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Bounds on the Non-real Eigenvalues of Nonlocal Indefinite Sturm–Liouville Problems with Coupled Boundary Conditions

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Abstract

The present paper deals with non-real eigenvalues of nonlocal indefinite Sturm–Liouville problems involving nonlocal potential terms associated to nonlocal coupled boundary conditions. A priori bounds on the imaginary parts and absolute values of these non-real eigenvalues in terms of the coefficients of the differential expression are obtained.

Keywords Indefinite Sturm–Liouville problem · Nonlocal potential · Nonlocal coupled boundary conditions · Non-real eigenvalue

Mathematics Subject Classification Primary 34B24 · 34L15; Secondary 47B50 · 34B05

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1 Introduction

This paper is concerned with the eigenvalue problem of nonlocal indefinite Sturm–Liouville differential equation

$$-y''(x)+q(x)y(x)+v(x)y(1)=\lambda w(x)y(x),\ \ x\in(0,1) \tag{1.1}$$

associated to nonlocal coupled boundary value conditions

$$\begin{pmatrix} y(1) \\ y'(1)+\int_0^1 v(x)y(x)\mathrm{d}x \end{pmatrix} = e^{i\gamma}C\begin{pmatrix} y(0) \\ y'(0)\end{pmatrix}, \tag{1.2}$$

where $q\in L^1([0,1],\mathbb{R})$, $v\in L^1([0,1],\mathbb{R})$ is called the *nonlocal potential*, $w\in L^1([0,1],\mathbb{R})$ changes its sign on $[0,1]$ in the meaning that

$$\mathrm{mes}\{x:w(x)>0\}>0,\ \ \mathrm{mes}\{x:w(x)<0\}>0$$

and

$$C=\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{pmatrix},\ c_{ij}\in\mathbb{R},\ i,j=1,2,\ \det C=1,\ \gamma\in[-\pi,\pi).$$

In this context a function y is called a solution of (1.1) if y and y' are in $AC_{\mathrm{loc}}(0,1)$ and y satisfies the differential Eq. (1.1) for almost all $x\in(0,1)$. A complex number λ is called an eigenvalue of the boundary value problem (1.1) and (1.2) if the equation (1.1) has a nontrivial solution satisfying the boundary conditions (1.2). Such a solution is called an eigenfunction of λ. If the weight function $w\in L^1[0,1]$ satisfies $w(x)>0$ a.e. $x\in[0,1]$, models similar to the nonlocal differential Eq. (1.1) have been studied in [2,9,11,24,25], and the authors in [12] and [1,18,19] investigate the reality of eigenvalues with Dirichlet boundary conditions and inverse spectral problems for the case $K(x,t)=v(x)u(t)$ with $v,u\in C([-1,1],\mathbb{R})$, $q\equiv 0$, $w\equiv 1$ and $K(x,t)=v(x)\delta(t-c)+\overline{v(t)}\delta(x-c)$ with $c\in(-1,1]$, $v\in L^2([-1,1],\mathbb{C})$, δ is Dirac's distribution, respectively.

It is well known that the (local)indefinite Sturm–Liouville eigenvalue problem, i.e., $v(x)\equiv 0$ in (1.1) with self-adjoint boundary conditions, has discrete, real eigenvalues unbounded both below and above, and the main difference from right-definite Sturm–Liouville problem was that the non-real eigenvalues may exist (see [3,13,15–17,21]). To determine the bounds of these non-real eigenvalues is a difficult problem since last century, however, this estimate problem was solved recently for (local)regular indefinite Sturm–Liouville problem with separated or coupled boundary conditions (see, for example, [4,14,20,26]) and for (local)singular case [5–7,23]. The nonlocal indefinite Sturm–Liouville problem occurs in some models, particularly in transport models, microwave propagation problems and quantum-mechanical theory. The spectral problems including a priori bounds and existence of non-real eigenvalues for the nonlocal indefinite Eq. (1.1) with separated self-adjoint boundary condition are well investigated in [22]. However, little is known for nonlocal indefinite Sturm–Liouville differential equation (1.1) under coupled boundary conditions.

The present paper will focus on the nonlocal indefinite Sturm–Liouville eigenvalue problems with coupled boundary conditions (1.1) and (1.2). Then the bounds of non-real eigenvalues for this nonlocal indefinite Sturm–Liouville problems are investigated. The rest of this paper is organized as follows. In Sect. 2, we state the main results about the bounds of non-real eigenvalues (see Theorems 2.1, 2.2 and 2.3) and give the proofs in Sect. 3.

