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Complex Analysis and Operator Theory

Band Edge Limit of the Scattering Matrix for Quasi-One-Dimensional Discrete Schrödinger Operators

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Abstract

This paper is about the scattering theory for one-dimensional matrix Schrödinger operators with a matrix potential having a finite first moment. The transmission coefficients are analytically continued and extended to the band edges. An explicit expression is given for these extensions. The limits of the reflection coefficients at the band edges are also calculated.

Keywords Jost solutions · Scattering matrix · Half-bound states

Mathematics Subject Classification 47A40 · 81U05 · 47B36

1 Introduction

We study scattering theory for the one-dimensional matrix Schrödinger Hamiltonians on $\mathbb{Z}$ of the form

$$H = H_0 + V, \tag{1}$$

where H_0 is (an energy shift of) the discrete Laplace operator and V is a self-adjoint matrix multiplication operator with finite first moment. See Sect. 1.1 below for a detailed definition of H. This model is widely used in the context of low-energy

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phenomena in solid state physics. Moreover, H is a tridiagonal operator, also called a Jacobi operator, which is the discrete analogue of a Sturm–Liouville operator. Its analysis is connected to orthogonal (matrix) polynomials and, via its spectral theory, to matrix-valued measures on the real line. Many authors have studied direct and inverse scattering theory for this class of operators, see for example Case and Kac [11], Serebryakov [33], Aptekarev and Nikishin [6, 29], Geronimo [20] and Guseinov [21–24]. More recent contributions are [17, 35].

The energy is parametrized by $E = z + 1/z$, for $z \in \overline{\mathbb{D}}\backslash\{0\}$ where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and this parametrization is used in the analysis of the scattering matrix which is constructed from the Jost solutions, namely asymptotically free generalized eigenfunctions. This permits to extend the transmission coefficients meromorphically to $\mathbb{D}\backslash\{0\}$ using the properties of the Wronskian. We compute the limit of the transmission coefficients when z tends to 1 and give an explicit expression for them (with a limit taken in $\overline{\mathbb{D}}$). We also prove that the limits of the reflection coefficients exist when z tends to 1 (these coefficients are only defined for $z \in \mathbb{S}^1$, and $\mathbb{S}^1$ is regarded as a subset of $\mathbb{C}$). The limits when z tends to -1 can be studied in a similar fashion and, for this reason, we omit them. The above results are useful for inverse scattering theory and the proof of Levinson's theorem, and we will address these problems in future works.

Scattering theory for matrix Schrödinger operators on $\mathbb{Z}$ of the form (1), but with compactly supported V, is studied in [7]. However, in [7] a different analytical approach is followed, namely the solutions of the eigenvalue equation of H are calculated in terms of the transfer matrix, whereas here the Volterra equation is used. This implies a major difference and there is practically no intersection between the proofs of the present manuscript and the proofs in [7]. Moreover, a compactly supported potential admits meromorphic continuations of the scattering matrix to the whole complex plane, something that is not possible in the present framework. Nevertheless, we essentially stick with the notations of [7] in this paper, but there are some new technical objects that are only addressed in the present analysis.

Let us briefly comment on other earlier contributions. In the continuous scalar case, [13] contains everything from Jost solutions to low-energy behavior of reflection and transmission coefficients (see Ch. XVII). In Sect. XVII4.4 and in page 381, some ideas and references for the matrix-valued case are presented. Much of this is now driven by interest in completely integrable systems, in particular, by studies of the matrix-valued KdV and Toda equations. The inverse scattering theory approach was developed in great detail in [9, 10, 15, 27, 31, 36], and the half-line case was addressed in [30]. For further references, see [8, 16, 18, 19].

The low-energy behavior of the scattering matrix is studied in [26] for the scalar continuous case and in [25] for the scalar discrete case. For continuous matrix-valued Schrödinger operators, there are works by Wadati and Kamijo [36], Martínez Alonso and Olmedilla [27, 31], Corona–Corona [15], Newton and Jost [30] as well as by Aktosun, Klaus, Van Der Mee and Weder [2–4]. We would like to stress here that our proof for the limit of the scattering matrix at the band edges is shorter and closely tied to a conceptual treatment of the Wronskian. Actually we do not use the Jordan decomposition as in [2].There are also works in the non-linear context, see for example [9, 10, 12, 28]. Bound states for the half-space version of the discrete matrix Schrödinger equation with compactly supported potentials have been analyzed in [5].

1.1 Mathematical Framework and Main Results

In this manuscript, $\mathbb{C}^L$ denotes the L-dimensional complex vector space and $\mathcal{M}_{d\times e}(\mathbb{C}) = \mathcal{M}_{d\times e}$ the vector space of matrices with d rows and e columns and with coefficients in $\mathbb{C}$. Furthermore, M^* denotes the adjoint of a matrix M, that is, its conjugate transpose.

1.1.1 Hamiltonians

Scattering theory compares asymptotic time evolution of two systems. The simpler one is called free system and the other is the interaction system. In this paper, the free system is described by the discrete Laplacian on the Hilbert space $\ell^2(\mathbb{Z}, \mathbb{C}^L)$. It is given by

$$(H_0\phi)(n) \ := \ \phi(n+1) \ + \ \phi(n-1), \quad \phi \in \ell^2(\mathbb{Z}, \mathbb{C}^L). \tag{2}$$

We denote by $V \in \mathcal{M}_{L\times L}(\mathbb{C})^{\mathbb{Z}}$ the interaction, which is a matrix-valued multiplication operator defined by

$$(V\Psi)(n) := V(n)\Psi(n), \quad \Psi \in \ell^2(\mathbb{Z}, \mathbb{C}^L), \tag{3}$$

and assume that $V(n) = V(n)^*$ for every $n \in \mathbb{Z}$, and that

$$\sum_{n\in\mathbb{Z}} \|nV(n)\| < \infty. \tag{4}$$

Now the interaction Hamiltonian is defined by (1), namely $H = H_0 + V$, with domain $\ell^2(\mathbb{Z}, \mathbb{C}^L)$. We denote by $\mathcal{F} : \ell^2(\mathbb{Z}, \mathbb{C}^L) \to L^2([-\pi, \pi], \mathbb{C}^L)$ the Fourier transform and by $\mathcal{F}^{-1}$ its inverse. They are given by

$$\mathcal{F}(\phi)(k) = \frac{1}{\sqrt{2\pi}} \sum_n e^{ikn}\phi(n), \quad \mathcal{F}^{-1}(\psi)(n) = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} e^{-ikn}\psi(k). \tag{5}$$

A direct calculation leads us to

$$\mathcal{F}H_0\mathcal{F}^{-1}\psi(k) = (e^{ik} + e^{-ik})\psi(k) = 2\cos(k)\psi(k). \tag{6}$$

As the Fourier transform is unitary, then the spectrum of H_0 is $\sigma(H_0) = [-2, 2]$ and it is purely absolutely continuous. The essential spectrum of H is thus $[-2, 2]$ (see Section XIII.4 in [32]).

This paper studies stationary scattering theory. Then, naturally, eigenvalue equations for H and H_0 are relevant in this manuscript. As usual, we study generalized eigenvalues and, moreover, do not only address real energies, but also study generalized eigenvectors corresponding to complex energies. We use the same symbols H and H_0 to denote the operators defined (with the same expressions as above) either on $(\mathcal{M}_{L\times L})^{\mathbb{Z}}$ or $(\mathbb{C}^L)^{\mathbb{Z}}$ and parametrize the eigenvalues in the form

$$E = z + 1/z\ , \qquad z \in \mathbb{C}\backslash\{0\}. \tag{7}$$

Then, the eigenvalue equations take the form

$$Hu = Eu, \tag{8}$$

$$H_0 u = Eu, \tag{9}$$

and generally we take $u \in (\mathcal{M}_{L\times L})^{\mathbb{Z}}$. In the context of this article we simply call solutions the functions u satisfying (8). The solutions of (9) are referred to as the free solutions. Of particular importance are the Jost solutions, which are solutions with prescribed data at $-\infty$ or ∞ given by free solutions. They are essential objects for the scattering matrix. We also study solutions with prescribed data at 0 and 1, because they are important in technical elements in our proofs.

1.1.2 Jost Solutions

Jost solutions are the key ingredient for the construction of the scattering matrix. They are solutions of the system that behave as free waves (plane waves) at infinity. Standard properties and the construction of these solutions can be found in the books [1, 13]. In (6), k might be seen as a momentum and the Fourier modes, namely the functions $n \mapsto e^{ink}$ represent plane waves. These are generalized eigenvalues of H_0 and satisfy the equation

$$H_0 e^{ink}\alpha = (e^{ik} + e^{-ik})e^{ink}\alpha = 2\cos(k)e^{ink}\alpha, \tag{10}$$

for every $\alpha \in \mathbb{C}^L$. In the previous equation we identify, as usual, the function $n \mapsto e^{ink}\alpha$ with $e^{ink}\alpha$. For $k \notin \{-\pi, 0, \pi\}$, the functions $\{e^{ink}\alpha, e^{-ikn}\alpha : \alpha \in \mathbb{C}^L\}$ define a $2L$-dimensional vector space and, as H_0 has only two discrete derivatives, they generate all solutions of (10) (see also (9)). Of course, $2\cos(k) = E$ is interpreted as the energy of the corresponding plane waves. In this generalized sense, the solutions of the time dependent Schrödinger equation

$$i\frac{d}{dt}\phi = H_0\phi$$

are of the form

$$e^{-i(Et-kn)}\alpha, \quad e^{-i(Et+kn)}\alpha.$$

For positive energies E, the first wave function above moves in the direction of k and the second in the direction of $-k$. For negative energies it is the other way around.

Heuristically, we understand a wave traveling to the right allowing n to be real (as in the continuous case) and looking at the equation $Et - kn = 0$ (taking the phase to be zero): for positive E and k (for example), a positive increment in time leads to a positive increment in position.

