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Blow-Up Theorems for *p*-Sub-Laplacian Heat Operators on Stratified Groups

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Abstract

In this paper, we present blow-up results for solutions to the Dirichlet initial value problem for the p-sup-Laplacian heat operators on the stratified (Lie) groups. A stratified group version of the Poincaré inequality from Ruzhansky and Suragan (J Differ Equ 262:1799–1821, 2017) plays a key role in our proofs.

Keywords Blow-up · p-Sub-Laplacian · Stratified group · Poincaré inequality

Mathematics Subject Classification 35H20 · 35K92 · 35B44 · 35R03

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1 Introduction

A Lie group $\mathbb{G} = (\mathbb{R}^n, \circ)$ is called a stratified (Lie) group if it satisfies the following conditions:

(a) For some integer numbers $N_1 + N_2 + \cdots + N_r = n$, the decomposition $\mathbb{R}^n = \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_r}$ is valid, and for any $\lambda > 0$ the dilation

$$\delta_\lambda(x) := (\lambda x', \lambda^2 x^{(2)}, \ldots, \lambda^r x^{(r)})$$

is an automorphism of $\mathbb{G}$. Here $x' \equiv x^{(1)} \in \mathbb{R}^{N_1}$ and $x^{(k)} \in \mathbb{R}^{N_k}$ for $k = 2, \ldots, r$.

(b) Let N_1 be as in (a) and let $X_1, \ldots, X_{N_1}$ be the left-invariant vector fields on $\mathbb{G}$ such that $X_k(0) = \frac{\partial}{\partial x_k}|_0$ for $k = 1, \ldots, N_1$. Then the Hömander rank condition must be satisfied, that is,

$$\operatorname{rank}(\operatorname{Lie}\{X_1, \ldots, X_{N_1}\}) = n,$$

for every $x \in \mathbb{R}^n$.

Then, we say that the triple $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ is a stratified (Lie) group.

Recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [8]). The left-invariant vector field X_j has an explicit form:

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', \ldots, x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{1.1}$$

see e.g. [8]. The following notations are used throughout this paper:

$$\nabla_{\mathbb{G}} := (X_1, \ldots, X_{N_1})$$

for the horizontal gradient,

$$\mathcal{L}_p f := \nabla_{\mathbb{G}} \cdot (|\nabla_{\mathbb{G}} f|^{p-2} \nabla_{\mathbb{G}} f), \quad 1 < p < \infty, \tag{1.2}$$

for the p-sub-Laplacian. When $p = 2$, that is, the second order differential operator

$$\mathcal{L} = \sum_{k=1}^{N_1} X_k^2, \tag{1.3}$$

is called the sub-Laplacian on $\mathbb{G}$. The sub-Laplacian $\mathcal{L}$ is a left-invariant homogeneous hypoelliptic differential operator and it is known that $\mathcal{L}$ is elliptic if and only if the step r of $\mathbb{G}$ is equal to 1.

Let $\Omega \subset \mathbb{G}$ be an open set, then we define the functional spaces

$$S^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R}; u, |\nabla_{\mathbb{G}} u| \in L^p(\Omega)\}. \tag{1.4}$$

We consider the following functional

$$J_p(u) := \left(\int_\Omega |\nabla_{\mathbb{G}} u(x)|^p dx\right)^{\frac{1}{p}}.$$

Thus, the functional class $\overset{\circ}{S}{}^{1,p}(\Omega)$ can be defined as the completion of $C_0^1(\Omega)$ in the norm generated by J_p, see e.g. [3].

Let Ω be a bounded domain on the stratified Lie groups with the smooth boundary $\partial\Omega$. We consider the Dirichlet initial value problem for the p-sub-Laplacian heat operator

$$\begin{cases} u_t(x,t) - \mathcal{L}_p u(x,t) = f(u(x,t)), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{\Omega}. \end{cases} \tag{1.5}$$

Here f is a locally Lipschitz continuous function on $\mathbb{R}$, $f(0) = 0$, and such that $f(u) > 0$ for $u > 0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $L^\infty(\Omega) \cap \overset{\circ}{S}{}^{1,p}(\Omega)$ and that $u_0(x) = 0$ on the boundary $\partial\Omega$ of Ω.