2 Main results

For the benefit of the reader and simplify our description of results, we fix some symbols at first. Let $L^2_{|w|}(0,1)$ be the linear space of functions $y : (0,1) \to \mathbb{C}$ such that $\int_0^1 |w||y|^2 < \infty$ and equip this space with the inner product $(y,z)_{|w|} = \int_0^1 |w(x)|y(x)\overline{z(x)}dx$. As usual the L^1, L^2 and L^∞ norm will be denoted by $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively. Setting $q_\pm = \max\{0, \pm q\}$ and

$$\Delta_c = \max\begin{cases} \Delta_{c_1} = \dfrac{|c_{22}| + |c_{11}| + 2}{|c_{12}|}, & c_{12} \neq 0, \\ \Delta_{c_2} = |c_{11}||c_{21}|, & c_{12} = 0, \end{cases}$$
$$\Delta_{q,v} = \Delta_c + \|q_-\|_1 + 2\|v\|_1, \quad \Delta = \sqrt{\Delta_{q,v}(1+\Delta_{q,v})} + \Delta_{q,v}. \tag{2.1}$$

Let real-valued function g satisfy

$$\begin{aligned} g \in H^1(0,1) := \{g \in L^2(0,1) : g \in AC_{\text{loc}}(0,1),\ g' \in L^2(0,1), \\ g(1) = g(0) = 0,\ \operatorname{sgn} g = \operatorname{sgn} w \text{ a.e. on } [0,1]\}. \end{aligned} \tag{2.2}$$

It follows from (2.2) and $w(x) \neq 0$ a.e. on $[0,1]$ that $gw > 0$ a.e. on $(0,1)$, hence we can choose $\alpha > 0,\ \beta > 0$ such that

$$A = \{x \in [0,1] : g(x)w(x) < \alpha\},\quad m(\alpha) = \text{mes } A \le \frac{1}{2(2\Delta+1)}, \tag{2.3}$$

$$B = \{x \in [0,1] : g(x)w(x) < \beta\},\quad m(\beta) = \text{mes } B \le \frac{1}{2\Delta_w^2\Delta_{q,v}}. \tag{2.4}$$

Theorem 2.1 *Let (2.1) and (2.2) hold. Suppose that λ is a non-real eigenvalue of (1.1) and (1.2). Then*

$$\begin{aligned} |\operatorname{Im}\lambda| &\le \frac{2}{\alpha}\|g'\|_2\left(\Delta\sqrt{1+2\Delta} + (1+2\Delta)\|v\|_1\right), \\ |\lambda| &\le \frac{2}{\alpha}\Big[\|g'\|_2\left(\Delta\sqrt{1+2\Delta} + (1+2\Delta)\|v\|_1\right) \\ &\quad + \|g\|_\infty\left(\Delta^2 + (1+2\Delta)\|q\|_1\right)\Big], \end{aligned} \tag{2.5}$$

where α *is defined in* (2.3).

If the weight function w satisfies

$$\int_0^1 w(x)\mathrm{d}x \neq 0.$$

Setting

$$\Gamma(x) = \int_0^x w(t)\mathrm{d}t,\ \ x \in [0, 1],\ \ \Delta_w = 2 + \frac{2\|w\|_1}{|\Gamma(1)|}. \tag{2.6}$$

Then $\Gamma(1) = \int_0^1 w(x)\mathrm{d}x \neq 0$, $\|\Gamma\|_\infty \leq \|w\|_1$ and Δ_w are well defined.

Theorem 2.2 *Let* (2.1) *and* (2.2) *hold. If* $\int_0^1 w(x)dx \neq 0$, *then for any non-real eigenvalue* λ *of problem* (1.1) *and* (1.2), *we have*

$$\begin{aligned} |\operatorname{Im}\lambda| &\leq \frac{2}{\beta}\Delta_w^2\Delta_{q,v}\|g'\|_2\left(\sqrt{\Delta_{q,v}} + \|v\|_1\right), \\ |\lambda| &\leq \frac{2}{\beta}\Delta_w^2\Delta_{q,v}\left[\|g'\|_2\left(\sqrt{\Delta_{q,v}} + \|v\|_1\right) + \|g\|_\infty\left(\Delta_{q,v} + \|q\|_1\right)\right], \end{aligned} \tag{2.7}$$

