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On the Location of Zeros of Polynomials

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Abstract

In this paper, we discuss the necessary and sufficient conditions for a polynomial $P(z)$ to have all its zeros inside the open unit disc. These results involve two associated polynomials namely, the derivative of the reciprocal polynomial of $P(z)$ and the reciprocal of the derivative of $P(z)$. We also derive some generalizations of the classical Theorem of Laguerre.

Keywords Polynomial · Reciprocal polynomial · Zero

Mathematics Subject Classification 30C15

1 Introduction and statements of results

In the literature, we can find a category of problems concerning the regions, mostly circular or annular, containing all the zeros of a polynomial. A classic solution to such a problem was first obtained by Cauchy [1], and subsequently, many related results appeared in the literature (see [13]). One such problem is to study, when and what class of polynomials has all their zeros in or outside a circular region.

Let $P(z)$ be a polynomial of degree n and z_0 be a complex number, and let $r > 0$. Then it is quite interesting to study the conditions on which $P(z)$ has some or all its zeros or no zeros in the disc $|z - z_0| \leq r$. For detailed information, we refer to

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the monograph by Marden [13]. By means of a suitable linear transformation, these problems can be reduced to the study of zeros in the unit disc centered at the origin. In other words, the number of zeros of $P(z)$ in $|z - z_0| \leq r$ equals the number of zeros of the polynomial $P^*(z) = P(z_0 + rz)$ satisfying $|z| \leq 1$. In this context, the study of location of zeros with respect to the unit disc is quite natural and simpler.

The location of zeros has been vastly studied with more focus on finding the number of zeros in a given domain. But we could not find any exclusive work on a necessary and sufficient condition for a polynomial to have all its zeros in the unit disc except the famous Schur–Cohn algorithm [2,14] which is very often used to decide whether a given polynomial is free of zeros in the closed unit disc. The Schur–Cohn algorithm with some additional hypotheses can also be used to determine the number of zeros in a circular region. Most of the results in this direction have used coefficients as the main parameters. The Eneström-Kakeya Theorem [5] and its generalizations like the one given by Govil and Rahman [7] are the classic and significant examples of this kind. In this paper, we derive two results on the necessary and sufficient conditions for a polynomial $P(z)$ to have all its zeros in the open unit disc in terms of a simple inequality involving the derivative of the reciprocal of a polynomial $P(z)$ and the reciprocal of the derivative of the polynomial $P(z)$.

Theorem 1.1 *Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n. If $P(z)$ has all its zeros in $|z| < 1$ then on $|z| = 1$,*

$$|R(z)| < |S(z)| \tag{1.1}$$

where

$$R(z) = \left(z^n P\left(\frac{1}{z}\right)\right)' + \left[\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right] z^{n-1} P\left(\frac{1}{z}\right),$$

and $S(z) = z^{n-1} P'\left(\frac{1}{z}\right)$.

If $P(z)$ is a polynomial of degree n having no zeros in $|z| \leq 1$, then $Q(z) = z^n P\left(\frac{1}{z}\right)$ has all its zeros in $|z| < 1$. Therefore applying Theorem 1.1 to $Q(z)$, we will obtain a necessary condition for a polynomial to have no zeros in the closed unit disc, which is given below.

Corollary 1.2 *Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n. If $P(z)$ has no zeros in $|z| \leq 1$ then on $|z| = 1$,*

$$|R^*(z)| < |S^*(z)| \tag{1.2}$$

where $R^*(z) = P'(z) + \left[\frac{|a_0| - |a_n|}{|a_o| + |a_n|}\right] z^{n-1} Q\left(\frac{1}{z}\right)$, $S^*(z) = z^{n-1} Q'\left(\frac{1}{z}\right)$ *and* $Q(z) = z^n P\left(\frac{1}{z}\right)$.

The sufficiency analogue of Theorem 1.1 can be established as follows.

Theorem 1.3 *Let $P(z)$ be a polynomial of degree n. Then $P(z)$ has all its zeros in $|z| < 1$ if in $|z| \leq 1$,*

$$|T(z)| < |S(z)| \tag{1.3}$$

where

$$T(z) = \left(z^n P\left(\frac{1}{z}\right)\right)',$$

and $S(z) = z^{n-1} P'\left(\frac{1}{z}\right)$.

Again applying Theorem 1.3 to the polynomial $Q(z) = z^n P\left(\frac{1}{z}\right)$, we will obtain the following result.