In the Euclidean setting, it is well-known that there often exists a solution of the p-Laplacian parabolic equation as the one in (1.5) for all times. There is a large literature on the sufficient conditions for the local existence of solutions to the p-Laplacian parabolic equation. For example, the sufficient conditions for the local existence of solutions to the p-Laplacian parabolic equations are derived by Ball [1] and Zhao [10] for $p = 2$ and $p > 2$, respectively. Then, the blow-up solutions have been investigated by many authors such as Levine [6], Philippin and Proytcheva [7], Ding and Hu [5], Bandle and Brunner [2], with a more detailed review of their works presented in [4].

In the present paper, our aim is to extend Abelian (Euclidean) to general stratified (Lie) groups. Thus, in this paper, the blow-up theorems for the p-sub-Laplacian heat operators are proved on the stratified Lie groups. We use the concavity method and the Poincaré inequality from [9] as the main tool. However, these results cannot be directly extended to the general stratified groups since there are no those spectral inequalities on the general stratified groups. On the other hand, in general, our stratified group theorems cannot completely cover the Heisenberg group results due to the restriction $N_1 \neq p$ (see Theorem 2.5).

2 Main Results

2.1 Blow-Up Solutions for the Sub-Laplacian Heat Operators

We consider the blow-up solutions to the sub-Laplacian heat equation on the stratified groups $\mathbb{G}$, that is,

$$\begin{cases} u_t(x,t) - \mathcal{L}u(x,t) = f(u(x,t)), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{2.1}$$

where f is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$, and such that $f(u) > 0$ for $u > 0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $C^1(\overline{\Omega})$ and that $u_0(x) = 0$ on the boundary $\partial\Omega$.

Theorem 2.1 *Let Ω be a bounded domain of the stratified group $\mathbb{G}$ with $N_1 \geq 3$ being the dimension of the first stratum and smooth boundary $\partial\Omega$. Let a function f satisfy the condition that there exist constants $\alpha > 2$ and γ such that for all $u > 0$ we have*

$$\alpha F(u) \leq uf(u) + \beta u^2 + \alpha\gamma, \tag{2.2}$$

where $F(u) = \int_0^u f(s)ds$ and $0 < \beta \leq \frac{(\alpha-2)(N_1-2)^2}{8R^2}$. If $u_0 \in C^1(\overline{\Omega})$ with $u_0 = 0$ on $\partial\Omega$ satisfies the inequality

$$-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}} u_0(x)|^2 dx + \int_\Omega \left(\int_0^{u_0(x)} f(s)ds - \gamma \right) dx > 0, \tag{2.3}$$

then the nonnegative solution to the Eq. (2.1) blows up at a finite time T^* for

$$M := \frac{\left(1 + \sqrt{\frac{\alpha}{2}}\right) \left(\int_\Omega u_0^2(x)dx\right)^2}{2(\alpha-2)[-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}} u_0|^2 dx + \int_\Omega (\int_0^{u_0(x)} f(s)ds - \gamma)dx]}, \tag{2.4}$$

such that

$$0 < T^* \leq \frac{M}{\sigma \int_\Omega u_0^2(x)dx}, \tag{2.5}$$

that is,

$$\lim_{t \to T^*} \int_0^t \int_\Omega u^2(x,\tau)dxd\tau = +\infty. \tag{2.6}$$

Lemma 2.2 [9] *Let Ω be a bounded domain on the stratified group $\mathbb{G}$ with $N_1 \geq 3$ being the dimension of the first stratum. For every function $u \in C_0^\infty(\Omega \backslash \{x' = 0\})$ we have*

$$\int_\Omega |\nabla_{\mathbb{G}} u|^2 dx \geq \frac{(N_1-2)^2}{4R^2} \int_\Omega |u|^2 dx, \tag{2.7}$$

where $R = \sup_{x \in \Omega} |x'|$.