where Δ_w *and* β *are defined in* (2.6) *and* (2.4), *respectively.*

Let λ be an eigenvalue of (1.1)–(1.2) and ψ be a corresponding eigenfunction. We say λ is either a *positive eigenvalue of negative type* or a *negative eigenvalue of positive type* if $\lambda \in \mathbb{R}$ and $\lambda\int_0^1 w|\psi|^2 < 0$ (cf. [17]). In the following, we will give the upper bounds on the eigenvalues corresponding to the non-real eigenvalues and non-zero real eigenvalues of a positive (negative) eigenvalues of negative (positive, resp.) type. That is we assume $\lambda\int_0^1 w|\psi|^2 \leq 0$ in the following theorem.

Theorem 2.3 *Let* (2.1), (2.2) *and* (2.3) *hold. Assume that* λ *corresponds to an eigenfunction* ψ *of problem* (1.1) *and* (1.2) *with* $\lambda\int_0^1 w|\psi|^2 \leq 0$, *then the eigenvalue* λ *satisfies*

$$|\lambda| \leq \frac{2}{\alpha}\left[\|g'\|_2\left(\Delta\sqrt{1+2\Delta} + (1+2\Delta)\|v\|_1\right) + \|g\|_\infty\left(\Delta^2 + (1+2\Delta)\|q\|_1\right)\right]. \tag{2.8}$$

3 The Proof of Theorems 2.1, 2.2 and 2.3

In order to prove Theorems 2.1, 2.2 and 2.3, we firstly give some lemmas as preparation. The operator associated to the nonlocal right-definite problem

$$\tau_v y := -y''(x) + q(x)y(x) + v(x)y(1) = \lambda|w(x)|y(x),\ \ \mathcal{B}_v y = 0 \tag{3.1}$$

is defined as $S_v y = \frac{1}{|w|}\tau_v y$ for $y \in D(S_v)$, where $\mathcal{B}_v y = 0$ is the boundary value condition (1.2) in Sect. 1. It is self-adjoint in the Hilbert space $(L^2_{|w|}, (\cdot,\cdot)_{|w|})$ and its spectra are real-valued and bounded from below, where

$$D(S_v) := \{y \in L^2_{|w|} : y, y' \in AC_{\mathrm{loc}}[0,1],\ \tau_v y/|w| \in L^2_{|w|},\ \mathcal{B}_v y = 0\}.$$

We consider the Krein space $\mathcal{K} = (L^2_{|w|}(0,1), [\cdot,\cdot]_w)$ with the inner product $[f,g]_w = \int_0^1 wf\overline{g}$, where $f, g \in L^2_{|w|}[0,1]$, and let $J = \operatorname{sgn} w$ be the fundamental symmetry operator. The operator T_v in $\mathcal{K}$ is defined as

$$T_v y = \frac{1}{w}\tau_v y = \frac{1}{w}\left(-y'' + qy + vy(1)\right), \quad y \in D(T_v) = D(S_v).$$

Then $S_v = JT_v$, $[T_v f, g]_w = (S_v f, g)_{|w|}$, $f, g \in D(T_v)$ and T_v is a self-adjoint operator in $\mathcal{K}$ with $D(T_v)$ (cf. [8,10]). Let φ be an eigenfunction of (1.1) and (1.2) corresponding to a non-real eigenvalue λ, that is $\mathcal{B}_v\varphi = 0$ and

$$-\varphi'' + q\varphi + v\varphi(1) = \lambda w\varphi. \tag{3.2}$$

Since the problem (1.1) with (1.2) is a linear system and φ is continuous, we can choose φ satisfies $\|\varphi\|_2 = 1$ in the following discussion.

Lemma 3.1 *Let Δ_c be defined in (2.1). Then for $\varphi \in D(T_v)$, it holds that*

$$\varphi'(1)\overline{\varphi(1)} - \varphi'(0)\overline{\varphi(0)} + \int_0^1 v\varphi\overline{\varphi(1)} \le \Delta_c \max\left\{|\varphi(0)|^2, |\varphi(1)|^2\right\}. \tag{3.3}$$