It is convenient to change our notation and take $z = e^{ik}$. With this notation, we take $z \in \mathbb{C}$ and not only $z = e^{ik}$ as before, *i.e.* we analytically continue the solutions. From the discussion above, we obtain that a complex number $z = e^{ik}$ represents a wave traveling to the right if its real and imaginary parts have the same sign. Otherwise, it travels to the left. This implies that if z represents a wave traveling to the right, then $1/z$ represents a wave traveling to the left, and vice versa. Notice that this holds true only when $z = e^{ik}$, if this is not fulfilled then there is no interpretation for the direction of traveling. We analyze matrix valued solutions in order to consider all vector valued solutions at once. We set (for $z \neq 0$)

$$u_0^z(n) \ := \ z^n \, \mathbf{1}, \tag{11}$$

where $\mathbf{1}$ is the identity in $\mathcal{M}_{L\times L}$. It satisfies the complex extension of (10), notably with $E = z + 1/z$,

$$H_0 u_0^z = E u_0^z, \tag{12}$$

where we use matrix multiplication, and the vector valued solutions of (10) are of the form $u_0^z a$. Notice that for any fixed value of the energy E and $z \notin \{-1, 0, 1\}$, the columns of u_0^z and $u_0^{1/z}$ generate all solutions. This is not the case for $z = 1$ or $z = -1$ (which correspond to $E = 2$, or $E = -2$, respectively) because in this situation $u_0^z = u_0^{1/z}$. In order to provide all solutions, also for $E = 2$ and $E = -2$, we define

$$v_0^{\pm}(n) := (\pm 1)^n \, n. \tag{13}$$

Then, the columns of the matrices $u_0^{\pm 1}$ and $v_0^{\pm}$ generate the space of free solutions (generalized eigenvectors) of (10) for $E = \pm 2$. As mentioned above, the generalized eigenvectors of H_0 are free waves (or plane waves). Jost solutions are generalized eigenvectors of H that behave as plane waves away from the interaction. They are introduced in the next definition. Their existence is proved in Sect. 2.2.

Definition 1 *(Jost Solutions)* For every $z \in \overline{\mathbb{D}} \backslash \{0\}$, we denote by u_+^z , $u_-^{1/z}$ the $\mathcal{M}_{L\times L}$-valued solutions of

$$H u_+^z = E u_+^z, \quad H u_-^{1/z} = E u_-^{1/z}, \quad E = z + 1/z, \tag{14}$$

satisfying, as $n \to +\infty$ and $n \to -\infty$ respectively,

$$u_+^z(n) = z^n(\mathbf{1} + o(1)), \quad u_-^{1/z}(n) = z^{-n}(\mathbf{1} + o(1)). \tag{15}$$

Moreover, for $z = 1$, we denote by $v^z_\pm$ the $\mathcal{M}_{L\times L}$-valued solutions of

$$Hv^z_\pm = Ev^z_\pm, \quad E = 2, \tag{16}$$

satisfying as $n \to \pm\infty$

$$v^z_\pm(n) = n(\mathbf{1} + o(1)). \tag{17}$$

1.1.3 The Scattering Matrix

Due to the asymptotic behavior of the Jost solutions (see Eq. (15)), the columns of the matrix $(u^z_\pm, u^{1/z}_\pm)$ are linearly independent for $z \in \mathbb{S}^1\backslash\{-1, 1\}$ and, therefore, they form a basis of solutions. The same holds true for $(u^1_\pm, v^1_\pm)$. This implies that there are matrices $M^z_\pm, N^z_\pm \in \mathcal{M}_{L\times L}$ such that

$$u^z_+ = u^z_- M^z_+ + u^{1/z}_- N^z_+, \quad u^{1/z}_- = u^z_+ N^z_- + u^{1/z}_+ M^z_-. \tag{18}$$

Moreover, it will be proved that the matrices $M^z_\pm$ have a meromorphic continuation to $\mathbb{D}$ [see Eq. (59)]. Assuming that $M^z_\pm$ are invertible, we can rewrite these equations as

$$u^z_+ T^z_+ = u^z_- - u^{1/z}_- R^z_+, \quad u^{1/z}_- T^z_- = u^{1/z}_+ - u^z_+ R^z_-, \tag{19}$$

where

$$T^z_\pm = (M^z_\pm)^{-1}, \quad R^z_\pm = -N^z_\pm (M^z_\pm)^{-1}, \tag{20}$$

are the transmission and reflection coefficients, respectively. The interpretation of (19) in the case that $z = e^{ik} \notin \{-1, 1\}$, corresponding to a wave traveling to the right, is the following (we only describe the first equation in (19)): the incoming wave u^z_- produces the outgoing wave $u^z_+ T^z_+$ traveling to the right (*i.e.*, a transmitted wave) and the outgoing wave $u^{1/z}_- R^z_+$ traveling to the left (*i.e.*, a reflected wave). The relation between transmitted and reflected waves is described by the scattering matrix.

Definition 2 For any $z \in \mathbb{S}^1\backslash\{-1, 1\}$, the scattering matrix $\mathcal{S}^z \in \mathcal{M}_{2L\times 2L}$ is defined by

$$\mathcal{S}^z = \begin{pmatrix} T^z_+ & R^z_- \\ R^z_+ & T^z_- \end{pmatrix} = \begin{pmatrix} (M^z_+)^{-1} & -N^z_-(M^z_-)^{-1} \\ -N^z_+(M^z_+)^{-1} & (M^z_-)^{-1} \end{pmatrix}.$$

Notice that matrices $M^z_\pm$ are indeed invertible, see Proposition 16. In the case that $z = e^{ik}$ represents a wave traveling to the right, then u^z_- and $u^{1/z}_+$ are incoming and u^z_+ and $u^{1/z}_-$ are outgoing. In this case, the scattering matrix expresses the incoming Jost solutions u^z_- and $u^{1/z}_+$ in terms of the outgoing ones u^z_+ and $u^{1/z}_-$:

$$\left(u^z_- \ u^{1/z}_+\right) = \left(u^z_+ \ u^{1/z}_-\right) \mathcal{S}^z . \tag{21}$$

1.2 Main Results

The following theorem is proved in Theorem 32 below.

Theorem 3 *There is a neighborhood of* 1 *such that, for every z in this neighborhood with $z \in \overline{\mathbb{D}}$, the matrices $M^z_\pm$ are invertible. Moreover, the limits*

$$T^1_\pm := \lim_{z \to 1} T^z_\pm \tag{22}$$

exist, where the limits are taken in $\overline{\mathbb{D}}$, and they have explicit expressions (see (106)). The kernels and images of $T^1_\pm$ can be explicitly calculated (see (107)) and are tightly connected to half-bound states. The limits

$$R^1_\pm := \lim_{z \to 1} R^z_\pm \tag{23}$$

of the reflection coefficients $R^z_\pm$ exist, where the limits are taken in $\mathbb{S}^1$, and they have explicit expressions (see (109)). The kernels and images of $\mathbf{1} - R^1_\pm$ can be explicitly calculated (see (110)).

Remark 4 Theorem 3 is also valid if one takes the limits $z \to -1$, and the proofs are the same. In order to simplify notations, we focus on the case $z \to 1$.

2 Solutions

2.1 Free Solutions with Prescribed Data on 0 and 1

Definition 5 For every $z \in \overline{\mathbb{D}} \backslash \{0\}$, we denote by s^z and τ^z the scalar solutions $s^z, \tau^z \in \mathbb{C}^{\mathbb{Z}}$ of (9) such that $s^z(0) = 0$, $s^z(1) = 1$ and $\tau^z(0) = 1 = \tau^z(1)$.

Explicitly, one can verify that

$$s^z(n) = \begin{cases} \frac{1}{z - z^{-1}}(z^n - z^{-n}) = \frac{z}{z+1} \sum_{j=-n}^{n-1} z^j, & z^2 \neq 1 \\ (\pm 1)^{n+1} n, & z = \pm 1 \end{cases} \tag{24}$$

and (for $z \neq -1$)

$$\tau^z(n) = \frac{z^n + z^{-n+1}}{z+1}. \tag{25}$$

Let us introduce the notation $D(w; r) := \{z \in \mathbb{C} \ : \ |z - w| < r\}$.

Lemma 6 *For all $z \in \overline{\mathbb{D}} \cap D(1; 1/2)$, the following holds true:*

(i)

$$|s^z(n)| \le C|n||z|^{-|n|}, \quad \frac{|s^z(n)-s^1(n)|}{|z-1|} \le Cn^2|z|^{-|n|}, \quad s^z(n)-s^1(n) = O(|z-1|^2), \tag{26}$$

as z tends to 1, where the O symbol does depend on n, but C does not.

(ii)

$$\left|\frac{s^z(n-j)-s^1(n-j)-(s^z(n)-s^1(n))}{z-1}\right| \le C|nj||z|^{-|n|}. \tag{27}$$

(iii)

$$|\tau^z(n)-\tau^1(n)| = O(|z-1|^2), \tag{28}$$

where the O symbol depends on n.

(iv)

$$|\tau^z(n)| \le C|z|^{-|n|}, \quad \left|\frac{\tau^z(n)-\tau^1(n)}{z-1}\right| \le C|n||z|^{-|n|}. \tag{29}$$

Proof Since

$$s^z(n) = \frac{z}{z+1}\sum_{j=-n}^{n-1} z^j, \tag{30}$$

the left bound in (26) is obvious. Equation (30) implies that $s^z(n)$ is analytic and its derivative is uniformly bounded by a constant times $n^2|z|^{-|n|}$. Then, an application of the mean value theorem yields the middle bound in (26). A direct calculation shows that the derivative of $s^z(n)$ vanishes at $z=1$, and Taylor's theorem implies the right bound in (26).

The derivative of $s^z(n-j)-s^z(n) = \frac{z}{z+1}\left(\sum_{j=-(n-j)}^{n-j-1} z^j - \sum_{j=-n}^{n-1} z^j\right)$ is uniformly bounded by a constant times $|z^{-n}|nj$ and, therefore, the mean value theorem shows (27).

An elementary calculation gives that $\frac{d}{dz}\tau^z(n)|_{z=1} = 0$, and this together with Taylor's theorem implies (28).

