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Hankel Operators Between Bergman Spaces with Variable Exponents on the Unit Ball of $\mathbb{C}^n$

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Abstract

We characterize boundedness and compactness of Hankel operators between Bergman spaces of variable exponent and the Lebesgue spaces of variable exponents. We also give some characterizations of the symbol class which is some BMO-type spaces with variable exponent on the unit ball of $\mathbb{C}^n$.

Keywords Hankel operators · Homogeneous spaces · Muckenhoupt weights · Variable exponent Bergman spaces · Variable exponent Lebesgue spaces

Mathematics Subject Classification Primary 32A36 · Secondary 47B35

1 Introduction and Statement of Results

Variable Lebesgue spaces are a generalization of the Lebesgue spaces that allow the exponents to be a measurable function and thus the exponent may vary. These spaces have many properties similar to the normal Lebesgue spaces, but they also differ in surprising and subtle ways. For this reason, the variable Lebesgue spaces have an intrinsic interest and also very important in applications to partial differential equations and variational integrals with non-standard growth conditions. See [5] for more details on the variable Lebesgue spaces.

Let $\mathbf{B}$ denote the unit ball in $\mathbb{C}^n$ and $d\nu$ the normalized Lebesgue measure on $\mathbf{B}$. If $z \in \mathbf{B}$ and $\alpha > -1$ we set $d\nu_\alpha(z) = (1-|z|^2)^\alpha d\nu$. For $1 \leq p < \infty$, the Bergman space $A^p_\alpha = A^p(\mathbf{B}, d\nu_\alpha)$ is the space of all analytic functions, f, on $\mathbf{B}$ such that

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$$\|f\|_p^p = \int_{\mathbf{B}} |f(z)|^p d\nu_\alpha(z) < \infty.$$

Let P_α be the Bergman projection from L^2 onto A^2. Then P_α is an integral operator given by

$$P_\alpha(f)(z) = \int_{\mathbf{B}} \frac{f(w)}{(1-\langle z, w\rangle)^{n+1+\alpha}} d\nu_\alpha(w), \tag{1}$$

for each $z \in \mathbf{B}$ and $f \in L^2$, where $\langle z, w\rangle = \sum_{i=1}^n z_i \overline{w_i}$. The function $K^\alpha(z, w) = K_w^\alpha(z) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}$ is the reproducing kernel for A_α^2. For $1 < p \le q < \infty$ and $f \in L_{loc}^p$ we define the Hankel operator $H_f^\alpha : A_\alpha^p \to L_\alpha^q$ by

$$H_f^\alpha(g) = fg - P_\alpha(fg) = (I - P_\alpha)(fg), \quad g \in A_\alpha^p.$$

Hankel operators are amongst the most widely studied classes of concrete operators and have attracted alot of interest in recent years. The behaviour of these operators on the Bergman spaces and Fock spaces have been studied widely and alot of results are available in the literature.

Given $\Omega \subset \mathbb{R}^n$, a measurable function $p : \Omega \to [1, \infty)$ will be called a variable exponent. We denote by

$$p_+ = p_\Omega^+ := \operatorname*{ess\,sup}_{x\in\Omega} p(x), \quad p_- = p_\Omega^- := \operatorname*{ess\,inf}_{x\in\Omega} p(x).$$

Let $\mathcal{P}(\Omega)$ denote the set of all variable exponents for which $p_+ < \infty$.

For a complex-valued measurable function $\phi : \Omega \to \mathbb{C}$ we define the modular $\rho_{p(\cdot)}$ by

$$\rho_{p(\cdot)}(\phi) := \int_\Omega |\phi(x)|^{p(x)} dx$$

and the norm

$$\|\phi\|_{p(\cdot)} := \inf\left\{\lambda > 0 : \rho_{p(\cdot)}\left(\frac{\phi}{\lambda}\right) \le 1\right\}. \tag{2}$$

If no confusion shall arise, we shall denote the modular $\rho_{p(\cdot)}$ simply by ρ. Let $p(\cdot) \in \mathcal{P}(\Omega)$. Then the Lebesgue variable exponent space $L^{p(\cdot)}$ is the set of all complex valued measurable functions $\phi : \Omega \to \mathbb{C}$ for which $\rho_{p(\cdot)}(\phi) < \infty$. If we equip $L^{p(\cdot)}$ with the norm given in (2), then $L^{p(\cdot)}$ becomes a Banach space. We note here that the condition $\rho_{p(\cdot)}(\phi) < \infty$ is not enough in general to define the variable exponent lebesgue space (see for example chapter 2 of [5]).

It is known, see for example chapter 2 of [5], that the dual of $L^{p(\cdot)}$ is $L^{p'(\cdot)}$ where $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$. A straight foward computation shows that

$$(p'(\cdot))_+ = (p_-)', \ (p'(\cdot))_- = (p_+)'. \tag{3}$$

For simplicity we will omit one set of parenthesis and write the left-hand side of each equality as $p'(\cdot)_+$ and $p'(\cdot)_-$. Throughout this work we shall use $p'(\cdot)$ as the conjugate exponent of $p(\cdot)$ and p' the conjugate exponent of p whenever p is a constant in $(1, \infty)$.

We will impose some regularity conditions on the variable exponents in order to have some "fruitful' theory (e.g boundedeness of the maximal operator). A function $p : \Omega \to \mathbb{C}$ is said to be log-Hölder continuous on Ω if there exists a positive constant $C_{\log}$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}}{\log(1/|x-y|)}, \tag{4}$$

for all $x, y \in \Omega$ with $|x - y| < 1/2$. It follows that

$$|p(x) - p(y)| \leq \frac{2lC_{\log}}{\log(2l/|x-y|)},$$

for all $x, y \in \Omega$ with $|x - y| < l$. We denote by $\mathcal{P}^{\log}(\Omega)$ the exponents in $\mathcal{P}(\Omega)$ that are log-Hölder continuous on Ω. It is well known that the condition (4) is equivalent to the condition

$$|B|^{p_B^- - p_B^+} \leq C, \tag{5}$$

for all balls, where $|\cdot|$ stands for the normalized Lebesgue measure. As a consequence of (5) we have that if $z, w \in B$ then

$$|B|^{p(z)} \approx |B|^{p(w)}, \tag{6}$$

for any ball B. We are going to use this relation several times in the paper.

The study of variable exponent Bergman spaces on the unit disc, $A^{p(\cdot)}(\Delta)$, which is the space of analytic functions in $L^{p(\cdot)}(\Delta)$, have been introduced in [4]. There, it was shown, amongst other things, that the Bergman projector P is bounded from $L^{p(\cdot)}(\Delta)$ onto $A^{p(\cdot)}(\Delta)$. Also in [1] the author studied compact operators on $A^{p(\cdot)}(\Delta)$ while Carleson measures on such spaces are given in [3]. Using a similar argument and with less difficulty one easily obtains the boundedness of the Bergman projector , P_α, from $L_\alpha^{p(\cdot)}(\mathbf{B})$ onto $A_\alpha^{p(\cdot)}(\mathbf{B})$ since the method used in [4] works in the case of the unit ball.

In this paper, we are interested in the mapping properties of Hankel operators between different Lebesgue spaces with variable exponent. The case with constant exponent has been of interest and has attracted a lot of research, see for example [7, 9, 11, 14], just to cite a few. We characterize functions $f \in L_\sigma^{q(\cdot)}$ such that both H_f^σ and $H_{\overline{f}}^\sigma$ are bounded and compact from $A_\alpha^{p(\cdot)}$ to $L_\sigma^{q(\cdot)}$, $1 < p(\cdot) \leq q(\cdot)$. We define two BMO-type spaces; $BMOS^{p(\cdot)}$ and $VMOS^{p(\cdot)}$ which will enable us characterize the symbols for boundedness and compactness of the Hankel operators. We also note that when the exponents are constant, then the spaces mentioned above becomes the well known BMO and VMO spaces. The main results of this paper are the following:

Theorem 1.1 *Suppose* $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $p(\cdot) \le q(\cdot)$, $\alpha, \sigma > -1$, $f \in L^{q(\cdot)}_{\sigma}$ *and*

$$\frac{n+1+\sigma}{q(\cdot)} = \frac{n+1+\alpha}{p(\cdot)}.$$

Let $p_0 \le q_0$ *be such that* $1 \le p_0 \le p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$ *and* $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{p_0} - \frac{1}{q_0}$. *Then* $H^{\sigma}_{f}, H^{\sigma}_{\overline{f}} : A^{p(\cdot)}_{\alpha} \to L^{q(\cdot)}_{\sigma}$ *are both bounded if and only if* $f \in BMOS^{q(\cdot)}$

Theorem 1.2 *Suppose* $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $p(\cdot) \le q(\cdot)$, $\alpha, \sigma > -1$, $f \in L^{q(\cdot)}_{\sigma}$ *and*

$$\frac{n+1+\sigma}{q(\cdot)} = \frac{n+1+\alpha}{p(\cdot)}.$$

Let $p_0 \le q_0$ *be such that* $1 \le p_0 \le p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$ *and* $\frac{1}{p(\cdot)} - \frac{1}{q(\cdot)} = \frac{1}{p_0} - \frac{1}{q_0}$.

Then $H^{\sigma}_{f}, H^{\sigma}_{\overline{f}} : A^{p(\cdot)}_{\alpha} \to L^{q(\cdot)}_{\sigma}$ *are both compact if and only if* $f \in VMOS^{q(\cdot)}$.

In other to obtain these results we will first characterize the symbol spaces, which is, the variable exponent BMO-type spaces, establish a weighted version of the Okikiolu Lemma, [10], that will enable us use extrapolation techniques to extend the results given in [11] to the variable exponent setting.

In the the following, the notation $A \lesssim B$ means there is a positive constant C such that $A \le CB$ and the notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

This paper is organized as follows: In Sect. 2 we will study some basic concepts that will be useful in the work. Sect. 3 deals with the variable Bergman spaces,in Sect. 4 we characterize the space $BMOS^{p(\cdot)}$ and then we give the Proof of Theorem 1.1 in Sect. 5. Sect. 6 deals with the space $VMOS^{p(\cdot)}$ and the Proof of Theorem 1.2 is given in Sect. 7.

2 Preliminaries

In this section we collect some preliminaries results that are needed for the proof of the main theorems. For $z = (z_1, \cdots, z_n)$, $w = (w_1, \cdots, w_n) \in \mathbb{C}^n$, we write

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i \overline{w_i}$$

and $|z| = \sqrt{\langle z, z \rangle}$. For $a \in \mathbf{B}$ with $a \ne 0$ we denote by φ_a the Möbius transformation on $\mathbf{B}$ that interchanges the points 0 and a. It is well known that

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle}, \quad z \in \mathbf{B},$$

where $s_a = 1 - |a|^2$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the one dimensional space $[a]$ generated by a and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto the complement of $[a]$.

For $z, w \in \mathbf{B}$, the Bergman metric distance between z and w is given by

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

Let $z \in \mathbf{B}$ and $r > 0$. Then the Bergman metric ball with centre z is given by

$$D(z, r) = \{w \in \mathbf{B} : \beta(z, w) < r\}.$$

It is known that if $w \in D(z, r)$ then $1 - \langle z, w\rangle \approx (1 - |w|^2) \approx (1 - |z|^2)$,

$$(1 - |z|^2)^{n+1+\alpha} \approx \nu_\alpha(D(z, r)) \approx \nu_\alpha(D(w, r)) \approx (1 - |w|^2)^{n+1+\alpha}, \tag{7}$$

and by (6) if $\beta(z, w) < r$,

$$(1 - |z|^2)^{(n+1+\alpha)p(z)} \approx (1 - |z|^2)^{(n+1+\alpha)p(w)}, \quad p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B}). \tag{8}$$

We will need the following result which is Theorem 2.23 of [16].

Lemma 2.1 *There exists a positve integer N such that for any $r \in (0, 1)$ we can find a sequence $\{a_k\}$ in $\mathbf{B}$ satisfying the following properties:*

(i) $\mathbf{B} = \bigcup_k D(a_k, r)$.
(ii) *The sets $D(a_k, r/4)$ are mutually disjoint.*
(iii) *Each point $z \in \mathbf{B}$ belongs to at most N of the sets $D(a_k, r)$.*

Recall, that any sequence $\{a_k\}$ satisfying the conditions of the lemma above is called an r-lattice in the Bergman metric. A sequence $\{a_k\}$ is said to be separated in the Bergman metric if there is a positive number δ such that $\beta(a_i, a_j) \geq \delta$ for all $i \neq j$.

We will need the following well-known estimates:

Lemma 2.2 *Let $t > -1$, $s > 0$, and $d \geq 0$. Then there is a positive constant C such that*

$$\int_{\mathbf{B}} \frac{(1 - |w|^2)^t \beta(z, w)^d}{|1 - \langle z, w\rangle|^{n+1+t+s}} d\nu(w) \leq C(1 - |z|^2)^s \tag{9}$$

for all $z \in \mathbf{B}$.

