1,\lambda,j}(0)$, $j \in I_\lambda$ form the spectrum of the compression of $A_1 A_2$ to the λ-eigenspace of A_1 and that e_j, $j \in I_\lambda$ form the corresponding eigenbasis. Of course, this implies the statement we are proving. $\qquad\square$

Proof of Theorem 1.20 Let $\lambda \neq 0$. Apply Theorem 1.15 to each of the pairs (A_1, A_2) and $(A_1, A_1 A_2)$ and use Lemma 3.39. We obtain

$$\mathcal{P}_{j,\lambda} A_2 T_\lambda A_2 \mathcal{P}_{j,\lambda} = -\frac{x''_{1,\lambda,j}(0)}{2} \mathcal{P}_{j,\lambda},$$

$$\mathcal{P}_{j,\lambda} A_1 A_2 T_\lambda A_1 A_2 \mathcal{P}_{j,\lambda} = -\frac{z''_{1,\lambda,j}(0)}{2} \mathcal{P}_{j,\lambda}.$$

It follows from the definition of the operator T_λ, (1.14), that

$$T_\lambda A_1 = A_1 T_\lambda = -\Big(\sum_{\mu \in \sigma(A_1), \mu \neq \lambda} \mathscr{P}_\mu \Big) + \lambda T_\lambda = -I + \mathscr{P}_\lambda + \lambda T_\lambda.$$

Since $\mathcal{P}_{j,\lambda} A_1 = \lambda \mathcal{P}_{j,\lambda}$, this implies

$$-\lambda \mathcal{P}_{j,\lambda} A_2^2 \mathcal{P}_{j,\lambda} + \lambda \mathcal{P}_{j,\lambda} A_2 \mathscr{P}_\lambda A_2 \mathcal{P}_{j,\lambda} + \lambda^2 \mathcal{P}_{j,\lambda} A_2 T A_2 \mathcal{P}_{j,\lambda}$$
$$= -\frac{z''_{1,\lambda,j}(0)}{2} \mathcal{P}_{j,\lambda},$$

$$-\lambda \mathcal{P}_{j,\lambda} A_2^2 \mathcal{P}_{j,\lambda} + \lambda \mathcal{P}_{j,\lambda} A_2 \Big(\sum_{l \in I_\lambda} \mathcal{P}_{l,\lambda} \Big) A_2 \mathcal{P}_{j,\lambda} = \frac{\lambda^2 x''_{1,\lambda,j}(0) - z''_{1,\lambda,j}(0)}{2} \mathcal{P}_{j,\lambda},$$

$$-\lambda \mathcal{P}_{j,\lambda} A_2^2 \mathcal{P}_{j,\lambda} + \lambda \mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} = \frac{\lambda^2 x''_{1,\lambda,j}(0) - z''_{1,\lambda,j}(0)}{2} \mathcal{P}_{j,\lambda}.$$

Now, $\mathcal{P}_{j,\lambda}^2 = \mathcal{P}_{j,\lambda}$ results in

$$\mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} = \Big(\mathcal{P}_{j,\lambda} A_2 \mathcal{P}_{j,\lambda} \Big)^2,$$

and (3.31) yields

$$\mathcal{P}_{j,\lambda} A_2^2 \mathcal{P}_{j,\lambda} = \frac{z''_{1j}(0) + 2\lambda \big(x'_{1j}(0) \big)^2 - \lambda^2 x''_{1j}(0)}{2\lambda} \mathcal{P}_{j,\lambda}.$$

The proof is complete.   □

4 Application to Representations of Coxeter Groups: Proof of Theorem 1.22

Recall that a Coxeter group is a finitely generated group G with generators $g_1, \ldots, g_n$ satisfy the following relations:

$$(g_i g_j)^{m_{ij}} = 1, \quad i, j = 1, \ldots, n,$$

where $m_{ii} = 1$ and $m_{ij} \in \mathbb{N} \cup \{\infty\}$, $m_{ij} \geq 2$ when $i \neq j$. It is easy to see that to avoid redundancies we must have $m_{ij} = m_{ji}$, and that $m_{ij} = 2$ means g_i and g_j commute. The set of generators $\{g_1, \ldots, g_n\}$ is called a *Coxeter set of generators*, and m_{ij} are called the *Coxeter exponents*. The matrix $\|m_{ij}\|$ is called a *Coxeter matrix*. A Coxeter group with 2 generators is called a *Dihedral* group. The monographs [1, 20, 27] are good sources for information on Coxeter groups.