The left identity of (29) is obvious from the definition of τ^z. The right identity of (29) follows again from the mean value theorem, since the derivative of τ^z is uniformly bounded by $C|n||z|^{-|n|}$. □

2.2 Jost Solutions

Lemma 7 (Jost Solutions) *The Jost solutions u_+^z, $u_-^{1/z}$ as defined in Definition 1 exist, for every $z \in \overline{\mathbb{D}}\backslash\{0\}$. Moreover, for every n, the functions $u_+^z(n)$, $u_-^{1/z}(n)$ are holomorphic on $\mathbb{D}\backslash\{0\}$ and continuous on $\overline{\mathbb{D}}\backslash\{0\}$. The following Volterra equations are* *satisfied:*

$$u_+^z(n) = z^n \mathbf{1} - \sum_{j=n+1}^{\infty} s^z(j-n)V(j)u_+^z(j), \qquad n \in \mathbb{Z},$$
$$u_-^{1/z}(n) = z^{-n}\mathbf{1} + \sum_{j=-\infty}^{n-1} s^{1/z}(j-n)V(j)u_-^{1/z}(j), \qquad n \in \mathbb{Z}, \tag{31}$$

where s^z is defined in Definition 5.

Proof The result follows from the Theorem 33: Taking $g = \mathbf{1}$, $K^z(n, j) = -z^{j-n}s^z(j-n)V(j)$ and $M(j) = j\|V(j)\|$, we obtain a solution $\tilde{u}_+^z$ to the equation (for $n \in \mathbb{N}$)

$$\tilde{u}_+^z(n) = \mathbf{1} - \sum_{j=n+1}^{\infty} s^z(j-n)V(j)z^{j-n}\tilde{u}_+^z(j). \tag{32}$$

A direct computation using (32) shows that $u_+^z(n) = z^n\tilde{u}_+^z(n)$ solves the Schrödinger equation $u_+^z(n-1) + V(n)u_+^z(n) + u_+^z(n+1) = (z + 1/z)u_+^z(n)$, for $n \geq 2$ (this is proved in Lemma 36). For $n \in \mathbb{Z}^- \cup \{0\}$, we recursively fit Eq. (8) defining : $u_+^z(n-1) = (z + 1/z)u_+^z(n) - V(n)u_+^z(n) - u_+^z(n+1)$. The construction of the other solution is similar. □

Lemma 8 *The solutions $v_\pm^1$ introduced in Definition 1 exist.*

Proof Theorem 33 implies that there is a solution $\tilde{v}_+^1$ to the Volterra equation (for $n \geq N$)

$$\tilde{v}_+^1(n) = \mathbf{1} + \frac{1}{n}\sum_{j=N}^{n} j^2 V(j)\tilde{v}_+^1(j) + \sum_{j=n+1}^{\infty} jV(j)\tilde{v}_+^1(j), \tag{33}$$

where $N \in \mathbb{N}$ is such that $\sum_{j=N}^{\infty} j\|V(j)\| < 1/2$. Here we set $g = \mathbf{1}$,

$$K(n, j) = \begin{cases} jV(j), & j \geq n+1 \\ \frac{j^2}{n}V(j), & N \leq j \leq n \end{cases}$$

and $M(j) = j\|V(j)\|$. A direct calculation using (33) shows that $v_+^1(n) = n\tilde{v}_+^1(n)$ solves the Schrödinger equation $v_+^1(n-1) + V(n)v_+^1(n) + v_+^1(n+1) = 2v_+^1(n)$, for $n \geq N+1$ (this is carried out using similar methods as in the proof of Lemma 36). For $n \leq N$, we recursively fit Eq. (8) defining $v_+^1(n-1) = 2v_+^1(n) - V(n)v_+^1(n) - v_+^1(n+1)$. The construction of the other solution is similar. □

A key point of this paper is the study of Jost solutions when the spectral parameter z tends to 1. It turns out to be more accessible to control the behavior of solutions with prescribed data on 0 and 1 as $z \to 1$ in a detailed manner. Via Wronskian identities this ultimately allows to deal with Jost solutions and the behavior of the scattering matrix as $z \to 1$.

2.3 Solutions with Prescribed Data at 0 and 1

Lemma 9 *Let $a, b \in \mathcal{M}_{L\times L}$. For every $z \in \overline{\mathbb{D}}\backslash\{0\}$, the solution Ψ^z of (8) such that $\Psi^z(0) = a$, $\Psi^z(1) = b$ satisfies the following equations: for every $n \in \mathbb{N}$,*

$$\Psi^z(n) = s^z(n)(b-a) + \tau^z(n)a - \sum_{j=1}^{n-1} s^z(n-j)V(j)\Psi^z(j), \tag{34}$$

and for every $n \in \mathbb{Z}^- \cup \{0\}$

$$\Psi^z(n) = s^z(n)(b-a) + \tau^z(n)a + \sum_{j=n+1}^{0} s^z(n-j)V(j)\Psi^z(j). \tag{35}$$

Moreover, for every fixed n, $\Psi^z(n)$ is holomorphic on $\mathbb{D}\backslash\{0\}$ and continuous on $\overline{\mathbb{D}}\backslash\{0\}$.

Proof The result follows from Lemma 35, taking $A = -(z + 1/z)\mathbf{1}$, $B(n) = -V(n)$, $S_1 = s^z$ and $S_2 = \tau^z$. The analyticity and continuity follows from the analyticity and continuity of $s^z(n)$ and $\tau^z(n)$. ◻

Lemma 10 *Let Ψ^z be as in Lemma 9. For all $z \in \overline{\mathbb{D}} \cap D(1; 1/2)$, the estimates*

$$\|\Psi^z(n)\| \le C|n||z|^{-|n|}, \tag{36}$$

$$\|\Psi^z(n) - \Psi^1(n)\| = O(|z-1|^2), \quad z \to 1, \tag{37}$$

hold, where the O symbol depends on a, b and n, but not on z and C depends on a and b, but not on z and n.

Proof Let $n \in \mathbb{N}$, it follows from (34) and Lemma 6 that

$$\begin{aligned}\|\Psi^z(n)\||z|^n\frac{1}{n} &= |z|^n\frac{1}{n}\left(|s^z(n)(b-a) + \tau^z(n)a - \sum_{j=1}^{n-1} s^z(n-j)V(j)\Psi^z(j)|\right)\\ &\le C\left(1 + \sum_{j=1}^{n-1}\|jV(j)\|\,\|\Psi^z(j)\|\,|z|^j\frac{1}{j}\right),\end{aligned} \tag{38}$$

and, therefore, Gronwall's Lemma (see Lemma 34) combined with (4) yields (36). For negative n, the argument is similar, using (35). Now we prove (37) for $n \in \mathbb{N}$, the case of n negative is proved similarly. The proof uses induction on n. By definition,

$\Psi^z(1) - \Psi^1(1) = 0$. Suppose that for every $j < n$, $\Psi^z(j) - \Psi^1(j) = O(|z-1|^2)$. It follows from Lemma 9 that

$$\begin{aligned}\|\Psi^z(n) - \Psi^1(n)\| \;\leq\; &\|s^z(n) - s^1(n)\| \, \|b-a\| + \|\tau^z(n) - \tau^1(n)\| \, \|a\| \\ &+ \sum_{j=1}^{n-1} \|s^z(n-j)V(j)\| \, \|\Psi^z(j) - \Psi^1(j)\| \\ &+ \sum_{j=1}^{n-1} \|s^z(n-j) - s^1(n-j)\| \, \|V(j)\| \, \|\Psi^1(j)\|. \qquad (39)\end{aligned}$$

Equation (39), Lemma 6 and (36) for $z = 1$ imply that there is a function $\iota(n, z) \geq 0$ such that $\iota(n, z) = O(|z-1|^2)$ and

$$\|\Psi^z(n) - \Psi^1(n)\| \leq \iota(n, z) + Cn|z|^{-n} \sum_{j=1}^{n-1} \|V(j)\| \, \|\Psi^z(j) - \Psi^1(j)\|. \qquad (40)$$

Since each element in the sum on the right is $O(|z-1|^2)$, it follows that $\|\Psi^z(n) - \Psi^1(n)\| = O(|z-1|^2)$, and the argument is completed by induction. □

Of particular importance are here solutions with the initial condition $a = u_+^1(0)$, $b = u_+^1(1)$.

Definition 11 For all $z \in \overline{\mathbb{D}} \cap D(1; 1/2)$, the solution described in Lemma 9 with $a = u_+^1(0)$ and $b = u_+^1(1)$ is denoted by $\Phi^z \in (\mathcal{M}_{L\times L})^{\mathbb{Z}}$.

Notice that $\Phi^1 = u_+^1$. In this case, we have that (see Lemma 7)

$$b - a = \sum_{j=1}^{\infty} V(j)u_+^1(j) = \sum_{j=1}^{\infty} V(j)\Phi^1(j). \qquad (41)$$

Let us set

$$d(n) := \sum_{j=n}^{\infty} V(j)\Phi^1(j).$$

It follows from Lemma 9 that for every $n \in \mathbb{N}$,

$$\begin{aligned}\Phi^z(n) = s^z(n)d(n) + \tau^z(n)a - &\sum_{j=1}^{n-1} s^z(n-j)V(j)(\Phi^z(j) - \Phi^1(j)) \\ &- \sum_{j=1}^{n-1} (s^z(n-j) - s^z(n))V(j)\Phi^1(j), \qquad (42)\end{aligned}$$

and for every $n \in \mathbb{Z}^- \cup \{0\}$

$$\Phi^z(n) = s^z(n)(b-a) + \tau^z(n)a + \sum_{j=n+1}^{0} s^z(n-j)V(j)\Phi^z(j), \tag{43}$$

The following result shows that Φ^z satisfies essentially the same bounds as s^z and τ^z given in Lemma 6.