Lemma 2.3 *Let $\{z_k\}$ be a separated sequence in $\mathbf{B}$ and $n < t < s$. Then*

$$\sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^t}{|1 - \langle z, z_k\rangle|^s} \leq C(1 - |z|^2)^{t-s}, \quad z \in \mathbf{B}.$$

The Proof of Lemma 2.3 can be deduced from Lemma 2.2 in the case when $d = 0$ and by noticing that if a sequence $\{a_k\}$ is separated, then there is a constant $r > 0$ such that the Bergman metric balls, $D(a_k, r)$ are pairwise disjoint

Definition 2.4 Let Ω be a set. Then the function $d : \Omega \times \Omega \to \mathbb{R}^+$ is said be a pseudo-distance on Ω if it satisties the following:

(1) $d(x, y) = 0$ if and only if $x = y$;
(2) $d(x, y) = d(y, x)$;
(3) There exists a positive constant $K \geq 1$, such that, for all $x, y, z \in \Omega$,

$$d(x, y) \leq K(d(x, z) + d(z, y)).$$

For $x \in \Omega$ and $r > 0$, the set $B(x, r) = \{y \in \Omega : d(x, y) < r\}$ is called a pseudo-ball with centre x and radius r. If μ is a measure on Ω then the triple (Ω, d, μ) is called a homogeneous space if Ω is endowed with the topology generated by the collection $\{B(x, r) : x \in \Omega,\ r > 0\}$ (that is the topology generated by the pseudo balls) and μ satisfies the doubling property: there exists a constant δ such that for all $x \in \Omega$ and $r > 0$, we have

$$0 < \mu(B(x, 2r)) \leq \delta\mu(B(x, r)) < \infty.$$

We now tend our attention to the case when $\Omega = \mathbf{B}$. By lemma 2.6 of [13], it is shown that the distance function d given on $\mathbf{B}$ by

$$d(z, w) = \begin{cases} ||z| - |w|| + |1 - \langle z, w\rangle/|z||w|| & \text{if } z, w \in \mathbf{B}^* \\ |z| + |w| & \text{otherwise} \end{cases}$$

is a pseudo-distance on $\mathbf{B}$, where $\mathbf{B}^* = \mathbf{B} \setminus \{0\}$. It is known (see for example [2]), that at the boundary of $\mathbf{B}$, d becomes the Koranyi distance. Also by Lemma 2 of [2], we have that for any pseudo-ball $B(w, r)$, $w \neq 0$, and $r \in (0, 2)$ we have that

$$\nu_\alpha(B(w, r)) \approx r^{n+1}. \tag{10}$$

Also, observe that the pseudo-ball $B(0, 1) = \mathbf{B}$. Also from [2] the space $(\mathbf{B}, d, d\nu_\alpha)$ is a homogeneous space. On the variable exponent setting we will need the Diening inequality, which will play the role of the Jensen integral inequality: For any Borel measure μ

$$\left(\frac{1}{\mu(Bx, r)}\int_{B(x,r)} |f(y)|d\mu(y)\right)^{p(x)} \leq C\left(1 + \frac{1}{\mu(B(x, r)}\int_{B(x,r)} |f(y)|^{p(y)}d\mu(y)\right) \tag{11}$$

which holds whenever

$$\int_{B(x,r)} |f(y)|^{p(y)}d\mu(y) \leq 1$$

and $p(\cdot)$ is log-Hölders continuous. For more details see [12].

Let f be a locally integrable function in $\mathbf{B}$. Then the Hardy–Littlewood maximal function relative to the pseudo-distance d is given by

$$Mf(z) = \sup_B \frac{1}{\nu_\alpha(B)} \int_B |f(w)| d\nu_\alpha(w)$$

where the supremum is taken over all pseudo-balls containing z.

Suppose $0 < \omega(z) < \infty$ almost everywhere on $\mathbf{B}$, we will set $\omega(B) = \int_B \omega(\xi) d\nu_\alpha(\xi)$. Then we say that ω is in the Muckenhoupt weight A_1 if

$$[\omega]_{A_1} = \text{ess} \sup_{z \in \mathbf{B}} \frac{M\omega(z)}{\omega(z)} < \infty$$

There are two equivalent definitions, of the class A_1, which are useful in practice. First, $\omega \in A_1$ if for almost every $z \in \mathbf{B}$,

$$M\omega(z) \leq [\omega]_{A_1} \omega(z). \tag{12}$$

It follows that if $\omega \in A_1$ then

$$[\omega]_{A_1} \omega(z) \geq M\omega(z) \geq \omega(\mathbf{B}).$$

Thus

$$1 \leq \frac{[\omega]_{A_1} \omega(z)}{\omega(\mathbf{B})}. \tag{13}$$

Alternatively $w \in A_1$ if for every pseudo-ball B we have that

$$\frac{\omega(B)}{\nu_\alpha(B)} \leq [\omega]_{A_1} \text{ ess} \inf_{u \in B} \omega(u). \tag{14}$$

For more details on the Muckenhoupt weights see Chapter 9 of [6] or Chapter 4 of [5].

We will need the following extrapolation result which is Theorem 5.24 of [5].

Proposition 2.5 *Given $\Omega \in \mathbb{R}^n$ and suppose for some p_0, q_0, $1 \leq p_0 \leq q_0$ the family $\mathcal{F}$ is such that for all $\omega \in A_1$,*

$$\left(\int_\Omega F(x)^{q_0} \omega(x) dx \right)^{\frac{1}{q_0}} \leq C_0 \left(\int_\Omega G(x)^{p_0} \omega(x)^{\frac{p_0}{q_0}} dx \right)^{\frac{1}{p_0}}, \quad (F, G) \in \mathcal{F}. \tag{15}$$

Given $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ such that $p_0 \leq p_- \leq p_+ < \frac{p_0 q_0}{q_0 - p_0}$, define $q(\cdot)$ by

$$\frac{1}{p(x)} - \frac{1}{q(x)} = \frac{1}{p_0} - \frac{1}{q_0}.$$

If the maximal operator is bounded on $L^{(q(\cdot)/q_0)'}(\Omega)$, *then*

$$\|F\|_{q(\cdot)} \le C_{p(\cdot)}\|G\|_{p(\cdot)}, \quad (F, G) \in \mathcal{F}, \tag{16}$$

where $C_{p(\cdot)} = CC_0$ *and* C *is some positive constant depending on the dimension of* Ω.

The following is Theorem 3.16 of [5].

Proposition 2.6 *Let* $p \in \mathcal{P}^{\log}(\Omega)$. *Then the Hardy–Littlewood maximal operator function is bounded in* $L^{p(\cdot)}(\Omega)$ *and we have*

$$\|Mf\|_{p(\cdot)} \le C\|f\|_{p(\cdot)}.$$

The next result, which establishes a relationship between the modular ρ and the norm on Lebesgue spaces with variable exponents will be very useful in the rest of the work, which is from Corollaries 2.22 and 2.23 of [5].

Lemma 2.7 *Suppose* $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ *and* $|\Omega| < \infty$. *If* $\|f\|_{p(\cdot)} > 1$ *then* $\|f\|_{p(\cdot)} \le \rho(f)$ *and*

$$\rho(f)^{1/p_+} \le \|f\|_{p(\cdot)} \le \rho(f)^{1/p_-}.$$

If $0 < \|f\|_{p(\cdot)} \le 1$ *then* $\rho(f) \le \|f\|_{p(\cdot)}$ *and*

$$\rho(f)^{1/p_-} \le \|f\|_{p(\cdot)} \le \rho(f)^{1/p_+}.$$

Finally, we will also use the following fact: If $a > 0, b > 0$ and $p(\cdot) \in \mathcal{P}^{\log}(\Omega)$ then

$$(a+b)^{p(w)} \le 2^{p(w)-1}(a^{p(w)} + b^{p(w)}) \tag{17}$$

For more details see Chapter 1 of [5].

3 Variable Exponent Bergman Spaces

Given $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, we define the variable exponent Bergman space $A^{p(\cdot)}(\mathbf{B}, d\nu_\alpha) = A_\alpha^{p(\cdot)}$ as the space of all analytic functions on $\mathbf{B}$ that belong to the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbf{B}, d\nu_\alpha) = L_\alpha^{p(\cdot)}$. It is known that $A_\alpha^{p(\cdot)}$ is a closed subspace of $L_\alpha^{p(\cdot)}$. On the unit disc, Δ, it is shown in Theorem 4.4 of [4], that the Bergman projection, P, given by (1) is bounded from $L^{p(\cdot)}$ onto $A^{p(\cdot)}$ for any $p(\cdot) \in \mathcal{P}^{\log}(\Delta)$. This result can easily be extended to the case of the unit ball in $\mathbb{C}^n$:

Lemma 3.1 *Let* $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ *and* $\alpha > -1$. *Then the Bergman projection,* P_α, *is bounded from* $L_\alpha^{p(\cdot)}$ *onto* $A_\alpha^{p(\cdot)}$.

The proof is an easy adaptation of the one dimensional case in [4] and so we leave it for the interested reader.

We shall denote the norm on $L^{p(\cdot)}_\alpha$ by $\|\cdot\|_{p(\cdot),\alpha}$ or simply by $\|\cdot\|_{p(\cdot)}$ if no confusion shall arise.

The following result will be useful throughout the paper.

Lemma 3.2 *Let $p(\cdot)\in\mathcal{P}^{\log}(\mathbf{B})$. Then for every $z\in\mathbf{B}$ we have*

$$\|K_z^{2/p(z)}\|_{p(\cdot)}\lesssim\frac{C}{(1-|z|^2)^{(n+1+\alpha)/p(z)}}.$$

Proof Let $J_z(w)=\frac{1-|z|^2}{(1-\langle w,z\rangle)^2}$. We will first estimate the following integral:

$$I_z:=\int_{\mathbf{B}}|J_z(w)|^{sp(w)/p(z)}d\nu_\alpha(w),$$

where $s=n+1+\alpha$. Using the change of variable $w=\varphi_z$ we have that

$$\begin{aligned}
I_z&=\int_{\mathbf{B}}\frac{(1-|z|^2)^{sp(\varphi_z(w))/p(z)}}{|1-\langle z,\varphi_z(w)\rangle|^{2sp(\varphi_z(w))/p(z)}}\frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{2s}}d\nu_\alpha(w)\\
&=\int_{\mathbf{B}}|J_z(w)|^{s-sp(\varphi_z(w))/p(z)}d\nu_\alpha(w)\\
&=\int_{|J_z(w)|\geq 1}|J_z(w)|^{s-sp(\varphi_z(w))/p(z)}d\nu_\alpha(w)\\
&\quad+\int_{|J_z(w)|\leq 1}|J_z(w)|^{s-sp(\varphi_z(w))/p(z)}d\nu_\alpha(w)\\
&=I_1+I_2
\end{aligned}$$

If $|J_z(w)|\geq 1$ then the function $t\mapsto|J_z(w)|^t$ is increasing. Thus $|J_z(w)|^{s-sp(\varphi_z(w))/p(z)}\leq|J_z(w)|^s$ and we have that $I_1\lesssim 1$. For I_2 since $J_z\neq 0$ we have that

$$\begin{aligned}
\left|\log|J_z(w)|^{s-sp(\varphi_z(w))/p(z)}\right|&=\frac{s}{p(z)}|p(z)-p(\varphi_z(w))|\log\frac{1}{|J_z(w)|}\\
&\leq\frac{s}{p_-}\frac{2C_{\log}}{\log\frac{4}{|z-\varphi_z(w)|}}\log\frac{1}{|J_z(w)|}
\end{aligned}$$

where we have used the log-Hölder condition to obtain the last inequality and the fact that $p_-\leq p(\cdot)$. Now , a simple calculation shows that

$$|z-\varphi_z(w)|=\Big(|J_z(w)|(|w|^2-|\langle z,w\rangle|^2)\Big)^{\frac{1}{2}}$$

and that

$$\frac{1}{|J_z(w)|} \leq \frac{4}{(|w|^2 - |\langle z, w\rangle|^2 |J_z(w)|}.$$

Using these estimates we have that

$$\left|\log |J_z(w)|^{s - sp(\varphi_z(w))/p(z)}\right| \leq \frac{4sC_{\log}}{p_-}.$$

It follows that I_z is bounded and the bound depends only on $p(\cdot)$. Thus there is a constant $C > 0$ depending on $p(\cdot)$ such that

$$\| |J_z|^{(n+1+\alpha)/p(z)} \|_{p(\cdot)} \leq C.$$

The previous estimate implies that

$$\| K_z^{2/p(z)} \|_{p(\cdot)} \leq \frac{C}{(1 - |z|^2)^{(n+1+\alpha)/p(z)}}$$

as required. □

Lemma 3.3 *Let $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $f \in A_\alpha^{p(\cdot)}$ and $a \in \mathbf{B}$. Then*

$$|f(a)| \approx \frac{\|f\|_{p(\cdot)}}{(1 - |a|^2)^{(n+1+\alpha)/p(a)}}.$$

Proof Suppose f is holomorphic on $\mathbf{B}$ and assume $\|f\|_{p(\cdot)} = 1$. Then by the mean value theorem we have

$$|f(z)| = |f(\varphi_z(0))| \lesssim \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w)| d\nu_\alpha(w).$$

By the Diening inequality we have that

$$\begin{aligned} |f(z)|^{p(z)} &\lesssim \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w)|^{p(w)} d\nu_\alpha(w) + 1 \\ &\lesssim \frac{\|f\|_{p(\cdot)}}{(1 - |z|^2)^{n+1+\alpha}}. \end{aligned}$$

Since $\|f\|_{p(\cdot)} = 1$, it follows that

$$|f(z)| \lesssim \frac{\|f\|_{p(\cdot)}}{(1 - |z|^2)^{(n+1+\alpha)/p(z)}}.$$

Now for general f, define $g = f/\|f\|_{p(\cdot)}$ and apply the previous result.