Corollary 1.4 *Let $P(z)$ be a polynomial of degree n. Then $P(z)$ has no zeros in $|z| \leq 1$ if in $|z| \leq 1$,*

$$|T^*(z)| < |S^*(z)| \tag{1.4}$$

where $T^*(z) = P'(z)$, $S^*(z) = z^{n-1} Q'\left(\frac{1}{z}\right)$ *and* $Q(z) = z^n P\left(\frac{1}{z}\right)$.

The well known Theorem of Laguerre [11] states that, if $P(z)$ is a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$$

has no zeros in $|z| < 1$ for any complex number α with $|\alpha| < 1$. Here some questions may arise; why only 'n' is sitting with $P(z)$ in the expression for $D_\alpha P(z)$? what happens if we replace 'n' by 'c' where 'c' is any positive real number? These questions appear genuine, intriguing and at the same time directed towards capturing more information on $D_\alpha P(z)$, and its possible generalization. In fact these thoughts motivated us to derive the following results. Without loss of generality we consider the class of monic polynomials in the next two results.

A set S_n of n distinct points in the complex plane can be associated with a class $\mathbb{P}_{n-1}$ of monic polynomials of degree $n-1$ having all their zeros in S_n. The classical Theorem of Laguerre for polynomials can now be extended to more general operator involving convex linear combination of polynomials in $\mathbb{P}_{n-1}$.

So let us take the set $S_n = \{z_1, z_2, \ldots, z_n\}$ of n complex numbers not necessarily distinct in the complex plane, let $\{P_k(z) | 1 \leq k \leq n\}$ denote the sequence of n polynomials given by

$$P_k(z) = \prod_{i=1, i \neq k}^{n} (z - z_i).$$

One can observe that a convex linear combination of members of $\{P_k(z)\}$ carries the essence of the ordinary derivative of a monic polynomial $P(z)$ of degree n whose zeros are $z_1, z_2, \ldots, z_n$ in the sense that

$$P'(z) = \sum_{k=1}^{n} \prod_{i=1, i\neq k}^{n} (z - z_i).$$

Thus the derivative of the monic polynomial $P(z)$ can be equivalently expressed as

$$\frac{1}{n} P'(z) = \sum_{k=1}^{n} \frac{1}{n} P_k(z) \tag{1.5}$$

where all the coefficients of the convex linear combination are $\frac{1}{n}$, and $\sum_{k=1}^{n} \frac{1}{n} = 1$.

In view of the above, the expression (1.5) can be extended to more generalized form as follows. If $\alpha_k > 0$, $1 \leq k \leq n$ with $\sum_{k=1}^{n} \alpha_k = 1$ then $\sum_{k=1}^{n} \alpha_k P_k(z)$ is a polynomial of degree $n-1$ in its general form and $P'(z)$ is one such representation. This behaviour of $\Lambda_{(\alpha_1,\ldots,\alpha_n)} P(z) = \sum_{k=1}^{n} \alpha_k P_k(z)$ leads us towards the extension of results involving the derivative of a polynomial $P(z)$ to the operator $\Lambda_{(\alpha_1,\ldots,\alpha_n)}$ on $P(z)$ as given above by $\Lambda_{(\alpha_1,\ldots,\alpha_n)} P(z)$. One important result involving the relative location of zeros and critical points of a polynomial is the Gauss-Lucas Theorem, and Díaz and Egozcue [3] generalized the Gauss-Lucas Theorem from the derivative operator to the $\Lambda_{(\alpha_1,\ldots,\alpha_n)}$ operator on polynomials. Here we present the similar extension for another significant result on the polar derivative of a polynomial, i.e., the Theorem of Laguerre, as follows.

Theorem 1.5 *Let $P(z) = \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$. Then the polynomial*

$$cP(z) + (\alpha - z) \sum_{k=1}^{n} \alpha_k P_k(z) \tag{1.6}$$

has no zeros in the disc $|z| < 1$, for all α with $|\alpha| < 1$, $c \geq 1$, where $P_k(z) = \prod_{i=1, i\neq k}^{n}(z - z_i)$, $1 \leq k \leq n$ and $\alpha_k > 0$ with $\sum_{k=1}^{n} \alpha_k = 1$.