Proof For $\varphi \in D(T_v)$, it follows from

$$\begin{pmatrix} \varphi(1) \\ \varphi'(1) + \int_0^1 v(x)\varphi(x)dx \end{pmatrix} = e^{i\gamma}\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}\begin{pmatrix} \varphi(0) \\ \varphi'(0) \end{pmatrix} \tag{3.4}$$

that

$$\varphi(1) = e^{i\gamma}c_{11}\varphi(0) + e^{i\gamma}c_{12}\varphi'(0). \tag{3.5}$$

From (3.4) and $\det C = 1$ one sees that

$$\begin{pmatrix} c_{22} & -c_{12} \\ -c_{21} & c_{11} \end{pmatrix}\begin{pmatrix} \varphi(1) \\ \varphi'(1) + \int_0^1 v(x)\varphi(x)dx \end{pmatrix} = e^{i\gamma}\begin{pmatrix} \varphi(0) \\ \varphi'(0) \end{pmatrix}. \tag{3.6}$$

Then

$$c_{22}\varphi(1) - c_{12}\varphi'(1) - c_{12}\int_0^1 v\varphi = e^{i\gamma}\varphi(0).$$

This together with (3.5) yields that

$$
\begin{aligned}
&c_{12}\left(\varphi'(1)\overline{\varphi(1)}-\varphi'(0)\overline{\varphi(0)}+\int_0^1 v\varphi\overline{\varphi(1)}\right)\\
&\quad = c_{22}|\varphi(1)|^2+c_{11}|\varphi(0)|^2-2\,\mathrm{Re}\left(e^{i\gamma}\varphi(0)\overline{\varphi(1)}\right).
\end{aligned}
$$

Therefore, if $c_{12}\neq 0$, by (2.1) we get

$$
\begin{aligned}
&\varphi'(1)\overline{\varphi(1)}-\varphi'(0)\overline{\varphi(0)}+\int_0^1 v\varphi\overline{\varphi(1)}\\
&= c_{12}^{-1}\left(c_{22}|\varphi(1)|^2+c_{11}|\varphi(0)|^2-2\,\mathrm{Re}\left(e^{i\gamma}\varphi(0)\overline{\varphi(1)}\right)\right)\\
&\le |c_{12}^{-1}|\left(|c_{22}||\varphi(1)|^2+|c_{11}||\varphi(0)|^2+2|e^{i\gamma}||\varphi(0)||\overline{\varphi(1)}|\right)\\
&\le \frac{|c_{22}|+|c_{11}|+2}{|c_{12}|}\max\left\{|\varphi(0)|^2,|\varphi(1)|^2\right\}\\
&\le \Delta_c\max\left\{|\varphi(0)|^2,|\varphi(1)|^2\right\}.
\end{aligned}
\tag{3.7}
$$

If $c_{12}=0$, then (3.4) and (3.6) give that

$$
\varphi'(1)+\int_0^1 v(x)\varphi(x)dx=e^{i\gamma}c_{21}\varphi(0)+e^{i\gamma}c_{22}\varphi'(0)\ \text{ and }\ c_{22}\varphi(1)=e^{i\gamma}\varphi(0),
$$

which together with $\det C=c_{11}c_{22}=1$ and (2.1) implies that

$$
\begin{aligned}
&\varphi'(1)\overline{\varphi(1)}-\varphi'(0)\overline{\varphi(0)}+\int_0^1 v(x)\varphi(x)\overline{\varphi(1)}\mathrm{d}x\\
&\quad = c_{11}c_{21}|\varphi(0)|^2\le |c_{11}||c_{21}|\max\left\{|\varphi(0)|^2,|\varphi(1)|^2\right\}\\
&\quad \le \Delta_c\max\left\{|\varphi(0)|^2,|\varphi(1)|^2\right\}.
\end{aligned}
\tag{3.8}
$$

It follows from (3.7) and (3.8) that (3.3) holds immediately. □

The following lemma is the estimates of $\|\varphi'\|_2$ and $\|\varphi\|_\infty$.

