Complex Analysis and Operator Theory (2022) 16:10
https://doi.org/10.1007/s11785-021-01185-5

Complex Analysis and Operator Theory

On Characterization of Hilbert Transform of Riemannian Surface with Boundary

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Received: 24 April 2021 / Accepted: 20 November 2021 / Published online: 30 December 2021

Abstract

Let (M, g) be a smooth compact orientable two-dimensional Riemannian manifold (*surface*) with a smooth metric tensor g and a smooth connected boundary Γ. The *Hilbert transform* H associated with (M, g) acts in $C(\Gamma; \mathbb{R})$ by $H : \Re\eta \mapsto \Im\eta$, where $\eta = w|_\Gamma$ is the trace of a function w holomorphic in M. We provide characteristic conditions on an operator H defined on a curve Γ to be the Hilbert transform of a surface. In fact, the characterization of H is reduced to one of the *Dirichlet-to-Neumann map* Λ_g of the surface (M, g), which is related to the Hilbert transform by $H = J\Lambda_g$, where J is integration along Γ. In contrast to the known characterization of Λ_g by Henkin and Michel in terms of multidimensional complex analysis, our one makes use of the Commutative Banach Algebra theory.

Keywords Riemann surface · Holomorphic function algebra · Hilbert transform · Characterization

Mathematics Subject Classification 35R30 · 46J15 · 46J20 · 30F15

In memory of S. N. Naboko.

Communicated by Jussi Behrndt.

This article is part of the topical collection "Spectral Theory and Operators in Mathematical Physics" edited by Jussi Behrndt, Fabrizio Colombo and Sergey Naboko.

M. I. Belishev and D. V. Korikov were supported by the RFBR Grant 20-01-627A.

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Hilbert Transform

• Let (M, g) be a smooth[1] compact orientable two-dimensional Riemannian manifold with a smooth metric tensor g and smooth connected boundary Γ. In what follows, we deal with the manifolds of this class only and, for short, call them *surfaces*. We denote the length element on Γ by ds and the set of continuous functions with zero mean on Γ by $\dot{C}(\Gamma;\mathbb{R})$, i. e., $\dot{C}(\Gamma;\mathbb{R}) := \{f \in C(\Gamma;\mathbb{R}) \mid \int_\Gamma f\,ds = 0\}$.

There are two continuous families $\{\Phi_x \in \operatorname{End} T_x M \mid x \in M\}$ of rotations on (M, g)

$$g(\Phi_x a, \Phi_x b) = g(a, b), \quad g(\Phi_x a, a) = 0, \qquad a, b \in T_x M,\ x \in M.$$

Each family Φ fixes the orientation on M and in the subsequent we deal with the *oriented surface* (M, g, Φ). The rotation Φ also orients the boundary Γ by the field of tangent unit vectors

$$\gamma := \Phi\nu, \tag{1}$$

where ν is a unit outward normal on Γ. In what follows, we denote by ∂_γ the derivative with respect to the length s in direction γ and by $J\ :\ \dot{C}(\Gamma;\mathbb{R}) \mapsto \dot{C}(\Gamma;\mathbb{R})$ the corresponding integration on Γ: $J\partial_\gamma f = \partial_\gamma J f$ for $f \in \dot{C}(\Gamma;\mathbb{R})$.

• A function $w \in C(M;\mathbb{C})$ is called *holomorphic* if the Cauchy–Riemann condition

$$\nabla_g \Im w = \Phi \nabla_g \Re w \qquad \text{in } M\backslash\Gamma \tag{2}$$

holds. The set $\mathfrak{A}(M)$ of all holomorphic continuous functions on M is a closed subalgebra of $C(M;\mathbb{C})$. Its smooth elements $\mathfrak{A}^\infty(M)$ are dense in $\mathfrak{A}(M)$. Due to the maximum modulus principle, the trace operator $\operatorname{Tr}\ :\ \ w \mapsto w|_\Gamma$ is isometric isomorphism between $\mathfrak{A}(M)$ and a (closed) subalgebra $\mathfrak{A}(\Gamma) := \operatorname{Tr}\mathfrak{A}(M)$ in $C(\Gamma;\mathbb{C})$. Moreover, $\mathfrak{A}^\infty(\Gamma) := \operatorname{Tr}\mathfrak{A}^\infty(M) = \mathfrak{A}(\Gamma) \cap C^\infty(\Gamma;\mathbb{C})$ is dense subalgebra of $\mathfrak{A}(\Gamma)$.

• There are several ways to define the Hilbert transform. By the first definition, a *Hilbert transform* of the oriented surface (M, g, Φ) is the closed operator $\mathcal{H}$ in $C(\Gamma;\mathbb{C})$ with the graph

$$\mathsf{G} = \{(\Re\zeta, \Im\zeta) \mid \zeta \in \mathfrak{A}(\Gamma),\ \ \Im\zeta \in \dot{C}(\Gamma;\mathbb{R})\} \subset C(\Gamma;\mathbb{R}) \times C(\Gamma;\mathbb{R})$$

(this graph is closed since $\mathfrak{A}(\Gamma)$ is closed in $C(\Gamma;\mathbb{C})$). However, it is more convenient to consider the Hilbert transform as an operator acting on smooth functions. Choose in G the following (dense) subgraph

$$\mathsf{G}' := \{(\Re\zeta, \Im\zeta) \mid \zeta \in \mathfrak{A}^\infty(\Gamma),\ \ \Im\zeta \in \dot{C}(\Gamma;\mathbb{R})\} \subset C^\infty(\Gamma;\mathbb{R}) \times C^\infty(\Gamma;\mathbb{R})$$

and assign with it the operator $\mathcal{H}'$. Then $\mathcal{H}'$ is closable $C(\Gamma;\mathbb{R})$ and its closure is $\mathcal{H}$. The domain of definition $\operatorname{Dom}\mathcal{H}'$ depends on topology of M: $\dim \frac{C^\infty(\Gamma;\mathbb{R})}{\operatorname{Dom}\mathcal{H}'} = 1 - \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ is Euler characteristics of M. Nevertheless, we can extend $\mathcal{H}'$ to all smooth functions using the following lemma.

[1] Throughout the paper *smooth* means C^∞-smooth.

Lemma 1 *The relation $\mathcal{H}' \subset J\Lambda_g$ holds, where $\Lambda_g : C^\infty(\Gamma;\mathbb{R}) \mapsto C^\infty(\Gamma;\mathbb{R})$ is the Dirichlet-to-Neumann operator (DN-map) of the surface (M, g) associated with the (elliptic) problem*

$$\Delta_g u = 0 \qquad \text{in } M\backslash\Gamma, \tag{3}$$

$$u = f \qquad \text{on } \Gamma \tag{4}$$

and acting by the rule

$$\Lambda_g f := \partial_\nu u^f \qquad \text{on } \Gamma,$$

where Δ_g is the Beltrami–Laplace operator and $u = u^f$ is the solution of (3)–(4).

Proof Note that $\text{Dom}\, J\Lambda_g \equiv C^\infty(\Gamma;\mathbb{R})$. Indeed, Green formula implies $\int_\Gamma \partial_\nu u^f = 0$ for any $f \in C^\infty(\Gamma;\mathbb{R})$. Thus, $\text{Ran}\Lambda_g \subset \dot{C}(\Gamma;\mathbb{R}) = \text{Dom}\, J$.

Suppose that $\{f, h\} \in \mathsf{G}'$. Then $f + ih$ is a trace on Γ of a holomorphic smooth function w. From (2) it follows that $\Delta_g \Re w = \Delta_g \Im w = 0$, whence $\Re w = u^f$ and $\Im w = u^h$. Restriction of (2) on Γ yields $\partial_\gamma h = \partial_\nu u^f = \Lambda_g f$ in view of (1). Since $h \in \dot{C}(\Gamma;\mathbb{R})$, after integration we get $h = J\Lambda_g f$. □

Lemma 1 motivates the following definition of Hilbert transform.

Definition 1 In what follows, the operator $H := J\Lambda_g$ is referred to as a Hilbert transform of the surface (M, g, Φ).

Note that if H is the Hilbert transform of (M, g, Φ), then $-H$ is the Hilbert transform of $(M, g, -\Phi)$.

Characterization

• Let (Γ, ds, γ) be a curve diffeomorphic to a circle in $\mathbb{R}^2$ and let H be a linear operator in $C^\infty(\Gamma;\mathbb{R})$. Our goal is to describe the necessary and sufficient conditions for H to be the Hilbert transform of some surface (M, g, Φ). In other words, we need to *characterize* the Hilbert transform of the surfaces. Our main result is the list of such conditions.

Lemma 1 shows that characterization of H is equivalent to characterization of the DN-map Λ_g.

• The characterization is directly related to the *inverse problem*: to determine the surface (M, g) from the (given) DN-map Λ_g (or the Hilbert transform H). In applications it is also known as the Electric Impedance Tomography problem.

In [8], M. Lassas and G. Uhlman show that the DN-map Λ_g determines the surface M up to conformal equivalence. In more detail, if (M, g) and (M', g') have the common boundary Γ and $\Lambda_g = \Lambda_{g'}$, then there exists a diffeomorphism $\psi : M' \to M$ and a smooth positive function ρ on M obeying $\psi|_\Gamma = \text{id}$, $\rho|_\Gamma \equiv 1$, and $g = \rho\, \psi_* g'$.

In [2], M. I. Belishev obtained the same result by using relation between the EIT problem and the holomorphic function algebra of the surface. Moreover, the expression for the topological invariants of the surface (Betti numbers) in terms of the DN-map is obtained. In [4] these formulas are generalized to the multidimensional case. The paper [3] extends the algebraic approach to nonorientable surfaces.

• In the papers above, it is a priori assumed that the given Λ_g is the DN-map of some surface. Thus, for such a Λ_g, the solvability of the EIT problem is guaranteed. However, an important question remains: what are the *necessary and sufficient* conditions on an operator Λ to provide the solvability? G. M. Henkin and V. Michel presented such a criterion in the paper [6] in terms of multidimensional complex analysis. In our paper we propose a characterization based on the connections of EIT problem with Banach algebras. So, the novelty is a new formulation of the solvability conditions. The list of them is rather long, however, we venture to claim that our formulation is more transparent and understandable.

• Our approach makes use of the classical result [1] on the existence of a complex structure on the Gelfand spectrum of a commutative Banach algebra. It is the result, which provides the sufficiency of the proposed characteristic conditions.

Main Result

• Let Γ be a smooth curve diffeomorphic to a circle, let $d\gamma$ be its length element, and let $\Lambda: C^\infty(\Gamma;\mathbb{R}) \mapsto C^\infty(\Gamma;\mathbb{R})$ be a linear map. With such Λ we associate the map $\Upsilon: C^\infty(\Gamma;\mathbb{C}) \mapsto C^\infty(\Gamma;\mathbb{C})$, given by the formula

$$\Upsilon\zeta := (\Lambda\Re\zeta - \partial_\gamma\Im\zeta) + i(\Lambda\Im\zeta + \partial_\gamma\Re\zeta), \tag{5}$$

where γ is a tangent field of unit vectors on Γ. It is easy to verify that Υ is a (complex) linear operator. For $\eta \in C^\infty(\Gamma;\mathbb{C})$ and $z \in \mathbb{C}\backslash\eta(\Gamma)$, we introduce the map $\Upsilon_{\eta,z}: C^\infty(\Gamma;\mathbb{C}) \mapsto C^\infty(\Gamma;\mathbb{C})$ as follows

$$\Upsilon_{\eta,z}\zeta := \Upsilon\frac{\zeta}{\eta - ze}, \tag{6}$$

where e is the function equal to 1 on Γ.