Remark 1.6 Theorem 1.5 is quite useful in generalizing Bernstein-type inequalities for the derivative of a polynomial to the more general class of operators of the type $\sum_{k=1}^{n} \alpha_k P_k(z)$ as given above. In many cases we need to proceed with the same proof available for the results on the derivative of a polynomial, with minor modifications and replacing $P'(z)/n$ by $\sum_{k=1}^{n} \alpha_k P_k(z)$. For example, Erdős-Lax inequality [12] for the derivative of a polynomial and its L^P version have been proved mainly by using the Theorem of Laguerre. Instead, if we use Theorem 1.5 in those results, we can establish their analogues in terms of $\sum_{k=1}^{n} \alpha_k P_k(z)$ for the given $P(z)$. As the proofs of these generalizations do not deviate much from that of their analogues on the derivative of a polynomial, we do not mention every such result and its proof over here, rather leave it to the readers.

The choice of $\alpha_k = \frac{1}{n}$, $1 \leq k \leq n$ in (1.6) reduces Theorem 1.5 to an interesting result that generalizes the Theorem of Laguerre as given below.

Corollary 1.7 *Let $P(z) = \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$. Then the polynomial*

$$cP(z) + (\alpha - z)P'(z)$$

has no zeros in the disc $|z| < 1$ for any α with $|\alpha| < 1$ and $c \geq n$.

Remark 1.8 For the case $c = n$, Corollary 1.7 reduces to the Theorem of Laguerre.

2 Lemmas

We need the following lemmas to prove our results. The first lemma is due to Karamata [8], and is given below.

Lemma 2.1 *Let I be an interval of the real line and f denote a real-valued, concave function defined on I. If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are numbers in I such that $(x_1, \ldots, x_n)$ majorizes $(y_1, \ldots, y_n)$, i.e.,*

$$x_1 \geq x_2 \geq \cdots \geq x_n \quad \text{and} \quad y_1 \geq y_2 \geq \cdots \geq y_n,$$
$$x_1 + \cdots + x_i \geq y_1 + \cdots + y_i \quad \text{for all} \quad i \in \{1, \ldots, n-1\},$$

and

$$x_1 + \cdots + x_n = y_1 + \cdots + y_n$$

hold, then

$$f(x_1) + \cdots + f(x_n) \leq f(y_1) + \cdots + f(y_n).$$

Our next lemma describes an estimate for the real value of the logarithmic derivative of a complex polynomial having all its zeros in the closed unit disc. Lemma 2.2 was first appeared in a paper of Dubinin [4] where it was proved using the Boundary Schwarz Lemma, and also recently reproved in a paper by Govil and Kumar [6] using the method of principle of mathematical induction on the degree of the underlying polynomial. Here we present a different, simpler as well as a shorter proof of it. A new observation on Lemma 2.2 is made in this paper on the equality case.

Lemma 2.2 *Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n = a_n \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$. Then for all z on $|z| = 1$ for which $P(z) \neq 0$, we have*

$$Re\left(\frac{zP'(z)}{P(z)}\right) \geq \frac{n}{2} + \frac{|a_n| - |a_0|}{2(|a_n| + |a_0|)}. \tag{2.1}$$

The result is sharp and equality holds for the polynomial $P(z) = (z^{n-1} + 1)(z + a)$, $0 \leq a \leq 1$.

Proof Since $|z_k| \le 1$, $1 \le k \le n$, we have

$$Re\left(\frac{zP'(z)}{P(z)}\right) = \sum_{k=1}^{n} Re\left(\frac{z}{z-z_k}\right) \ge \sum_{k=1}^{n} \frac{1}{1+|z_k|} \tag{2.2}$$

for all z on $|z| = 1$ for which $P(z) \ne 0$.

To prove (2.1) in view of (2.2), it suffices to establish that

$$\sum_{k=1}^{n} \frac{1}{1+|z_k|} \ge \frac{n-1}{2} + \frac{1}{1+\left|\frac{a_0}{a_n}\right|},$$

or equivalently,

$$\sum_{k=1}^{n} \left(\frac{1}{1+|z_k|} - \frac{1}{2}\right) \ge \frac{1}{1+\prod_{k=1}^{n}|z_k|} - \frac{1}{2},$$

which with further simplification gets the form and thus essentially we need to show that

$$\sum_{k=1}^{n} \frac{1-|z_k|}{1+|z_k|} \ge \frac{1-\prod_{k=1}^{n}|z_k|}{1+\prod_{k=1}^{n}|z_k|}. \tag{2.3}$$

If any of the $z_k = 0$ for $1 \le k \le n$, then result holds true and hence let us assume that $z_k \ne 0$, for each $k = 1, 2, \ldots, n$.

$$\text{Let} \quad f(x) = \frac{1-e^{-x}}{1+e^{-x}}.$$

Then for $x \ge 0$, we have

$$f''(x) = \frac{2e^{-x}(e^{-x}-1)}{(1+e^{-x})^3} \le 0,$$

and hence $f(x)$ is concave in $[0, \infty)$. Now let $|z_k| = e^{-a_k}$, $1 \le k \le n$. Thus each $a_k \ge 0$, and without loss of generality let us assume that $|z_n| \ge |z_{n-1}| \ge \ldots \ge |z_1|$.