Remark 2.3 Note that Lemma 2.2 will be useful in the proof of Theorem 2.1.

Proof of Theorem 2.1 Let Γ be denoted by

$$\Gamma(t) := \int_0^t \int_\Omega u^2(x,\tau)dxd\tau + M, \ \ t \geq 0, \tag{2.8}$$

where $M > 0$ is a constant to be determined later. By Leibniz's integral rule we get

$$\Gamma'(t) = \frac{d}{dt}\Gamma(t) = \frac{d}{dt}\left(\int_0^t \int_\Omega u^2(x,\tau)dxd\tau\right) = \int_\Omega u^2(x,t)dx,$$

and

$$\begin{aligned}\int_\Omega \int_0^t 2u(x,\tau)u_\tau(x,\tau)d\tau dx &= \int_\Omega \int_0^t \frac{d}{d\tau}u^2(x,\tau)d\tau dx \\ &= \int_\Omega u^2(x,t)dx - \int_\Omega u_0^2(x)dx.\end{aligned}$$

This gives the relation

$$\Gamma'(t) = \int_\Omega u^2(x,t)dx = \int_\Omega \int_0^t 2u(x,\tau)u_\tau(x,\tau)d\tau dx + \int_\Omega u_0^2(x)dx. \tag{2.9}$$

Using the above computations, the condition (2.2) and Lemma 2.2, we compute the second derivative of $\Gamma(t)$ with respect to time

$$\begin{aligned}\Gamma''(t) &= 2\int_\Omega u(x,t)u_t(x,t)dx \\ &= 2\int_\Omega u(x,t)\mathcal{L}u(x,t) + 2\int_\Omega u(x,t)f(u(x,t))dx \\ &\geq -2\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^2dx + 2\int_\Omega \Big[\alpha F(u(x,t)) - \beta u^2(x,t) - \alpha\gamma\Big]dx \\ &\geq 2\alpha\left[-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^2dx + \int_\Omega (F(u(x,t)) - \gamma)dx\right] \\ &\quad + \left(\frac{(\alpha-2)(N_1-2)^2}{4R^2} - 2\beta\right)\int_\Omega u^2(x,t)dx \\ &\geq 2\alpha\left[-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^2dx + \int_\Omega (F(u(x,t)) - \gamma)dx\right] \\ &= 2\alpha\left[-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}}u_0|^2dx + \int_\Omega (F(u_0) - \gamma)dx\right] + 2\alpha\int_0^t \int_\Omega u_\tau^2(x,\tau)dxd\tau.\end{aligned}$$

Then we have

$$\Gamma''(t)\Gamma(t) \geq 2\alpha\left(\left[-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}}u_0|^2dx + \int_\Omega (F(u_0)-\gamma)dx\right] + \int_0^t \int_\Omega u_\tau^2(x,\tau)dxd\tau\right)$$

$$
\begin{aligned}
&\times \left(\int_0^t \int_\Omega u^2(x,\tau)dxd\tau + M \right) \\
&\geq 2\alpha M \left[-\frac{1}{2} \int_\Omega |\nabla_{\mathbb{G}} u_0|^2 dx + \int_\Omega (F(u_0) - \gamma) dx \right] \\
&+ 2\alpha \left(\int_\Omega \int_0^t u^2(x,\tau) d\tau dx \right) \left(\int_\Omega \int_0^t u_\tau^2(x,\tau) d\tau dx \right).
\end{aligned}
$$