For the other inequality, let us take

$$f_z(w) = \frac{(1-|z|^2)^{(n+1+\alpha)/p(z)}}{(1-\langle w, z\rangle)^{2(n+1+\alpha)/p(z)}}.$$

Then by Lemma 3.2, $\|f_z\|_{p(\cdot)} \leq C$. Observe that

$$f_z(z) = \frac{1}{(1-|z|^2)^{(n+1+\alpha)/p(z)}}$$

and this completes the proof. ◻

From Lemma 3.3 we see that

$$\|K_z\|_{p(\cdot)} \approx (1-|z|^2)^{(n+1+\alpha)(1/p(z)-1)}.$$

We have the following:

Proposition 3.4 *Suppose $\alpha > -1$ and $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$. Then for every Borel measure μ in $\mathbf{B}$ the following two statements are equivalent:*

(a) *There is a positive constant C such that*

$$\int_{\mathbf{B}} |f(z)|^{p(z)} d\mu(z) \leq C \int_{\mathbf{B}} |f(z)|^{p(z)} d\nu_\alpha(z),$$

for every $f \in A^{p(\cdot)}$.

(b) *There is a positive constant C_r such that*

$$\mu(D(z,r)) \leq C_r (1-|z|^2)^{n+1+\alpha}.$$

Proof Suppose (a) holds and let

$$f_z(w) = \frac{(1-|z|^2)^{(n+1+\alpha)/p(z)}}{(1-\langle w, z\rangle)^{2(n+1+\alpha)/p(z)}}.$$

Then by (a) and Lemma (3.2) there is a constant C such that

$$\int_{\mathbf{B}} \frac{(1-|z|^2)^{(n+1+\alpha)p(w)/p(z)}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)p(w)/p(z)}} d\mu(w)$$
$$\lesssim \int_{\mathbf{B}} \frac{(1-|z|^2)^{(n+1+\alpha)p(w)/p(z)}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)p(w)/p(z)}} d\nu_\alpha(w) \lesssim C.$$

Thus

$$\int_{\mathbf{B}} \frac{(1-|z|^2)^{(n+1+\alpha)p(w)/p(z)}}{|1-\langle w, z\rangle|^{(n+1+\alpha)p(w)/p(z)}} d\mu(w) \lesssim C.$$

It follows that

$$C \gtrsim \int_{D(z,r)} \frac{(1-|z|^2)^{(n+1+\alpha)p(w)/p(z)}}{|1-\langle w,z\rangle|^{(n+1+\alpha)p(w)/p(z)}} d\mu(w)$$
$$\approx \int_{D(z,r)} \frac{(1-|z|^2)^{(n+1+s+\alpha)p(w)/p(z)}}{(1-|z|^2)^{2(n+1+\alpha)p(w)/p(z)}} d\mu(w)$$
$$\approx \frac{\mu(D(z,r))}{(1-|z|^2)^{(n+1+\alpha)}}$$

where we have used the identity (8) to obtain the last equation.

Conversely, suppose (b) holds. We first observe that if $z \in D(a,r)$ and $w \in D(z,r)$ then $w \in D(a,2r)$. Now, if $f \in A^{p(\cdot)}$ and $z \in D(a,r)$ then there is a positive constant C such that

$$|f(z)| \leq \frac{C}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)| d\nu_\alpha(w)$$
$$\leq \frac{C}{(1-|a|^2)^{n+1+\alpha}} \int_{D(a,2r)} |f(w)| d\nu_\alpha(w)$$
$$\approx \frac{C}{\nu_\alpha(D(a,2r))} \int_{D(a,2r)} |f(w)| d\nu_\alpha(w)$$

Suppose $\int_{\mathbf{B}} |f(z)|^{p(z)} d\nu_\alpha(z) = 1$. Then by the Diening inequality (11) we have that

$$|f(z)|^{p(z)} \leq \frac{C}{\nu_\alpha(D(a,2r))} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) + 1. \tag{18}$$

Thus, we have

$$\int_{D(a,r)} |f(z)|^{p(z)} d\mu(z) \leq \frac{C}{\nu_\alpha(D(a,2r))} \int_{D(a,r)} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) d\mu(z)$$
$$+\mu(D(a,r))$$
$$\approx \frac{\mu(D(a,2r))}{\nu_\alpha(D(a,2r))} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) d\mu(z) + \mu(D(a,r))$$
$$\leq C \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) d\mu(z) + \mu(D(a,r)).$$

where we have used the hypothesis to obtain the last inequality. Now let the sequence $\{a_k\}$ be an r-lattice on $\mathbf{B}$ then

$$\int_{\mathbf{B}} |f(z)|^{p(z)} d\mu(z) \leq \sum_{k=1}^{\infty} \int_{D(a_k,r)} |f(z)|^{p(z)} d\mu(z)$$
$$\leq \sum_{k=1}^{\infty} \int_{D(a_k,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) + \sum_{k=1}^{\infty} \mu(D(a_k,r))$$
$$\leq N \int_{\mathbf{B}} |f(w)|^{p(w)} d\nu_\alpha(w) + \mu(\mathbf{B}) \leq N(1+\mu(\mathbf{B})).$$

Thus,

$$\int_{\mathbf{B}} |f(z)|^{p(z)} d\mu(z) \lesssim \int_{\mathbf{B}} |f(z)|^{p(z)} d\nu_\alpha(z).$$

Now, if $\int_{\mathbf{B}} |f(z)|^{p(z)} d\nu_\alpha(z) \neq 1$ we set $g = f/\|f\|_{p(\cdot)}$. Then Proposition 2.21 of [5] implies that $\int_{\mathbf{B}} |g(z)|^{p(z)} d\nu_\alpha(z) = 1$ and by the preceeding argument

$$\int_{\mathbf{B}} |g(z)|^{p(z)} d\mu(z) \lesssim \int_{\mathbf{B}} |g(z)|^{p(z)} d\nu_\alpha(z).$$

This completes the proof.

□

It is relatively easy to prove the little-oh version of Proposition 3.4, which we now state.

Proposition 3.5 *Suppose* $\alpha > -1$ *and* $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$. *Then for every Borel measure* μ *in* $\mathbf{B}$ *the following two statements are equivalent:*

(a) If $\{f_k\}$ *is a bounded sequence in* $A^{p(\cdot)}$ *that convergences uniformly on every compact subset of* $\mathbf{B}$ *then*

$$\lim_{k\to\infty} \int_{\mathbf{B}} |f_k(z)|^{p(z)} d\mu(z) = 0.$$

(b)

$$\lim_{|z|\to 1} \frac{\mu(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} = 0.$$

Proposition 3.6 *Suppose* $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ *with* $1 \leq p(\cdot) \leq q(\cdot)$. *Then the following are equivalent:*

(a) *There is a positive constant* C *such that*

$$\int_{\mathbf{B}} |f(z)|^{q(z)} d\mu(z) \leq C \int_{\mathbf{B}} |f(z)|^{p(z)} d\nu_\alpha(z),$$

for every $f \in A^{p(\cdot)}$.

(b) *There is a positive constant* C_r *such that*

$$\mu(D(z,r)) \leq C_r (1-|z|^2)^{(n+1+\alpha)q(z)/p(z)}.$$

Proof Suppose (a) holds. Let $f_z(w) = \frac{(1-|z|^2)^{(n+1+\alpha)/p(z)}}{(1-\langle w,z\rangle)^{2(n+1+\alpha)/p(z)}}$, $z \in \mathbf{B}$. Then as it was in the Proof of Lemma 3.2 we have that

$$\int_{\mathbf{B}} |f_z(w)|^{p(w)} d\nu_\alpha(w) = \int_{\mathbf{B}} |J_z(w)|^{(n+1+\alpha)p(w)/p(z)} d\nu_\alpha(w) \leq C$$

for some positive constant C. It follows from (a) that

$$\int_{\mathbf{B}} \frac{(1-|z|^2)^{(n+1+\alpha)q(w)/p(z)}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)q(w)/p(z)}} d\mu(w) \leq C \tag{19}$$

Thus, using the estimate (8), we obtain

$$\begin{aligned} C &\geq \int_{D(z,r)} \frac{(1-|z|^2)^{(n+1+\alpha)q(w)/p(z)}}{|1-\langle w, z\rangle|^{2(n+1+\alpha)q(w)/p(z)}} d\mu(w) \\ &\approx \int_{D(z,r)} \frac{(1-|z|^2)^{(n+1+\alpha)q(w)/p(z)}}{(1-|z|^2)^{2(n+1+\alpha)q(w)/p(z)}} d\mu(w) \\ &\approx \frac{\mu(D(z,r))}{(1-|z|^2)^{(n+1+\alpha)q(z)/p(z)}} \end{aligned}$$

which gives (b).

Conversely, suppose (b) holds. Suppose $\int_{\mathbf{B}} |f(z)|^{p(\cdot)} d\nu_\alpha(z) = 1$ and let $d\mu_{f,\alpha}(w) = |f(w)|^{p(w)} d\nu_\alpha(w)$. Then by Eq. (18) we have that

$$|f(z)|^{p(z)} \lesssim \frac{1}{\nu_\alpha(D(a, 2r))} \int_{D(a,2r)} d\mu_{f,\alpha}(w) + 1$$

whenever $z, w \in D(a, r)$, $a \in \mathbf{B}$. Now if $p(z) < q(z)$ we write $q(z) = s(z)p(z)$, where $s(\cdot)$ is log-Hölder continuous, then

$$\begin{aligned} |f(z)|^{q(z)} = |f(z)|^{p(z)s(z)} &\lesssim \left(\frac{1}{\nu_\alpha(D(a, 2r))} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) + 1 \right)^{s(z)} \\ &\lesssim \left(\left(\frac{1}{\nu_\alpha(D(a, 2r))} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) \right)^{s(z)} + 1 \right) \\ &\leq \left(\frac{1}{(\nu_\alpha(D(a, 2r)))^{s(z)}} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) + 2 \right) \\ &\approx \left(\frac{1}{\nu_\alpha(D(a, 2r))^{s(a)}} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) + 2 \right) \end{aligned} \tag{20}$$

where we have used the identity (8) to obtain the last equation. It follows that

$$\begin{aligned} \int_{D(a,r)} |f(z)|^{q(z)} d\mu(z) \lesssim \frac{\mu(D(a, 2r))}{(1-|a|^2)^{(n+1+\alpha)q(a)/p(a)}} \int_{D(a,2r)} |f(w)|^{p(w)} d\nu_\alpha(w) \\ +\mu(D(a, 2r)) \end{aligned}$$

Now using the same argument used in the proof (b) implies (a) of Proposition 3.4 we obtain

$$\int_{\mathbf{B}} |f(z)|^{q(z)} d\mu(z) \lesssim \int_{\mathbf{B}} |f(z)|^{p(z)} d\nu_\alpha(z),$$

which gives the proof for the case $p(\cdot) < q(\cdot)$. The case when $q(\cdot) = p(\cdot)$ is just Proposition 3.4. □

Similarly we obtain little-oh version of Proposition 3.6.

Proposition 3.7 *Suppose $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ with $1 \leq p(\cdot) \leq q(\cdot)$ Then for every Borel measure μ in $\mathbf{B}$ the following two statements are equivalent:*

(a) *If $\{f_k\}$ is a bounded sequence in $A^{p(\cdot)}$ that convergences uniformly on every compact subset of $\mathbf{B}$ then*

$$\lim_{k\to\infty} \int_{\mathbf{B}} |f_k(z)|^{q(z)} d\mu(z) = 0.$$

(b)

$$\lim_{|z|\to 1} \frac{\mu(D(z,r))}{(1-|z|^2)^{(n+1+\alpha)q(z)/p(z)}} = 0.$$

Let $f \in L_\alpha^{p(\cdot)}$, $\alpha > -1$ and set

$$Tf(z) = \int_{\mathbf{B}} f(w)K(z,w)d\nu_\alpha(w),\ z \in \mathbf{B}, \tag{21}$$

where $K : \mathbf{B} \times \mathbf{B} \to \mathbb{C}^n$ is a kernel function.

We give an Okikiolu type theorem that will be useful in our work.