Then $a_1 \ge a_2 \ge \ldots \ge a_n$.

Note that $(a_1 + \cdots + a_n, 0, \ldots, 0)$ majorizes $(a_1, a_2, \ldots, a_n)$ in $[0, \infty)$, and hence applying Lemma 2.1, we have

$$\sum_{k=1}^{n} \frac{1-|z_k|}{1+|z_k|} = \sum_{k=1}^{n} \frac{1-e^{-a_k}}{1+e^{-a_k}} \ge \frac{1-e^{-a_1-\cdots-a_n}}{1+e^{-a_1-\cdots-a_n}} = \frac{1-\prod_{k=1}^{n}|z_k|}{1+\prod_{k=1}^{n}|z_k|},$$

which establishes (2.3).

The inequality (2.1) is best possible and the equality holds for some special class of polynomials. By considering the circle or line onto which $|z| = 1$ is mapped by the Möbius transformation $T(z) = \dfrac{z}{z-a}$, one may easily check that if $0 \le a \le 1$ and

$|z| = 1$, then $Re\left(\frac{z}{z-a}\right) \geq \frac{1}{1+|a|}$ as presented in the proof of Lemma 2.2 with equality if and only if either $a = 0$, or $|a| = 1$, or $z = -\frac{a}{|a|}$ whenever $a \neq 0$. In view of this, equality holds in the inequality (2.1) only when all the zeros of $P(z)$ lie on the unit circle apart from one simple zero say a such that $0 < |a| \leq 1$, and $z = -\frac{a}{|a|}$. When $a = 0$, the equality anyway holds. Therefore it is possible to have the equality in (2.1) for the polynomial $P(z) = z^n + az^{n-1} + z + a$ at $z = 1$ whenever $0 \leq a \leq 1$. □

Lemma 2.2 was used to obtain the sharpened version of Turán's inequality [15]

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{2.4}$$

for the polynomial $P(z)$ having all its zeros in the closed unit disc, in the paper of Govil and Kumar [6] (see also [4]) and they proved that, if $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is a polynomial of degree $n \geq 1$ having all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{n}{2} + \frac{|a_n| - |a_0|}{2(|a_n| + |a_0|)}\right) \max_{|z|=1} |P(z)|. \tag{2.5}$$

The paper of Govil and Kumar [6] says that the result is best possible and equality in (2.5) holds for polynomials having all their zeros on $|z| = 1$. As this is the same example cited by Turán [15] for the equality case in (2.4), one can observe that this example does not reflect the sharpness of the inequality (2.5) over (2.4). In other words, the example cited above does not consider the second term in the right hand side of (2.5), but the polynomial $P(z) = (z^{n-1} + 1)(z + a)$ where $0 \leq a \leq 1$, carries this information and gives an equality in (2.5), and thus makes a non-trivial example to justify the sharpness of (2.5).

The more generalized versions of (2.5) can be seen in the recent papers due to Kumar [9] and, Kumar and Dhankhar [10].

We need the following result to prove Theorem 1.5.

Lemma 2.3 *Let $P(z) = \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial*

$$cP(z) + (\gamma - 1)z \sum_{k=1}^{n} \alpha_k P_k(z) \tag{2.6}$$

has no zeros in the disc $|z| < 1$, for any complex number γ with $|\gamma| \leq 1$, where $P_k(z) = \prod_{i=1, i\neq k}^{n}(z - z_i)$, $c \geq 1$ and $\alpha_k > 0$ with $\sum_{k=1}^{n} \alpha_k = 1$.