Lemma 12 (Regularity 1) *There is a constant $C \in \mathbb{R}$ independent of n and z such that, for all $z \in \overline{\mathbb{D}} \cap D(1; 1/2)$,*

$$\left\| \frac{\Phi^z(n) - \Phi^1(n)}{z-1} \right\| \le Cn|z|^{-n}, \qquad \forall n \in \mathbb{N}. \tag{44}$$

Proof It follows from (42), that

$$\begin{aligned} &\Phi^z(n) - \Phi^1(n) \\ &\quad = (s^z(n) - s^1(n))d(n) + (\tau^z(n) - \tau^1(n))a - \sum_{j=1}^{n-1} s^z(n-j)V(j)(\Phi^z(j) - \Phi^1(j)) \\ &\qquad - \sum_{j=1}^{n-1} \Big(s^z(n-j) - s^1(n-j) - \big(s^z(n) - s^1(n)\big)\Big) V(j)\Phi^1(j). \end{aligned} \tag{45}$$

Equation (45), Lemma 6, (4), and the fact that $nd(n)$ is uniformly bounded (see (4)) implies that

$$\frac{|z|^n}{n} \left\| \frac{\Phi^z(n) - \Phi^1(n)}{z-1} \right\| \le C\Big[1 + \sum_{j=1}^{n-1} j\|V(j)\| \, \frac{|z|^j}{j} \, \left\| \frac{\Phi^z(j) - \Phi^1(j)}{z-1} \right\| \Big], \tag{46}$$

here we use that $\Phi^1(j) = u_+^1(j)$ is uniformly bounded for $j \ge 0$ (which is a consequence of the definition of the Jost solution in question). Equation (46) and Gronwall's Lemma (see Lemma 34) combined with (4) imply (44). □

Lemma 10 (with Φ^1 playing the role of Ψ^1) implies that the series

$$\sum_{j=-\infty}^{\infty} V(j)\Phi^1(j) = \sum_{j=-\infty}^{\infty} V(j)u_+^1(j), \tag{47}$$

converges to a matrix in $\mathcal{M}_{L\times L}$. This matrix plays an important role in the proofs as it is connected to the Wronskian of u_+^1 and v_-^1, see Lemma 20. In Lemma 28 (see Definition 21), we prove that for every vector ξ that belongs to its kernel, the sequence $(u_+^1(j)\xi)_{j\in\mathbb{Z}}$ is bounded. The corresponding states $(u_+^1(j)\xi)_{j\in\mathbb{Z}}$ are called half-bound states.

Lemma 13 (Regularity 2) *Let ξ belong to the kernel of (47). There is a constant $C \in \mathbb{R}$, independent of n and z, such that, for all $z \in \overline{\mathbb{D}} \cap D(1, 1/2)$,*

$$\left\| \frac{1}{z-1}(\Phi^z(n) - \Phi^1(n))\xi \right\| \leq C|n||z|^n, \qquad \forall n \in \mathbb{Z}^- \cup \{0\}. \tag{48}$$

Proof In this case, (see Eq. (41) and recall that ξ belongs to the kernel of (47))

$$(b-a)\xi = \sum_{j=1}^{\infty} V(j)\Phi^1(j)\xi = -\sum_{j=-\infty}^{0} V(j)\Phi^1(j)\xi. \tag{49}$$

Hence let us set

$$d_-(n) := -\sum_{j=-\infty}^{n} V(j)\Phi^1(j).$$

Due to (4) one has

$$|nd_-(n)| \leq \sum_{j=-\infty}^{n} |j| \, \|V(j)\| \, \|\Phi^1(j)\| < \infty. \tag{50}$$

It follows from Lemma 9 applied to Φ^z and (49) that for every $n \in \mathbb{Z}^- \cup \{0\}$

$$\Phi^z(n)\xi = s^z(n)d_-(n)\xi + \tau^z(n)a\xi + \sum_{j=n+1}^{0} s^z(n-j)V(j)(\Phi^z(j) - \Phi^1(j))\xi$$
$$+ \sum_{j=n+1}^{0} \big(s^z(n-j) - s^z(n)\big)V(j)\Phi^1(j)\xi. \tag{51}$$

Then, we have that

$$\left\| \frac{|z|^{-n}}{n} \frac{1}{z-1}(\Phi^z(n)\xi - \Phi^1(n)\xi) \right\| \leq \frac{|z|^{-n}}{n} \frac{1}{z-1} \Big[\|(s^z(n) - s^1(n))d_-(n)\xi\|$$
$$+ \|(\tau^z(n) - \tau^1(n))a\xi\| + \sum_{j=n+1}^{0} \|s^z(n-j)V(j)(\Phi^z(j) - \Phi^1(j))\xi\|$$
$$+ \sum_{j=n+1}^{0} \|\big(s^z(n-j) - s^1(n-j) - (s^z(n) - s^1(n))\big)V(j)\Phi^1(j)\xi\| \Big]. \tag{52}$$

Bounding the first summand on the right is bounded using the second estimate from (26) and (50), the second summand with (28) and the fourth summand with (27), one deduces

$$\frac{|z|^{-n}}{n}\left\|\frac{\Phi^z(n)-\Phi^1(n)}{z-1}\xi\right\| \leq C\Big[1+\sum_{j=n+1}^{0} j\|V(j)\|\,\frac{|z|^{-j}}{j}\,\left\|\frac{\Phi^z(j)-\Phi^1(j)}{z-1}\xi\right\|\Big], \tag{53}$$

because $\Phi^1(j)\xi$ is uniformly bounded (see Lemma 28 and Definition 21). Equation (53) and Gronwall's (see Lemma 34) imply (48). □

3 The Scattering Matrix

Definition 14 *(Wronskian)* For two functions $u, v : \mathbb{Z} \to \mathcal{M}_{L\times L}$ and $n \in \mathbb{Z}$, the Wronskian is defined by

$$W(u, v)(n) \;=\; i\,\big(u(n+1)^*v(n) \;-\; u(n)^*v(n+1)\big) \;\in \mathcal{M}_{L\times L}\,. \tag{54}$$

It is easy to see that $W(u, v)^* = W(v, u)$ and that for matrix solutions satisfying $Hu = Eu$ and $Hv = \overline{E}v$, the Wronskian $W(u, v)$ is independent of n. In these cases we omit the argument n. For $z \in \overline{\mathbb{D}}$, the Wronskians of the Jost solutions can be evaluated using that they are independent of n: taking the limits, either $n \to \infty$ or $n \to -\infty$, we find that

$$W(u_+^{\overline{z}}, u_+^z) = 0 = W(u_-^{1/\overline{z}}, u_-^{1/z}), \tag{55}$$

and if, additionally, $z \in \mathbb{S}^1\backslash\{-1, 1\}$

$$W(u_\pm^z, u_\pm^z) \;=\; (\nu^z)^{-1}\,\mathbf{1}\,, \tag{56}$$

where

$$\nu^z \;=\; \frac{i}{z-z^{-1}}\,. \tag{57}$$

Next recall that for $z \in \mathbb{S}^1\backslash\{-1, 1\}$, it is possible to decompose the states u_+^z and $u_-^{1/z}$ on the basis $(u_-^z, u_-^{1/z})$ and $(u_+^z, u_+^{1/z})$, respectively, and that the matrices $M_\pm^z$ and $N_\pm^z$ are defined by

$$u_+^z \;=\; u_-^z M_+^z \;+\; u_-^{1/z}N_+^z\,, \qquad u_-^{1/z} \;=\; u_+^z N_-^z \;+\; u_+^{1/z}M_-^z\,. \tag{58}$$

Equation (56) leads to

$$\begin{aligned} M_+^z &= \nu^z\, W(u_-^{1/\overline{z}}, u_+^z)\,,\\ N_+^z &= -\,\nu^z\, W(u_-^{1/z}, u_+^z)\,,\\ N_-^z &= \nu^z\, W(u_+^z, u_-^{1/z})\,,\\ M_-^z &= -\,\nu^z\, W(u_+^{\overline{z}}, u_-^{1/z})\,. \end{aligned} \tag{59}$$

This shows that $M^z_\pm$ can be extended to analytic functions on $\mathbb{D}\backslash\{0\}$ which are continuous on $\overline{\mathbb{D}}\backslash\{-1, 0, 1\}$. Equations (59) imply that

$$(N^z_+)^* \ = -N^z_- \ , \qquad (M^z_+)^* \ = \ M^{\overline{z}}_- \ , \tag{60}$$

where the first equation holds true for $z \in \mathbb{S}^1\backslash\{-1, 1\}$ and the second can be extended to $\overline{\mathbb{D}}\backslash\{-1, 0, 1\}$.

Lemma 15 *For every $z \in \mathbb{S}^1\backslash\{-1, 1\}$, the following identities hold true:*

$$(M^z_-)^* M^z_- \ = \ \mathbf{1} \ + \ (N^z_-)^* N^z_- \ , \tag{61}$$

$$M^z_+ N^z_- \ = \ - N^{1/z}_+ M^z_- \ , \tag{62}$$

$$(M^z_+)^* M^z_+ \ = \ \mathbf{1} \ + \ (N^z_+)^* N^z_+, \tag{63}$$

$$M^z_- N^z_+ \ = \ - N^{1/z}_- M^z_+ \ . \tag{64}$$

Proof We notice that for every matrix $M \in \mathcal{M}_{L\times L}$ and every solutions u, v,

$$\begin{aligned} W(uM, (v + w)) &= M^*(W(u, v) + W(u, w)),\\ W(u + v, wM) &= (W(u, w) + W(v, w))M. \end{aligned} \tag{65}$$

First we prove (63). It follows from Eqs. (56) and (58) that

$$(\nu^z)^{-1}\mathbf{1} = W(u^z_+, u^z_+) = W(u^z_- M^z_+ \ + \ u^{1/z}_- N^z_+, u^z_- M^z_+ \ + \ u^{1/z}_- N^z_+). \tag{66}$$

Expanding the right hand side of (66) and using Eqs. (55), (56) and (65), we get

$$(\nu^z)^{-1}\mathbf{1} = (\nu^z)^{-1}(M^z_+)^* M^z_+ - (\nu^z)^{-1}(N^z_+)^* N^z_+,$$

where $\nu^{1/z} = -\nu^z$ was used. This implies (63). Equation (61) is obtained in similar manner by expanding $W(u^z_-, u^z_-)$. Now let us prove (64). It follows from Eqs. (55) and (58) that

$$0 = W(u^{1/z}_+, u^z_+) = W(u^{1/z}_- M^{1/z}_+ \ + \ u^z_- N^{1/z}_+, u^z_- M^z_+ \ + \ u^{1/z}_- N^z_+). \tag{67}$$

Expanding the right hand side of (67) and using Eqs. (55), (56) and (65), we get

$$\begin{aligned} 0 &= -(\nu^z)^{-1}(M^{1/z}_+)^* N^z_+ + (\nu^z)^{-1}(N^{1/z}_+)^* M^z_+\\ &= -(\nu^z)^{-1} M^z_- N^z_+ - (\nu^z)^{-1} N^{1/z}_- M^z_+, \end{aligned}$$

where the last equality follows from (60). Equation (62) is obtained in similar manner expanding $W(u^{1/z}_-, u^z_-)$. □