Lemma 3.8 *Let $\omega \in A_1$ and suppose p and r are positive numbers such that $1 < p \leq r$,*

$$\frac{1}{p'} + \frac{1}{r} = 1.$$

If there exist positive constants C_1 and C_2 depending on $[\omega]_{A_1}$ and non-negative measurable functions, h_1 and h_2, such that

$$\int_{\mathbf{B}} |K(\xi,z)| h_1(z)^{p'} d\nu_\alpha(z) \leq C_1 h_2(\xi)^{p'} \tag{22}$$

for almost every $\xi \in \mathbf{B}$ and

$$\omega(\mathbf{B})^{-1} \int_{\mathbf{B}} |K(\xi,z)| h(\xi)^r \omega(\xi) d\nu_\alpha(\xi) \leq C_2 h(z)^r \tag{23}$$

for almost every $z \in \mathbf{B}$, then

$$\left(\int_{\mathbf{B}} |Tf(z)|^r \omega(z) d\nu_\alpha(z)\right)^{\frac{1}{r}} \leq CC_1C_2 \left(\int_{\mathbf{B}} |f(z)|^p \omega(z)^{p/r} d\nu_\alpha(z)\right)^{\frac{1}{p}}, \tag{24}$$

Proof We suppose $r > p$ and let $c = \frac{1}{p} - \frac{1}{r}$. Then $c > 0$, $\frac{1}{p'} + \frac{1}{r} + c = 1$ and $\frac{p(1+cr)}{r^2} = \frac{1}{r}$. It follows by the three term Hölder's inequality with indices p', r and $1/c$ that

$$\begin{aligned}
|Tf(z)| &\leq \int_{\mathbf{B}} |K(\xi,z)||f(\xi)|d\nu_\alpha(\xi) \\
&= \int_{\mathbf{B}} \left(|K(\xi,z)|^{1/p'} h_1(\xi)\right)\left(h_1(\xi)^{-1}|K(\xi,z)|^{1/r}|f(\xi)|^{p/r}\right)|f(\xi)|^{pc} d\nu(\xi) \\
&\leq \left(\int_{\mathbf{B}} |K(\xi,z)| h_1(\xi)^{p'} d\nu_\alpha(\xi)\right)^{1/p'} \left(\int_{\mathbf{B}} |K(\xi,z)| h_1(\xi)^{-r}|f(\xi)|^p d\nu_\alpha(\xi)\right)^{1/r} \\
&\quad \left(\int_{\mathbf{B}} |f(\xi)|^p d\nu_\alpha(\xi)\right)^{c} \\
&\leq C h_2(z) \left(\int_{\mathbf{B}} |K(\xi,z)| h_1(\xi)^{-r}|f(\xi)|^p d\nu_\alpha(\xi)\right)^{1/r} \left(\int_{\mathbf{B}} |f(\xi)|^p d\nu_\alpha(\xi)\right)^{c} \\
&\leq C \left(\frac{[\omega]_{A_1}}{\omega(\mathbf{B})}\right)^{\frac{p(1+cr)}{r^2}} h_2(z) \left(\int_{\mathbf{B}} |K(\xi,z)| h_1(\xi)^{-r}|f(\xi)|^p \omega(\xi)^{p/r} d\nu_\alpha(\xi)\right)^{\frac{1}{r}} \\
&\quad \left(\int_{\mathbf{B}} |f(\xi)|^p \omega(\xi)^{p/r} d\nu_\alpha(\xi)\right)^{c} \\
&= C' \omega(\mathbf{B})^{-1/r} h_2(z) \left(\int_{\mathbf{B}} |K(\xi,z)| h_1(\xi)^{-r}|f(\xi)|^p \omega(\xi)^{p/r} d\nu_\alpha(\xi)\right)^{\frac{1}{r}} \|f\omega^{1/r}\|_p^{pc}
\end{aligned}$$

where the third inequality comes from (22) and the fourth inequality is from (13). Now, Fubini's theorem gives

$$\begin{aligned}
&\int_{\mathbf{B}} |Tf(z)|^r \omega(z) d\nu_\alpha(z) \\
&\leq C_1 \|f\omega^{1/r}\|_p^{prc} \int_{\mathbf{B}} \frac{h_2(z)^r}{\omega(\mathbf{B})} \left\{\int_{\mathbf{B}} |K(\xi,z)| h_1(\xi)^{-r}|f(\xi)|^p \omega(\xi)^{p/r} d\nu_\alpha(\xi)\right\} \omega(z) d\nu_\alpha(z) \\
&= C_1 \|f\omega^{1/r}\|_p^{prc} \int_{\mathbf{B}} |f(\xi)|^p h_1(\xi)^{-r} \int_{\mathbf{B}} |K(\xi,z)| h_2(z)^r \frac{\omega(z)}{\omega(\mathbf{B})} d\nu_\alpha(z) \omega(\xi)^{p/r} d\nu_\alpha(\xi) \\
&\leq C_1 C_2 \int_{\mathbf{B}} |f(\xi)|^p \omega(\xi)^{p/r} d\nu_\alpha(\xi),
\end{aligned}$$

where we have used (23) to get the last inequality. □

Proposition 3.9 *Let* $1 < p_0 \leq q_0$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, *be such that* $p_0 \leq p_- \leq p_+ < \frac{p_0 q_0}{q_0 - p_0}$. *Suppose* $K : \mathbf{B} \times \mathbf{B} \to \mathbb{C}$ *satisfies the hypothesis of Lemma 3.8. Then for* $q(\cdot)$ *defined by*

$$\frac{1}{p(z)} - \frac{1}{q(z)} = \frac{1}{p_0} - \frac{1}{q_0}$$

we have that

$$\|Tf\|_{q(\cdot)} \lesssim \|f\|_{p(\cdot)}.$$

Proof Set $p_0 = p$ and $q_0 = r$ in Lemma 3.8 and consider the family $\{(|Tf|, |f|) : f \in L_\alpha^{p_0}(\mathbf{B})\}$. Also, by Theorem 3.16 of [5], maximal function Mf is bounded on $L_\alpha^{(q(\cdot)/q_0)'}$. The conclusion then follows from Proposition 2.5. □

We also observe that , if $\sigma > \alpha$ then the Bergman projection P_σ is bounded on $L_\alpha^{p(\cdot)}$. Indeed, if $h \in L_\alpha^{p(\cdot)}$ then

$$\begin{aligned} |(P_\sigma h)(z)| &= \left| \int_{\mathbf{B}} K_w^\sigma(z) h(w) d\nu_\sigma(w) \right| \\ &\leq \int_{\mathbf{B}} |K_w^\alpha(z)||h(w)| d\nu_\alpha(w). \end{aligned}$$

Now, we can use Lemma 3.8, with $p(\cdot) = q(\cdot)$ and Proposition 3.9 to see that the statement holds.

4 Variable Exponent BMO Type Spaces

For a continuous function on $\mathbf{B}$ we let

$$\omega_r(f)(z) = \sup\{|f(z) - f(\xi)| : \xi \in D(z, r)\}.$$

The function $\omega_r(f)(z)$ is called the oscillation of f at the point z in the Bergman metric. For any $r > 0$, let BO_r denote the space of continuous functions f on $\mathbf{B}$ such that

$$\|f\|_{BO_r} = \sup_{z \in \mathbf{B}} \omega_r(f)(z) < \infty.$$

We have the following charcterization of BO_r functions which is proved in [11].

Lemma 4.1 *Let $r > 0$ and f be a continuous function on $\mathbf{B}$. Then $f \in BO_r$ if and only if there is a constant $C > 0$ such that*

$$|f(z) - f(w)| \leq C(\beta(z, w) + 1)$$

for all $z, w \in \mathbf{B}$.

We say that $f \in BA_r^{p(\cdot)}$ if $f \in L_{loc}^1$ and

$$\sup_{z \in \mathbf{B}} \widehat{|f|_r^{p(\cdot)}}(z) < \infty,$$

where

$$\widehat{f}_r(z) = \frac{1}{D(z,r)} \int_{D(z,r)} f(w) d\nu_\alpha(w).$$

We proceed to show that the space $BA_r^{p(\cdot)}$ is also independent of r. For $\alpha > -1$ and $c > 0$, the generalized Berezin transform $\mathcal{B}_{c,\alpha}$ of a function $g \in L^1$ is defined by

$$\mathcal{B}_{c,\alpha}(g)(z) = (1-|z|^2)^c \int_{\mathbf{B}} \frac{g(w)}{|1-\langle w, z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w). \tag{25}$$

When $c = n+1+\alpha$ then $B_\alpha g(z) = \langle g k_z^\alpha, k_z^\alpha\rangle_\alpha$, where $\langle\cdot,\cdot\rangle_\alpha$ denote the inner product of A_α^2, is the standard Berezin transform.

Observe also that by the Diening inequality (11), if $\|f\|_{p(\cdot)} = 1$ then

$$|\widehat{f}_r(z)|^{p(z)} \le \widehat{|f_r|^{p(\cdot)}}(z) + 1, \quad w, z \in \mathbf{B}. \tag{26}$$

Lemma 4.2 *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ with $p(z) \le q(z)$ and $f \in L_{loc}^{q(\cdot)}$. Suppose $\sigma, \alpha > -1$ be such that $(n+1+\sigma) = (n+1+\alpha)q(z)/p(z)$. Then the following are equivalent:*

(i) *The embedding $i : A_\alpha^{p(\cdot)} \to L_\sigma^{q(\cdot)}(\mathbf{B}, d\mu_{f,\sigma})$ is bounded, where $d\mu_{f,\sigma}(z) = |f(z)|^{q(z)} d\nu_\sigma(z)$.*
(ii) *$\mathcal{B}_{c,\alpha}(|f|^{q(\cdot)}) \in L^\infty$ for all $\alpha > -1$ and all $c > 0$.*
(iii) *$f \in BA_r^{q(\cdot)}$ for some (or all) $r > 0$.*

Proof Observe that by Proposition 3.6 the condition (i) is equivalent to the condition

$$\mu_{f,\sigma}(D(z,r)) \le C(1-|z|^2)^{(n+1+\alpha)q(z)/p(z)} \tag{27}$$

On the other hand,

$$\widehat{|f|_r^{p(\cdot)}}(z) \approx \frac{\mu_{f,\sigma}(D(z,r))}{(1-|z|^2)^{n+1+\sigma}},$$

Thus if $n+1+\sigma = (n+1+\alpha)p(\cdot)/q(\cdot)$ we see that (i) and (ii) are equivalent.

Suppose (ii) holds. Then for $z \in \mathbf{B}$ we have that

$$\begin{aligned}
\mathcal{B}_{c,\alpha}(|f|^{q(\cdot)})(z) &= (1-|z|^2)^c \int_{\mathbf{B}} \frac{|f(w)|^{q(w)}}{|1-\langle w,z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \\
&\ge (1-|z|^2)^c \int_{D(z,r)} \frac{|f(w)|^{q(w)}}{|1-\langle w,z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \\
&\approx \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w)|^{q(w)} d\nu_\alpha(w) = \widehat{|f|_r^{q(\cdot)}}(z),
\end{aligned}$$

which gives (iii).

Suppose (iii) holds and let $\{a_j\}$ be an $r-lattice$ on $\mathbf{B}$. Then

$$\mathcal{B}_{c,\alpha}(|f|^{q(\cdot)})(z) = (1-|z|^2)^c \int_{\mathbf{B}} \frac{|f(w)|^{q(w)}}{|1-\langle w,z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)$$
$$\leq (1-|z|^2)^c \sum_{j=1}^{\infty} \int_{D(a_j,r)} \frac{|f(w)|^{q(w)}}{|1-\langle w,z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w).$$

Now if $\beta(w,a_j) < r$ then equation (2.20) on page 63 of [16] implies that

$$|1-\langle z,w\rangle| \approx |1-\langle z,a_j\rangle|.$$

It follows that,

$$\mathcal{B}_{c,\alpha}(|f|^{q(\cdot)})(z) \leq (1-|z|^2)^c \sum_{j=1}^{\infty} \frac{1}{|1-\langle a_j,z\rangle|^{n+1+c+\alpha}} \int_{D(a_j,r)} |f(w)|^{q(w)} d\nu_\alpha(w)$$
$$= (1-|z|^2)^c \sum_{j=1}^{\infty} \frac{\nu_\alpha(D(a_j,r))}{|1-\langle a_j,z\rangle|^{n+1+c+\alpha}} \widehat{|f|_r^{q(\cdot)}}(a_j)$$
$$\lesssim (1-|z|^2)^c \sum_{j=1}^{\infty} \frac{(1-|a_j|^2)^{n+1+\alpha}}{|1-\langle a_j,z\rangle|^{n+1+c+\alpha}}.$$

Here, we have used the fact that $f \in BA^{p(\cdot)}$ to obtain the last inequality. By Lemma 2.3 we see that (ii) holds, which completes the proof. □

We let $BMOS^{p(\cdot)}$ denote the space of functions f in $\mathbf{B}$ such that

$$\|f\|_{BMOS_r^{p(\cdot)}} = \sup_{z\in\mathbf{B}} MO_{p(\cdot),r}(f)(z) < \infty$$

where

$$MO_{p(\cdot),r}(f)(z) = \left\| \frac{1}{(\nu_\alpha(D(z,r)))^{1/p(\cdot)}} \chi_{D(z,r)}(f - \widehat{f}_r(z)) \right\|_{p(\cdot)}. \tag{28}$$

We note here that if $p(\cdot) = p$ is a constant, then it is shown in [8] that the spaces $BMOS^p$ are equivalent to the well known BMO^p spaces.

Lemma 4.3 *Let $f \in L^{p(\cdot)}_{loc}$ and $r > 0$. Then $f \in BMOS_r^{p(\cdot)}$ if and only if there is a constant $C > 0$ such that for any $z \in \mathbf{B}$, there exists a function λ_z such that*

$$\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^{p(w)} d\nu_\alpha(w) \leq C. \tag{29}$$

Proof If $f \in BMOS_r^{p(\cdot)}$, then (29) holds with $\lambda_z = \widehat{f}_r(z)$ and by Lemma 2.7, $C^{1/l} = \|f\|_{BMOS_r^{p(\cdot)}}$ for some $l \in \{p_-, p_+\}$.