Let I be the identity operator on $C^\infty(\Gamma;\mathbb{R})$, let $\partial_\gamma C^\infty(\Gamma;\mathbb{R})$ be the space of smooth real-valued functions with zero mean value on Γ, and let $J: \partial_\gamma C^\infty(\Gamma;\mathbb{R}) \to \partial_\gamma C^\infty(\Gamma;\mathbb{R})$ be the integration on Γ: $J\partial_\gamma = \partial_\gamma J = I$. By $\sharp S$ we denote the cardinality of S.

Our main result is the following.

Theorem 1 *The operator Λ is the DN-map of a surface if and only if it satisfies the conditions:*

i $e \in \operatorname{Ker}\Upsilon$ *and* $\zeta_1\zeta_2 \in \operatorname{Ker}\Upsilon$ *for any* $\zeta_1, \zeta_2 \in \operatorname{Ker}\Upsilon$*;*
ii *if* $\zeta_1, \zeta_2 \in \operatorname{Ker}\Upsilon$, $\zeta_1/\zeta_2 \in C^\infty(\Gamma;\mathbb{C})$*, and there exists a polynomial* P, $\deg P \geq 1$ *such that* $P(\zeta_1/\zeta_2) \in \operatorname{Ker}\Upsilon$*, then* $\zeta_1/\zeta_2 \in \operatorname{Ker}\Upsilon$*;*
iii $\overline{\operatorname{Ker}\Upsilon} \cap C^\infty(\Gamma;\mathbb{C}) = \operatorname{Ker}\Upsilon$ *(the closure in* $C(\Gamma;\mathbb{C})$*);*
iv $\dim(\partial_\gamma + \Lambda J\Lambda)C^\infty(\Gamma;\mathbb{R}) < \infty$*;*

v *if* $\eta \in \mathrm{Ker}\Upsilon$ *and* $z \in \mathbb{C}\backslash\eta(\Gamma)$*, then*

$$\dim[\Upsilon_{\eta,z}\mathrm{Ker}\Upsilon] = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\partial_\gamma \eta}{\eta - ze} d\gamma\,; \tag{7}$$

vi *for any* $x \in \Gamma$*, there exist a function* $\eta_x \in \mathrm{Ker}\Upsilon$ *and a neighborhood* $U_x \ni \eta_x(x)$ *diffeomorphic to an open disk* $D \subset \mathbb{C}$*, such that*

1 $\partial_\gamma \eta_x(x) \neq 0$ *is valid and there is no points on* Γ*, at which all derivatives* $\partial_\gamma^k \eta_x$*,* $k \geq 1$ *vanish simultaneously, whereas* $\sharp\, \eta_x^{-1}(\{z\}) < \infty$ *holds for all* $z \in \mathbb{C}$*;*
2 $\Upsilon_{\eta_x,z} e = 0$ *holds on one connected component of* $U_x \backslash \eta_x(\Gamma)$*, whereas* $\Upsilon_{\eta_x,z} e \neq 0$ *holds on the other connected component;*
3 *the equation*

$$\Upsilon\Big(\frac{\zeta - ce}{\eta_x - ze}\Big) = 0 \text{ on } \Gamma \tag{8}$$

has a solution $c = c(z,\zeta) \in \mathbb{C}$ *for any* $z \in U_x$ *and* $\zeta \in \mathrm{Ker}\Upsilon$*;*

vii *if* $\zeta, 1/\zeta \in \mathrm{Ker}\Upsilon$*, then* $\Lambda \log|\zeta| = \partial_\gamma \arg\zeta$.

Remark 1 Conditions i–vii do not depend on the choice of the field γ.

The characterisation of the Hilbert transform immediately follows from Theorem 1.

Corollary 1 *The linear map* $H : C^\infty(\Gamma;\mathbb{R}) \mapsto C^\infty(\Gamma;\mathbb{R})$ *is the Hilbert transform of a surface if and only if one of the operators* $\Lambda = \partial_\gamma H$*,* $\Lambda = -\partial_\gamma H$ *satisfies the conditions i–vii of* Theorem 1.

As a comment, note the following. Condition i means that $\mathrm{Ker}\Upsilon$ is an algebra, whereas vi shows that this algebra must be rich enough to contain the functions η_x with the required properties. By condition vi 2, invertibility of $\eta_x - ze$ in the algebra $\mathrm{Ker}\Upsilon$ depends on the position of z on the complex plane. Condition vi 3, from an algebraic point of view, means that $(\eta_x - ze)\,\mathrm{Ker}\Upsilon$ is an ideal in $\mathrm{Ker}\Upsilon$ of codimension 1, i.e., it is a maximal ideal. It is easy to see that the embedding $g \in \mathrm{Ker}\Upsilon$ implies $\partial_\gamma \Re g, \partial_\gamma \Im g \in \mathrm{Ker}[I + (\Lambda J)^2]$. The operator $I + (\Lambda J)^2$ is the key object of the papers [2–4].

The rest of the paper is devoted to the proof of Theorem 1.

Necessity

Here we show that any DN-map satisfies conditions i–vii.

• Suppose that $\Lambda = \Lambda_g$ is the DN-map of some surface (M, g). Recall that γ and ν are the tangent and normal unit vector fields at the boundary Γ.

Choose a continuous family of rotations Φ such that $\Phi\nu = \gamma$ on Γ. Recall that a function $w \in C^\infty(M;\mathbb{C})$ is called holomorphic if the Cauchy–Riemann condition (2) holds in M. Let w be holomorphic and let $\zeta = w|_\Gamma$ be its trace on the boundary. The real functions $\Re w$ and $\Im w$ are harmonic in M and provide the solutions $\Re w = u^{\Re\zeta}$ and $\Im w = u^{\Im\zeta}$ to (3), (4). Restricting the Cauchy–Riemann conditions on Γ, we obtain

$$\Lambda\Re\zeta = \partial_\nu\Re w = \partial_\gamma\Im\zeta, \qquad \Lambda\Im\zeta = \partial_\nu\Im w = -\partial_\gamma\Re\zeta \quad \text{on } \Gamma, \tag{9}$$

which implies $\Upsilon(\zeta) = 0$ according to (5).

Now, suppose that $\zeta \in C^\infty(\Gamma; \mathbb{C})$ and $\Upsilon(\zeta) = 0$. Then the function $w := u^{\Re\zeta} + iu^{\Im\zeta}$ is holomorphic in int$M$. Indeed, since $\Upsilon(\zeta) = 0$, we have (9), i.e., $\nabla_g\Im w = \Phi\nabla_g\Re w$ holds on Γ. Let U be an arbitrary neighborhood in M diffeomorphic to the disc, and let $\partial U \cap \Gamma$ contain a segment Γ' of non-zero length. Since $\partial_\nu\Re w = u^{\Re\zeta}$ is harmonic in U, there exists a function v such that $\nabla_g v = \Phi\nabla_g\Re w$ in U. Thus, $\partial_\nu\Re w = \partial_\gamma v$ and $\partial_\nu v = -\partial_\gamma\Re w$ on Γ'. Comparing with (9), we obtain $v = \Im w + \text{const}$, $\partial_\nu v = \partial_\nu\Im w$ on Γ'. So, v and $\Im w + \text{const}$ are harmonic in U and have the same Cauchy data on Γ'. Due to uniqueness of the solution to the Cauchy problem for the second order elliptic equations, v coincides with $\Im w + \text{const}$ in U, and $\nabla_g\Im w = \nabla_g v = \Phi\nabla_g\Re w$ in U. Since U is arbitrary, $\nabla_g\Im w = \Phi\nabla_g\Re w$ holds in M, and w is holomorphic. So, we have proved that KerΥ coincides with the algebra $\mathfrak{A}^\infty(\Gamma)$ of traces on Γ of all holomorphic smooth functions on M. This yields i.

• Suppose that $\zeta_1, \zeta_2 \in \text{Ker}\Upsilon$, $\zeta = \zeta_1/\zeta_2 \in C^\infty(\Gamma; \mathbb{C})$, and $P(\zeta) \in \text{Ker}\Upsilon$, where P is a polynomial of degree $p \geq 1$. In view of the already proven, there exist holomorphic functions w_1, w_2, w_P such that $w_1|_\Gamma = \zeta_1$, $w_2|_\Gamma = \zeta_2$, and $w_P|_\Gamma = P(\zeta)$. Then the function $w := w_1/w_2$ is meromorphic in int$M$ and $w|_\Gamma = \zeta \in C^\infty(\Gamma; \mathbb{C})$. The last implies that the poles of w do not accumulate to Γ and the number of them is finite. The function $P(w)$ is also meromorphic and its poles coincide with those of w, while their multiplicities are p times greater than those of w. Since $P(w) = P(\zeta) = w_P$ on Γ, the function $P(w)$ coincides with w_P outside the poles of w due to uniqueness of analytic continuation. Then $P(w) = w_P$ everywhere on M. Thus, w is holomorphic and its trace ζ belongs to KerΥ. This proves ii.

• Recall that $\overline{\text{Ker}\Upsilon} = \overline{\mathfrak{A}^\infty(\Gamma)}$ coincides with the set $\mathfrak{A}(\Gamma)$ of traces on Γ of all holomorphic continuous functions. Since $\mathfrak{A}(\Gamma) \cap C^\infty(\Gamma; \mathbb{C}) = \mathfrak{A}^\infty(\Gamma)$, we obtain iii.

The property iv follows from the equality

$$\dim(\partial_\gamma + \Lambda J\Lambda)C^\infty(\Gamma; \mathbb{R}) = 1 - \mathscr{X}(M) \tag{10}$$

(see formula (1.6), [2]), where $\mathscr{X}(M)$ is the Euler characteristics of M.

• Suppose that $\eta \in \text{Ker}\Upsilon$ and $z \in \mathbb{C}\backslash\eta(\Gamma)$. Then there exists the holomorphic in int$M$ function w_0 such that $w_0|_\Gamma = \eta$. Denote by $x_1, \ldots, x_l$ all the zeroes of $w_0 - z$ and by $m_1, \ldots, m_l$ their multiplicities. We make use of the argument principle[2]:

$$\frac{1}{2\pi i}\int\limits_\Gamma \frac{\partial_\gamma\eta}{\eta - ze}d\gamma = \sum_{k=1}^{l} m_k. \tag{11}$$

[2] For a compact Riemannian surface with boundary, the argument principle can be obtained by simple modification of the proofs of Theorem 3.17 and Corollary 3.18, [7].