Lemma 3.2 *Let φ and λ be defined as above. Then*

$$
\|\varphi'\|_2\le\Delta,\quad \|\varphi\|_\infty\le\sqrt{1+2\Delta}.
\tag{3.9}
$$

Proof Multiplying both sides of (3.2) by $\overline{\varphi}$ and integrating by parts over the interval $[x,1]$, we have

$$
\varphi'(x)\overline{\varphi(x)}-\varphi'(1)\overline{\varphi(1)}+\int_x^1(|\varphi'|^2+q|\varphi|^2)+\int_x^1 v\overline{\varphi}\varphi(1)=\lambda\int_x^1 w|\varphi|^2.
\tag{3.10}
$$

Separating imaginary parts and the absolute values of λ on both sides of (3.10) yields

$$\operatorname{Im}\lambda\int_x^1 w|\varphi|^2=\operatorname{Im}\left(\varphi'(x)\overline{\varphi(x)}-\varphi'(1)\overline{\varphi(1)}\right)+\operatorname{Im}\left(\int_x^1 v\overline{\varphi}\varphi(1)\right), \tag{3.11}$$

$$|\lambda|\left|\int_x^1 w|\varphi|^2\right|=\left|\varphi'(x)\overline{\varphi(x)}-\varphi'(1)\overline{\varphi(1)}+\int_x^1\left(|\varphi'|^2+q|\varphi|^2\right)+\int_x^1 v\overline{\varphi}\varphi(1)\right|. \tag{3.12}$$

Let $x=0$, then from (3.11), $\det C=1$ and $\mathcal{B}_v\varphi=0$ one sees that

$$\begin{aligned}
&\operatorname{Im}\lambda\int_0^1 w|\varphi|^2\\
&=\operatorname{Im}\left[\varphi'(0)\overline{\varphi(0)}+\left(-e^{i\gamma}C\varphi'(0)+\int_0^1 v\varphi\right)e^{-i\gamma}C\overline{\varphi(0)}+\int_0^1 v\overline{\varphi}\varphi(1)\right]\\
&=\operatorname{Im}\left(\varphi'(0)\overline{\varphi(0)}-C^2\varphi'(0)\overline{\varphi(0)}+e^{-i\gamma}C\overline{\varphi(0)}\int_0^1 v\varphi+\int_0^1 v\overline{\varphi}\varphi(1)\right)\\
&=\operatorname{Im}\left(\overline{\varphi(1)}\int_0^1 v\varphi+\int_0^1 v\overline{\varphi}\varphi(1)\right)=\operatorname{Im}\left[2\operatorname{Re}\left(\int_0^1 v\overline{\varphi}\varphi(1)\right)\right]=0.
\end{aligned}$$

This together with $\operatorname{Im}\lambda\neq 0$ yields that $\int_0^1 w|\varphi|^2=0$. Then from (3.12) we get

$$\left|\varphi'(0)\overline{\varphi(0)}-\varphi'(1)\overline{\varphi(1)}+\int_0^1\left(|\varphi'|^2+q|\varphi|^2\right)+\int_0^1 v\overline{\varphi}\varphi(1)\right|=0. \tag{3.13}$$

For $x,\ y\in[0,1],\ y<x$,

$$\begin{aligned}
|\varphi(x)|^2-|\varphi(y)|^2&=\int_y^x\left(|\varphi(t)|^2\right)'\mathrm{d}t=\int_y^x\left(\varphi'(t)\overline{\varphi(t)}+\varphi(t)\overline{\varphi'(t)}\right)\mathrm{d}t\\
&=\int_y^x 2\operatorname{Re}\left(\varphi'(t)\overline{\varphi(t)}\right)\mathrm{d}t\le 2\int_0^1|\varphi'(t)\overline{\varphi(t)}|\mathrm{d}t\\
&\le 2\|\varphi'\|_2.
\end{aligned}$$

Integrating the above inequality over $[0,1]$ with respect to y gives

$$\int_0^1|\varphi(x)|^2\mathrm{d}y-\int_0^1|\varphi(y)|^2\mathrm{d}y\le\int_0^1 2\|\varphi'\|_2\mathrm{d}y.$$

Then $|\varphi(x)|^2 \le 2\|\varphi'\|_2 + 1$, and hence $\|\varphi\|_\infty^2 \le 2\|\varphi'\|_2 + 1$. These facts together with (2.1), (3.3) in Lemma 3.1, (3.13) and $q = q_+ - q_-$ lead to