Let G be a Coxeter group with Coxter generators $g_1, \ldots, g_n$, and $\rho : G \to V$ be a finite dimensional unitary representation of G. Suppose that $A_1, \ldots, A_n$ is a

tuple of $N \times N$ matrices. In this section we investigate what information about relations between $A_1, \ldots, A_n$ can be obtained from the fact that the joint spectrum $\sigma_p(\rho(g_1), \ldots, \rho(g_n))$ is contained in $\sigma_p(A_1, ..., A_n)$.

Of course, for every pair of generators g_i, g_j the representation ρ generates a unitary representation of the Dihedral group generated by g_i and g_j. It is well-known that every irreducible unitary representation of a Dihedral group is either 1- or 2-dimensional, and by Maschke's Theorem ([31, 32]) that every representation is a sum of irreducible ones. The one dimensional representations of a Dihedral group are:

$$\rho(g_1) = \rho(g_2) = I, \text{ or } \rho(g_1) = \rho(g_2) = -I$$

for an odd order group. For an even order group there is an additioal one-dimensional representation

$$\rho(g_1) = I, \ \rho(g_2) = -I.$$

Two-dimensional irreducible representations are equivalent to those generated by

$$\rho(g_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \ \rho(g_2) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{bmatrix},$$

where $0 < \alpha < \pi$. If the group is finite, α is a rational multiple of π. It is easy to see (cf [8]) that the proper joint spectrum of images of the Coxeter generators of a Dihedral group under an irreducible representation could be either a line of the form (one-dimensional)

$$\{(x_1, x_2) : \ x_1 \pm x_2 = \pm 1\},$$

or a "complex ellips" (two-dimensional)

$$\{(x_1, x_2) : \ x_1^2 + 2(\cos\alpha)x_1 x_2 + x_2^2 = 1\},$$

and the joint spectrum of $(\rho(g_1)$ and $\rho(g_1)\rho(g_2))$ could be

$$\{x_1 \pm x_2 = \pm 1\}$$

for one-dimensional representations, and

$$\{x_1^2 - x_2^2 + 2(\cos\alpha)x_2 = 1\}$$

for two-dimensional.

Proof of Theorem 1.22 Consider $2 \leq i \leq n$. Condition $(*)$ in the statement of Theorem 1.22 implies that every line and ellipse in the joint spectrum of $\rho(g_1)$ and $\rho(g_i)$ has multiplicity one, and, therefore, by condition (II) in Theorem 1.22, $\sigma_p(A_1, A_i)$ and $\sigma_p(A_1, A_1 A_i)$ satisfy Theorems 1.15 and 1.20 at $(\pm 1, 0)$. Details of the local analysis

near each of them are similar, so we concentrate on $(1, 0)$. Following the notations of the previous section we denote by $\mathcal{P}_{j,1}$ the component projections from Theorem 1.11. We will apply Theorem 1.20.