Since Γ is smooth, the manifold (M, g) can be embedded into a larger non-compact smooth manifold (M', g'), $g'|_M = g$. For each $k = 1, \dots, l$ and $s = 0, \dots, m_k - 1$, choose a function $w_{k,s}$ holomorphic on M' and such that $x_1, \dots, x_l$ are all zeroes of $w_{k,s}$ on M' and multiplicity of x_j is equal to s if $j = k$ and to m_j if $j \neq k$. The existence of such $w_{k,s}$ follows from Proposition 26.5 in [5]. The linear combination $\sum_{k,s} \frac{c_{k,s} w_{k,s}}{w_0 - z}$ has no poles in M only if all $c_{k,s}$ equal to zero. Denote $\eta_{k,s} := w_{k,s}|_\Gamma$; then $\Upsilon_{\eta,z}(\sum_{k,s} c_{k,s}\eta_{k,s}) = 0$ only if all $c_{k,s}$ are zeros. Hence, the functions

$$\Upsilon_{\eta,z}(\eta_{k,s}), \quad k = 1, \dots, l, \quad s = 0, \dots, m_k - 1 \tag{12}$$

are linearly independent.

Now, suppose that $w \in C^\infty(\Gamma; \mathbb{C})$ is holomorphic in M and $\zeta = w|_\Gamma$. For any $k = 1, \dots, l$, there exist $d_{k,s} \in \mathbb{C}$ $(l = 1, \dots, m_k)$ such that $w - \sum_s d_{k,s} w_{k,s}$ has a zero of multiplicity not less than m_k at $x = x_k$. In view of this and the definition of $w_{k,s}$, the function $w - \sum_{k,s} d_{k,s} w_{k,s}$ has a zero of multiplicity not less than m_k at each x_k. Therefore, the ratio $\frac{w - \sum_{k,s} d_{k,s} w_{k,s}}{w_0 - z}$ is holomorphic in M and, hence, $\Upsilon_{\eta,z}(\zeta - \sum_{k,s} d_{k,s}\eta_{k,s}) = 0$. This means that (12) is a basis in $\Upsilon_{\eta,z}\mathrm{Ker}\Upsilon$. In particular, $\dim \Upsilon_{\eta,z}\mathrm{Ker}\Upsilon = \sum_{k=1}^{l} m_k$. Comparing with (11), we arrive at v.

• Let x be an arbitrary point of Γ. According to Proposition 26.5, [5], there exists the holomorphic in M' function w_x such that x is unique zero of w_x and its multiplicity is equal to one. For any $c \in \mathbb{C}$, the function $w_x - c$ has only finite number of zeros on M (otherwise, there would be an accumulation point of such zeros due to the compactness of M) and each zero of $w_x - c$ is of finite multiplicity. This implies vi 1 for the function $\eta_x := w_x|_\Gamma \in \mathrm{Ker}\Upsilon$.

Next, since $\nabla\Re w_x(x) \neq 0$, the map $w_x : M \to \mathbb{C}$ is a bijection of a neighborhood V_0 of x and neighborhood $w_x(V_0)$ of the zero, and $|w_x(x')| > 0$ holds for any $x' \in M'\backslash\{x\}$. Let K be a compact in M' that contains $M \cup V_0$. Then the set $K\backslash V_0$ is also compact and $|w_x(x')| > c_0 > 0$ for any $x' \in K\backslash V$. Choose a neighborhood $V_1 \subset V_0$ sufficiently small to obey $|w_x(x')| < c_0/2$ for any $x \in V_1$. Then the pre-image $w_x^{-1}(\{z\})$ of any $z \in w_x(V_1)$ is contained in V_0 and, since w_x is a bijection of V_0 and $w_x(V_0)$, it consists of a single element. Denote $U_x := w_x(V_1)$, $U_{x,1} := U_x\backslash w_x(M)$, and $U_{x,2} := U_x \cap w_x(M)$. The function $\frac{1}{w_x - z}$ has no poles on M for any $z \in U_{x,1}$ and has a simple pole on M for any $z \in U_x\backslash U_{x,1}$. Thus, $\Upsilon_{\eta_x,z}e = \Upsilon(\frac{e}{\eta_x - ze}) = 0$ for all $z \in U_{x,1}$ and $\Upsilon(\frac{e}{\eta_x - ze}) \neq 0$ for any $z \in U_{x,2}$. This yields vi 2.

Finally, suppose that $w \in C^\infty(M; \mathbb{C})$ is holomorphic in M and $\zeta := w|_\Gamma$. If $z \in U_{x,1}$ and $c \in \mathbb{C}$, then the function $\frac{w-c}{w_x - z}$ is holomorphic in M. Hence, any $c \in \mathbb{C}$ is a solution of (8). Now, suppose that $z \in U_{x,2}$. Since $\frac{1}{w_x - z}$ has a simple pole at the point $w_x^{-1}(z)$ and no other poles on M, the function $\frac{w-c}{w_x - z}$ is holomorphic in M if and only if $c = w(w_x^{-1}(z))$. So, (8) has a unique solution $c = w(w_x^{-1}(z))$ for any $z \in U_{x,1}$. This proves vi 3.

• Suppose that $\zeta, 1/\zeta \in \mathrm{Ker}\Upsilon$. Then $\zeta = w|_\Gamma$, where $w, 1/w$ are holomorphic functions in M. Let U be an arbitrary simply connected neighborhood in M. Since w has no zeroes in U, each branch of $\log w$ is a holomorphic function in U. In particular, $\log|w| = \Re \log w$ is harmonic in U. In addition, $\log|w|$ is a single-valued function on the whole M. Then $\log|w| = u^{\log|\zeta|}$ is a solution of (3), (4) with $f = \log|\zeta|$. Hence, $\partial_\nu \log|w| = \Lambda \log|\zeta|$ on Γ. Now choose U in such a way that $\overline{U} \cap \Gamma$ is a segment $\Gamma' \subset \Gamma$ of nonzero length. Since each branch of $\log w$ is holomorphic in U and smooth up to Γ', it follows from Cauchy–Riemann conditions that

$$\partial_\nu \log|w| = \partial_\nu \Re \log w = \partial_\gamma \Im \log w = \partial_\gamma \arg w = \partial_\gamma \arg \zeta$$

on Γ. Therefore, $\Lambda \log|\zeta| = \partial_\gamma \arg \zeta$, i.e., vii does hold.

The necessity is proved.

Sufficiency

Here we assume that Λ obeys i–vii and construct a Riemannian surface (M, g) such that its DN-map is Λ, i.e., $\Lambda = \Lambda_g$ holds. Before that, we recall some known facts and definitions that will be used in the construction.

• A *commutative Banach algebra* is a (complex) Banach space $(\mathfrak{A}, \|\cdot\|)$ equipped with multiplication satisfying $\eta\zeta = \zeta\eta$, $\|\eta\zeta\| \leq \|\eta\|\|\zeta\|$ for all $\eta, \zeta \in \mathfrak{A}$. Algebra $\mathfrak{A}$ is *unital* if there exists $e \in \mathfrak{A}$ such that $e\eta = \eta$ holds for all $\eta \in \mathfrak{A}$. Element $\eta \in \mathfrak{A}$ is *invertible* if there exists $\eta^{-1} \in \mathfrak{A}$ such that $\eta^{-1}\eta = e$. The set of all $z \in \mathbb{C}$, for which $\eta - ze$ is noninvertible, is called the *spectrum* of η and is denoted by $\mathrm{Sp}_{\mathfrak{A}}\eta$, such a set being compact.

A *character* of the commutative Banach algebra $\mathfrak{A}$ is a nonzero homomorphism $\chi : \mathfrak{A} \mapsto \mathbb{C}$. Each character χ is a continuous map: we have

$$|\chi(\eta)| \leq \|\eta\|, \qquad \eta \in \mathfrak{A}. \tag{13}$$

The set of characters $\widehat{\mathfrak{A}}$ is called the *spectrum* of the algebra $\mathfrak{A}$. For an $\eta \in \mathfrak{A}$, its *Gelfand transform* $\hat{\eta} : \widehat{\mathfrak{A}} \mapsto \mathbb{C}$ is defined as

$$\hat{\eta}(\chi) := \chi(\eta), \qquad \chi \in \widehat{\mathfrak{A}}.$$

For any $\hat{\eta}$, the image $\hat{\eta}(\widehat{\mathfrak{A}}) \subset \mathbb{C}$ coincides with the spectrum $\mathrm{Sp}_{\mathfrak{A}}\eta$.

Spectrum $\widehat{\mathfrak{A}}$ is endowed with the canonical Gelfand ($*$-weak) topology, with respect to which it is a compact Hausdorff space. The Gelfand transforms $\{\hat{\eta} \mid \eta \in \mathfrak{A}\}$ constitute a subalgebra in $C(\widehat{\mathfrak{A}})$, which separates points of $\widehat{\mathfrak{A}}$. The space $\widehat{\mathfrak{A}}$ is connected if and only if there is no nontrivial idempotents $\eta = \eta^2$, $\eta \neq 0, e$ in $\mathfrak{A}$.

A closed subset $B \subset \widehat{\mathfrak{A}}$ is called a *boundary* of $\mathfrak{A}$ if $\max_B |\hat{\eta}| = \max_{\widehat{\mathfrak{A}}} |\hat{\eta}|$ for any $\eta \in \mathfrak{A}$. The intersection of all boundaries is called *the Shilov boundary* of $\mathfrak{A}$ and denoted by $\mathfrak{b}\mathfrak{A}$.

The key fact that we use in the proof of sufficiency is the fundamental Bishop–Aupetit–Wermer analytic structure theorem.

Theorem 2 (see Theorem 2.2, [1] or Chapter 11, [9]) *Assume that* $\eta \in \mathfrak{A}$*, the set* $\hat{\eta}(\widehat{\mathfrak{A}}) \backslash \hat{\eta}(\mathfrak{b}\mathfrak{A})$ *is non-empty, and* V *is its connected component. Assume also that the set* $\{z \in V \mid \sharp\, \hat{\eta}^{-1}(\{z\}) < \infty\}$ *has nonzero Lebesgue measure. Then* $\sharp\, \hat{\eta}^{-1}(\{z\}) \leq N < \infty$

for any $z \in V$ and the subset $\hat{\eta}^{-1}(V) \subset \widehat{\mathfrak{A}}$ has the structure of 1-dim complex analytic manifold, on which all functions $\hat{\zeta}$ ($\zeta \in \mathfrak{A}$) are holomorphic.

The rest of the proof of Theorem 1 is as follows. We construct a Riemann surface M as the spectrum $\widehat{\mathfrak{A}}$ of some Banach function algebra $\mathfrak{A}$ that is provided by conditions i and iii. Then, using Theorem 2 and the condition v, we endow a part $\Omega_\eta \subset \widehat{\mathfrak{A}}$ with the structure of Riemannian surface, and this part depends on the element $\eta \in \mathfrak{A}$. The condition vi enables us, by varying the elements $\eta \in \mathfrak{A}$, to cover the whole spectrum $\widehat{\mathfrak{A}}$ by parts Ω_η and thus endow the $\widehat{\mathfrak{A}}$ with the structure of Riemannian surface. Due to Theorem 2, the Gelfand transforms of elements of $\mathfrak{A}$ form a subalgebra in the algebra of holomorphic smooth functions on M. By conditions ii and iv, this subalgebra coincides with the set of all holomorphic smooth functions on M. As a consequence, we see that Λ coincides with the DN-map of the surface M for all $f = \Re\, w|_\Gamma$ such that w is holomorphic on M. The set of such f is of finite codimension. To check that Λ coincides with the DN-map of M on *all other functions* from $C^\infty(\Gamma; \mathbb{R})$, we use the remaining condition vii.

