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A Question About Invariant Subspaces and Factorization

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Abstract

We ask whether a connection between isometric functional calculus and factorization of linear functionals, known to hold for the case of a single contraction operator, persists in the case of commuting pairs—or, more generally, n-tuples—of contractions. A positive answer has consequences concerning the jointly invariant subspaces of the commuting operators.

We recall first that an algebra $\mathcal{A}$ of bounded linear operators on a complex Hilbert space $\mathcal{H}$ is said to have property $(\mathbb{A}_1)$ if every weak*-continuous functional $\varphi : \mathcal{A} \to \mathbb{C}$ can be represented as

$$\varphi(T) = \langle Tx, y\rangle, \quad T \in \mathcal{H},$$

for some vectors $x, y \in \mathcal{H}$. When one proves that a given algebra has property $(\mathbb{A}_1)$, one usually obtains an estimate of the form $\|x\|\|y\| \le C\|\varphi\|$, for some constant C, independent of φ.

Suppose now that $G \subset \mathbb{C}^d$ is a bounded open set for some $d \in \mathbb{N}$, and denote by $H^\infty(G)$ the Banach algebra consisting of all bounded holomorphic functions defined in G. The algebra $H^\infty(G)$ has a natural weak* topology such that a sequence in

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$H^\infty(G)$ converges if and only it converges pointwise and is uniformly bounded. We are interested in unital representations Φ from $H^\infty(G)$ to the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$. Recall that $\mathcal{B}(\mathcal{H})$ also has a natural weak* topology, arising from its duality with the trace class.

Problem Suppose that $\Phi : H^\infty(G) \to \mathcal{B}(\mathcal{H})$ is a unital algebra representation such that

1. $\|\Phi(u)\| = \|u\|_\infty$ for every $u \in H^\infty(G)$, and
2. Φ is weak*-to-weak* continuous.

Does it follow that the algebra $\{\Phi(u) : u \in H^\infty(G)\}$ has property $(\mathbb{A}_1)$?

The answer is known to be in the affirmative when $G \subset \mathbb{C}$ is a disk [4, 5]. Versions of this problem, usually with the stronger hypothesis of the existence of a dominating spectrum, were proved by several authors. We only mention here Eschmeier [6] for a rather general setting and Ambrozie–Müller [1] for a Banch space version in case G is a polydisk.

A particular case arises from considering a pair (T_1, T_2) of commuting contractions on $\mathcal{H}$. When these contractions are completely nonunitary, it was shown in [3] that there is a version of the Sz.-Nagy–Foias functional calculus that yields a representation of $H^\infty(\mathbb{D}^2)$, where $\mathbb{D} \subset \mathbb{C}$ is the unit disk. A different argument, along with a dilation of this representation, is given in [2] in the special case in which T_1 and T_2 are of class C_{00} in the sense of [8]. Our purpose in [2] was to give an affirmative answer to the above problem in this special case. Eschmeier [7] pointed out a subtle error in our argument, and therefore the problem must be considered to be open even in this particular case. For the record, we comment briefly on the nature of this error.

The argument of [2] relies on the fact that one can consider that there is some measure ν on the distinguished boundary $\mathbb{T}^2$ of $\mathbb{D}^2$ such that $\mathcal{H}$ can be viewed as a subspace of $L^2(\nu) \otimes \ell^2$ in such a way that the operators T_1 and T_2 are compressions to $\mathcal{H}$ of the operators of multiplication by the two coordinates on $\mathbb{T}^2$. Lemma 4.2 of [2] shows that, given $\varepsilon > 0$ and a Borel set $\sigma \subset \mathbb{T}^2$ with $\nu(\sigma) > 0$, there exists a function $u \in H^\infty(\mathbb{D}^2)$ such that

$$\nu(\{\zeta \in \mathbb{T}^2 : |u(\zeta) - \chi_\sigma(\zeta)| > \varepsilon\} < \varepsilon.$$

In other words, $|u|$ is close to 1 on most of σ and close to 0 on most of $\mathbb{T}^2 \backslash \sigma$. If $f \in \mathcal{H}$ is such that $\|\Phi(u)f\|$ is very close to $\|u\|_\infty \|f\|$, it follows that f is concentrated mostly on the set on which $|u|$ is close to 1. Then [2,Proposition 4.3] asserts, incorrectly, that f must be concentrated mostly on σ itself. Indeed, it may well be that much of f lives on a set $\omega \subset \mathbb{T}^2 \backslash \sigma$ where $|u|$ is close to 1. Finding an argument along these lines would require either finding u such that $|u| < \varepsilon$ almost everywhere on $\mathbb{T}^2 \backslash \sigma$, or an improvement in the basic factorization [2,Theorem 2.3]. The first alternative seems unlikely to succeed because σ may be, for instance, a one dimensional arc.

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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