Also, we compute by making use of Hölder and Schwartz's inequalities,

$$
\begin{aligned}
(\Gamma'(t))^2 &\leq 4(1+\delta) \left(\int_\Omega \int_0^t u(x,\tau) u_\tau(x,\tau) d\tau dx \right)^2 + \left(1 + \frac{1}{\delta}\right) \left(\int_\Omega u_0^2(x) dx \right)^2 \\
&\leq 4(1+\delta) \left(\int_\Omega \left(\int_0^t u^2(x,\tau) d\tau \right)^{\frac{1}{2}} \left(\int_0^t u_\tau^2(x,\tau) d\tau \right)^{\frac{1}{2}} dx \right)^2 \\
&+ \left(1 + \frac{1}{\delta}\right) \left(\int_\Omega u_0^2(x) dx \right)^2 \\
&\leq 4(1+\delta) \left(\int_\Omega \int_0^t u^2(x,\tau) d\tau dx \right) \left(\int_\Omega \int_0^t u_\tau^2(x,\tau) d\tau dx \right) \\
&+ \left(1 + \frac{1}{\delta}\right) \left(\int_\Omega u_0^2(x) dx \right)^2,
\end{aligned}
$$

where $\delta > 0$. Then by combining the above expressions and taking $\sigma = \delta = \sqrt{\alpha/2} - 1 > 0$, we establish the following estimate

$$
\begin{aligned}
&\Gamma''(t)\Gamma(t) - (1+\sigma)(\Gamma'(t))^2 \\
&\quad \geq 2\alpha M \left[-\frac{1}{2} \int_\Omega |\nabla_{\mathbb{G}} u_0|^2 dx + \int_\Omega (F(u_0) - \gamma) dx \right] - \frac{\alpha}{\sqrt{2\alpha} - 2} \left(\int_\Omega u_0^2(x) dx \right)^2.
\end{aligned}
$$

Then we choose $M > 0$ as large enough to satisfy

$$
\Gamma''(t)\Gamma(t) - (1+\sigma)(\Gamma'(t))^2 > 0. \tag{2.10}
$$

We can see that the above expression for $t \geq 0$ implies

$$
\frac{d}{dt}\left[\frac{\Gamma'(t)}{\Gamma^{\sigma+1}(t)} \right] > 0 \Rightarrow \begin{cases} \Gamma'(t) \geq \left(\frac{\int_\Omega u_0^2(x)dx}{M^{\sigma+1}} \right) \Gamma^{1+\sigma}(t), \\ \Gamma(0) = M. \end{cases}
$$

Then we arrive at

$$
\Gamma(t) \geq \left(\frac{1}{M^\sigma} - \frac{\int_\Omega u_0^2(x)dx}{M^{\sigma+1}} t \right)^{-\frac{1}{\sigma}}.
$$

From here we see that the solutions blow up in the finite time T^* which is

$$0 < T^* \leq \frac{M}{\sigma \int_\Omega u_0^2(x)dx},$$

where M can be estimated from (2.10), that is,

$$M := \frac{\left(1+\sqrt{\frac{\alpha}{2}}\right)\left(\int_\Omega u_0^2(x)dx\right)^2}{2(\alpha-2)[-\frac{1}{2}\int_\Omega |\nabla_{\mathbb{G}} u_0|^2 dx + \int_\Omega (F(u_0)-\gamma)dx]}.$$

Therefore, it follows that $\Gamma(t)$ cannot remain finite for all $t > 0$. In other words, the solutions u blows up in finite time T^*. □

2.2 Blow-Up Solutions for p-Sub-Laplacian Heat Operators

We consider now the blow-up solutions to the p-sub-Laplacian heat equation on the stratified group $\mathbb{G}$, that is,

$$\begin{cases} u_t(x,t) - \mathcal{L}_p u(x,t) = f(u(x,t)), & (x,t) \in \Omega \times (0,+\infty), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(x) \geq 0, & x \in \overline{\Omega}, \end{cases} \tag{2.11}$$

where f is locally Lipschitz continuous on $\mathbb{R}$, $f(0) = 0$, and such that $f(u) > 0$ for $u > 0$. Furthermore, we suppose that u_0 is a non-negative and non-trivial function in $L^\infty(\Omega) \cap \mathring{S}^{1,p}(\Omega)$ and that $u_0(x) = 0$ on the boundary $\partial\Omega$.