So, we proceed to prove the sufficiency of the conditions of Theorem 1.

- In view of i, the set $\mathrm{Ker}\Upsilon$ is a unital (sub)algebra in $C(\Gamma; \mathbb{C})$. The closure

$$\mathfrak{A} := \overline{\mathrm{Ker}\Upsilon} \subset C(\Gamma; \mathbb{C})$$

is a unital commutative Banach algebra with the norm $\| \zeta \| := \max_\Gamma |\zeta|$. We denote its spectrum $\widehat{\mathfrak{A}}$ by M. Recall that M is a compact Hausdorff space. In addition, M is connected: indeed, if $\eta^2 = \eta$ on Γ, then $\eta(x) = 0$ or 1 for any $x \in \Gamma$; since η is continuous and Γ is connected, this means that either $\eta = 0$ or $\eta = 1$. Note that the set of smooth elements of $\mathfrak{A}$ coincides with $\mathrm{Ker}\Upsilon$ due to property iii.

The set $\delta_\Gamma := \{\delta_x \mid x \in \Gamma\}$ of the Dirac measures $\delta_x(\zeta) := \zeta(x)$ is a subset of M. In view of (13) and the definition of the norm $\| \cdot \|$, we have $|\hat{\eta}(\chi)| = |\chi(\eta)| \le \|\eta\| = \max_{x\in\Gamma} |\delta_x(\eta)|$ for any $\chi \in M$ and $\eta \in \mathfrak{A}$. Hence, δ_Γ is a boundary of $\mathfrak{A}$ and thus it contains the Shilov boundary $\mathfrak{b}\mathfrak{A}$ of $\mathfrak{A}$.

- Our first goal is to endow $M \backslash \delta_\Gamma$ with the structure of an analytic manifold by means of Theorem 2. To verify the conditions of Theorem 2, we prove that $\hat{\eta}^{-1}(\{z\})$ is finite for any $\eta \in \mathrm{Ker}\Upsilon = \mathfrak{A} \cap C^\infty(\Gamma; \mathbb{C})$ and $z \in \hat{\eta}(\widehat{\mathfrak{A}}) \backslash \eta(\Gamma)$. The proof is based on a bijection between the characters from $\hat{\eta}^{-1}(\{z\})$ and the characters over a certain finite-dimensional factor-algebra $\mathfrak{A}_{\eta,z}$ which is constructed below. Let us get down to implementation of this plan.

Let $\eta \in \mathrm{Ker}\Upsilon$ and $z \in \mathbb{C} \backslash \eta(\Gamma)$; then the function $\eta - ze$ is invertible in $C^\infty(\Gamma; \mathbb{C})$ (but not necessarily in $\mathfrak{A}$). Consider the main ideal $\mathcal{I}_{\eta,z} := (\eta - ze)\,\mathfrak{A}$ in $\mathfrak{A}$. It is closed in $\mathfrak{A}$: indeed, if $\zeta_k \in \mathcal{I}_{\eta,z}$ (i.e., $\frac{\zeta_k}{\eta - ze} \in \mathfrak{A}$) and $\zeta_k \to \zeta$ in $C(\Gamma; \mathbb{C})$, then $\frac{\zeta_k}{\eta - ze} \to \frac{\zeta}{\eta - ze}$ in $C(\Gamma; \mathbb{C})$ since $\frac{1}{\eta - ze} \in C^\infty(\Gamma; \mathbb{C})$. Therefore we have $\frac{\zeta}{\eta - ze} \in \mathfrak{A}$ and $\zeta \in \mathcal{I}_{\eta,z}$. Since $\mathrm{Ker}\Upsilon$ is dense in $\mathfrak{A}$, the set $\mathcal{I}^\infty_{\eta,z} := (\eta - ze)\,\mathrm{Ker}\Upsilon$ is dense in $\mathcal{I}_{\eta,z}$. The function $\zeta \in \mathrm{Ker}\Upsilon$ belongs to $\mathcal{I}^\infty_{\eta,z}$ if and only if $0 = \Upsilon(\frac{\zeta}{\eta - ze}) = \Upsilon_{\eta,z}(\zeta)$.

Introduce the factor-algebra

$$\mathfrak{A}_{\eta,z} := \mathfrak{A}/\mathcal{I}_{\eta,z}$$

with the factor-norm $\|\zeta+\mathcal{I}_{\eta,z}\|_{\eta,z}:=\inf_{\tilde\zeta\in\mathcal{I}_{\eta,z}}\|\zeta+\tilde\zeta\|$; here and in what follows we denote by $\zeta+\mathcal{I}_{\eta,z}$ the equivalence class in $\mathfrak{A}_{\eta,z}$ of element $\zeta\in\mathfrak{A}$. Due to definition of the factor-norm and the equality $\overline{\mathrm{Ker}\Upsilon}=\mathfrak{A}$, the set $\mathfrak{A}^\infty_{\eta,z}:=\{\zeta+\mathcal{I}_{\eta,z}\mid\zeta\in\mathrm{Ker}\Upsilon\}$ is dense in $\mathfrak{A}_{\eta,z}$. Let us prove that the algebra $\mathfrak{A}_{\eta,z}$ is finite-dimensional. To this end, we consider a linear map $\mathscr{G}_{\eta,z}:\ \mathfrak{A}^\infty_{\eta,z}\mapsto C^\infty(\Gamma;\mathbb{C})$ defined by the rule

$$\mathscr{G}_{\eta,z}(\zeta+\mathcal{I}_{\eta,z})=\Upsilon_{\eta,z}(\zeta).$$

The map $\mathscr{G}_{\eta,z}$ is well-defined and its kernel is trivial. Indeed, if $\zeta_1+\mathcal{I}_{\eta,z}=\zeta_2+\mathcal{I}_{\eta,z}\in\mathfrak{A}^\infty_{\eta,z}$, then $\zeta_1-\zeta_2\in\mathcal{I}_{\eta,z}\cap C^\infty(\Gamma;\mathbb{C})=\mathcal{I}^\infty_{\eta,z}$ and $\Upsilon_{\eta,z}(\zeta_1)-\Upsilon_{\eta,z}(\zeta_2)=0$. Similarly, if $\mathscr{G}_{\eta,z}(\zeta+\mathcal{I}_{\eta,z})=0$, then $\Upsilon_{\eta,z}(\zeta)=0$ and thus $\zeta\in\mathcal{I}^\infty_{\eta,z}\subset\mathcal{I}_{\eta,z}$, i.e., $\zeta+\mathcal{I}_{\eta,z}$ is the zero element in $\mathfrak{A}_{\eta,z}$. Note that $\mathscr{G}_{\eta,z}\mathfrak{A}^\infty_{\eta,z}=\Upsilon_{\eta,z}\mathrm{Ker}\Upsilon$. Since the map $\mathscr{G}_{\eta,z}$ is a bijection of $\mathfrak{A}^\infty_{\eta,z}$ and $\mathscr{G}_{\eta,z}\mathfrak{A}^\infty_{\eta,z}$, we have

$$\dim\mathfrak{A}^\infty_{\eta,z}=\dim[\Upsilon_{\eta,z}\mathrm{Ker}\Upsilon].$$

In view of condition v, the right-hand side is equal to the integral

$$\frac{1}{2\pi i}\int\limits_\Gamma\frac{\partial_\gamma\eta}{\eta-ze}\,d\gamma\,.$$

Since the functions η and $\frac{1}{\eta-ze}$ are smooth, this integral is finite. So, $\dim\mathfrak{A}^\infty_{\eta,z}$ is finite and, since $\mathfrak{A}^\infty_{\eta,z}$ is dense in $\mathfrak{A}_{\eta,z}$, we have

$$\dim\mathfrak{A}_{\eta,z}=\frac{1}{2\pi i}\int\limits_\Gamma\frac{\partial_\gamma\eta}{\eta-ze}\,d\gamma\,.$$

Note that the right-hand side is the winding number $d(z)$ of the image $\eta(\Gamma)\subset\mathbb{C}$ with respect to the point z; this number depends only on the connected component V of $\mathbb{C}\backslash\eta(\Gamma)$ that contains z. If $d(z)=0$, then $\dim\mathfrak{A}_{\eta,z}=0$ and $e\in\mathcal{I}_{\eta,z}=\mathfrak{A}$. This means that $\eta-ze$ is invertible in $\mathfrak{A}$ and $z\notin\mathrm{Sp}_{\mathfrak{A}}\eta=\hat\eta(M)$. Thus,

$$\hat\eta(M)\backslash\eta(\Gamma)=\{z\in\mathbb{C}\backslash\eta(\Gamma)\mid d(z)>0\}.$$

- Now, we show that the set $\hat\eta^{-1}(\{z\})$ is finite, $z\in V$ being the same as before. Let $\tilde\chi$ be a character on the algebra $\mathfrak{A}_{\eta,z}$; then the rule

$$\chi(\zeta):=\tilde\chi(\zeta+\mathcal{I}_{\eta,z}) \tag{14}$$

defines a character $\chi\in M$ that vanishes on $\mathcal{I}_{\eta,z}$. Hence, we have $\chi(\eta-ze)=0$ and $\chi(\eta)=z$. Conversely, suppose that $\chi\in M$ and $\chi(\eta)=z$ (then, obviously, $\chi(\mathcal{I}_{\eta,z})=\{0\}$). Then the same rule (14) defines the character $\tilde\chi$ on $\mathfrak{A}_{\eta,z}$. Thus, we have

$$\sharp\,\widehat{\mathfrak{A}}_{\eta,z}=\sharp\,\hat\eta^{-1}(\{z\}). \tag{15}$$

Suppose that $\tilde{\chi}_1, \ldots, \tilde{\chi}_N$ are the distinct characters in $\widehat{\mathfrak{A}}_{\eta,z}$. Since the Gelfand transforms of the elements $\zeta + \mathcal{I}_{\eta,z} \in \mathfrak{A}_{\eta,z}$ separate the points of $\widehat{\mathfrak{A}}_{\eta,z}$, there exists $\zeta_{ij} \in \mathfrak{A}$ such that $\tilde{\chi}_i(\zeta_{ij} + \mathcal{I}_{\eta,z}) = 1$ and $\tilde{\chi}_j(\zeta_{ij} + \mathcal{I}_{\eta,z}) = 0$. Denote $q_i := \Pi_{j \neq i}(\zeta_{ij} + \mathcal{I}_{\eta,z})$, then $\tilde{\chi}_j(q_i) = \delta_{ij}$. In particular, q_i, $i = 1, \ldots, N$, are linearly independent in $\mathfrak{A}_{\eta,z}$. Therefore, $N \leq \dim \mathfrak{A}_{\eta,z} = d(z)$. So, we see that $\sharp\, \widehat{\mathfrak{A}}_{\eta,z} \leq d(z)$ and, by (15), we conclude that $\sharp\, \hat{\eta}^{-1}(\{z\}) \leq d(z) < \infty$.