First suppose that the jth component of $\sigma_p(A_1, A_i)$ which passes through $(1, 0)$ is the line

$$x_1 \pm x_2 = 1.$$

Then the corresponding component of the joint spectrum of A_1 and $A_1 A_2$ is also a line, and $x_{1j}(x_2) = 1 \pm x_2$, $z_{1j}(x_2) = 1 \pm z_2$, so that $x''_{1j} = z''_{1j} \equiv 0$, $(x'_{1j})^2 \equiv 1$. Theorem 1.20 now implies

$$\mathcal{P}_{j,1} A_2^2 \mathcal{P}_{j,1} = \mathcal{P}_{j,1}. \tag{4.1}$$

Let the jth component is the ellipse

$$x_1^2 + 2\left(\cos\alpha\right)x_1 x_2 + x_2^2 = 1.$$

Then the corresponding component of $\sigma_p(A_1, A_1 A_2)$ is given by

$$z_1^2 - z_2^2 + 2\left(\cos\alpha\right)z_2 = 1.$$

In this case

$$x'_{1j}(0) = z'_{1j}(0) = -\cos\alpha, \ x''_{1j}(0) = -1 + \cos^2\alpha,$$
$$z''_{1j}(0) = 1 - \cos^2\alpha,$$

and Theorem 1.20 for $\lambda = 1$ shows that (4.1) holds in this case too.

Thus, (3.3) shows that the compression of A_2^2 to the 1-eigenspace of A_1 is the identity. Since A_1 is normal, the projection $\mathcal{P}_1$, onto this subspace is orthogonal. Since the norm of A_2 is equal to 1, this implies that every 1-eigenvector of A_1 is a 1-eigenvector of A_2^2.

A similar proof shows that every (-1)-eigenvector of A_1 is a 1-eigenvector of A_2^2.

Further, every component of the joint spectrum of Coxeter generators under a representation of a Dihedral group, either an ellipse, or a straight line, passes the same number of times through $(\pm 1, 0)$ and through $(0, \pm 1)$. That is, if t_1 and t_2 are the multiplicities of 1 and -1 as eigenvalues of $\rho(g_1)$, and u_1 and u_2 are the same numbers for $\rho(g_2)$, then

$$t_1 + t_2 = u_1 + u_2.$$

Now, condition II) in the statement of Theorem 1.22 implies that the sum of multiplicities of eigenvalues 1 and -1 of A_1 and A_i are the same. This sum is equal of the sum of dimensions of all Jordan cells in the Jordan representation of A_i corresponding to eigenvalues ± 1 and, of course, to the multiplicity of eigenvalue 1 for A_i^2. Thus, the sum of 1- and (-1)-eigenspaces of A_1 is exactly the invariant subspace for

A_i^2 corresponding to eigenvalue 1, and the restriction of A_i^2 to this subspace is the identity. Since the square of a Jordan cell which corresponds to a non-zero eigenvalue and has dimension higher than 1 is never diagonal, we see that ± 1-eigenvectors for A_1 and ± 1-eigenvectors of A_i span the same subspace, which we call L, that is invariant under the action of both A_1 and A_i. As mentioned above, every component of the joint spectrum of $\rho(g_1)$ and $\rho)g_2$ passes through $(\pm 1, 0)$, so that the dimension of L is equal the dimension of ρ. Of course, L being spanned by ± 1-eigenvectors of A_1 is independent of $i = 2, ..., n$, and (1) is proved.

The fact that $A_i \big|_L$ are unitary and self-adjoint is straightforward. Indeed, it follows from L being spanned by ± 1-eigenvectors of A_i and from a simple fact that for an operator of norm 1 any two eigenvectors corresponding to different unimodular eigenvalues are orthogonal. Since every component of the joint spectrum of Coxeter generators of a representation of a Coxeter group passes through $(\pm 1, 0, ..., 0)$, we see that (1.23) holds. The fact that restrictions of $A_i \big|_L$ generate a representation of G follows from the Theorem 1.1 in [8] stating that for a representation of a pair of unitary matrices U_1, U_2 their joint spectrum determines a number m such that $(U_1 U_2)^m = 1$ (m is infinite, if there is no such relation). Thus, the Coxeter matrices of ρ and $\tilde{\rho}$ are the same, and (2) is proved.

Finally, (3) follows from (1.23) and Theorem 1.2 in [8]. We are done.

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