Lemma 2.4 [9] *Let Ω be a bounded domain on the stratified group $\mathbb{G}$ with N_1 being the dimension of the first stratum. Let $1 < p < \infty$ with $p \neq N_1$. For every function $u \in C_0^\infty(\Omega \backslash \{x' = 0\})$ we have*

$$\int_\Omega |\nabla_{\mathbb{G}} u|^p dx \geq \frac{|N_1 - p|^p}{(pR)^p} \int_\Omega |u|^p dx, \tag{2.12}$$

where $R = \sup_{x\in\Omega} |x'|$.

Theorem 2.5 *Let Ω be a bounded domain of the stratified group $\mathbb{G}$ with N_1 being the dimension of the first stratum and smooth boundary $\partial\Omega$. Let a function f satisfy the condition that there exist constants $\alpha > p$ and γ such that for all $u > 0$ we have*

$$\alpha F(u) \leq uf(u) + \beta u^p + \alpha\gamma, \tag{2.13}$$

where $F(u) = \int_0^u f(s)ds$ *and* $0 < \beta \leq \frac{(\alpha-p)|N_1-p|^p}{p(pR)^p}$ *with* $N_1 \neq p$. *If* $u_0 \in L^\infty(\Omega) \cap \mathring{S}^{1,p}(\Omega)$ *satisfies*

$$-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u_0(x)|^p dx + \int_\Omega \left(\int_0^{u_0(x)} f(s)ds - \gamma \right) dx > 0, \tag{2.14}$$

then the nonnegative solution to the Eq. (2.11) blows up at a finite time T^* *for*

$$M := \frac{\left(1+\sqrt{\frac{\alpha}{2}}\right)\left(\int_\Omega u_0^2(x)dx\right)^2}{2(\alpha-2)\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u_0|^p dx + \int_\Omega (\int_0^{u_0(x)} f(s)ds - \gamma)dx\right]}, \tag{2.15}$$

such that

$$0 < T^* \leq \frac{M}{\sigma \int_\Omega u_0^2(x)dx}, \tag{2.16}$$

that is,

$$\lim_{t \to T^*} \int_0^t \int_\Omega u^2(x,\tau)dxd\tau = +\infty. \tag{2.17}$$

Lemma 2.6 *Let* u *be the weak solution to the Eq. (2.11) with* $|\nabla_{\mathbb{G}} u_0| \in L^p(\Omega)$. *Then*

$$\begin{aligned} \frac{1}{2}\int_0^t \int_\Omega (u^2(x,\tau))_\tau dxd\tau &= \frac{1}{2}\int_\Omega [u^2(x,t) - u_0^2(x)]dx \\ &= \int_0^t \int_\Omega [-|\nabla_{\mathbb{G}} u(x,t)|^p + u(x,t) f(u(x,t))]dxd\tau, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \int_0^t \int_\Omega u_\tau^2(x,\tau)dxd\tau = &-\frac{1}{p}\int_\Omega [|\nabla_{\mathbb{G}} u(x,t)|^p - |\nabla_{\mathbb{G}} u_0(x)|^p]dx \\ &+ \int_\Omega [F(u(x,t)) - F(u_0(x))]dx, \end{aligned} \tag{2.19}$$

where $F(u) := \int_0^u f(s)ds$.

Proof of Lemma 2.6 We first prove the equality (2.18) by using the equation (2.11), that is,

$$\frac{1}{2}\int_0^t \int_\Omega (u^2(x,\tau))_\tau dxd\tau = \frac{1}{2}\int_\Omega u^2(x,\tau)|_0^t dx = \frac{1}{2}\int_\Omega [u^2(x,t) - u_0^2(x)]dx,$$

and

$$\begin{aligned}\frac{1}{2}\int_0^t\int_\Omega (u^2(x,\tau))_\tau dxd\tau &= \int_\Omega\int_0^t u(x,\tau)u_\tau(x,\tau)dxd\tau \\ &= \int_0^t\int_\Omega \mathcal{L}_p u(x,\tau)u(x,\tau)dxd\tau + \int_0^t\int_\Omega f(u(x,\tau))u(x,\tau)dxd\tau \\ &= -\int_0^t\int_\Omega |\nabla_{\mathbb{G}} u(x,\tau)|^p dxd\tau + \int_0^t\int_\Omega f(u(x,\tau))u(x,\tau)dxd\tau,\end{aligned}$$