• Suppose that $z \in \hat{\eta}(M) \backslash \eta(\Gamma)$ (to obtain $\hat{\eta}(M) \backslash \eta(\Gamma) \neq \varnothing$, we can take $\eta = \eta_x$ for any function η_x obeying vi1). Denote by V the connected component of $\mathbb{C} \backslash \eta(\Gamma)$ that contains z. Obviously, V is open and, hence, it has a nonzero Lebesgue measure. Moreover, $1 \leq d(z) < \infty$ and $d(z') = d(z)$ is valid for any $z' \in V$. For such a z', the dimension $d(z')$ of algebra $\mathfrak{A}_{\eta,z'}$ is finite and nonzero. Hence, $\mathcal{I}_{\eta,z'} \neq \mathfrak{A}$, i.e., $\eta - z'e$ is noninvertible. Therefore $z' \in \hat{\eta}(M)$ and, thus, the embedding $V \subset \hat{\eta}(M) \backslash \eta(\Gamma)$ holds.

Since $\eta(\Gamma) = \hat{\eta}(\delta_\Gamma)$ and $\mathfrak{b}\mathfrak{A} \subset \delta_\Gamma$, the set V does not intersect with $\hat{\eta}(\mathfrak{b}\mathfrak{A})$. In view of Theorem 2, the set $\hat{\eta}^{-1}(V) \subset \widehat{\mathfrak{A}}$ has the structure of 1-dim complex analytic manifold on which all functions $\hat{\zeta}$ ($\zeta \in \mathfrak{A}$) are analytic. Thus, for any character $\chi \in \hat{\eta}^{-1}(V)$ there exist an open (in the Gelfand topology) neighborhood $U \ni \chi$ and a homeomorphism $\kappa : U \to D$ onto an open disk $D \subset \mathbb{C}$ such that any function $\hat{\zeta} \circ \kappa^{-1}$ ($\zeta \in \mathfrak{A}$) is holomorphic on D. In other words, every $\chi \in \hat{\eta}^{-1}(V)$ does possess a local analytic coordinate $\hat{\eta}$.

As a result, we can represent the spectrum M as the following disjoint union:

$$M = M' \cup \delta_\Gamma \cup \widetilde{M},$$

where

$$M' := \bigcup_{\eta \in \operatorname{Ker}\Upsilon} \hat{\eta}^{-1}(\mathbb{C} \backslash \eta(\Gamma))$$

is the set of characters that can be provided with the local coordinate by the choice of a suitable $\eta \in \operatorname{Ker}\Upsilon$, and $\widetilde{M} := M \backslash (M' \cup \delta_\Gamma)$.

• Let us show that $\widetilde{M} = \varnothing$. Suppose, on the contrary, that $\chi \in \widetilde{M}$ and η satisfies condition vi 1 (as such η, we can choose any η_x from vi). Then, $z := \hat{\eta}(\chi) \in \eta(\Gamma)$ and the set $\delta_\Gamma \cap \hat{\eta}^{-1}(\{z\}) = \{\delta_x \mid \eta(x) = z\}$ is finite. Denote all characters from $\delta_\Gamma \cap \hat{\eta}^{-1}(\{z\})$ by $\delta_{x_1}, \ldots, \delta_{x_l}$. Choose η_{x_k} from condition vi in such a way that $\eta_{x_k}(x_k) = z$ (this condition can always be satisfied since η_x in vi are determined up to a constant). In view of vi 3, any $\zeta \in \operatorname{Ker}\Upsilon$ can be represented as $\zeta = c_{\zeta,k}e + \zeta'_k$, where $\Upsilon_{\eta_{x_k},z}(\zeta'_k) = 0$. Then $\tilde{\zeta}'_k := \frac{\zeta'_k}{\eta_{x_k} - ze}$ belongs to $\operatorname{Ker}\Upsilon = \mathfrak{A} \cap C^\infty(\Gamma; \mathbb{C})$, whence $\zeta'_k(x_k) = 0$ and $c_{\zeta,k} = \zeta(x_k)$. Thus, $\zeta = \zeta(x_k)e + (\eta_{x_k} - ze)\tilde{\zeta}'_k$ with $\tilde{\zeta}'_k \in \operatorname{Ker}\Upsilon$ and we have

$$\chi(\zeta) - \zeta(x_k) = [\chi(\eta_{x_k}) - z]\, \chi(\tilde{\zeta}'_k).$$

In particular, if $\chi(\eta_{x_k}) = z$, then χ coincides with δ_{x_k} on the dense set $\operatorname{Ker}\Upsilon$ in $\mathfrak{A}$ and, hence, on the whole $\mathfrak{A}$. Thus, the embedding $\chi \in \widetilde{M}$ shows that $\chi(\eta_{x_k}) \neq z$ for any

$k = 1, \ldots, l$. Then the element

$$\theta = \prod_{k=1}^{l} [\, \eta_{x_k} - \eta_{x_k}(x_k)\, e\,]$$

satisfies $\chi(\theta) \neq 0$ and $\theta(x_k) = 0$ for any $k = 1, \ldots, l$.

Denote

$$\eta_{\varepsilon,\varphi} := \eta + \varepsilon^2 e^{i\varphi} \chi(\theta)^{-1}\theta,$$

where $\varepsilon > 0$ and $\varphi \in [0, 2\pi)$. Due to condition vi 1, the function $\eta - ze$ has a zero of multiplicity not more than $m < \infty$ at each x_k, and there are no other zeros of $\eta - ze$ on Γ. Thus, the pre-image $\eta^{-1}(B_\varepsilon)$ of the $\varepsilon-$neighborhood U_ε of z is contained in $O(\varepsilon^{1/m})$-neighborhood of the set $\{x_1, \ldots, x_l\}$ in Γ. Therefore,

$$\begin{aligned} |\eta_{\varepsilon,\varphi}(x) - \eta(x)| &= \varepsilon^2 |\chi(\theta)^{-1}\theta(x)| = \varepsilon^2 |\,\theta(x_k) + O(\mathrm{dist}\{x, x_k\})| \\ &= \varepsilon^2 |0 + O(\varepsilon^{1/m})| = O(\varepsilon^{2+1/m}), \end{aligned} \tag{16}$$

where x_k is chosen to be the nearest point to x. For $x \in \Gamma \backslash \eta^{-1}(B_\varepsilon)$, we have $|\eta(x)| \geq \varepsilon$, whence

$$|\eta_{\varepsilon,\varphi}(x)| = |\eta(x) + \varepsilon^2 O(1)| \geq \varepsilon - O(\varepsilon^2). \tag{17}$$

Estimates (16),(17) show that, for sufficiently small ε, the set $B_{\varepsilon/2}$ does not intersect with the fragment $\eta_{\varepsilon,\varphi}(\Gamma \backslash \eta^{-1}(B_\varepsilon))$ of $\eta_{\varepsilon,\varphi}(\Gamma)$ while the fragment $\eta_{\varepsilon,\varphi}(\eta^{-1}(B_\varepsilon))$ is contained in $O(\varepsilon^{2+1/m})-$neighborhood of $\eta(\eta^{-1}(B_\varepsilon))$. Thus, it is possible to choose ε and φ such that $\hat{\eta}_{\varepsilon,\varphi}(\chi) = \chi(\eta_{\varepsilon,\varphi}) = z + \varepsilon^2 e^{i\varphi} \notin \eta_{\varepsilon,\varphi}(\Gamma)$. This means that $\chi \in \hat{\eta}_{\varepsilon,\varphi}^{-1}(\mathbb{C} \backslash \eta(\Gamma)) \subset M'$, so that we arrive at the contradiction and prove that $\widetilde{M} = \varnothing$.

- Now, we endow δ_Γ with coordinates. Let δ_x ($x \in \Gamma$) be an arbitrary character from δ_Γ. Consider the map

$$\hat{\eta}_x \,:\, \hat{\eta}_x^{-1}(U_x) \mapsto \mathbb{C},$$

where η_x, U_x are the same as in condition vi. Due to condition vi 2, for any z from one connected component $U_{x,1}$ of $U_x \backslash \eta_x(\Gamma)$ we have $\frac{1}{\eta_x - ze} \in \mathrm{Ker}\Upsilon \subset \mathfrak{A}$. Hence $z \notin \mathrm{Sp}_{\mathfrak{A}}\eta_x = \hat{\eta}_x(M)$. Now, suppose that $z \in U_x \backslash U_{x,1}$ is arbitrary and $\zeta \in \mathrm{Ker}\Upsilon$. By vi 2, we have $\frac{1}{\eta_x - ze} \notin \mathrm{Ker}\Upsilon$, i.e., either $z \in \eta_x(\Gamma)$ or $\frac{1}{\eta_x - ze}$ is smooth on Γ and does not belong to $\mathfrak{A}$. Thus, $z \in \mathrm{Sp}_{\mathfrak{A}}\eta_x = \hat{\eta}_x(M)$. In addition, due to vi 3, there exists a unique $c_{\zeta,z} \in \mathbb{C}$ such that

$$\frac{\zeta - c_{\zeta,z}e}{\eta_x - ze} =: \tilde{\zeta}_z \in \mathrm{Ker}\Upsilon \subset \mathfrak{A}. \tag{18}$$

By the same reason, there exists a unique $c'_{\zeta,z} \in \mathbb{C}$ such that

$$\frac{\tilde{\zeta}_z - c'_{\zeta,z}e}{\eta_x - ze} =: \tilde{\zeta}'_z \in \mathrm{Ker}\Upsilon \subset \mathfrak{A}. \tag{19}$$

If $\chi \in \hat{\eta}_x^{-1}(\{z\})$, then we have

$$\begin{aligned}\hat{\zeta}(\chi) = \chi(\zeta) = \chi(c_{\zeta,z}e + (\eta_x - ze)\tilde{\zeta}_z) &= c_{\zeta,z} + (\chi(\eta_x) - z)\chi(\tilde{\zeta}_z) \\ = c_{\zeta,z} + (\hat{\eta}_x(\chi) - z)\chi(\tilde{\zeta}_z) &= c_{\zeta,z}.\end{aligned} \tag{20}$$

Since $\zeta \in \mathrm{Ker}\Upsilon$ is arbitrary and $\mathrm{Ker}\Upsilon$ is dense in $\mathfrak{A}$, (20) means that $\chi(\zeta) = \chi'(\zeta)$ for any $\chi, \chi' \in \hat{\eta}_x^{-1}(z)$ and $\zeta \in \mathfrak{A}$. Thus, $\sharp\, \hat{\eta}_x^{-1}(\{z\}) = 1$ and the map $\hat{\eta}_x : \hat{\eta}_x^{-1}(U_x) \mapsto U_x \backslash U_{x,1}$ is a bijection. In addition,

$$c_{\zeta,z} = \hat{\zeta} \circ \hat{\eta}_x^{-1}(z) = [\hat{\eta}_x^{-1}(z)](\zeta)$$

and $z \mapsto c_{\zeta,z}$ is a bounded function on $U_x \backslash U_{x,1}$ due to (13). By the same reason, for fixed $z \in U_x \backslash U_{x,1}$, the function

$$z' \mapsto [\hat{\eta}_x^{-1}(z')](\tilde{\zeta}'_z)$$

is also bounded on $U_x \backslash U_{x,1}$. Now, (18) implies

$$\begin{aligned}c_{\zeta,z'} = [\hat{\eta}_x^{-1}(z')](\zeta) &= [\hat{\eta}_x^{-1}(z')](c_{\zeta,z}e + (\eta_x - ze)\tilde{\zeta}_z) \\ &= c_{\zeta,z} + (z' - z)[\hat{\eta}_x^{-1}(z')](\tilde{\zeta}_z).\end{aligned}$$