Proposition 16 *For $z \in \mathbb{S}^1\backslash\{-1, 1\}$, $M^z_\pm$ is invertible and $\mathcal{S}^z$ is unitary*

Proof The invertibility of $M^z_\pm$ follows from Eqs. (61) and (63). Now we prove the unitarity. The off diagonal terms of $(\mathcal{S}^z)^*\mathcal{S}^z$ are (see Definition 2)

$$-((M^z_+)^{-1})^* N^z_-(M^z_-)^{-1} - ((M^z_+)^{-1})^*(N^z_+)^*(M^z_-)^{-1}, \tag{68}$$
$$-((M^z_-)^{-1})^*(N^z_-)^*(M^z_+)^{-1} - ((M^z_-)^{-1})^* N^z_+(M^z_+)^{-1} \tag{69}$$

and they vanish by (60). The diagonal terms are

$$\begin{aligned}&((M^z_+)^{-1})^*(1+(N^z_+)^*N^z_+)(M^z_+)^{-1},\\ &((M^z_-)^{-1})^*(1+(N^z_-)^*N^z_-)(M^z_-)^{-1},\end{aligned} \tag{70}$$

and they are both equal to $\mathbf{1}$, see (63) and (61). This proves the unitary of $\mathcal{S}^z$. □

4 Analysis of the Wronskian

It follows from (59) and Definition 2 that the Wronskian is tightly connected to the scattering matrix. In particular, the invertibility of $M^z_\pm$ is essential for its definition. Since the purpose of the present paper is the analysis of the scattering matrix as z tends to 1, it is crucial to study the behavior of $W(u^{1/\bar z}_-, u^z_+) = (\nu^z)^{-1}M^z_+$ as z tends to 1 (the study of M^z_- is carried out using (60)). Regularity properties of u^z_+ as z tends to 1 are thus relevant. As stated above, this will be deduced from the regularity results on Φ^z. Indeed, it turns out that these properties of Φ^z allow to identify lower order terms of $W(u^{1/\bar z}_-, u^z_+)$ with respect to $|z-1|$. This holds because $W(u^{1/\bar z}_-, u^z_+)$ can be written in terms of $W(\Phi^{\bar z}, u^z_+)$ and $W(u^{1/\bar z}_-, \Phi^z)$ (see Lemma 18). Then, most of this section is devoted to the study of $W(\Phi^{\bar z}, u^z_+)$ and $W(u^{1/\bar z}_-, \Phi^z)$.

From the definition of u^1_+, we know that $u^1_+(j)$ tends to $\mathbf{1}$ as j tends to infinity. Then, for large enough j, $u^1_+(j)$ is invertible. In order to simplify notations, we assume that $u^1_+(1)$ is already invertible. This does not imply any restriction because, translating the origin, we can always take it for granted.

Assumption 17 We assume, without loss of generality, that $u^1_+(1)$ is invertible.

Lemma 18 *For every $z \in \overline{\mathbb{D}}\backslash\{0\}$, it follows that*

$$W(u^{1/\bar z}_-, u^z_+) = u^{1/\bar z}_-(1)^*(u^1_+(1)^*)^{-1}W(\Phi^{\bar z}, u^z_+) + W(u^{1/\bar z}_-, \Phi^z)u^1_+(1)^{-1}u^z_+(1). \tag{71}$$

Proof The result follows from an expansion of the right hand side of (71) using Definition 11 of Φ^z and the definition of the Wronskians, evaluated on $n=0$, and the identity (see (55))

$$(u^1_+(1)^*)^{-1}u^1_+(0)^* = u^1_+(0)u^1_+(1)^{-1}.$$

Notice that, by definition, $\Phi^z(j) = u^1_+(j)$, for $j \in \{0, 1\}$. □

Proposition 19 *The following formula holds true*

$$W(\Phi^{\overline{z}}, u_+^z) = i(1-z)\mathbf{1} + o(|z-1|), \tag{72}$$

as z *tends to* 1 *in* $\overline{\mathbb{D}}$.

Proof Equation (34) for $\Psi^z = \Phi^z$ together with the bounds from Lemma 6 imply that for $n \in \mathbb{N}$

$$|z|^n \|\Phi^z(n)\| \leq C_z + \sum_{j=1}^{n-1} C_z |z|^j \, \|V(j)\| \, \|\Phi^z(j)\|, \tag{73}$$

for some positive function C_z depending on z (that blows up as z tends to 1 because in Eq. (24) for s^z a cancellation of the factor $\frac{1}{1-z}$ implies growth in n). Then, Gronwall's lemma implies that $|z|^n|\Phi^z(n)|$ is bounded (with respect to n, for positive n). This implies that (see Eq. (15))

$$\begin{aligned} W(\Phi^{\overline{z}}, u_+^z) &= \lim_{n\to\infty} i(\Phi^{\overline{z}}(n+1)^* z^n - \Phi^{\overline{z}}(n)^* z^{n+1}) \\ &= i(b-a)^* - i(z-1)a^* - i\sum_{j=1}^{\infty} z^j \Phi^{\overline{z}}(j)^* V(j), \end{aligned}$$

where in the second step we used (34) and Definition 5 of s^z and τ^z. On the other hand,

$$\sum_{j=1}^{\infty} z^j (\Phi^{\overline{z}} - \Phi^1)(j)^* V(j) = o(|1-z|). \tag{74}$$

Indeed, Eqs. (44) and (4) imply that the series multiplied by $\frac{1}{1-z}$ is bounded by a summable function that does not depend on z, and (37) implies that each term of the sum multiplied by $\frac{1}{1-z}$ tends to zero. Hence interpreting the series as an integral with respect to a counting measure, Lebesgue's dominated convergence theorem shows (74). Furthermore, since Φ^1 is bounded for $j > 0$ (because $\Phi^1(j) = u_+^1(j)$, for $j > 1$) it follows that

$$\sum_{j=1}^{\infty} (z^j - 1 - j(z-1))\Phi^1(j)^* V(j) = o(|1-z|). \tag{75}$$

Then, we obtain that

$$\begin{aligned} W(\Phi^{\overline{z}}, u_+^z) = {} & i(b-a)^* - i(z-1)a^* - i \\ & \sum_{j=1}^{\infty} (1 + j(z-1))\Phi^1(j)^* V(j) + o(|1-z|). \end{aligned}$$

This last equation and the fact that (see (31))

$$(b-a)=\sum_{j=1}^{\infty} V(j)u_+^1(j)=\sum_{j=1}^{\infty} V(j)\Phi^1(j),$$
$$a=\mathbf{1}-\sum_{j=1}^{\infty} jV(j)u_+^1(j)=\mathbf{1}-\sum_{j=1}^{\infty} jV(j)\Phi^1(j), \tag{76}$$

imply the desired result. □

Lemma 20 *The following formula holds true*

$$W(u_-^1,u_+^1)=-i\sum_{j=-\infty}^{\infty} V(j)u_+^1(j). \tag{77}$$

Proof It follows from Lemma 7 that

$$u_+^1(n)=\mathbf{1}-\sum_{j=n+1}^{\infty}(j-n)V(j)u_+^1(j), \qquad n\in\mathbb{Z},$$
$$u_-^1(n)=\mathbf{1}+\sum_{j=-\infty}^{n-1}(j-n)V(j)u_-^1(j), \qquad n\in\mathbb{Z}. \tag{78}$$

Since $|jV(j)|$ tends to zero as j tends to minus infinity (see (4)) and there is a constant C such that $|u_+^1(j)|\le C|j|$, for $j\le 0$, (see Lemma 10), it follows that for $n\to\infty$

$$\begin{aligned}
&W(u_-^1,u_+^1)(n)\\
&=i\left(\mathbf{1}+\sum_{j=-\infty}^{n}(j-n-1)u_-^1(j)^*V(j)\right)\left(\mathbf{1}-\sum_{j=n+1}^{\infty}(j-n)V(j)u_+^1(j)\right)\\
&\quad-i\left(\mathbf{1}+\sum_{j=-\infty}^{n-1}(j-n)u_-^1(j)^*V(j)\right)\left(\mathbf{1}-\sum_{j=n+2}^{\infty}(j-n-1)V(j)u_+^1(j)\right)\\
&=-i\sum_{j=n+2}^{\infty}V(j)u_+^1(j)+i\sum_{j=-\infty}^{n}u_-^1(j)^*V(j)\sum_{j=n+1}^{\infty}(j-n)V(j)u_+^1(j)+o(1),
\end{aligned}$$

Here, we used that $u_\pm^1(\pm j)$ is bounded for $j\in\mathbb{N}$ and that $\lim_{m\to\infty}\sum_{j=-\infty}^{-m}|j|\|V(j)\| + \sum_{j=m}^{\infty}|j|\|V(j)\|=0$. Taking the limit $n\to-\infty$ yields the desired result. □

We define some technical objects that will be used in Sect. 5.1.

Definition 21 We introduce the notations

$$\mathcal{N}:=\mathrm{Ker}(W(u_-^1,u_+^1)),\quad \mathcal{L}:=\mathrm{Ker}(W(u_+^1,u_-^1)). \tag{79}$$

The generic case is referred as $\mathcal{N} = \{0\}$, otherwise one speaks of the exceptional case. In Lemma 28, a characterization of $\mathcal{N}$ and $\mathcal{L}$ in terms of half-bound states is presented, in particular we prove that $(u_+^1(j)\xi)_{j\in\mathbb{Z}}$ is bounded, for every $\xi \in \mathcal{N}$ (*i.e.*, it is a half-bound state, see Sect. 5.1). For $\xi \in \mathcal{N}$, let us define

$$\Gamma\xi := \left(\xi - \sum_{j=-\infty}^{\infty} jV(j)u_+^1(j)\xi\right). \tag{80}$$

In Lemma 29 another characterization of Γ is given and in Lemma 30 it is shown that Γ is a bijection from $\mathcal{N}$ onto $\mathcal{L}$. We denote by

$$P_{\mathcal{N}}, \quad P_{\mathcal{L}}, \quad P_{\mathcal{N}^\perp}, \quad P_{\mathcal{L}^\perp} \tag{81}$$

the orthogonal projections onto $\mathcal{N}$, $\mathcal{L}$, $\mathcal{N}^\perp$ and $\mathcal{L}^\perp$ respectively.