Conversely, suppose (29) holds. Then by the triangle inequality for variable exponent Lebesgue spaces we have that

$$
\begin{aligned}
MO_{p(\cdot),r}(f)(z) &\leq \left\| \frac{\chi_{D(z,r)}}{(\nu_\alpha(D(z,r)))^{1/p(\cdot)}}(f - \lambda_z) \right\|_{p(\cdot)} \\
&\quad + \left\| \frac{\chi_{D(z,r)}}{(\nu_\alpha(D(z,r)))^{1/p(\cdot)}}(\lambda_z - \widehat{f}_r(z)) \right\|_{p(\cdot)} \\
&= I(z) + J(z)
\end{aligned}
$$

The boundedness of $I(z)$ follows immediately from (29). Now, using Lemma 2.7 and the Hölder's inequality we have

$$
\begin{aligned}
|\lambda_z - \widehat{f}_r(z)| &\leq \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z| d\nu_\alpha(w) \\
&\lesssim \left\| \frac{\chi_{D(z,r)}}{(\nu_\alpha(D(z,r)))^{1/p(\cdot)}}(\lambda_z - \widehat{f}_r(z)) \right\|_{p(\cdot)} \\
&\leq \left(\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^{p(w)} d\nu_\alpha(w) \right)^{1/l}
\end{aligned}
$$

for some $l \in \{p_-, p_+\}$. Also, by Lemma 2.7 there is an $l_1 \in \{p_-, p_+\}$, such that

$$
\begin{aligned}
J(z) &= \left\| \frac{\chi_{D(z,r)}}{(\nu_\alpha(D(z,r)))^{1/p(\cdot)}}(\lambda_z - \widehat{f}_r(z)) \right\|_{p(\cdot)} \\
&\leq \left(\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |\widehat{f}_r(z) - \lambda_z|^{p(w)} d\nu_\alpha(w) \right)^{1/l_1} \\
&\leq \left(\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |\widehat{f}_r(z) - \lambda_z|^{p_+} + |\widehat{f}_r(z) - \lambda_z|^{p_-} d\nu_\alpha(w) \right)^{1/l_1} \\
&= |\widehat{f}_r(z) - \lambda_z|^{p_+/l_1} + |\widehat{f}_r(z) - \lambda_z|^{p_-/l_1} \\
&\leq \left(\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^{p(w)} d\nu_\alpha(w) \right)^{p_+/ll_1} \\
&\quad + \left(\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^{p(w)} d\nu_\alpha(w) \right)^{p_-/ll_1} \lesssim C,
\end{aligned}
$$

by (29). This shows that

$$
\sup_{z \in \mathbf{B}} MO_{p(\cdot),r}(f)(z) < \infty,
$$

as required. □

Theorem 4.4 *Suppose* $r > 0$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $f \in L^{p(\cdot)}_{loc}(\mathbf{B})$. *Then the following are equivalent:*

(a) $f \in BMOS^{p(\cdot)}$,
(b) $f = f_1 + f_2$ *with* $f_1 \in BO$ *and* $f_2 \in BA^{p(\cdot)}$.
(c) *For all* $c > 0$, $\alpha > -1$ *we have*

$$\sup_{z\in\mathbf{B}} \int_{\mathbf{B}} |f(w) - \widehat{f}_r(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) < \infty. \tag{30}$$

(d) *For all* $c > 0$, $\alpha > -1$, *there is a function* λ_z *such that, we have*

$$\sup_{z\in\mathbf{B}} \int_{\mathbf{B}} |f(w) - \lambda_z|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) < \infty. \tag{31}$$

Proof (c) implies (d) is obvious. (d) implies (a) follows by using the identities $|1 - \langle z,w\rangle| \approx 1-|z|^2$ for $w \in D(z,r)$ and $\nu_\alpha(D(z,r)) \approx (1-|z|^2)^{n+1+\alpha}$ to obtain

$$\begin{aligned} &\frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(w)-\lambda_z|^{p(w)} d\nu_\alpha(w) \\ &\le \int_{\mathbf{B}} |f(w)-\lambda_z|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w). \end{aligned}$$

(a) implies (b). It suffices to show that $BMOS^{p(\cdot)}_{2r} \subset BO_r + BA^{p(\cdot)}_r$. Given $f \in BMOS^{p(\cdot)}_{2r}$ and let $z, w \in \mathbf{B}$ with $\beta(z,w) < r$. Then we have

$$\begin{aligned} |\widehat{f}_r(z) - \widehat{f}_r(w)| &\le |\widehat{f}_r(z) - \widehat{f_{2r}}(z)| + |\widehat{f_{2r}}(z) - \widehat{f}_r(w)| \\ &\le \frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(u)-\widehat{f_{2r}}(z)| d\nu_\alpha(u) \\ &\quad + \frac{1}{\nu_\alpha(D(w,r))}\int_{D(w,r)} |f(u)-\widehat{f_{2r}}(z)| d\nu_\alpha(u) \end{aligned} \tag{32}$$

Now using Hölder's inequality, we have that

$$\begin{aligned} &\frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(u)-\widehat{f_{2r}}(z)| d\nu_\alpha(u) \\ &\le \frac{\nu_\alpha(D(z,2r))}{\nu_\alpha(D(z,r))} \frac{1}{\nu_\alpha(D(z,2r))}\int_{D(z,2r)} |f(u)-\widehat{f_{2r}}(z)| d\nu_\alpha(u) \\ &\lesssim \|f\|_{BMOS^{p(\cdot)}}. \end{aligned}$$

Now, since for $w \in D(z,r)$, $\nu_\alpha(D(z,r)) \approx \nu_\alpha(D(w,r))$ and $D(w,r) \subset D(z,2r)$ we see immediately that the two integral summands in (32) are both bounded by some constant times $\|f\|_{BMOS^{p(\cdot)}}$. Since $\widehat{f}_r$ is continuous we see that $\widehat{f}_r \in BO_r$. We are left to show that $g = f - \widehat{f}_r$ is in $BA_r^{p(\cdot)}$ whenever $f \in BMOS_{2r}^{p(\cdot)} = BMOS_r^{p(\cdot)}$. Now

$$\begin{aligned}\widehat{|g|_r^{p(\cdot)}}(z) &= \frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(w) - \widehat{f}_r(z) + \widehat{f}_r(z) - \widehat{f}_r(w)|^{p(w)} d\nu_\alpha(w)\\ &\le \frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} \left(|f(w) - \widehat{f}_r(z)| + |\widehat{f}_r(z) - \widehat{f}_r(w)|\right)^{p(w)} d\nu_\alpha(w)\\ &\le \frac{2^{p_+-1}}{\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(w) - \widehat{f}_r(z)|^{p(w)} + |\widehat{f}_r(z) - \widehat{f}_r(w)|^{p(w)} d\nu_\alpha(w)\end{aligned}$$

where we have used the modular triangle inequality to obtain the last inequality. Now for a fixed $z \in \mathbf{B}$, and $w \in D(z,r)$ we have

$$|\widehat{f}_r(z) - \widehat{f}_r(w)|^{p(w)} \le \omega_r(\widehat{f}_r)(z)^{p(w)} \le (\omega_r(\widehat{f}_r)(z)^{p_+} + \omega_r(\widehat{f}_r)(z)^{p_-},\ w \in \mathbf{B}$$

and thus

$$\begin{aligned}\widehat{|g|_r^{p(\cdot)}}(z) \ \le\ & \frac{2^{p_+-1}}{\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(w) - \widehat{f}_r(z)|^{p(w)} d\nu_\alpha(w)\\ & + \omega_r(\widehat{f}_r)(z)^{p_+} + \omega_r(\widehat{f}_r)(z)^{p_-}.\end{aligned}$$

Since $\widehat{f}_r \in BO_r$ and $f \in BMOS^{p(\cdot)}$ we deduce that $g \in BA^{p(\cdot)}$.

To show that (b) implies (c), we assume $\|f\|_{p(\cdot)} = 1$ and observe that

$$\begin{aligned}&\int_{\mathbf{B}} |f(w) - \widehat{f}_r(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)\\ &\le \int_{\mathbf{B}} 2^{p_+-1}(|f(w)|^{p(w)} + |\widehat{f}_r(z)|^{p(w)}) \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)\\ &\le 2^{p_+-1} B_{c,\alpha}(|f|^{p(\cdot)})(z) + 2^{p_+-1}\int_{\mathbf{B}} |\widehat{f}_r(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)\\ &\le 2^{p_+-1} B_{c,\alpha}(|f|^{p(\cdot)})(z) + 2^{p_+-1}(\widehat{|f_r|^{p(\cdot)}}(z) + 1)\int_{\mathbf{B}} \frac{(1-|z|^2)^c}{|1-\langle z,w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)\\ &\lesssim B_{c,\alpha}(|f|^{p(\cdot)})(z) + \widehat{|f_r|^{p(\cdot)}}(z) + 1\end{aligned}$$

where we have used the estimate (26) to obtain third inequality. Thus we have (c) whenever $f \in BA^{p(\cdot)}$. Now, if $f \in BO_r$ then

$$\begin{aligned}f(w) - \widehat{f}_r(z) &= \frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} (f(w) - f(\xi)) d\nu_\alpha(\xi)\\ &= \frac{1}{\nu_\alpha(D(z,r))}\int_{D(z,r)} (f(w) - f(z)) + (f(z) - f(\xi)) d\nu_\alpha(\xi).\end{aligned}$$

By Lemma 4.1 and the triangle inequality we have that

$$|f(w) - \widehat{f_r}(z)| \lesssim \beta(z, w) + 1.$$

It follows that

$$\begin{aligned} &\int_{\mathbf{B}} |f(w) - \widehat{f_r}(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \\ &\lesssim \int_{\mathbf{B}} (\beta(z, w) + 1)^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \end{aligned}$$

Also if $z, w \in \mathbf{B}$, we have that $\beta(z, w) + 1 \geq 1$ and thus $(\beta(z, w) + 1)^{p(w)} \leq (\beta(z, w) + 1)^{p_+} \leq C_{p_+}(\beta(z, w)^{p_+} + 1)$. Thus, the last integral above is bounded by some constant times

$$\int_{\mathbf{B}} (\beta(z, w)^{p_+} + 1) \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)$$

which is finite by Lemma 2.2. This shows that (c) holds whenever $f \in BO_r$ and this completes the proof. □

5 Bounded Hankel Operators

Proof of Theorem 1.1

The Proof of Theorem 1.1 will be given by the following two Lemmas and three Propositions. We will first begin with the necessary part.

Lemma 5.1 *Suppose $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ and $-1 < \sigma < \alpha$. Then*

$$\|H_f^\sigma g\|_{p(\cdot),\alpha} \lesssim \|H_f^\alpha g\|_{p(\cdot),\alpha}$$

for all polynomials g and all $f \in L^1_\sigma$.

Proof Since $\alpha > \sigma > -1$, it follows from Theorem 2.11 of [16] that P_α is bounded from L^1_σ onto A^1_σ. Since P_σ reproduces A^1_σ we have that $P_\alpha P_\sigma(fg) = P_\alpha(fg)$ for $f \in L^1_\sigma$ and any polynomial g. Thus

$$(P_\sigma - P_\alpha)(fg) = (P_\sigma - P_\sigma P_\alpha)(fg) = P_\sigma(I - P_\alpha)(fg) = P_\sigma H_f^\alpha g.$$

We use this to get

$$\begin{aligned} \|H_f^\sigma g\|_{p(\cdot),\alpha} &= \|(I - P_\sigma(fg)\|_{p(\cdot),\alpha} \\ &\leq \|(I - P_\alpha(fg)\|_{p(\cdot),\alpha} + \|(P_\alpha - P_\sigma(fg)\|_{p(\cdot),\alpha} \end{aligned}$$

$$= \|H_f^\alpha g\|_{p(\cdot),\alpha} + \|P_\sigma H_f^\alpha g\|_{p(\cdot),\alpha}$$
$$\leq (1 + \|P_\sigma\|_{p(\cdot)})\|H_f^\alpha g\|_{p(\cdot),\alpha},$$

as required. □

Let $z \in \mathbf{B}$, $h_z := K_z/\|K_z\|_{p(\cdot)}$ and

$$g_z(w) = \frac{P_\alpha(\overline{f}h_z)(w)}{h_z(w)}, \quad w \in \mathbf{B}.$$

We consider the function $LO_{q(\cdot),\alpha}f$ defined by

$$LO_{q(\cdot),\alpha}f(z) = \|f(w)h_z(w) - \overline{g_z(z)}h_z(w)\|_{q(\cdot)}.$$

Lemma 5.2 *Suppose $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $p(z) \leq q(z)$ and $\alpha > -1$. If $LO_{q(\cdot),\alpha}f \in L^\infty$ then $f \in BMOS^{q(\cdot)}$.*