Hence, the function $z \mapsto c_{\zeta,z}$ is continuous on $U_x \backslash U_{x,1}$. In view of (19),

$$\begin{aligned}\frac{c_{\zeta,z'} - c_{\zeta,z}}{z' - z} = [\hat{\eta}_x^{-1}(z')](\tilde{\zeta}_z) &= [\hat{\eta}_x^{-1}(z')](c'_{\zeta,z}e + (\eta_x - ze)\tilde{\zeta}'_z) \\ &= c'_{\zeta,z} + (z' - z)[\hat{\eta}_x^{-1}(z')](\tilde{\zeta}'_z).\end{aligned}$$

Therefore, there exists $\lim\limits_{z' \to z} \frac{c_{\zeta,z'} - c_{\zeta,z}}{z' - z} = c'_{\zeta,z}$. So, it is proved that, for any $\zeta \in \mathrm{Ker}\Upsilon$, the function $z \mapsto c_{\zeta,z} = \hat{\zeta} \circ \hat{\eta}_x^{-1}(z)$ is holomorphic on $U_x \backslash \overline{U_{x,1}}$ and continuous on $U_x \backslash U_{x,1}$. By the definition of the Gelfand topology, the map $\eta_x : \hat{\eta}_x^{-1}(U_x) \mapsto U_x \backslash U_{x,1}$ is homeomorphism. So, any character $\delta_x \in \delta_\Gamma$ is coordinatizable in the following sense: there exists a neighborhood $V := \hat{\eta}_x^{-1}(U_x)$ (in the Gelfand topology) of δ_x and the local coordinate $\hat{\eta}_x : V \mapsto U_x \backslash U_{x,1}$ in which all functions $\hat{\zeta}$ ($\zeta \in \mathrm{Ker}\Upsilon$) are holomorphic on $U_x \backslash \overline{U_{x,1}}$ and continuous up to the (smooth) curve $U_x \cap \eta_x(\Gamma)$. Note that, in view of vi 1, the map $\eta^{-1}(U_x) \ni x' \to (\Re\eta_x(\delta_{x'}), \Im\eta_x(\delta_{x'})$ is a diffeomorphism.

- We have proved above that for each character $\chi \in M$ there exist an open neighborhood V_χ and a homeomorphism $\kappa_\chi : V_\chi \mapsto U_\chi \subset \mathbb{C}$ such that

1. the set $U'_\chi := \kappa_\chi(V_\chi\backslash\chi(\delta_\Gamma))$ is open,
2. $U_\chi\backslash U'_\chi \subset \partial U_\chi$ is empty or it is the fragment of smooth curve,
3. each function $\hat{\zeta}\circ\kappa_\chi^{-1}$ ($\zeta\in\mathrm{Ker}\Upsilon$) is holomorphic on U'_χ and continuous differentiable up to $U'_\chi\subset\partial U_\chi$.

Now, we construct a biholomorphic atlas on M using $\{V_\chi,\kappa_\chi\}_{\chi\in M}$. The collection $\{V_\chi\}_{\chi\in M}$ is an open cover of M and, since M is compact, there exists a finite subcover $\{V_{\chi_k}\}_{k=1}^L$. Denote $V_k := V_{\chi_k}$ and $\kappa_k := \kappa_{\chi_k}$. Suppose that $V_k\cap V_l\neq\varnothing$ and denote $W_k := \kappa_k((V_k\cap V_l)\backslash\delta_\Gamma)$, $W_l := \kappa_l((V_k\cap V_l)\backslash\delta_\Gamma)$. Choose an arbitrary nonconstant $\zeta\in\mathrm{Ker}\Upsilon$ (for example, one of η_x from condition iii), then $\hat{\zeta}\circ\kappa_k^{-1}$, $\hat{\zeta}\circ\kappa_l^{-1}$ are holomorphic on W_k and W_l, respectively. In particular, any zero of $\nabla\Re(\hat{\zeta}\circ\kappa_k^{-1})$ on W_k is isolated. If $\kappa_k(\chi)\in W_k$ does not coincide with zero of $\nabla\Re(\hat{\zeta}\circ\kappa_k^{-1})$, then there exists the neighborhood V' of χ such that $\zeta\circ\chi_k^{-1}:\ \kappa_k(V')\mapsto\zeta\circ\chi_k^{-1}(W')$ is biholomorphic map. So, the function

$$\kappa_k\circ\kappa_l^{-1}=\kappa_k\circ\zeta^{-1}\circ\zeta\circ\kappa_l=(\zeta\circ\kappa_k^{-1})^{-1}\circ(\zeta\circ\kappa_l^{-1})$$

is holomorphic on $\kappa_l(V')$. So, $\kappa_k\circ\kappa_l^{-1}$ is holomorphic on W_l except for only some isolated points. Since $\kappa_k\circ\kappa_l^{-1}$ is continuous on W_l, we find that $\kappa_k\circ\kappa_l^{-1}$ is holomorphic on the whole W_l. The same reasoning shows that $\kappa_l\circ\kappa_k^{-1}$ is holomorphic on W_k and, thus, the transition function $\kappa_k\circ\kappa_l^{-1}$ is biholomorphic. So, we have proved that $\{V_k := V_{\chi_k},\kappa_k := \kappa_{\chi_k}\}_{k=1}^L$ is a biholomorphic atlas on M. Endowed with this atlas, M is a Riemann surface with boundary δ_Γ. Moreover, the map $\delta\colon x\mapsto\delta_x$ is a diffeomorphism from Γ to δ_Γ. In what follows, we identify Γ and δ_Γ by means of the map δ.

• Now, we introduce a metric g and a rotation Φ on M that are consistent with the metrics and the tangent field γ on Γ. Endow M with the metric tensor $g'=\sum_{k=1}^L\psi_k g_k$, where $g_k^{ij}=\delta^{ij}$ in local coordinates κ_k, and $\{\psi_k\}_{k=1}^L$ is a partition of unity on M: $\psi_k\circ\kappa_l^{-1}$ is smooth for any l, $\psi_k\geq 0$, $\operatorname{supp}\psi_k\subset V_k$, and $\sum_{k=1}^L\psi_k=1$. Since the transition functions are biholomorphic, the tensor g' is of the form $\sum_{k=1}^L\psi_k|\nabla\Re(\kappa_k\circ\kappa_l^{-1})|^2\delta^{ij}$ in any local coordinates κ_k. Tensor g' induces a new metrics $d\gamma'=q(x)d\gamma$ on $\Gamma\equiv\delta_\Gamma$, where the function $q>0$ is smooth on Γ due to condition **vi** 1. Introducing a smooth conformal multiplier ρ, such that $\rho=q^{-1}$ on Γ, we obtain the new metric tensor $g=\rho g'$ which is consistent with the original metric on Γ.

Choose a continuous family of rotations $\{\Phi_x\in\mathrm{End}T_xM\mid x\in M\}$,

$$g(\Phi_x a,\Phi_x b)=g(a,b),\quad g(\Phi_x a,a)=0,\qquad\forall a,b\in T_xM,\ x\in M$$

such that $\Phi_1^1=\Phi_2^2=0$, $\Phi_2^1=-\Phi_1^2=1$ in local coordinates $x_1=\Re\kappa_k$, $x_2=\Im\kappa_k$. For any k and $\zeta\in\mathrm{Ker}\Upsilon$, the function $\hat{\zeta}\circ\kappa_k$ is holomorphic, whence

$$\nabla\Im\hat{\zeta}=\Phi\nabla\Re\hat{\zeta}\ \text{in}\ M\backslash\Gamma. \tag{21}$$

So, any function $\hat{\zeta}$ ($\zeta\in\mathrm{Ker}\Upsilon$) is holomorphic on $M\backslash\Gamma$ (in the sense of Cauchi–Riemann conditions (21)) and continuous up to Γ. In particular, any functions $u=\Re\hat{\zeta}$

and $v = \Im\hat{\zeta}$ are harmonic in $M\backslash\Gamma$ and continuous up to Γ. Let us show that u, v are smooth up to Γ. Denote by u^f and u^h the solutions to (3), (4) with $f = \Re\zeta$ and $h = \Im\zeta$; these solutions are smooth because $\zeta \in C^\infty(\Gamma; \mathbb{C})$. Since $\hat{\zeta} = \zeta$ on Γ, the functions $u - u^f$, $v - v^h$ are harmonic in $M\backslash\Gamma$, continuous up to Γ and have zero traces on Γ. Thus, due to the maximum principle, $u = u^f$, $v = u^h$, and $\hat{\zeta} = u^f + iu^h$ is smooth up to Γ.

Let ν be the outward normal on Γ. Then $\gamma' = \Phi\nu$ is the tangent field on Γ and, hence, it coincides with $s\gamma$, where $s = 1$ or -1. Choose some $\eta \in \mathrm{Ker}\Upsilon$ and $z \in \mathrm{Sp}_{\mathfrak{A}}\eta\backslash\eta(\Gamma)$, then $\hat{\eta} - ze$ have at least one zero on M. Consider the integral

$$\int\limits_\Gamma \frac{1}{2\pi i}\frac{\partial_{\gamma'}\hat{\eta}}{\hat{\eta} - z}d\gamma = s\int\limits_\Gamma \frac{1}{2\pi i}\frac{\partial_\gamma \eta}{\eta - ze}d\gamma.$$

In view of the argument principle, the integral in the left-hand side coincides with the number of zeroes of $\hat{\eta}$ counted with their multiplicities, and, thus, it is positive. The integral in the right-hand side is positive in view of (7). Therefore, $s = 1$ and $\gamma = \gamma'$.

- Suppose that $f \in \mathrm{Ker}(\partial_\gamma + \Lambda J\Lambda)$. Denote $h := J\Lambda f (= J\Lambda J\partial_\gamma f)$ and $\zeta = f + ih$. Then $\partial_\gamma h = \Lambda f$, $\partial_\gamma f = -\Lambda h$ and, hence, $\zeta \in \mathrm{Ker}\Upsilon$. The proved above implies $\Re\hat{\zeta} = u^f$, $\Im\hat{\zeta} = u^h$, where u^f, u^h are solutions to (3), (4). In particular, $\partial_\nu u = \Lambda_g f$, where Λ_g is the DN-map of the above constructed (M, g). Moreover, the Cauchy–Riemann condition (21) holds. Passing in (21) to the trace on Γ, we obtain

$$\Lambda_g f = \partial_\nu u^f = \partial_\gamma h = \Lambda f, \qquad \partial_\nu u^h = -\partial_\gamma f.$$

Since f is arbitrary, we have proved that $\mathrm{Ker}(\partial_\gamma + \Lambda J\Lambda) \subset \mathrm{Ker}(\partial_\gamma + \Lambda_g J\Lambda_g)$ and $\Lambda f = \Lambda_g f$ for any $f \in \mathrm{Ker}(\partial_\gamma + \Lambda J\Lambda)$.