Proposition 22 *The following formula holds true*

$$W(u_-^{1/\bar{z}}, \Phi^z) = W(u_-^1, u_+^1) + o(1), \tag{82}$$

as z tends to 1 in $\overline{\mathbb{D}}$. Moreover, if $\xi \in \mathcal{N}$, then

$$W(u_-^{1/\bar{z}}, \Phi^z)\xi = i(1-z)\Gamma\xi + o(|z-1|), \tag{83}$$

as z tends to 1 in $\overline{\mathbb{D}}$.

Proof Equation (82) follows from the continuity of the functions $z \mapsto u_-^{1/\bar{z}}$ and $z \mapsto \Phi^z$, and the fact that $\Phi^1 = u_+^1$. Now we prove (83). Equation (35) implies that

$$|z^{-n}||\Phi^z(n)| \le C_z + \sum_{j=n+1}^{0} C_z|z^{-j}|\,\|V(j)\|\,\|\Phi^z(j)\|, \tag{84}$$

for some positive function C_z of z (that blows up as z tends to 1). Then, Gronwall's lemma implies that $|z^{-n}||\Phi^z(n)|$ is bounded (with respect to n, for $n \le 0$). This implies that (see Eq. (15))

$$\begin{aligned} W(u_-^{1/\bar{z}}, \Phi^z) &= \lim_{n\to-\infty} i(z^{-n-1}\Phi^z(n) - z^{-n}\Phi^z(n+1)) \\ &= -i(b-a) + i\frac{1-z}{z}a - i\sum_{j=-\infty}^{0} z^{-j}V(j)\Phi^z(j), \end{aligned}$$

where (35) was used. Utilizing (37), (48) and (4) we deduce that (here we use again Lebesgue's dominated convergence theorem)

$$\sum_{j=-\infty}^{0} z^{-j} V(j)(\Phi^z(j) - \Phi^1(j))\xi = o(|1-z|). \tag{85}$$

Since $\Phi^1(j)\xi$ is bounded (see Lemma 28), it follows that for $j \leq 0$

$$\sum_{j=-\infty}^{0} (z^{-j} - 1 - j(1-z))V(j)\Phi^1(j)\xi = o(|1-z|). \tag{86}$$

Then

$$W(u_-^{1/\bar{z}}, \Phi^z)\xi = -i(b-a)\xi + i\frac{1-z}{z}a\xi - i\sum_{j=-\infty}^{0}(1 + j(1-z))V(j)\Phi^1(j)\xi + o(|1-z|).$$

The desired result from this last equation and (76). Notice that we use that $\frac{1-z}{z} - (1-z) = o(|1-z|)$ and that by assumption (see Lemma 20 and Definition 21) $\sum_{j=-\infty}^{\infty} V(j)\Phi^1(j)\xi = 0$ and recall that $u_+^1 = \Phi^1$. □

The operator $u_-^{1/\bar{z}}(1)^*(u_+^1(1)^*)^{-1}$ that appears in (71) plays an important role because, when $z = 1$, it operates on half-bound states (see Remark 31) and it is present in the scattering matrix, in the limit when z tends to 1. For this reason, we also introduce a notation for this object.

Definition 23 We denote

$$\Omega := u_+^1(1)^{-1}u_-^1(1) \in \mathcal{M}_{L\times L}. \tag{87}$$

A different characterization of Ω is given in Remark 31.

Proposition 24 *There exist functions $X, Y : \mathbb{D} \to \mathcal{M}_{L\times L}$ such that*

$$W(u_-^{1/\bar{z}}, u_+^z) = \Big(i(1-z)(\Omega^* + \Gamma)P_{\mathcal{N}} + W(u_-^1, u_+^1)P_{\mathcal{N}^\perp} + X(z) + Y(z)P_{\mathcal{N}^\perp}\Big) \cdot u_+^1(1)^{-1}u_+^z(1), \tag{88}$$

and $X(z) = o(|z-1|)$ and $Y(z) = o(1)$, as z tends to 1 in $\overline{\mathbb{D}}$.

Proof The result follows from the continuity of $u_-^{1/\bar{z}}(1)$ and $u_+^z(1)$, Propositions 19 and 22 and Lemma 18. □

5 Proof of the Main Result

5.1 Half-Bound States

In this section we study half-bound states at the threshold energy corresponding to the spectral parameter $z = 1$. They are solutions of (8) with $E = 2$ that are bounded. These solutions play a fundamental role in the limit of the scattering matrix as z tends to 1.

Lemma 25 *Let $u \in (\mathbb{C}^L)^{\mathbb{Z}}$ be a solution of (8), for $z = 1$. The following items are equivalent:*

(i) u is $o(n)$ for $n \to +\infty$.
(ii) u is bounded as n tends to $+\infty$.
(iii) u converges as n tends to $+\infty$.

Moreover, u is $o(1)$ for $n \to +\infty$, if and only if $u = 0$.

Proof It follows from Eqs. (15) and (17) that the columns of u_+^1 and v_+^1 form a basis of all solutions. Then, there are $\alpha, \beta \in \mathbb{C}^L$ such that

$$u = v_+^1 \alpha + u_+^1 \beta.$$

The asymptotic behavior of u_+^1 and v_+^1 at ∞ yields the desired result. □

Remark 26 The previous lemma remains valid if we replace $+\infty$ by $-\infty$.

Lemma 27 *The next equations hold true:*

$$u_+^1(n) = n(iW(u_-^1, u_+^1) + o(1)), \qquad n \to -\infty.$$
$$u_-^1(n) = n(-iW(u_+^1, u_-^1) + o(1)), \qquad n \to +\infty.$$

Proof It was proved in Lemma 20 that

$$W(u_-^1, u_+^1) = -i \sum_{j=-\infty}^{\infty} V(j) u_+^1(j), \tag{89}$$

and, similarly we deduce that

$$W(u_+^1, u_-^1) = i \sum_{j=-\infty}^{\infty} V(j) u_-^1(j). \tag{90}$$

Equation (31) implies that

$$u_+^1(n+1) - u_+^1(n) = -\sum_{j=n+2}^{\infty} (j-n-1) V(j) u_+^1(j) + \sum_{j=n+1}^{\infty} (j-n) V(j) u_+^1(j)$$

$$
\begin{aligned}
&= \sum_{j=n+1}^{\infty} V(j)u_+^1(j) \\
&= iW(u_-^1, u_+^1) - \sum_{j=-\infty}^{n} V(j)u_+^1(j).
\end{aligned}
$$

Thus, $u_+^1(n+1) - u_+^1(n) \to iW(u_-^1, u_+^1)$, as $n \to -\infty$ (recall that $|u_+^1(n)| \leq C|n|$ due to Eq. (36)). This implies that

$$
\frac{1}{n}(u_+^1(n+1) - u_+^1(1)) = \frac{1}{n}\sum_{j=1}^{n} u_+^1(j+1) - u_+^1(j) \to iW(u_-^1, u_+^1)
$$

and, therefore,

$$
\frac{u_+^1(n)}{n} \to iW(u_-^1, u_+^1), \qquad n \to \infty.
$$

This proves the first equality. The proof of the second is similar. □

The next result establishes a connection between the subspace $\mathcal{N}$ and $\mathcal{L}$ introduced in Definition 21 and the half-bound states.

Lemma 28 (Half-Bound States) *The next equations hold true:*

$$
\mathcal{N} = \{\xi \in \mathbb{C}^L : u_+^1\xi \textit{ is bounded}\}, \quad \mathcal{L} = \{\chi \in \mathbb{C}^L : u_-^1\chi \textit{ is bounded}\}. \tag{91}
$$

Moreover, since $W(u_-^1, u_+^1)^* = W(u_+^1, u_-^1)$ by definition, it follows that

$$
\mathcal{N} = (W(u_+^1, u_-^1)\mathbb{C}^L)^\perp, \quad \mathcal{L} = (W(u_-^1, u_+^1)\mathbb{C}^L)^\perp, \tag{92}
$$

and, therefore, $\dim(\mathcal{N}) = \dim(\mathcal{L})$.

Proof Take $\xi \in \mathcal{N}$, then Lemmas 25 and 27 yield that $u_+^1\xi$ is bounded. Now taking $\xi \in \mathbb{C}^L$ such that $u_+^1\xi$ is bounded, Lemma 27 implies that

$$
0 = \lim_{n\to-\infty} \frac{u_+^1(n)\xi}{n} = iW(u_-^1, u_+^1)\xi.
$$

Therefore the first equality follows. The proof of the second equality is similar. □

Lemma 29 *For every $\xi \in \mathcal{N}$,*

$$
\Gamma\xi = \lim_{n\to-\infty} u_+^1(n)\xi. \tag{93}
$$

Proof Let $\xi \in \mathcal{N} = \mathrm{Ker}(W(u_-^1, u_+^1))$, we calculate using Lemma 20 and Eq. (31):

$$\begin{aligned}
\xi - u_+^1(n)\xi &= \sum_{j=n+1}^{\infty} (j-n)V(j)u_+^1(j)\xi \\
&= \sum_{j=n+1}^{\infty} jV(j)u_+^1(j)\xi - n \sum_{j=n+1}^{\infty} V(j)u_+^1(j)\xi \\
&= \sum_{j=n+1}^{\infty} jV(j)u_+^1(j)\xi - n \left(iW(u_-^1, u_+^1) - \sum_{j=-\infty}^{n} V(j)u_+^1(j) \right) \xi \\
&= \sum_{j=n+1}^{\infty} jV(j)u_+^1(j)\xi + n \sum_{j=-\infty}^{n} V(j)u_+^1(j)\xi.
\end{aligned}$$