$$\begin{aligned}
\int_0^1 |\varphi'|^2 =& \varphi'(1)\overline{\varphi(1)} - \varphi'(0)\overline{\varphi(0)} - \int_0^1 q|\varphi|^2 - \int_0^1 v\overline{\varphi}\varphi(1) \\
\le& \left|\varphi'(1)\overline{\varphi(1)} - \varphi'(0)\overline{\varphi(0)}\right| + \int_0^1 q_-|\varphi|^2 - \left|\int_0^1 v\overline{\varphi}\varphi(1)\right| \\
=& \left|\varphi'(1)\overline{\varphi(1)} - \varphi'(0)\overline{\varphi(0)}\right| - \left|\int_0^1 v\varphi\overline{\varphi(1)}\right| + \int_0^1 |q_-||\varphi|^2 \\
&- \left|\int_0^1 v\overline{\varphi}\varphi(1)\right| + \left|\int_0^1 v\varphi\overline{\varphi(1)}\right| \\
\le& \left|\varphi'(1)\overline{\varphi(1)} - \varphi'(0)\overline{\varphi(0)} + \int_0^1 v\varphi\overline{\varphi(1)}\right| + \int_0^1 |q_-||\varphi|^2 \\
&+ 2\int_0^1 |v||\overline{\varphi}||\varphi(1)| \\
\le& \left|\Delta_c \max\left\{|\varphi(0)|^2, |\varphi(1)|^2\right\}\right| + \int_0^1 |q_-||\varphi|^2 + 2\int_0^1 |v||\overline{\varphi}||\varphi(1)| \\
\le& \|\varphi\|_\infty^2 \left(\Delta_c + \|q_-\|_1 + 2\|v\|_1\right) \le \left(2\|\varphi'\|_2 + 1\right)\Delta_{q,v}.
\end{aligned} \tag{3.14}$$

Then $\left(\|\varphi'\|_2 - \Delta_{q,v}\right)^2 \le \Delta_{q,v}(1 + \Delta_{q,v})$, and hence $\|\varphi'\|_2 \le \left(\sqrt{\Delta_{q,v}(1 + \Delta_{q,v})} + \Delta_{q,v}\right)$. Therefore, $\|\varphi\|_\infty^2 \le 2\Delta + 1$, where $\Delta,\ \Delta_{q,v}$ are defined in (2.1). □

With the aids of Lemmas 3.1 and 3.2, we prove the first main result in Sect. 2.

The Proof of Theorem 2.1 Let φ and λ be defined as above. It follows from (2.3), Lemma 3.2, $A^c = [0, 1]\backslash A$ and $\int_0^1 w(x)|\varphi(x)|^2 \mathrm{d}x = 0$ that

$$\begin{aligned}
\int_0^1 g'(x) \int_x^1 w(t)|\varphi(t)|^2 \mathrm{d}t\mathrm{d}x &= \int_0^1 g(t)w(t)|\varphi(t)|^2 \mathrm{d}t \\
&\ge \alpha\left(\int_0^1 |\varphi(t)|^2 \mathrm{d}t - \int_A |\varphi(t)|^2 \mathrm{d}t\right) \\
&\ge \alpha\left(1 - \|\varphi\|_\infty^2 m(\alpha)\right) \ge \alpha/2.
\end{aligned} \tag{3.15}$$

This together with (3.11) and Lemma 3.2 yields that

$$\begin{aligned}
\frac{\alpha}{2}|\operatorname{Im}\lambda| \le& |\operatorname{Im}\lambda| \int_0^1 g' \int_x^1 w|\varphi|^2 \\
&\le \left|\int_0^1 g'\left[\operatorname{Im}(-\varphi'(1)\overline{\varphi(1)} + \varphi'\overline{\varphi}) + \operatorname{Im}\left(\int_x^1 v\varphi(1)\overline{\varphi}\right)\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&= \left| \int_0^1 g' \,\mathrm{Im}(\varphi'\overline{\varphi}) + \int_0^1 g' \,\mathrm{Im}\left(\int_x^1 v\varphi(1)\overline{\varphi} \right) \right| \\
&\leq \|\varphi\|_\infty \|g'\|_2 \|\varphi'\|_2 + \|\varphi\|_\infty^2 \|g'\|_2 \|v\|_1 \\
&\leq \Delta\sqrt{1+2\Delta}\|g'\|_2 + (1+2\Delta)\|g'\|_2\|v\|_1. \qquad (3.16)
\end{aligned}
$$