- Let us show that $\mathrm{Ker}(\partial_\gamma + \Lambda J\Lambda) = \mathrm{Ker}(\partial_\gamma + \Lambda_g J\Lambda_g)$. By iv, the dimension q of the factor-space $\mathrm{Ker}(\partial_\gamma + \Lambda_g J\Lambda_g)/\mathrm{Ker}(\partial_\gamma + \Lambda J\Lambda)$ is finite. In view of (5), we have

$$\mathrm{Ker}\Upsilon = \{f + iJ\Lambda f + ic \mid f \in \mathrm{Ker}(\partial_\gamma + \Lambda J\Lambda),\ c \in \mathbb{R}\}. \tag{22}$$

Denote by $\mathfrak{A}^\infty$ the algebra of traces of all holomorhic smooth functions on M; obviously, $\mathrm{Ker}\Upsilon$ is a subalgebra of $\mathfrak{A}^\infty$. From Cauchy–Riemann conditions on Γ, the representation

$$\mathfrak{A}^\infty := \{f + iJ\Lambda f + ic \mid f \in \mathrm{Ker}(\partial_\gamma + \Lambda_g J\Lambda_g),\ c \in \mathbb{R}\}$$

is valid. Comparison of the last two formulas shows that the algebra $\mathfrak{A}^\infty$ is a finite-dimensional extension of the algebra $\mathrm{Ker}\Upsilon$, and dimension of the factor-space $\mathfrak{A}^\infty/\mathrm{Ker}\Upsilon$ is equal to q.

Suppose that $q > 0$ and choose the elements $\theta_1, \dots, \theta_q \in \mathfrak{A}^\infty$ linearly independent modulo $\operatorname{Ker}\Upsilon$. Then any $\theta \in \mathfrak{A}^\infty$ can be represented as

$$\theta = \sum_{k=1}^{q} c_k(\theta)\theta_k + \tilde{\theta}, \tag{23}$$

where $c_q(\theta) \in \mathbb{C}$ and $\tilde{\theta} \in \operatorname{Ker}\Upsilon$. Take any nonconstant $\eta \in \operatorname{Ker}\Upsilon$. Representation (23) implies

$$\eta\theta_l = \sum_{k=1}^{q} T_{kl}\,\theta_k + \tilde{\theta}_l, \tag{24}$$

where T is a complex $q \times q-$matrix and $\tilde{\theta}_l \in \operatorname{Ker}\Upsilon$. Choose an arbitrary eigenpair $\lambda, X = (X_1, \dots, X_q)^{\mathrm{tr}}$ of T and denote

$$\Theta := \sum_{k} X_l\theta_l, \quad \tilde{\Theta} := \sum_{l=1}^{q} X_l\tilde{\theta}_l.$$

Relation (24) yields

$$\eta\Theta = \sum_{l=1}^{q} X_l\eta\theta_l = \sum_{k=1}^{q}\left(\sum_{l=1}^{q} T_{kl}X_k\right)\theta_k = \lambda\sum_{k=1}^{q} X_k\theta_k + \sum_{l=1}^{q} X_l\tilde{\theta}_l = \lambda\Theta + \tilde{\Theta}.$$

Note that $\eta - \lambda e$ does not vanish identically on any segment Γ' of Γ of non-zero length (indeed, since $\hat{\eta}$ is holomorphic and smooth on M, the equality $\eta = \lambda e$ on Γ' implies $\eta = \lambda e$ on the whole Γ). So,

$$\Theta := \frac{\tilde{\Theta}}{\eta - \lambda e} \tag{25}$$

holds on Γ, where both numerator and denominator are elements of $\operatorname{Ker}\Upsilon$. Note that $X \neq 0$ and $\Theta \notin \operatorname{Ker}\Upsilon$. Similarly, representation (23) yields

$$\Theta^l = \sum_{k=1}^{q} N_{kl}\theta_k + \tilde{\Theta}_l, \quad l = 1, \dots, q, \tag{26}$$

where N is a complex $q \times q-$matrix and $\tilde{\Theta}_l \in \operatorname{Ker}\Upsilon$.

If $\det N = 0$, then there exists a non-zero $Y = (Y_1, \dots, Y_q)^{\mathrm{tr}} \in \operatorname{Ker} N$ and the polynomial $P(\Theta) = \sum_{l=1}^{q} Y_l\Theta^l$ admits the following representation

$$P(\Theta) = \sum_{k=1}^{q}\left(\sum_{l=1}^{q} N_{kl}Y_l\right)\theta_k + \sum_{l=1}^{q} Y_l\tilde{\Theta}_l = 0 + \sum_{l=1}^{q} Y_l\tilde{\Theta}_l.$$

Therefore $P(\Theta) \in \operatorname{Ker}\Upsilon$ and, due to (25) and condition ii, we have $\Theta \in \operatorname{Ker}\Upsilon$, which leads to a contradiction.

If $\det N \neq 0$ and N' is the matrix inverse to N, then (26) implies

$$\theta_s - \sum_{l=1}^{q} N'_{ls}\Theta^l = \sum_{l=1}^{q} N'_{ls}\tilde{\Theta}_l \in \operatorname{Ker}\Upsilon.$$

This means that $\Theta, \Theta^2 \ldots, \Theta^q$ are linearly independent modulo $\operatorname{Ker}\Upsilon$. So, we can assume that $\theta_k = \Theta_k$. Now, formula (23) provides

$$\mathcal{R}(\Theta) := \Theta^{q+1} - \sum_{k=1}^{q} c_k\theta_k \in \operatorname{Ker}\Upsilon,$$

where $c_k \in \mathbb{C}$. Since the polynomial $\mathcal{R}$ is of degree $q + 1 > 0$, the inclusion $\mathcal{R}(\Theta) \in \operatorname{Ker}\Upsilon$, formula (25) and condition **ii** yield $\Theta \in \operatorname{Ker}\Upsilon$. This contradiction means that $\mathfrak{A}^\infty = \operatorname{Ker}\Upsilon$ and $q = 0$. Thus, it is proved that $\operatorname{Ker}\Upsilon$ is the algebra of traces of all holomorhic smooth functions on M and $\operatorname{Ker}(\partial_\gamma + \Lambda J\Lambda) = \operatorname{Ker}(\partial_\gamma + \Lambda_g J\Lambda_g)$. In particular, from (10) it follows that $\dim(\partial_\gamma + \Lambda J\Lambda)C^\infty(\Gamma;\mathbb{R}) = 1 - \mathscr{X}(M)$, where $\mathscr{X}(M)$ is the Euler characteristic of M.

• Thus, we have proved that Λ coincides with DN-map Λ_g of the surface (M, g) on the subspace

$$\mathfrak{K} := \operatorname{Ker}(\partial_\gamma + \Lambda J\Lambda) = \operatorname{Ker}(\partial_\gamma + \Lambda_g J\Lambda_g) \tag{27}$$

of codimension $r := 1 - \mathscr{X}(M)$ in $C^\infty(\Gamma;\mathbb{R})$. To complete the proof of sufficiency, it remains to show that $\Lambda f_1 = \Lambda_g f_1, \ldots, \Lambda f_r = \Lambda_g f_r$, where $f_1, \ldots, f_r$ are some functions from $C^\infty(\Gamma;\mathbb{R})$ linearly independent modulo $\mathfrak{K}$. Before that, recall the terminology associated with vector fields on the Riemannian manifolds and some well-known facts.

The vector fields are the TM_x-valued functions on M (the sections of TM). A field of the form $b = \nabla_g\varphi$ is called *potential*, φ being a potential. A field a is *harmonic* if $\operatorname{div}_g a = \operatorname{div}_g(\Phi a) = 0$ holds. The rotation Φ preserves harmonicity. Each harmonic field is *locally* potential. If $b = \nabla_g\varphi$ is harmonic then the potential φ is a harmonic function: $\Delta_g\varphi = 0$, the opposite being also true.

So, let $f_1, \ldots, f_r$ be linearly independent modulo $\mathfrak{K}$. Denote the solution of problem (3), (4) with $f = f_j$ by u_j. The vector fields $a_j := \Phi\nabla_g u_j$ are harmonic in M. Note that any non-zero linear combination of a_j is not a potential field in M. Indeed, if $\sum_{j=1}^r c_j a_j = \nabla_g v$, then the function $w := u + iv$, where $u := \sum_{j=1}^r c_j u_j$, is holomorphic in M. Then $w|_\Gamma \in \mathfrak{A}^\infty(M) = \operatorname{Ker}\Upsilon$ and $\Re w|_\Gamma = \sum_{j=1}^r c_j f_j \in \mathfrak{K}$ in view of (22). Since f_k are linearly independent modulo $\mathfrak{K}$, all c_j equal zero.

Although a_j are not potential on M, they can be represented as gradients of some multi-valued functions V_j which are defined on an appropriate covering $\mathbb{M}$ of the surface M. The covering $\mathbb{M}$ is constructed in the following way. Let $\mathcal{D}$ be a surface diffeomorphic to an open disk in $\mathbb{R}^2$ and such that $\partial\mathcal{D} = \Gamma$. Identifying the boundaries of M and $\mathcal{D}$, we obtain the closed compact surface $M' = M \cup \mathcal{D}$ of genus

$$\operatorname{gen} M' = 1 - \frac{\mathscr{X}(M')}{2} = 1 - \frac{\mathscr{X}(M)+1}{2} = \frac{r}{2}.$$

It is well known that the metric tensor g and rotation Φ on M can be extended to the (smooth) metric tensor g' and rotation Φ' on the whole M'.

Let $\mathbb{M}'$ be the universal covering of M' (see, for definition, §5, [5]), that is a simply connected Riemann surface, and let $\pi' : \mathbb{M}' \mapsto M'$ be the projection, that is a local homeomorphism. Tensor g' and rotation Φ' on M' induce the tensor $\mathrm{g}' := \pi'_* g'$ and the rotation $\tilde{\Phi} := \pi'_* \Phi'$ on $\mathbb{M}'$. As a result, $\pi' : (\mathbb{M}', \mathrm{g}') \mapsto (M', g')$ turns out to be a local isometry. At last, we get the required covering for (M, g, Φ) as the collection $(\mathbb{M}, \pi, \mathrm{g}, \dot{\Phi})$, where $\mathbb{M} := \mathbb{M}' \backslash \pi'^{\,-1}(\mathcal{D})$ is the surface with the boundary $\partial\mathbb{M} = \pi'^{\,-1}(\Gamma)$, $\mathrm{g} := \mathrm{g}'|_{\mathbb{M}}$, $\dot{\Phi} := \tilde{\Phi}|_{\mathbb{M}}$, and $\pi := \pi'|_{\mathbb{M}}$.