Notice that $u_+^1(j)\xi$ is bounded because $\xi \in \mathcal{N}$ (see Lemma 28). From the last equation and definition of Γ (see Definition 21), we have that (see also (4))

$$\begin{aligned}
\lim_{n\to-\infty} u_+^1(n)\xi &= \xi - \lim_{n\to-\infty} \sum_{j=n+1}^{\infty} (j-n)V(j)u_+^1(j)\xi \\
&= \xi - \sum_{j=-\infty}^{\infty} jV(j)u_+^1(j)\xi = \Gamma\xi,
\end{aligned} \tag{94}$$

completing the proof. □

Lemma 30 *Γ is a linear isomorphism from $\mathcal{N}$ to $\mathcal{L}$.*

Proof Taking $\xi \in \mathcal{N}$ and $\chi = \Gamma\xi$, it follows from Lemma 29 and Eq. (15) that

$$\lim_{n\to-\infty} u_+^1(n)\xi - u_-^1(n)\chi = 0.$$

Then, Remark 26 implies that

$$u_+^1(n)\xi = u_-^1(n)\chi, \; n \in \mathbb{Z}. \tag{95}$$

We deduce that $u_-^1\chi$ is bounded and, therefore, $\chi \in \mathcal{L}$ and it follows that $\Gamma\mathcal{N} \subset \mathcal{L}$. Let $\chi \in \mathcal{L}$ and $\xi = \lim_{n\to\infty} u_-^1(n)\chi$. As above, we obtain that $u_+^1(n)\xi = u_-^1(n)\chi$ for $n \in \mathbb{Z}$ and, therefore, $\Gamma\xi = \chi$. This proves the subjectivity and as $\mathcal{L}$ and $\mathcal{N}$ have the same dimension, that Γ is bijective. □

Remark 31 It follows from (95) that

$$\Gamma = (u_-^1(n)^{-1}u_+^1(n))\big|_{\mathcal{N}}, \tag{96}$$

whenever $u_-^1(n)^{-1}$ exists. As explained above it can be assumed without loss of generality that this is the case when $n = 1$. Let us also recall that

$$\Omega = u_+^1(1)^{-1} u_-^1(1), \tag{97}$$

and notice that

$$\Omega\big|_{\mathcal{L}} = \Gamma^{-1}. \tag{98}$$

5.2 Band Edge Limit of the Scattering Matrix

Let $\{e_1, \ldots, e_L\}$ be an orthonormal basis of $\mathbb{C}^L$ such that the first d vectors form a basis of $\mathcal{L}$ and the last $L-d$ vectors form a basis of $\mathcal{L}^\perp = W(u_-^1, u_+^1)\mathbb{C}^L$, see Lemma 28. We take another orthonormal basis $\{v_1, \ldots, v_L\}$ of $\mathbb{C}^L$ such that the first d vectors form a basis of $\mathcal{N}$ and the last $L-d$ vectors form a basis of $\mathcal{N}^\perp$.

Then define

$$P := \begin{pmatrix} e_1 & e_2 & \cdots & e_L \end{pmatrix}^*, \quad Q := \begin{pmatrix} v_1 & v_2 & \cdots & v_L \end{pmatrix}. \tag{99}$$

We recall that $P_{\mathcal{L}}$ and $P_{\mathcal{L}^\perp}$ are the projections onto $\mathcal{L}$ and $\mathcal{L}^\perp$, respectively. Then

$$P_{\mathcal{L}}\Big(i(1-z)(\Omega^* + \Gamma)P_{\mathcal{N}} + W(u_-^1, u_+^1)P_{\mathcal{N}^\perp}\Big)P_{\mathcal{N}} = i(1-z)P_{\mathcal{L}}(\Omega^* + \Gamma)P_{\mathcal{N}} \tag{100}$$

and $P_{\mathcal{L}}(\Omega^* + \Gamma)P_{\mathcal{N}}$ defines a bijection between $\mathcal{N}$ and $\mathcal{L}$ (see Lemma 28): in view of Lemma 28 it is enough to prove that it is injective. This holds true because $\Gamma : \mathcal{N} \to \mathcal{L}$ is a bijection and, for every $\xi \in \mathcal{N}$, (see Eq. (98))

$$\begin{aligned} \langle \Gamma\xi, P_{\mathcal{L}}(\Omega^* + \Gamma)\xi\rangle &= \langle P_{\mathcal{L}}\Gamma\xi, (\Omega^* + \Gamma)\xi\rangle = \langle \Gamma\xi, (\Omega^* + \Gamma)\xi\rangle \\ &= \langle \Omega\Gamma\xi, \xi\rangle + \|\Gamma\xi\|^2 = \|\xi\|^2 + \|\Gamma\xi\|^2. \end{aligned}$$

Moreover,

$$P_{\mathcal{L}^\perp}\Big(i(1-z)(\Omega^* + \Gamma)P_{\mathcal{N}} + W(u_-^1, u_+^1)P_{\mathcal{N}^\perp}\Big)P_{\mathcal{N}^\perp} = P_{\mathcal{L}^\perp}W(u_-^1, u_+^1)P_{\mathcal{N}^\perp} \tag{101}$$

defines a bijection between $\mathcal{N}^\perp$ and $\mathcal{L}^\perp$ (see Lemma 28 and Definition 21). It follows that there are matrix-valued functions $A(z)$, $B(z)$, $C(z)$, $D(z)$ such that

$$P\Big(i(1-z)(\Omega^* + \Gamma)P_{\mathcal{N}} + W(u_-^1, u_+^1)P_{\mathcal{N}^\perp} + X(z) + Y(z)P_{\mathcal{N}^\perp}\Big)Q = \begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \tag{102}$$

(recall the definitions of X and Y in Proposition (24)) and

$$A(z) = i(1-z)\boldsymbol{A} + o(|1-z|), \quad B(z) = o(1),$$

$$C(z) = O(|1-z|), \quad D(z) = \boldsymbol{D} + o(1), \tag{103}$$

where

$$\boldsymbol{A} := [P_{\mathcal{L}}(\Omega^* + \Gamma)P_{\mathcal{N}}]^{\beta}_{\alpha}, \quad \boldsymbol{D} := [P_{\mathcal{L}^\perp} W(u^1_-, u^1_+)P_{\mathcal{N}^\perp}]^{\delta}_{\gamma}, \tag{104}$$

are invertible where $[T]^{\eta}_{\theta}$ denotes the matrix representation of a linear transformation T in terms of the bases θ, η. Here $\alpha = \{v_1, \dots, v_d\}$, which is a basis of $\mathcal{N}$, $\beta = \{e_1, \dots, e_d\}$, which is a basis of $\mathcal{L}$, and $\gamma = \{v_{d+1}, \dots, v_L\}$ and $\delta = \{e_{d+1}, \dots, e_L\}$, which are bases of $\mathcal{N}^\perp$ and $\mathcal{L}^\perp$, respectively.

Theorem 32 *The transmission coefficients satisfy the following properties:*

$$T^z_+ = T^1_+ + o(1), \quad T^z_- = T^1_- + o(1), \tag{105}$$

as z tends to 1 *in $\overline{\mathbb{D}}$, where*

$$T^1_+ := Q\begin{pmatrix} 2\boldsymbol{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} P, \quad T^1_- := \left(Q\begin{pmatrix} 2\boldsymbol{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} P\right)^*. \tag{106}$$

Moreover,

$$\begin{aligned} T^1_+\mathbb{C}^L &= \mathcal{N}, \quad \mathrm{Ker}(T^1_+) = W(u^1_-, u^1_+)\mathbb{C}^L, \\ T^1_-\mathbb{C}^L &= \mathcal{L}, \quad \mathrm{Ker}(T^1_-) = W(u^1_+, u^1_-)\mathbb{C}^L. \end{aligned} \tag{107}$$

The reflection coefficients satisfy the following properties:

$$R^z_+ = R^1_+ + o(1), \quad R^z_- = R^1_- + o(1), \tag{108}$$

as z tends to 1 *in $\mathbb{S}^1$, where*

$$R^1_+ := \mathbf{1} - \Gamma T^1_+, \quad R^1_- := \mathbf{1} - \Gamma^{-1} T^1_-. \tag{109}$$

Moreover,

$$\begin{aligned} \mathcal{L} &= (\mathbf{1} - R^1_+)\mathbb{C}^L, \quad \mathrm{Ker}(T^1_+) = \mathrm{Ker}(\mathbf{1} - R^1_+), \\ \mathcal{N} &= (\mathbf{1} - R^1_-)\mathbb{C}^L, \quad \mathrm{Ker}(T^1_-) = \mathrm{Ker}(\mathbf{1} - R^1_-). \end{aligned} \tag{110}$$

Proof Equation (103) implies that the matrices $D(z)$ and $A(z) - B(z)D(z)^{-1}C(z)$ are invertible for $z \in \overline{\mathbb{D}}$ in a neighborhood of 1, so using the Schur complement formula, it follows that (recall (57))

$$(\nu^z)^{-1}\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix}^{-1}$$

$$
\begin{aligned}
&= (\nu^z)^{-1} \begin{pmatrix} \mathbf{1} & 0 \\ -D(z)^{-1}C(z) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \left(A(z) - B(z)D(z)^{-1}C(z)\right)^{-1} & 0 \\ 0 & D(z)^{-1} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \mathbf{1} & -B(z)D(z)^{-1} \\ 0 & \mathbf{1} \end{pmatrix} \\
&= \begin{pmatrix} 2\boldsymbol{A}^{-1} & 0 \\ 0 & 0 \end{pmatrix} + o(1), \qquad (111)
\end{aligned}
$$

where (57) and (103) were used. The first equation in (106) follows from Eq. (111), Proposition 24, the continuity of $u_+^z(1)$, (59) and Definition 2. The second equation in (106) is a consequence of the first equation, Definition 2 and (60).

Due to the definition (99) of P, the kernel of T_+^1 is generated by $\{e_{d+1}, \ldots, e_L\}$, and they are a basis of $W(u_-^1, u_+^1)\mathbb{C}^L$: since P is unitary, $P^* = P^{-1} = \left(e_1 \ldots e_L\right)$. Therefore, the kernel at stake equals the kernel of $\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix} \left(e_1 \cdots e_L\right)^{-1}$. Due to the definition (99) of Q, the image of T_+^1 is generated by $\{v_1, \ldots, v_d\}$, and these vectors form a basis of $\mathcal{N}$. This proves the first line in (107). The second one follows from the fact that $T_-^1 = (T_+^1)^*$ (which can be deduced from (60) and (20)).