Proof Since $q(z) \geq p(z)$ and $h_z(w) = (1-|z|^2)^{(n+1+\alpha)(1-1/p(z))}K_z(w)$, by Lemma 2.7 there is $l \in \{q_+, q_-\}$ such that

$$\begin{aligned}
\left(LO_{q(\cdot),\alpha}f(z)\right)^l &\geq \int_{\mathbf{B}} |h_z(w)|^{q(w)}|f(w) - g_z(z)|^{q(w)}d\nu_\alpha(w)\\
&\geq \int_{D(z,r)} |h_z(w)|^{q(w)}|f(w) - g_z(z)|^{q(w)}d\nu_\alpha(w)\\
&\approx (1-|z|^2)^{-(n+1+\alpha)q(z)/p(z)} \int_{D(z,r)} |f(w) - g_z(z)|^{q(w)}d\nu_\alpha(w)\\
&\geq (1-|z|^2)^{-(n+1+\alpha)} \int_{D(z,r)} |f(w) - g_z(z)|^{q(w)}d\nu_\alpha(w)\\
&\approx \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - g_z(z)|^{q(w)}d\nu_\alpha(w),
\end{aligned}$$

where we have used (8) to obtain the third equation. It follows from Lemma 4.3, with $\lambda_z = g_z(z)$, that $f \in BMOS^{p(\cdot)}$, which completes the proof. □

Proposition 5.3 *Suppose $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ and $\alpha > -1$. Then for any $f \in L^{q(\cdot)}$ we have*

$$LO_{q(\cdot),\alpha}f(z) \lesssim \|H_f^\alpha h_z\|_{q(\cdot)} + \|H_{\overline{f}}^\alpha h_z\|_{q(\cdot)}.$$

Proof By the triangle inequality and the definition of Hankel operators, we have

$$\begin{aligned}
LO_{q(\cdot),\alpha}f(z) &= \|fh_z - \overline{g_z(z)}h_z\|_{q(\cdot)}\\
&\leq \|fh_z - P_\alpha(fh_z)\|_{q(\cdot)} + \|P_\alpha(fh_z) - \overline{g_z(z)}h_z\|_{q(\cdot)}\\
&= \|H_f^\alpha h_z\|_{q(\cdot)} + \|P_\alpha(fh_z) - \overline{g_z(z)}h_z\|_{q(\cdot)}.
\end{aligned}$$

Now, for any $g \in A^1_\alpha$ we have that

$$\overline{g(z)} h_z = P_\alpha(\overline{g} h_z).$$

This together with the boundeness of the Bergman projection P_α on $L^{q(\cdot)}_\alpha$ gives

$$\begin{aligned}\|P_\alpha(fh_z) - \overline{g_z(z)} h_z\|_{q(\cdot)} &= \|P_\alpha\left(P_\alpha(fh_z) - \overline{g_z} h_z\right)\|_{q(\cdot)} \\ &\leq ||P_\alpha\| \|P_\alpha(fh_z) - \overline{g_z} h_z\|_{q(\cdot)}.\end{aligned}$$

Also, by the triangle inequality we have

$$\begin{aligned}\|P_\alpha(fh_z) - \overline{g_z} h_z\|_{q(\cdot)} &\leq \|P_\alpha(fh_z) - fh_z\|_{q(\cdot)} + \|fh_z - \overline{g_z} h_z\|_{q(\cdot)} \\ &= \|H^\alpha_f h_z\|_{q(\cdot)} + \|fh_z - \overline{g_z} h_z\|_{q(\cdot)} \\ &= \|H^\alpha_f h_z\|_{q(\cdot)} + \|fh_z - P(\overline{f} h_z)|_{q(\cdot)} \\ &= \|H^\alpha_f h_z\|_{q(\cdot)} + \|H^\alpha_{\overline{f}} h_z\|_{q(\cdot)}.\end{aligned}$$

This proves the result. □

This gives the proof of the necessity part of the Theorem 1.1. We now proceed to prove of sufficiency part.

Proposition 5.4 *Suppose* $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $p(z) \leq q(z)$, $\alpha, \sigma > -1$ *and*

$$(n + 1 + \sigma)/q(\cdot) = (n + 1 + \alpha)/p(\cdot).$$

If $f \in BA^{q(\cdot)}$, *then* $H^\sigma_f, H^\sigma_{\overline{f}} : A^{p(\cdot)}_\alpha \to L^{q(\cdot)}_\sigma$ *is bounded.*

Proof Since the Bergman projection P_σ is bounded on $L^{q(\cdot)}_\sigma$, we have by the triangle inequality and Lemma 4.2 that

$$\|H^\sigma_f g\|_{q(\cdot),\sigma} \leq \|fg\|_{q(\cdot),\sigma} + \|P_\sigma(fg)\|_{q(\cdot),\sigma} \lesssim \|fg\|_{q(\cdot),\sigma} = \|g\|_{L^{q(\cdot)}(d\mu_{f,\sigma})},$$

which completes the proof of the result. □

Proposition 5.5 *Suppose* $f \in BO$, $1 < p \leq r$ *be such that* $1/p' + 1/r = 1$, $\alpha, \sigma > -1$ *and* $\omega \in A_1$ *and* $g \in H^\infty$. *Then*

$$\left(\int_{\mathbf{B}} |H^\sigma_f g(z)|^r \omega(z) d\nu_\alpha(z)\right)^{\frac{1}{r}} \lesssim \left(\int_{\mathbf{B}} |g(z)|^p \omega(z)^{p/r} d\nu_\alpha(z)\right)^{\frac{1}{p}}, \tag{33}$$

Proof Firstly we let $\sigma = \alpha$ and write

$$(H_f^\alpha g)(z) = \int_{\mathbf{B}} \frac{(f(z) - f(\xi))g(\xi)}{(1 - \langle z, \xi\rangle)^{n+1+\alpha}} d\nu_\alpha(\xi)$$
$$= \int_{\mathbf{B}} K_f(z, \xi) g(\xi) d\nu_\alpha(\xi) \tag{34}$$

where $K_f(z, \xi) = \frac{(f(z) - f(\xi))}{(1 - \langle z, \xi\rangle)^{n+1+\alpha}}$. By Lemma 3.8 we are require to find non-negative functions h_1 and h_2 such that

$$\int_{\mathbf{B}} |K_f(\xi, z)| h_1(z)^{p'} d\nu_\alpha(z) \le C_1 h_2(\xi)^{p'} \tag{35}$$

for almost every $\xi \in \mathbf{B}$ and

$$\omega(\mathbf{B})^{-1} \int_{\mathbf{B}} |K_f(\xi, z)| h_2(\xi)^r \omega(\xi) d\nu(\xi) \le C_2 h(z)^r \tag{36}$$

for almost every $z \in \mathbf{B}$, where $\omega \in A_1$. Now we let $h_1(z) = h_2(z) = (1 - |z|^2)^{-\epsilon}$, for $0 < \epsilon < 1$. Since $f \in BO$, it follows from Lemmas 2.2 and 4.1 that

$$\int_{\mathbf{B}} |K_f(\xi, z)| h_1(z)^{p'} d\nu_\alpha(z) \le \int_{\mathbf{B}} \frac{(\beta(z, \xi) + 1)(1 - |z|^2)^{-\epsilon p'}}{|1 - \langle z, \xi\rangle|^{n+1+\alpha}} d\nu_\alpha(z)$$
$$\lesssim (1 - |\xi|^2)^{-\epsilon p'},$$

which gives the estimate (35).

To get the estimate (36), we first recall that for $\omega \in A_1$, $\omega(\xi) \le M\omega(\xi)$ and for any $\epsilon > 0$ with $\epsilon < \omega(\mathbf{B})$ there is a pseudo ball B containing ξ such that

$$M\omega(\xi) \le \frac{\omega(B)}{\nu_\alpha(B)} + \epsilon \le \frac{\omega(\mathbf{B})}{\nu_\alpha(B)} + \omega(\mathbf{B}) = \omega(\mathbf{B}) \left(\frac{1}{\nu_\alpha(B)} + 1 \right).$$

Using this together with Lemma 2.2 and the estimate (5) we have

$$\omega(\mathbf{B})^{-1} \int_{\mathbf{B}} |K_f(\xi, z)| (1 - |\xi|^2)^{-\epsilon r} \omega(\xi) d\nu_\alpha(\xi)$$
$$\le \omega(\mathbf{B})^{-1} \int_{\mathbf{B}} \frac{(\beta(z, \xi) + 1)(1 - |\xi|^2)^{-\epsilon r}}{|1 - \langle z, \xi\rangle|^{n+1+\alpha}} M\omega(\xi) d\nu_\alpha(\xi)$$
$$\lesssim \int_{\mathbf{B}} \frac{(\beta(z, \xi) + 1)(1 - |\xi|^2)^{-\epsilon r}}{|1 - \langle z, \xi\rangle|^{n+1+\alpha}} d\nu_\alpha(\xi) \lesssim (1 - |z|^2)^{-\epsilon r}$$

which is (36). If $\sigma > \alpha$ then we have that

$$(H_f^\sigma g)(z) = \int_{\mathbf{B}} K'_f(z, \xi) g(\xi) d\nu_\alpha(\xi)$$

where $K'_f(z,\xi) = \frac{(f(z)-f(\xi))(1-|\xi|^2)^{\sigma-\alpha}}{(1-\langle z,\xi\rangle)^{n+1+\sigma}}$. Since $|K'_f(z,\xi)| \le |K_f(z,\xi)|$ the estimates (35) and (36) hold, and thus the result in the case $\sigma > \alpha$. The case $\sigma < \alpha$ follows from Lemma 5.1. This completes the proof of the proposition. □

Proposition 5.6 *Suppose $f \in BO$, $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, $\alpha, \sigma > -1$, $1 < p_0 \le q_0$ be such that*

$$\frac{1}{p(z)} - \frac{1}{q(z)} = \frac{1}{p_0} - \frac{1}{q_0}$$

with $1 < p_0 \le p_- \le p_+ < \frac{p_0 q_0}{q_0 - p_0}$. Then $H_f^\sigma, H_{\bar{f}}^\sigma : A_\alpha^{p(\cdot)} \to L_\sigma^{q(\cdot)}$ is bounded.

Proof Set $p_0 = p, r = q_0$ in Proposition 5.5 and let $p(\cdot), q(\cdot), p_0, q_0$ satisfy the conditions of the Proposition. Then we apply Proposition 3.9 to the family $\{|H_f^\sigma g|, |g| : g \in L^{p_0}\}$ to obtain the required result. Similarly the result holds when we replace the family by $\{|H_{\bar{f}}^\sigma g|, |g| : g \in L^{p_0}\}$. □

6 Variable Exponent *VMO* Type Spaces

For $r > 0$, we define the space VO_r to be the space of functions f in BO_r such that

$$\lim_{|z|\to 1} \omega_r(f)(z) = 0$$

and $VA_r^{p(\cdot)}$ as the space of functions $f \in BA^{p(\cdot)}$ such that

$$\lim_{|z|\to 1} \widehat{|f_r|^{p(\cdot)}}(z) = 0.$$

Lemma 6.1 *Let $p(\cdot), q(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$ be such that $p(\cdot) \le q(\cdot)$ $\alpha, \sigma > -1$, $f \in L_{loc}^{q(\cdot)}$ and*

$$\frac{(n+1+\sigma)}{q(\cdot)} = \frac{(n+1+\alpha)}{p(\cdot)}, \quad d\mu_{f,\sigma} = |f|^{q(\cdot)} d\nu_\sigma.$$

Then the following are equivalent:

(i) *If $\{f_k\}$ is a bounded sequence in $A_\alpha^{p(\cdot)}$ and $f_k \to 0$ uniformly on every compact subset of $\mathbf{B}$, then*

$$\lim_{k\to\infty} \int_{\mathbf{B}} |f_k(w)|^{q(w)} d\mu_{f,\alpha}(w) = 0.$$

(ii) *$f \in VA_r^{p(\cdot)}$ for some (or all) $r > 0$.*

(iii) *For* $\alpha > -1$ *and* $c > 0$ *we have that*

$$\lim_{|z|\to 1} \mathcal{B}_{c,\alpha}(|f|^{q(\cdot)})(z) = 0.$$

Proof By Proposition 3.7 we have that (i) is equivalent to

$$\lim_{|z|\to 1} \frac{\mu_{f,\sigma}(D(z,r))}{(1-|z|^2)^{(n+1+\alpha)q(z)/p(z)}} = 0$$

for some (or all) $r > 0$. The equivalence of (i) and (ii) is a consequence of this result and the fact that

$$\widehat{|f|_r^{q(\cdot)}}(z) \approx \frac{\mu_{f,\sigma}(D(z,r))}{(1-|z|^2)^{n+1+\alpha}}.$$

(iii) implies (ii) follows from the inequality

$$\widehat{|f|_r^{q(\cdot)}}(z) \lesssim \mathcal{B}_{c,\alpha}(|f|^{q(\cdot)})(z)$$

as it was shown in the Proof of Lemma 4.1.