Recall that the solutions u_j and fields a_j correspond to the functions $f_1, \dots, f_r$ which are linearly independent modulo $\mathfrak{K}$ (see (27)). Introduce the vector fields $A_j := \pi_* a_j = \dot{\Phi}\nabla_{\mathrm{g}}(u_j \circ \pi)$ and the functions V_j on $\mathbb{M}$ such that

$$V_j(x) = \int_{\mathcal{L}} \mathrm{g}(A_j, l)\, dl \in \mathbb{R},$$

where $\mathcal{L}$ is an arbitrary curve in $\mathbb{M}$ that connects a fixed point $x_0 \in \mathbb{M}$ with a point x. In what follows, we denote by l and dl the unit tangent vector and the length element on the curve, respectively. Since $\mathbb{M} = \mathbb{M}' \backslash \pi'^{\,-1}(\mathcal{D})$ is no longer simply connected, we need to check that V_j are single-valued on $\mathbb{M}$. To this end, it suffices to show that $\int_{\tilde{\Gamma}} \mathrm{g}(A_j, l)\, dl = 0$ for any connected component $\tilde{\Gamma}$ of $\pi^{\,-1}(\Gamma)$. Since $\tilde{\Gamma}$ is isometric to Γ, we need to check only that $\int_{\Gamma} g(a_j, \gamma)\, d\gamma = 0$. By the Green formula, we have

$$\int_{\Gamma} g(a_j, \gamma)\, d\gamma = \int_{\Gamma} g(\Phi\nabla_g u_j, \gamma)\, d\gamma = \int_{\Gamma} \partial_\nu u_j\, d\gamma = \int_M \Delta_g u_j\, dx = 0$$

in view of harmonicity of u_j. So, we have constructed the functions V_j such that $\nabla_{\mathrm{g}} V_j = A_j = \dot{\Phi}\nabla_{\mathrm{g}}(u_j \circ \pi)$ holds on $\mathbb{M}$. This means that the functions

$$W_j := u_j \circ \pi + i\, V_j, \qquad j = 1, \dots, r \tag{28}$$

are holomorphic on $(\mathbb{M}, \mathrm{g})$, whereas the Cauchy–Riemann conditions $\nabla_{\mathrm{g}} \Im W_j = \dot{\Phi}\nabla_{\mathrm{g}} \Re W_j$ hold.

• We are going to show that the functions $f_1, \dots, f_r$ can be chosen in such a way that $e^{W_j} = w_j \circ \pi$, where w_j are holomorphic functions in M.

Introduce the groups

$$\begin{aligned}
\mathrm{Deck}(\mathbb{M}/M) &:= \{\phi \mid \phi \text{ is automorphism of } \mathbb{M},\ \pi \circ \phi = \pi\},\\
\mathrm{Deck}(\mathbb{M}'/M') &:= \{\phi' \mid \phi \text{ is automorphism of } \mathbb{M}',\ \pi' \circ \phi' = \pi'\}
\end{aligned}$$

of fiber-wise automorphisms of $\mathbb{M}$ and $\mathbb{M}'$, respectively (see, e.g., [5], 5.4). Obviously, if $\phi' \in \mathrm{Deck}(\mathbb{M}'/M')$, then $\phi'|_{\mathbb{M}} \in \mathrm{Deck}(\mathbb{M}/M)$. Conversely, if $\phi \in \mathrm{Deck}(\mathbb{M}/M)$, then it can be lifted to $\phi' \in \mathrm{Deck}(\mathbb{M}'/M')$ such that $\phi'|_{\mathbb{M}} = \phi$. Indeed, if x belongs to a connected component $\tilde{\mathcal{D}}$ of $\pi'^{-1}(\mathcal{D})$, then $\phi'(x)$ is uniquely determined by its projection $\pi'(\phi'(x)) = \pi'(x)$ and by the fact that the boundary of the connected component of $\pi'^{-1}(\mathcal{D})$ containing $\phi'(x)$ must coincide with $\phi(\partial\tilde{\mathcal{D}})$. So, the map $\phi' \mapsto \beta\phi' = \phi'|_{\mathbb{M}}$ is an isomorphism of the groups $\mathrm{Deck}(\mathbb{M}'/M')$ and $\mathrm{Deck}(\mathbb{M}/M)$.

Denote by $\pi_1(M')$ the fundamental group of M' and by $[L]$ the homotopy class of a closed curve L in M'. In view of Proposition 5.6, [5], the groups $\pi_1(M')$ and $\mathrm{Deck}(\mathbb{M}'/M')$ are isomorphic. The isomorphism

$$\alpha : \mathrm{Deck}(\mathbb{M}'/M') \mapsto \pi_1(M')$$

is constructed as follows. Let $\phi' \in \mathrm{Deck}(\mathbb{M}'/M')$. Choose an arbitrary point $x \in \mathbb{M}'$ and a curve $\mathcal{L}_{\phi'}$ which connects x to $\phi'(x)$. Then $\pi'(\mathcal{L}_{\phi'})$ is a closed curve in M' due to the equality $\pi'(\phi'(x)) = \pi'(x)$. It turns out that the homotopy class $[\pi'(\mathcal{L}_{\phi'})]$ of the curve $\pi'(\mathcal{L}_{\phi'})$ does not depend on the choice of $\mathcal{L}_{\phi'}$ and x. The required isomorphism α is defined by the rule

$$\alpha(\phi') := [\pi'(\mathcal{L}_{\phi'})].$$

The map $\alpha \circ \beta^{-1}$ is an isomorphism of groups $\mathrm{Deck}(\mathbb{M}/M)$ and $\pi_1(M')$.

Since M' is a surface of the genus $\mathrm{gen} M' = r/2$, there are $2\,\mathrm{gen} M' = r$ generators $[L_1], \dots, [L_r]$ of the fundamental group $\pi_1(M')$. Note that, since $\mathcal{D}$ is simply connected, we can deform the curves L_j, preserving their homotopy class, in such a way that any L_j does not intersect $\mathcal{D}$. Thus, we assume that $L_1, \dots, L_r \subset M$. Since the groups $\mathrm{Deck}(\mathbb{M}/M)$ and $\pi_1(M')$ are isomorphic, the automorphisms $\phi_j := \beta \circ \alpha^{-1}([L_j])$, $j = 1, \dots, r$, generate the group $\mathrm{Deck}(\mathbb{M}/M)$. Therefore, a function V on $\mathbb{M}$ can be represented as $V = v \circ \pi$ if and only if $V \circ \phi_j = V$, $j = 1, \dots, r$.

Suppose that V is a function on M such that $\nabla_{\mathrm{g}} V = A := \pi_* a$, where a is a vector field on M. Then

$$V(\phi_j(x)) - V(x) = \int_{\mathcal{L}_j} \mathrm{g}(A, l)\, dl,$$

where $\mathcal{L}_j$ connects x to $\phi_j(x)$. Since the field $A = \pi_* a$ is invariant under action of the group $\mathrm{Deck}(\mathbb{M}/M)$, the right-hand side does not depend on x and we can choose $\mathcal{L}_j$ to provide $\pi(\mathcal{L}_j) = L_j$. Then the difference $V(\phi_j(x)) - V(x)$ is equal to

$$T_j(a) := \int_{L_j} \mathrm{g}(a, l)\, dl.$$

Thus, $V = v \circ \pi$ and $a = \nabla_g v$ if and only if $T_1(a) = \cdots = T_r(a) = 0$.

Introduce the $r \times r$-matrix T with entries $T_{ij} = T_i(a_j)$. Recall that any non-zero linear combination $\sum_{j=1}^{r} c_j a_j$ is not potential field in M. This means that all

$T_i(\sum_{j=1}^r c_j a_j) = \sum_j T_{ij} c_j$ are zero if and only if $c_1 = \cdots = c_r = 0$. Thus, T is invertible. Denote $f'_s := 2\pi \sum_{l=1}^r R_{ls} f_l$, where $R = T^{-1}$. Then $f'_1, \ldots, f'_r$ are linear independent modulo $\mathfrak{K}$. Introduce the following new functions

$$V'_s = 2\pi \sum_{l=1}^r R_{ls} V_l, \quad W'_s = 2\pi \sum_{l=1}^r R_{ls} W_l,$$

that are determined by f'_s in the same way as V_s and W_s are determined by f_s (see (28)). Then $\nabla_g V'_s = 2\pi \sum_{l=1}^r R_{ls} A_l = \pi_* a'_s$, where $a'_s = 2\pi \sum_{l=1}^r R_{ls} a_l$ and

$$T_j(a'_s) = 2\pi \sum_{l=1}^r T_{jl} R_{ls} = 2\pi \delta_{js}$$

holds. By the latter, we have

$$V'_s \circ \phi_j - V'_s = W'_s \circ \phi_j - W'_s = 2\pi \delta_{js}, \qquad j = 1, \ldots, r.$$

Hence,

$$e^{W'_s \circ \phi_j} = e^{W'_s}$$

for any $j, s = 1, \ldots, r$. This means that $e^{W'_s}$ can be represented as $e^{W'_s} = w_s \circ \pi$, where w_s is a function on M. Since W'_s is holomorphic in $\mathbb{M}$, the function w_s is holomorphic in M. Replacing f_s by f'_s (what is the same, omitting 'prime' everywhere in the notation), we obtain $e^{W_j} = w_j \circ \pi$.

So, we have constructed the functions $f_1, \ldots, f_r$ with the properties claimed at the beginning of the paragraph.

• Since w_j is holomorphic on M, the function $\zeta_j := w_j|_\Gamma$ is an element of $\mathrm{Ker}\Upsilon$. We have

$$\log|w_j(\pi(x))| = \Re W_j(x) = u_j(\pi(x)), \qquad x \in \mathbb{M}.$$

In particular,

$$\log|w_j| = u_j \quad \text{and} \quad \log|\zeta_j| = f_j$$

holds on M and Γ respectively.

Since W_j is holomorphic on $\mathbb{M}$, the Cauchy–Riemann conditions yield

$$\begin{aligned}(\partial_\nu u_j) \circ \pi &= \partial_\nu (u_j \circ \pi) = \partial_\nu \Re W_j = \partial_\gamma \Im W_j = \partial_\gamma \Im \log(w_j \circ \pi) \\ &= \partial_\gamma \arg(w_j \circ \pi) = (\partial_\gamma \arg w_j) \circ \pi\end{aligned}$$

on $\pi^{-1}(\Gamma)$. This means that

$$\Lambda_g f_j = \partial_\nu u_j = \partial_\gamma \arg\zeta_j \tag{29}$$

holds on Γ. In the mean time, condition vii implies

$$\Lambda f_j = \Lambda \log|\zeta_j| = \partial_\gamma \arg\zeta_j. \tag{30}$$

Comparing (29) and (30), we obtain $\Lambda f_j = \Lambda_g f_j$ for any $j = 1, \dots, r$. Together with what was proved above, this means that $\Lambda = \Lambda_g$ and, hence, Λ is the DN-map of the surface (M, g).

The sufficiency of the conditions i–vii is established.

Theorem 1 is proved.

Acknowledgements The authors are grateful to the Referee for useful comments and criticism.

Declarations

Data Availibility Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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