For the absolute value part we exploit (3.12), (3.15) and Lemma 3.2 to derive

$$
\begin{aligned}
\frac{\alpha}{2}|\lambda| &\leq |\lambda| \int_0^1 g' \int_x^1 w|\varphi|^2 \\
&\leq \int_0^1 |g'| \left| \varphi'\overline{\varphi} - \varphi'(1)\overline{\varphi(1)} + \int_x^1 (|\varphi'|^2 + q|\varphi|^2) + \int_x^1 v\overline{\varphi}\varphi(1) \right| \\
&\leq \int_0^1 |g'||\varphi'||\overline{\varphi}| + \int_0^1 g(|\varphi'|^2 + q|\varphi|^2) + \int_0^1 |g'| \int_0^1 |v||\overline{\varphi}||\varphi(1)| \\
&\leq \|\varphi\|_\infty \|g'\|_2 \|\varphi'\|_2 + \|g\|_\infty (\|\varphi'\|_2^2 + \|q\|_1 \|\varphi\|_\infty^2) + \|g'\|_2 \|v\|_1 \|\varphi\|_\infty^2 \\
&\leq \|g'\|_2 \left(\Delta\sqrt{1+2\Delta} + (1+2\Delta)\|v\|_1 \right) + \|g\|_\infty \left(\Delta^2 + (1+2\Delta)\|q\|_1 \right). \qquad (3.17)
\end{aligned}
$$

So the inequalities in (2.5) follow from (3.16) and (3.17) immediately. □

Now we give the Proof of Theorem 2.2. It follows from $\int_0^1 w(x)\mathrm{d}x \neq 0$ and (2.6) that we can prove the following lemma.

Lemma 3.3 *Let φ and λ be defined as above. Then*

$$
\|\varphi'\|_2 \leq \Delta_w \Delta_{q,v}, \quad \|\varphi\|_\infty \leq \Delta_w \sqrt{\Delta_{q,v}}.
$$

Proof Since φ is an eigenfunction corresponding to the non-real eigenvalue λ, we still can make use of the result $\int_0^1 w(x)|\varphi(x)|^2\mathrm{d}x = 0$ in Lemma 3.2. This together with $\Gamma(x) = \int_0^x w(t)\mathrm{d}t$, $x \in [0, 1]$ yields that

$$
\begin{aligned}
\int_0^1 \Gamma(|\varphi|^2)' &= \Gamma(1)|\varphi(1)|^2 - \int_0^1 \Gamma'|\varphi|^2 \\
&= \Gamma(1)|\varphi(1)|^2 - \int_0^1 w|\varphi|^2 = \Gamma(1)|\varphi(1)|^2.
\end{aligned}
$$

And hence $\Gamma(1)|\varphi(x)|^2 = -\Gamma(1)\left(|\varphi(1)|^2 - |\varphi(x)|^2\right) + \int_0^1 \Gamma(|\varphi|^2)'$. This fact together with (2.6) implies

$$\begin{aligned}
|\varphi(x)|^2 =& |\varphi(x)|^2 - |\varphi(1)|^2 + \frac{1}{\Gamma(1)}\int_0^1 \Gamma(|\varphi|^2)' \\
=& -\int_x^1 (|\varphi|^2)' + \frac{1}{\Gamma(1)}\int_0^1 \Gamma(|\varphi|^2)' \\
\leq& \int_0^1 \left|\varphi'\overline{\varphi} + \varphi\overline{\varphi'}\right| + \frac{1}{|\Gamma(1)|}\int_0^1 |\Gamma|\left(|\varphi'||\overline{\varphi}| + |\varphi||\overline{\varphi'}|\right) \\
\leq& 2\int_0^1 |\varphi'||\overline{\varphi}| + 2\frac{\|\Gamma\|_\infty}{|\Gamma(1)|}\int_0^1 |\varphi'||\overline{\varphi}| \leq \Delta_w \|\varphi'\|_2.
\end{aligned}$$

Hence $\|\varphi\|_\infty^2 \leq \Delta_w \|\varphi'\|_2$, which together with (3.14) gives that

$$\int_0^1 |\varphi'|^2 \leq \|\varphi\|_\infty^2 (\Delta_c + \|q_-\|_1 + 2\|v\|_1) \leq \Delta_w \Delta_{q,v} \|\varphi'\|_2.$$

Therefore, $\|\varphi'\|_2 \leq \Delta_w \Delta_{q,v}$ and $\|\varphi\|_\infty \leq \Delta_w \sqrt{\Delta_{q,v}}$. □

Now we prove the second result in this paper, i.e., Theorem 2.2.