which proves the expression (2.18). Now we prove inequality (2.19) by using the Leibniz integral rule, as follows

$$\begin{aligned}\int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau &= \int_0^t\int_\Omega \mathcal{L}_p u(x,\tau)u_\tau(x,\tau)dxd\tau \\ &\quad + \int_0^t\int_\Omega f(u(x,\tau))u_\tau(x,\tau)dxd\tau \\ &= -\int_0^t\int_\Omega \langle|\nabla_{\mathbb{G}} u(x,\tau)|^{p-2}\nabla_{\mathbb{G}} u(x,\tau), \nabla_{\mathbb{G}} u_\tau(x,\tau)\rangle dxd\tau \\ &\quad + \int_0^t\int_\Omega f(u(x,\tau))u_\tau(x,\tau)dxd\tau \\ &= -\frac{1}{2}\int_\Omega\int_0^t \frac{d}{d\tau}\left(\int_0^{|\nabla_{\mathbb{G}} u(x,\tau)|^2} s^{\frac{p-2}{2}}ds\right)d\tau dx \\ &\quad + \int_\Omega\int_0^t \frac{d}{d\tau}\left(\int_0^{u(x,\tau)} f(s)ds\right)d\tau dx \\ &= -\frac{1}{2}\int_\Omega\int_0^{|\nabla_{\mathbb{G}} u(x,\tau)|^2} s^{\frac{p-2}{2}}dsdx|_0^t + \int_\Omega F(u(x,\tau))dx|_0^t \\ &= -\frac{1}{2}\int_\Omega\left(\frac{2}{p}s^{\frac{p}{2}}|_0^{|\nabla_{\mathbb{G}} u(x,\tau)|^2}\right)dx|_0^t + \int_\Omega F(u(x,\tau))dx|_0^t \\ &= -\frac{1}{p}\int_\Omega\left[|\nabla_{\mathbb{G}} u(x,t)|^p - |\nabla_{\mathbb{G}} u_0(x)|^p\right]dx \\ &\quad + \int_\Omega [F(u(x,t)) - F(u_0(x))]dx,\end{aligned}$$

which proves the expression (2.19). □

Proof of Theorem 2.5 Let us define the function Γ_p by

$$\Gamma_p(t) := \int_0^t\int_\Omega u^2(x,\tau)dxd\tau + M,\ \ t \geq 0, \tag{2.20}$$

where M is a positive constant. Then we compute the derivative of $\Gamma(t)$ with respect to time, which gives that

$$\begin{aligned}\Gamma_p'(t) &= \frac{d}{dt}\int_0^t\int_\Omega u^2(x,\tau)dxd\tau = \int_\Omega \frac{d}{dt}\left(\int_0^t u^2(x,\tau)d\tau\right)dx\\ &= \int_\Omega u^2(x,t)dx\\ &= \int_0^t\int_\Omega 2u(x,\tau)u_\tau(x,\tau)d\tau dx + \int_\Omega u_0^2(x)dx.\end{aligned}$$

The second derivative of $\Gamma_p(t)$ with respect to time t can be calculated by Lemma 2.6, using the condition (2.13), and Lemma 2.4, so that we get the estimate for $\Gamma_p''(t)$ as follows:

$$\begin{aligned}\Gamma_p''(t) &= 2\int_\Omega u(x,t)u_t(x,t)dx\\ &= -2\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^p dx + 2\int_\Omega u(x,t)f(u(x,t))dx\\ &\geq -2\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^p dx + 2\int_\Omega \big(\alpha F(u(x,t)) - \beta u^p(x,t) - \alpha\gamma\big)dx\\ &\geq 2\alpha\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^p dx + \int_\Omega [F(u(x,t))-\gamma]dx\right]\\ &\quad + 2\left(\frac{(\alpha-p)|N_1-p|^p}{p^{p+1}R^p} - \beta\right)\int_\Omega u^p(x,t)dx\\ &\geq 2\alpha\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}}u(x,t)|^p dx + \int_\Omega [F(u(x,t))-\gamma]dx\right]\\ &= 2\alpha\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}}u_0|^p dx + \int_\Omega [F(u_0)-\gamma]dx\right] + 2\alpha\int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau.\end{aligned}$$