We are left to show that (ii) implies (iii). Let $f \in VA_r^{q(\cdot)}$. Then

$$\mathcal{B}_{c,\alpha}(\widehat{|f|_r^{q(\cdot)}})(z) = (1-|z|^2)^c \int_{\mathbf{B}} \frac{\widehat{|f|_r^{q(\cdot)}}(w)}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w).$$

For $0 < s < 1$, let

$$I_1(s) = (1-|z|^2)^c \int_{|w|\le s} \frac{\widehat{|f|_r^{q(\cdot)}}(w)}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)$$

and

$$I_2(s) = (1-|z|^2)^c \int_{s<|w|<1} \frac{\widehat{|f|_r^{q(\cdot)}}(w)}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w).$$

Let $\epsilon > 0$ be given. Then by (ii) we have that there is an $s > 0$ such that

$$I_2(s) \le \epsilon(1-|z|^2)^c \int_{s<|w|<1} \frac{1}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w).$$

Now, since $1-|w|^2 \le |1-\langle z, w\rangle|$ and $f \in BA^{q(\cdot)}$ we see that

$$I_1(s) \lesssim (1-|z|^2)^c \int_{|w|\le s} \frac{1}{(1-|w|^2)^{n+1+c+\alpha}} d\nu_\alpha(w) \\ \lesssim \frac{(1-|z|^2)^c}{(1-s^2)^{n+1+c}}.$$

Hence, we can find an $l \in (0,1)$ such that $I_1(s) \lesssim \epsilon$ whenever $1-l < |z| < 1$. Combining the two inequalities for $I_1(s)$ and $I_2(s)$ it follows that for $1-l < |z| < 1$ we have

$$\mathcal{B}_{c,\alpha}(\widehat{|f|_r^{q(\cdot)}})(z) < C\epsilon,$$

for some positive constant C. Thus,

$$\lim_{|z|\to 1} \mathcal{B}_{c,\alpha}(\widehat{|f|_r^{q(\cdot)}})(z) = 0. \tag{37}$$

Also,

$$\widehat{|f|_r^{q(\cdot)}}(z) = \frac{1}{\nu_\sigma(D(z,r))}\int_{D(z,r)} |f(w)|^{q(w)} d\nu_\sigma(w) \\ = \frac{\mu_{f,\sigma}(D(z,r))}{\nu_\sigma(D(z,r))} \\ \approx \frac{\mu_{f,\sigma}(D(z,r))}{(1-|z|^2)^{n+1+\sigma}} = \widehat{\mu_{f,\sigma}}(z)$$

It follows that Eq. (37) is equivalent to

$$\lim_{|z|\to 1} \mathcal{B}_{c,\sigma}(\widehat{\mu_{f,\sigma}})(z) = 0. \tag{38}$$

By Lemma 52 of [15] ,

$$\mathcal{B}_{c,\alpha}(\mu_{f,\sigma})(z) \le \mathcal{B}_{c,\sigma}(\widehat{\mu_{f,\sigma}})(z)$$

and thus

$$\lim_{|z|\to 1} \mathcal{B}_{c,\sigma}(\mu_{f,\sigma})(z) = 0.$$

It follows that

$$\lim_{|z|\to 1} \mathcal{B}_{c,\sigma}(|f|_r^{q(\cdot)})(z) = 0.$$

This proves that (ii) implies (iii) and completes the prove of the Lemma. □

Theorem 6.2 *Suppose $r > 0$, $p(\cdot) \in \mathcal{P}^{\log}(\mathbf{B})$, and $f \in BMOS^{p(\cdot)}$. Then the following are equivalent:*

(a) $f \in VMOS_r^{p(\cdot)}$.
(b) $f = f_1 + f_2$ *with* $f_1 \in VO_r$ *and* $f_2 \in VA_r^{p(\cdot)}$.
(c) *For some (or all)* $\alpha > -1$ *and each* $c \geq n+1+\alpha$, *we have*

$$\lim_{|z|\to 1^-} \int_{\mathbf{B}} |f(w) - \widehat{f}_r(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle w, z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) = 0$$

(d) *For some (or all)* $\alpha > -1$ *and each* $c \geq n+1+\alpha$, *there is a function* λ_z *such that*

$$\lim_{|z|\to 1^-} \int_{\mathbf{B}} |f(w) - \lambda_z|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle w, z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) = 0$$

(e) *For some (or all)* $\alpha > -1$ *there is a function* λ_z *such that*

$$\lim_{|z|\to 1^-} \frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^{p(w)} d\nu_\alpha(w) = 0.$$

Proof That (c) implies (d) is obvious. That (d) implies (e) follows from the inequality

$$\begin{aligned}
&\frac{1}{\nu_\alpha(D(z,r))} \int_{D(z,r)} |f(w) - \lambda_z|^{p(w)} d\nu_\alpha(w) \\
&\lesssim \int_{\mathbf{B}} |f(w) - \lambda_z|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle w, z\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w).
\end{aligned}$$

That (a) implies (b) follows from the proof of (a) implies (b) of Theorem 4.4. Thus we only need to prove that (b) implies (c). Now, as in the proof of (b) implies (c) of Theorem 4.4, we have that

$$\begin{aligned}
&\int_{\mathbf{B}} |f(w) - \widehat{f}_r(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \\
&\leq 2^{p_+-1} B_{c,\alpha}(|f|^{p(\cdot)})(z) + 2^{p_+-1} \int_{\mathbf{B}} |\widehat{f}_r(z)|^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \\
&\leq 2^{p_+-1} B_{c,\alpha}(|f|^{p(\cdot)})(z) + 2^{p_+-1} \int_{\mathbf{B}} (\widehat{|f_r|}(z))^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w)
\end{aligned}$$

since $|\widehat{f}_r(z)| \leq \widehat{|f_r|}(z)$ for $z \in \mathbf{B}$. Now, suppose $f \in VA_r^{p(\cdot)}$. Then by the Hölders inequality and Corollary 2.23 of [4] we have

$$\begin{aligned}
\widehat{|f_r|}(z) &= \int_{\mathbf{B}} \nu_\alpha(D(z,r))^{-1/p(w)+1/p'(w)} |f(w)| \chi_{D(z,r)}(w) d\nu_\alpha(w) \\
&\lesssim \|\nu_\alpha(D(z,r))^{-1/p(\cdot)} f \chi_{D(z,r)}\|_{p(\cdot)} \| \|\nu_\alpha(D(z,r))^{-1/p'(\cdot)} \chi_{D(z,r)}\|_{p'(\cdot)} \\
&\lesssim \left(\widehat{|f_r|^{p(\cdot)}}(z)\right)^{1/l}
\end{aligned}$$

for some $l = \{p_-, p_+\}$. Thus $\widehat{|f_r|}(z) \to 0$ as $|z| \to 1$ whenever $f \in BA_r^{p(\cdot)}$. Thus, given $\epsilon > 0$, there is $\delta \in (0,1)$ such that if $\delta < |z| < 1$ we have that $\widehat{|f_r|}(z) < \epsilon^{\frac{1}{p_-}}$. Now, we fix $\delta < |z| < 1$ and have that

$$\begin{aligned}\int_{\mathbf{B}} (\widehat{|f_r|}(z))^{p(w)} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \\ < \epsilon \int_{\mathbf{B}} \frac{(1-|z|^2)^c}{|1-\langle z, w\rangle|^{n+1+c+\alpha}} d\nu_\alpha(w) \lesssim \epsilon.\end{aligned}$$

Now, suppose $f \in VO$. Then $f \in BO$ and we let

$$I(z) = \int_{\mathbf{B}} |f(\xi) - \widehat{f}_r(z)|^{p(\xi)} \frac{(1-|z|^2)^c}{|1-\langle z, \xi\rangle|^{n+1+c+\sigma}} d\nu_\sigma(\xi).$$

Then by making the change of variable $\xi = \varphi_z$ we have that

$$I(z) = \int_{\mathbf{B}} |f(\varphi_z(\xi)) - \widehat{f}_r(z)|^{p(\varphi_z(\xi))} \frac{1}{|1-\langle z, \xi\rangle|^{n+1+\sigma-c}} d\nu_\sigma(\xi).$$

Also, by the invariance of the Bergman metric we have

$$\begin{aligned}|f(\varphi_z(\xi)) - \widehat{f}_r(z)| &= \frac{1}{\nu_\sigma(D(z,r))} \int_{D(z,r)} |f(\varphi_z(\xi)) - f(w)| d\nu_\sigma(w) \\ &\lesssim \frac{1}{\nu_\sigma(D(z,r))} \int_{D(z,r)} \beta(\varphi_z(\xi), w) + 1 d\nu_\sigma(w) \\ &\lesssim \beta(\varphi_z(\xi), z) + 1 = \beta(\xi, 0) + 1\end{aligned}$$

Also, direct calculation gives

$$\begin{aligned}|f(\varphi_z(\xi)) - \widehat{f}_r(z)|^{p(\varphi_z(\xi))} &\lesssim |f(\varphi_z(\xi)) - f(z)|^{p(\varphi_z(\xi))} + |f(z) - \widehat{f}_r(z)|^{p(\varphi_z(\xi))} \\ &\lesssim |f(\varphi_z(\xi)) - f(z)|^{p(\varphi_z(\xi))} \\ &\quad + |f(z) - \widehat{f}_r(z)|^{p_+} + |f(z) - \widehat{f}_r(z)|^{p_-} \\ &\lesssim |f(\varphi_z(\xi)) - f(z)|^{p(\varphi_z(\xi))} + \omega_r(f)(z)^{p_-} + \omega_r(f)(z)^{p_+}.\end{aligned}$$

On the other hand, if we let $t = \beta(\xi, 0) = \beta(\varphi_z(\xi), z)$ then for $f \in VO$ we have

$$|f(\varphi_z(\xi)) - f(z)| \lesssim \omega_t(f)(z) \to 0$$

as $|z| \to 1^-$ and thus

$$|f(\varphi_z(\xi)) - f(z)|^{p(\varphi_z(\xi))} \lesssim \omega_t(f)(z)^{p(\varphi_z(\xi))} \lesssim \omega_t(f)(z)^{p_-} \to 0$$

as $|z| \to 1^-$. It follows that if $f \in VO$ we have that

$$\lim_{|z| \to 1^-} |f(\varphi_z(\xi)) - \widehat{f}_r(z)|^{p(\varphi_z(\xi))} = 0.$$

Now, since

$$\int_{\mathbf{B}} (\beta(\xi,0)+1)^{p(\varphi_z(\xi))} d\nu_\sigma(\xi) \lesssim \int_{\mathbf{B}} (\beta(\xi,0)^{p_+} + 1 d\nu_\sigma(\xi) < \infty$$

and the fact that $c \geq n+1+\sigma$ we have that

$$I_z \lesssim \int_{\mathbf{B}} (\beta(\xi,0)+1)^{p_+} d\nu_\sigma(\xi).$$

It thus follows by the Lebesgue dominated convergence theorem that $I(z) \to 0$ as $|z| \to 1^-$ which gives (c). This completes the proof of the Theorem. □

7 Compact Hankel Operators

Proof of Theorem 1.2

We are going to prove Theorem 1.2 with the following three lemmas. We begin with necessity.

Lemma 7.1 *Let $p(\cdot), q(\cdot), \alpha$ and σ be as in Theorem 1.2. If both $H_f^\sigma, H_{\overline{f}}^\sigma : A_\alpha^{p(\cdot)} \to L_\sigma^{q(\cdot)}$ are compact, then $f \in VMOS^{q(\cdot)}$*

Proof Firstly, observe that if $f \in A^{p'(\cdot)}$ then $\langle f, K_z/\|K_z\|_{p(\cdot)}\rangle \equiv f(z)(1-|z|^2)^{(n+1+\alpha)/p(z)}$. It follows that if f is a bounded holomorphic function in $\mathbf{B}$, then

$$|\langle f, K_z/\|K_z\|_{p(\cdot)}\rangle| \lesssim (1-|z|^2)^{(n+1+\alpha)(1-1/p_+)} \to 0$$

as $|z| \to 1$. Thus, the function $h_z = K_z/\|K_z\|_{p(\cdot)} \to 0$ weakly in $A^{p(\cdot)}$ as $|z| \to 1$. By the compactness of H_f^σ we have that

$$\lim_{|z|\to 1^-} \|H_f^\sigma h_z\|_{q(\cdot)} = 0.$$

The same is true if f is replaced by $\overline{f}$. By Proposition 5.3 we have that

$$\lim_{|z|\to 1^-} LO_{q(\cdot),\alpha} f(z) = 0.$$

That is,

$$\lim_{|z|\to 1^-} \int_{\mathbf{B}} \frac{(1-|z|^2)^{(n+1+\alpha)q(w)(1-1/p(z))}}{|1-\langle w,z\rangle|^{(n+1+\alpha)q(w)}} |f(w) - \overline{g_z(z)}|^{q(w)} d\nu_\alpha(w) = 0.$$