Next let us take small enough n such that $(u_-^1(n))^{-1}$ exists. Then, by continuity, $(u_-^{1/z}(n))^{-1}$ exists, for z in a neighborhood of 1. Using (19) leads to

$$
\begin{aligned}
R_+^z &= (u_-^{1/z}(n))^{-1} u_-^z(n) - (u_-^{1/z}(n))^{-1} u_+^z(n) T_+^z \to (u_-^1(n))^{-1} u_-^1(n) \\
&\quad - (u_-^1(n))^{-1} u_+^1(n) T_+^1, \qquad (112)
\end{aligned}
$$

as z tends to 1. Taking the limit $n \to -\infty$ in the right hand side of (112), we arrive at the first equation in (109) (see also Lemma 29 and (15)). The second equation is obtained similarly. Equations (110) follow from (107), (109) and the fact that Γ is a bijection from $\mathcal{N}$ onto $\mathcal{L}$. □

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Declarations

Data availability statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

A Appendix

Let $\mathcal{M} = \mathcal{M}_{L\times L}$ and $l^p(\mathbb{N}, \mathcal{M})$ the space of sequences $h : \mathbb{N} \to \mathcal{M}$ with $\sum_n \|h(n)\|^p < \infty$ for $p \in [1, \infty)$ and $\sup_n \|h(n)\| < \infty$ for $p = \infty$.

Theorem 33 (Lemma 7.8 [34], Volterra Equation) *Let $g \in l^\infty(\mathbb{N}, \mathcal{M})$ and $K(n,m) \in \mathcal{M}$ for each $m, n \in \mathbb{N}$. Consider the Volterra sum equation*

$$f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m) f(m), \tag{113}$$

and suppose there is a sequence $M \in l^1(\mathbb{N}, \mathbb{R})$ such that $\|K(n,m)\| \leq M(m)$ for each $m, n \in \mathbb{N}$. Then, Equation (113) has a unique solution $f \in l^\infty(\mathbb{N}, \mathcal{M})$. Moreover, if $g(n)$ and $K(n,m)$ depend continuously (resp. holomorphically) on a parameter z (for every n), M does not depend on z, and $g(n)$ is uniformly bounded with respect to n and z, then the same is true for $f(n)$.

Proof For each $k \in \mathbb{N}$, if one finds a solution $f \in l^\infty(\mathbb{N} \cap [k, \infty), \mathcal{M})$, then it can be extended to a solution in $l^\infty(\mathbb{N}, \mathcal{M})$ by defining recursively $f(n) = g(n) + \sum_{m=n+1}^{\infty} K(n,m) f(m)$ for each $n < k$. Since $M \in l^1(\mathbb{N}, \mathbb{R})$, there exists $k \in \mathbb{N}$ such that $\sum_{m=k+1}^{\infty} M(m) < 1/2$. Then, w.l.o.g., we can assume that $k = 0$, *i.e.*, $\sum_{m=1}^{\infty} M(m) < 1/2$. Then let us introduce the operator $T : l^\infty(\mathbb{N}, \mathcal{M}) \to l^\infty(\mathbb{N}, \mathcal{M})$ by

$$(Tf)(n) = \sum_{m=n+1}^{\infty} K(n,m) f(m)$$

which is well-defined because

$$\sum_{m=n+1}^{\infty} \|K(n,m) f(m)\| \leq \sum_{m=n+1}^{\infty} \|K(n,m)\| \|f(m)\| \leq 1/2 \|f\|_\infty.$$

Moreover, the last equation also implies that T is bounded and $\|T\| < 1/2$, therefore $I - T$ is invertible and $f := (I - T)^{-1} g$ is a solution to the equation on $l^\infty(\mathbb{N}, \mathcal{M})$.

Now we assume that $g(n) \equiv g^z(n)$ and $K(n,m) = K^z(n,m)$ depend continuously (resp. holomorphically) on a parameter z (for every n), M does not depend on z, and $g^z(n)$ is uniformly bounded with respect to n and z. Since the series $(Tg^z)(n) = \sum_{m=n+1}^{\infty} K^z(n,m) g^z(m)$ converges uniformly, $(Tg^z)(n)$ is then continuous (holomorphic) for each $n \in \mathbb{N}$ and it is uniformly bounded with respect to n and z. Repeating the argument, one obtains that the same holds true for $T^j g^z$, for every natural number j. Using that $\|T^j g\| \leq (1/2)^j \sup_{n,z}\{\|g(n)\|\}$, it follows that the series $f^z(n) = \sum_{j=0}^{\infty} (T^j g)^z(n)$ converges uniformly. This implies that the map $z \mapsto f^z(n)$ is continuous (holomorphic). □

The following well-known result is recalled without proof (see [14] for a proof).

Lemma 34 (Gronwall's Lemma) *Let $(u_n)_{n \in \mathbb{N}}$, $(w_n)_{n \in \mathbb{N}}$ be non-negative real sequences and α a real number such that for $n \in \mathbb{N}$*

$$u_n \leq \alpha + \sum_{i=1}^{n-1} u_i w_i.$$

Then, for $n \in \mathbb{N}$ we have that

$$u_n \leq \alpha \exp\left(\sum_{i=1}^{n-1} w_i\right).$$

Lemma 35 (Variation of parameters) *Consider the following difference equation*

$$X(n-1) + AX(n) + X(n+1) = B(n)X(n), \tag{114}$$

where $A, B(n) \in \mathcal{M}$ for each $n \in \mathbb{Z}$. Suppose that S_1, S_2 are solutions of the equation

$$X(n-1) + AX(n) + X(n+1) = 0, \tag{115}$$

such that $S_1(0) = 0$, $S_1(1) = \mathbf{1}$ and $S_2(0) = S_2(1) = \mathbf{1}$. Then, for $C, D \in \mathcal{M}$, the solution S to Equation (114), with initial conditions $S(0) = C$, $S(1) = D$, satisfies for $n \in \mathbb{N}$

$$S(n) = S_1(n)(D - C) + S_2(n)C + \sum_{j=1}^{n-1} S_1(n-j)B(j)S(j), \tag{116}$$

and for $n \in \mathbb{Z}^- \cup \{0\}$

$$S(n) = S_1(n)(D - C) + S_2(n)C - \sum_{j=n+1}^{0} S_1(n-j)B(j)S(j), \tag{117}$$

where we identify $\sum_{j=1}^{0} S_1(-j)B(j)S(j) \equiv 0$ and $\sum_{j=1}^{0} S_1(1-j)B(j)S(j) \equiv 0$.

Proof Equations (116) and (117) together with the initial conditions define recursively a matrix valued function that is denoted by S. Now we prove that S satisfies Equation (114). The proof is carried out only for $n \geq 2$, the other cases are similarly treated. For every $X \in \mathcal{M}^{\mathbb{Z}}$, we use the notation

$$h_n(X) = X(n-1) + AX(n) + X(n+1).$$

Using Equation (116) and $n \geq 2$ we get (here recall that S_1 ans S_2 satisfy (115) and, therefore, $h_n(S_1) = 0 = h_n(S_2)$; moreover $S_1(0) = 0$, $S_1(1) = \mathbf{1}$)

$$\begin{aligned} h_n(S) =& h_n(S_1)(D - C) + h_n(S_2)C + \sum_{j=1}^{n-2} h_{n-j}(S_1)B(j)S(j) \\ &+ AS_1(n-(n-1))B(n-1)S(n-1) \\ &+ S_1(n+1-(n-1))B(n-1)S(n-1) + S_1(n+1-n)B(n)S(n) \\ =& AS_1(1)B(n-1)S(n-1) + S_1(2)B(n-1)S(n-1) + B(n)S(n) \end{aligned}$$

$$
\begin{aligned}
&= \Big(AS_1(1) + S_1(2)\Big) B(n-1)S(n-1) + B(n)S(n) \\
&= -\,S_1(0)B(n-1)S(n-1) + B(n)S(n) = B(n)S(n).
\end{aligned}
$$

We obtain that $h_n(S) = B(n)S(n)$, which is (114). ◻

Lemma 36 *Consider the set-up of Theorem 33 with $g = \mathbf{1}$, $K^z(n, j) = -z^{j-n}s^z(j-n)V(j)$ and $M(j) = j\|V(j)\|$ where s^z is as in Definition 5. Furthermore let $\tilde{u}^z_+$ be the corresponding solution to the Volterra equation (for $n \in \mathbb{N}$)*

$$
\tilde{u}^z_+(n) = \mathbf{1} - \sum_{j=n+1}^{\infty} s^z(j-n)V(j)z^{j-n}\tilde{u}^z_+(j). \tag{118}
$$

Finally denote by $u^z_+(n) = z^n\tilde{u}^z_+(n)$, $n \in \mathbb{N}$. It follows that

$$
u^z_+(n-1) + V(n)u^z_+(n) + u^z_+(n+1) = (z + 1/z)u^z_+(n), \quad n \geq 2. \tag{119}
$$

Proof Let us take $n \geq 2$. We use the notation of Lemma 35 and its proof, taking $A = -(z+1/z)\mathbf{1}$, and set $\gamma(n) = z^n$. A direct calculation shows that $h_m(s^z) = 0$ and $h_m(\gamma) = 0$, for every m. Further let us note that

$$
u^z_+(n) = \gamma(n) + \sum_{j=n+1}^{\infty} s^z(n-j)V(j)u^z_+(j). \tag{120}
$$

It follows from (120) by an algebraic calculation using the definition of h_m that

$$
\begin{aligned}
h_n(u^z_+) = &h_n(\gamma) + \sum_{j=n+2}^{\infty} h_{n-j}(s^z)V(j)u^z_+(j) \\
&+ As^z(n-(n+1))V(n+1)u^z_+(n+1) \\
&+ s^z(n-1-(n+1))V(n+1)u^z_+(n+1) + s^z(n-1-n)V(n)u^z_+(n).
\end{aligned} \tag{121}
$$

Using (121), the facts that $h_m(s^z) = 0$, $h_n(\gamma) = 0$ and $s^z(m) = -s^z(-m)$ together with $s^z(0) = 0$, $s^z(1) = 1$, one gets that

$$
\begin{aligned}
h_n(u^z_+) &= -\,(As^z(1) + s^z(2))V(n+1)u^z_+(n+1) - s^z(1)V(n)u^z_+(n) \\
&= s^z(0)V(n+1)u^z_+(n+1) - V(n)u^z_+(n) \\
&= -\,V(n)u^z_+(n),
\end{aligned}
$$

which is (119). ◻

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