Then we have

$$\begin{aligned}\Gamma_p''(t)\Gamma_p(t) &\geq 2\alpha\left(\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}}u_0|^p dx + \int_\Omega [F(u_0)-\gamma]dx\right] + \int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau\right)\\ &\quad \times\left(\int_0^t\int_\Omega u^2(x,\tau)dxd\tau + M\right)\\ &\geq 2\alpha M\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}}u_0|^p dx + \int_\Omega [F(u_0)-\gamma]dx\right]\\ &\quad + 2\alpha\left(\int_0^t\int_\Omega u_\tau^2(x,\tau)dxd\tau\right)\left(\int_0^t\int_\Omega u^2(x,\tau)dxd\tau\right).\end{aligned}$$

By making use of Schwartz's inequality and for arbitrary $\delta > 0$ as in the case $p = 2$ we arrive at

$$\left(\Gamma_p'(t)\right)^2 \leq 4(1+\delta)\left(\int_\Omega \int_0^t u^2(x,\tau)d\tau dx\right)\left(\int_\Omega \int_0^t u_\tau^2(x,\tau)d\tau dx\right) + \frac{1+\delta}{\delta}\left(\int_\Omega u_0^2(x)dx\right)^2. \tag{2.21}$$

Now we use estimates of $\Gamma_p'(t)$ and $\Gamma_p''(t)$ to obtain

$$\Gamma_p''(t)\Gamma_p(t) - (1+\sigma)(\Gamma_p'(t))^2 \geq 2\alpha M\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u_0|^p dx + \int_\Omega (F(u_0)-\gamma)dx\right] - \frac{\alpha}{\sqrt{2\alpha}-2}\left(\int_\Omega u_0^2(x)dx\right)^2. \tag{2.22}$$

Note that from (2.14) we have

$$-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u_0|^p dx + \int_\Omega (F(u_0)-\gamma)dx > 0,$$

and taking $M > 0$ as large as necessary, we obtain the following estimate

$$\Gamma_p''(t)\Gamma_p(t) - (1+\sigma)(\Gamma_p'(t))^2 > 0. \tag{2.23}$$

For $t \geq 0$ the above expression can be written as

$$\frac{d}{dt}\left(\frac{\Gamma_p'(t)}{\Gamma_p^{1+\sigma}(t)}\right) > 0,$$

which implies

$$\begin{cases} \Gamma_p'(t) \geq \left(\frac{\int_\Omega u_0^2(x)dx}{M^{\sigma+1}}\right)\Gamma_p^{1+\sigma}(t), \ t > 0, \\ \Gamma_p(0) = M. \end{cases} \tag{2.24}$$

Then we arrive at

$$\Gamma_p(t) \geq \left(\frac{1}{M^\sigma} - \frac{\int_\Omega u_0^2(x)dx}{M^{\sigma+1}}t\right)^{-\frac{1}{\sigma}},$$

From here we see that the solutions blow up in the finite time T^* which is

$$0 < T^* \leq \frac{M}{\sigma \int_\Omega u_0^2(x)dx},$$

where M can be estimated from (2.22) as follows

$$M = \frac{\left(1+\sqrt{\frac{\alpha}{2}}\right)\left(\int_\Omega u_0^2(x)dx\right)^2}{2(\alpha-2)\left[-\frac{1}{p}\int_\Omega |\nabla_{\mathbb{G}} u_0|^p dx + \int_\Omega (F(u_0)-\gamma)dx\right]}. \tag{2.25}$$

□

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