Now, using the identity (8) we have

$$
\begin{aligned}
&\int_{\mathbf{B}} \frac{(1-|z|^2)^{(n+1+\alpha)q(w)(1-1/p(z))}}{|1-\langle w,z\rangle|^{(n+1+\alpha)q(w)}}|f(w)-\overline{g_z(z)}|^{q(w)}d\nu_\alpha(w)\\
&\geq \int_{D(z,r)} \frac{(1-|z|^2)^{(n+1+\alpha)q(w)(1-1/p(z))}}{|1-\langle w,z\rangle|^{(n+1+\alpha)q(w)}}|f(w)-\overline{g_z(z)}|^{q(w)}d\nu_\alpha(w)\\
&\approx \int_{D(z,r)} \frac{1}{(1-|z|^2)^{(n+1+\alpha)q(w)/p(z)}}|f(w)-\overline{g_z(z)}|^{q(w)}d\nu_\alpha(w)\\
&\approx \frac{1}{(1-|z|^2)^{(n+1+\alpha)q(z)/p(z)}}\int_{D(z,r)} |f(w)-\overline{g_z(z)}|^{q(w)}d\nu_\alpha(w)\\
&\gtrsim \frac{1}{|\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(w)-\overline{g_z(z)}|^{q(w)}d\nu_\alpha(w),
\end{aligned}
$$

which shows that

$$\lim_{|z|\to 1^-} \frac{1}{|\nu_\alpha(D(z,r))}\int_{D(z,r)} |f(w)-\overline{g_z(z)}|^{q(w)}d\nu_\alpha(w)=0,$$

that is, $f\in VMOS^{q(\cdot)}$ by Theorem 6.2. □

Let $f\in L^{p(\cdot)}_\alpha$ and for $0<\delta<1$ we set

$$T_\delta f(z)=\int_{1>|w|>\delta}\frac{f(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}d\nu_\alpha(w).$$

Then it is obvious by Proposition 3.9 that T_δ is bounded from $L^{p(\cdot)}_\alpha$ into $L^{q(\cdot)}_\alpha$, for $p(\cdot)\leq q(\cdot)$ such that $1/p(\cdot)-1/q(\cdot)=1/p_0-1/q_0$, $1\leq p_0\leq q_0$ and $p_+<p_0q_0/(q_0-p_0)$. We have the following:

Lemma 7.2 *Let $p(\cdot), q(\cdot), \alpha$ and σ be as in Theorem 1.2 and let $f\in VO$. Then $H^\sigma_f, H^\sigma_{\overline{f}}: A^{p(\cdot)}_\alpha\to L^{q(\cdot)}_\sigma$ is compact.*

Proof Let $\{g_n\}$ be a bounded sequence in $A^{p(\cdot)}$ converging to zero uniformly on compact subsets of $\mathbf{B}$. By Lemma 5.1 it suffices to show that

$$\lim_{n\to\infty}\|H^\eta_f g_n\|_{q(\cdot)}=0,$$

for $\eta\geq n+1+\sigma$. Now, given $\epsilon>0$ we fine $\delta_1\in(0,1)$ such that $\omega_r(f)(z)<\epsilon$ whenever $|z|<\delta_1<1$ since $f\in VO$. Let $\eta\geq n+1+\alpha+\sigma$ then since $f\in VMOS^{q(\cdot)}$ there is a $\delta_2\in(0,1)$ such that

$$\int_{\mathbf{B}}|f(w)-\widehat{f}_r(z)|^{p(z)}\frac{(1-|z|^2)^\eta}{|1-\langle w,z\rangle|^{n+1+\eta+\alpha}}d\nu_\alpha(w)<\epsilon, \tag{39}$$

whenever $|z|<\delta_2<1$. Now, let $\delta=\min(\delta_1,\delta_2)$. Then

$$H^\eta_f g_n(z)=I_n(z)+J_n(z),$$

where

$$I_n(z) = \int_{|w|>\delta} K_f^\eta(z,w) g_n(w) d\nu_\eta(w), \;\; J_n(z) = \int_{|w|\le\delta} K_f^\eta(z,w) g_n(w) d\nu_\eta(w)$$

and

$$K_f^\eta(z,w) = (f(w) - \widehat{f}_r(z)) \frac{(1-|z|^2)^\eta}{(1-\langle w, z\rangle)^{n+1+\eta}}.$$

Since,

$$|f(z) - f(w)| \le |f(z) - \widehat{f}_r(w)| + |\widehat{f}_r(w) - f(w)|$$

we have that

$$I_n(z) \le s_n(z) + t_n(z)$$

where

$$s_n(z) = \int_{|w|>\delta} |f(z) - \widehat{f}_r(w)||K_z^\eta(w)||g_n(w)| d\nu_\eta(w)$$

and

$$t_n(z) = \int_{|w|>\delta} |\widehat{f}_r(w) - f(w)||K_z^\eta(w)||g_n(w)| d\nu_\eta(w).$$

Since,

$$f(w) - \widehat{f}_r(w) = \frac{1}{\nu_\eta(D(w,r))} \int_{D(w,r)} (f(w) - f(\xi)) d\nu_\eta(\xi),$$

we have that for $|w| > \delta$

$$|\widehat{f}_r(w) - f(w)| \le \omega_r(f)(w) < \epsilon.$$

Thus,

$$t_n(z) < \epsilon T_\delta |g_n|(z).$$

Now, for $z, w \in \mathbf{B}$ set $G(z,w) = (f(z) - \widehat{f}_r(w))|g_n(w)|\chi_{\{|w|>\delta\}}$. Then by Hölder's inequality we have that

$$s_n(z) \lesssim \|G(z,\cdot)K_z^{1/q(\cdot)}\|_{q(\cdot)} \|K_z^{1/q'(\cdot)}\|_{q'(\cdot)} \approx \|G(z,\cdot)K_z^{1/q(\cdot)}\|_{q(\cdot)}.$$

Now, if $\|G(z,\cdot)\|_{q(\cdot)} > 1$ then by Corollary 2.22 of [5]

$$\|G(z,\cdot)\|_{q(\cdot)} \leq \int_{\mathbf{B}} |G(z,w)|^{q(w)} |K_z^{\beta}(w)| d\nu_{\beta}(w) \tag{40}$$

and if $\|G(z,\cdot)\|_{q(\cdot)} \leq 1$ by Corollary 2.23 of [5]

$$\|G(z,\cdot)\|_{q(\cdot)} \leq \left(\int_{\mathbf{B}} |G(z,w)|^{q(w)} |K_z^{\beta}(w)| d\nu_{\beta}(w)\right)^{\frac{1}{q+}}. \tag{41}$$

It follows that

$$\begin{aligned} |H_f^{\eta} g_n(z)|^{q(z)} &\lesssim |I_n(z)|^{q(z)} + |J_n(z)|^{q(z)} \lesssim s_n(z)^{q(z)} + t_n(z)^{q(z)} + |J_n(z)|^{q(z)} \\ &< s_n(z)^{q(z)} + \epsilon^{1/q_+} T_{\delta}|g_n|(z)^{q(z)} + |J_n(z)|^{q(z)}. \end{aligned}$$

and thus

$$\begin{aligned} &\int_{\mathbf{B}} |H_f g_n(z)|^{q(z)} d\nu_{\alpha}(w) \\ &< \int_{\mathbf{B}} s_n(z)^{q(z)} d\nu_{\alpha}(z) + \epsilon^{1/q_+} \int_{\mathbf{B}} T_{\delta}|g_n|(z)^{q(z)} d\nu_{\alpha}(z) + \int_{\mathbf{B}} |J_n(z)|^{q(z)} d\nu_{\alpha}(z) \\ &\lesssim \int_{\mathbf{B}} \|G(z,\cdot)\|_{q(\cdot)}^{q(z)} d\nu_{\alpha}(z) + \epsilon^{1/q_+} \int_{\mathbf{B}} T_{\delta}|g_n|(z)^{q(z)} d\nu_{\alpha}(z) + \int_{\mathbf{B}} |J_n(z)|^{q(z)} d\nu_{\alpha}(z) \end{aligned} \tag{42}$$

Now, by Eqs. (40) and (41) we write

$$\int_{\mathbf{B}} \|G(z,\cdot)\|_{q(\cdot)}^{q(z)} d\nu_{\alpha}(z) \leq \int_{\mathbf{B}\cap\{\|G(z,\cdot)\|_{q(\cdot)}>1\}} + \int_{\mathbf{B}\cap\{\|G(z,\cdot)\|_{q(\cdot)}\leq 1\}} \|G(z,\cdot)\|_{q(\cdot)}^{q(z)} d\nu_{\alpha}(z).$$

Firstly, we observe that

$$\nu_{\alpha}(\{\|G(z,\cdot)\|_{q(\cdot)} > 1\}) = 0.$$

Indeed, using (40) we see that for any $\epsilon > 0$ we can fine $\delta > 0$ such that

$$\begin{aligned} &\nu_{\alpha}(\{\|G(z,\cdot)\|_{q(\cdot)} > 1\}) \\ &\leq \int_{\{\|G(z,\cdot)\|_{q(\cdot)}>1\}} \|G(z,\cdot)\|_{q(\cdot)} d\nu_{\alpha}(z) \\ &\leq \int_{\{\|G(z,\cdot)\|_{q(\cdot)}>1\}} \int_{\{|w|>\delta\}} \frac{|f(z) - \widehat{f}_r(w)|^{q(w)} |g_n(w)|^{q(w)}}{|1 - \langle z, w\rangle|^{n+1+\eta}} d\nu_{\eta}(w) d\nu_{\alpha}(z) \\ &\leq \int_{\{|w|>\delta\}} |g_n(w)|^{q(w)} \int_{\mathbf{B}} \frac{|f(z) - \widehat{f}_r(w)|^{q(w)}}{|1 - \langle z, w\rangle|^{n+1+\eta}} d\nu_{\alpha}(z) d\nu_{\eta}(w) \end{aligned}$$

$$= \int_{\{|w|>\delta\}} |g_n(w)|^{q(w)}(1-|w|^2)^{\eta-\alpha} \int_{\mathbf{B}} \frac{|f(z)-\widehat{f_r}(w)|^{q(w)}}{|1-\langle z, w\rangle|^{n+1+\alpha+(\eta-\alpha)}} d\nu_\alpha(z) d\nu_\alpha(w)$$
$$< \epsilon \|g_n\|_{q(\cdot)}$$

It follows that

$$\begin{aligned}
&\int_{\mathbf{B}} \|G(z,\cdot)\|_{q(\cdot)}^{q(z)} d\nu_\alpha(z) \\
&= \int_{\mathbf{B}\cap\{\|G(z,\cdot)\|_{q(\cdot)}\le 1\}} \|G(z,\cdot)\|_{q(\cdot)}^{q(z)} d\nu_\alpha(z) \\
&\le \int_{\mathbf{B}\cap\{\|G(z,\cdot)\|_{q(\cdot)}\le 1\}} \|G(z,\cdot)\|_{q(\cdot)} d\nu_\alpha(z) \\
&\le \int_{\mathbf{B}} \left(\int_{\{|w|>\delta\}} \frac{|f(z)-\widehat{f_r}(w)|^{q(w)} |g_n(w)|^{q(w)}}{|1-\langle z, w\rangle|^{n+1+\eta}} d\nu_\eta(w) \right)^{1/q_+} d\nu_\alpha(z) \\
&\le \left(\int_{\mathbf{B}} \int_{\{|w|>\delta\}} \frac{|f(z)-\widehat{f_r}(w)|^{q(w)} |g_n(w)|^{q(w)}}{|1-\langle z, w\rangle|^{n+1+\eta}} d\nu_\eta(w) d\nu_\alpha(z) \right)^{1/q_+} \\
&= \left(\int_{\{|w|>\delta\}} |g_n(w)|^{q(w)}(1-|w|^2)^{\eta-\alpha} \int_{\mathbf{B}} \frac{|f(z)-\widehat{f_r}(w)|^{q(w)}}{|1-\langle z, w\rangle|^{n+1+\alpha+(\eta-\alpha)}} d\nu_\alpha(z) d\nu_\alpha(w) \right)^{1/q_+} \\
&< \epsilon^{1/q_+} \|g_n\|_{q(\cdot)}^{1/q_+}.
\end{aligned}$$

Also, we know that $\|T_\delta|g_n|\|_{q(\cdot)} \lesssim \|g_n\|_{q(\cdot)} < \infty$ which implies that

$$\int_{\mathbf{B}} T_\delta|g_n|(z)^{q(z)} d\nu_\alpha(z) < \infty.$$

Also, because of the uniform convergence of g_n on compact subsets of **B** there is an $N \in \mathbb{N}$ such that for all $n \ge N$, $|J_n(z)| < \epsilon^{1/q_-}$. That is, for $n \ge N$ we have

$$\int_{\mathbf{B}} |J_n(z)|^{q(z)} d\nu_\alpha(z) \le \int_{\mathbf{B}} |J_n(z)|^{q_-} d\nu_\alpha(z) < \epsilon.$$

It follows from (42) that,

$$\lim_{n\to\infty} \|H_f^\sigma g_n\|_{q(\cdot)} = 0,$$

which completes the proof. □

Lemma 7.3 *Let $p(\cdot), q(\cdot), \alpha$ and σ be as in Theorem 1.2 and let $f \in VA^{q(\cdot)}$. Then $H_f^\sigma, H_{\overline{f}}^\sigma : A_\alpha^{p(\cdot)} \to L_\sigma^{q(\cdot)}$ is compact.*

Proof Let $\{g_n\}$ be a bounded sequence in $A_\alpha^{p(\cdot)}$ converging to zero uniformly on compact subsets of $\mathbf{B}$. We must prove that $\|H_f^\sigma g_n\|_{q(\cdot),\sigma} \to 0$ as $n \to \infty$. By Proposition 5.4 we know that

$$\|H_f^\sigma g_n\|_{q(\cdot),\sigma} \lesssim \|g_n\|_{L^{q(\cdot)}(d\mu_{f,\sigma})},$$

where $d\mu_{f,\sigma} = |f|^{q(\cdot)} d\nu_\sigma$. The result follows from Lemma 6.1. □

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