The Proof of Theorem 2.2. It follows from (2.4) and $\int_0^1 w(t)|\varphi(t)|^2 \mathrm{d}t = 0$ that

$$\begin{aligned}
&\int_0^1 g' \int_x^1 w|\varphi|^2 = \int_0^1 g w |\varphi|^2 \geq \beta\left(\int_0^1 |\varphi|^2 - \int_B |\varphi|^2\right) \\
&\geq \beta\left(1 - \|\varphi\|_\infty^2 m(\beta)\right) \geq \frac{\beta}{2}.
\end{aligned}$$

This fact with Lemma 3.3 and (3.16) in the Proof of Theorem 2.1 lead to

$$\begin{aligned}
\frac{\beta}{2}|\operatorname{Im}\lambda| &\leq \|\varphi\|_\infty \|g'\|_2 \|\varphi'\|_2 + \|\varphi\|_\infty^2 \|g'\|_2 \|v\|_1 \\
&\leq \Delta_w^2 \Delta_{q,v}^{3/2} \|g'\|_2 + \Delta_w^2 \Delta_{q,v} \|g'\|_2 \|v\|_1.
\end{aligned} \tag{3.18}$$

For the real part, it follows from (3.17) and Lemma 3.3 that

$$\begin{aligned}
\frac{\beta}{2}|\lambda| &\leq \|\varphi\|_\infty \|g'\|_2 \|\varphi'\|_2 + \|g\|_\infty \|\varphi'\|_2^2 + \|g\|_\infty \|\varphi\|_\infty^2 \|q\|_1 + \|g'\|_2 \|\varphi\|_\infty^2 \|v\|_1 \\
&\leq \Delta_w^2 \Delta_{q,v} \|g'\|_2 \left(\sqrt{\Delta_{q,v}} + \|v\|_1\right) + \Delta_w^2 \Delta_{q,v} \|g\|_\infty \left(\Delta_{q,v} + \|q\|_1\right).
\end{aligned} \tag{3.19}$$

So the inequalities in (2.7) follow from (3.18) and (3.19) immediately. □

Since ψ is the eigenfunction of (1.1) and (1.2) corresponding to an eigenvalue λ with $\lambda \int_0^1 w|\psi|^2 \le 0$, that is

$$-\psi'' + q\psi + v\psi(1) = \lambda w\psi, \quad \mathcal{B}_v\psi = 0. \tag{3.20}$$

Similar with Lemma 3.2, we have

Lemma 3.4 *Assume that λ is an eigenvalue of* (3.20) *and ψ is the corresponding eigenfunction with $\lambda \int_0^1 w|\psi|^2 \le 0$. Then*

$$\|\psi'\|_2 \le \Delta\|\psi\|_2, \quad \|\psi\|_\infty \le \sqrt{1+2\Delta}\|\psi\|_2.$$

Proof The proof of this result is quite similar to that given earlier for the estimates of $\|\psi'\|_2$ and $\|\psi\|_\infty$ in Lemma 3.2 and so is omitted. □

With the help of the preceding Lemma 3.4, we can now prove the Theorem 2.3.

The Proof of Theorem 2.3 Since the problem (3.20) is a linear system and ψ is continuous, we can choose ψ satisfies $\|\psi\|_2 = 1$. Multiplying both sides of (3.20) by $\overline{\psi}$ and integrating over the interval $[x, 1]$ we have

$$-\psi'(1)\overline{\psi(1)} + \psi'(x)\overline{\psi(x)} + \int_x^1 (|\psi'|^2 + q|\psi|^2) + \int_x^1 v\overline{\psi}\psi(1) = \lambda \int_x^1 w|\psi|^2. \tag{3.21}$$

Similar with the argument in the Proof of Theorem 2.1 and 2.2, one sees that

$$\begin{aligned}
\frac{\alpha}{2}|\lambda| &\le |\lambda| \int_0^1 g' \int_x^1 w|\psi|^2 \le \left|\lambda \int_0^1 g' \int_x^1 w|\psi|^2\right| \\
&= \left|\int_0^1 g' \left(-\psi'(1)\overline{\psi(1)} + \psi'(x)\overline{\psi(x)} + \int_x^1 |\psi'|^2 + \int_x^1 q|\psi|^2 + \int_x^1 v\overline{\psi}\psi(1)\right)\right| \\
&\le \|g'\|_2 \left(\Delta\sqrt{1+2\Delta} + (1+2\Delta)\|v\|_1\right) + \|g\|_\infty \left(\Delta^2 + (1+2\Delta)\|q\|_1\right)
\end{aligned}$$

by Lemma 3.4. So the inequality in (2.8) follows immediately. □

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