Proof If $\gamma = 1$, then the result follows directly. So let us assume that $\gamma \neq 1$, and take $w_j = \frac{1}{z_j}$, $1 \leq j \leq n$. Since $c \geq 1$, we can write $c = 1 + a$, for some $a \geq 0$. Now for all z with $|z| < 1$ and using the fact that $\sum_{k=1}^{n} \alpha_k = 1$, we will have

$$\begin{aligned}
\frac{z\sum_{k=1}^{n}\alpha_k P_k(z)}{P(z)} - \frac{c}{1-\gamma} &= \sum_{k=1}^{n}\alpha_k\left(\frac{z}{z-z_k}\right) - \frac{1+a}{1-\gamma}\\
&= \sum_{k=1}^{n}\alpha_k\left(\frac{zw_k}{zw_k-1}\right) - \frac{1+a}{1-\gamma}\\
&= \frac{1}{2}\sum_{k=1}^{n}\alpha_k\left(1-\frac{1+zw_k}{1-zw_k}\right) - \frac{1+a}{1-\gamma}\\
&= \frac{1}{2}\sum_{k=1}^{n}\alpha_k - \frac{1}{2}\sum_{k=1}^{n}\alpha_k\left(\frac{1+zw_k}{1-zw_k}\right) - \frac{1+a}{1-\gamma}\\
&= -\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left(\frac{1+zw_k}{1-zw_k}\right) + \frac{1}{2} - \frac{1}{1-\gamma} - \frac{a}{1-\gamma}\\
&= -\frac{1}{2}\sum_{k=1}^{n}\alpha_k\left(\frac{1+zw_k}{1-zw_k}\right) - \frac{(1+\gamma)}{2(1-\gamma)} - \frac{a}{1-\gamma}.
\end{aligned}$$

Therefore

$$Re\left(\frac{z\sum_{k=1}^{n}\alpha_k P_k(z)}{P(z)} - \frac{c}{1-\gamma}\right)$$

$$= -\frac{1}{2}\sum_{k=1}^{n} Re\ \alpha_k\left(\frac{1+zw_k}{1-zw_k}\right) - Re\left(\frac{(1+\gamma)}{2(1-\gamma)}\right) - Re\left(\frac{a}{1-\gamma}\right).$$

It is easy to verify that for any complex number γ with $|\gamma| \le 1$, $Re\left(\frac{1+\gamma}{1-\gamma}\right) \ge 0$, and in the same way, since $|zw_k| < 1$, and each $\alpha_k > 0$ for $0 \le k \le n$, we must have $Re\ \alpha_k\left(\frac{1+zw_k}{1-zw_k}\right) > 0$. Further since $a \ge 0$ and $|\gamma| \le 1$, we have $Re\left(\frac{a}{1-\gamma}\right) \ge 0$. Thus we have shown that

$$Re\left(\frac{z\sum_{k=1}^{n}\alpha_k P_k(z)}{P(z)} - \frac{c}{1-\gamma}\right) < 0,$$

which establishes the conclusion of the Lemma 2.3. □

As an immediate consequence of Lemma 2.3, by taking $\alpha_k = 1/n, \quad 1 \le k \le n$, in (2.6), we will obtain the following analogous result for the derivative of a complex polynomial.

Corollary 2.4 *Let $P(z) = \prod_{k=1}^{n}(z - z_k)$ be a polynomial of degree n having no zeros in the disc $|z| < 1$, then the polynomial*

$$cP(z) + (\gamma - 1)zP'(z) \tag{2.7}$$

has no zeros in the disc $|z| < 1$ for any complex number γ with $|\gamma| \le 1$ and $c \ge n$.

3 Proofs of Theorems

Proof of Theorem 1.1 Suppose $P(z)$ has all its zeros in $|z| < 1$. By Gauss Lucas Theorem, $P'(z)$ has all its zeros in $|z| < 1$. This implies that $S(z)$ has no zeros in $|z| \le 1$.

Note that
$$\begin{aligned} R(z) &= nz^{n-1}P\left(\frac{1}{z}\right) - z^{n-2}P'\left(\frac{1}{z}\right) + \left[\frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right] z^{n-1}P\left(\frac{1}{z}\right) \\ &= n\left[1 + \frac{|a_n| - |a_0|}{n(|a_n| + |a_0|)}\right] z^{n-1}P\left(\frac{1}{z}\right) - z^{n-2}P'\left(\frac{1}{z}\right). \end{aligned}$$

Also on $|z| = 1$, we have

$$\begin{aligned} \frac{zR(z)}{S(z)} &= \frac{z\left\{n\left[1 + \frac{|a_n| - |a_0|}{n(|a_n| + |a_0|)}\right] z^{n-1}P\left(\frac{1}{z}\right) - z^{n-2}P'\left(\frac{1}{z}\right)\right\}}{z^{n-1}P'\left(\frac{1}{z}\right)} \\ &= \frac{nz\left[1 + \frac{|a_n| - |a_0|}{n(|a_n| + |a_0|)}\right] P\left(\frac{1}{z}\right) - P'\left(\frac{1}{z}\right)}{P'\left(\frac{1}{z}\right)} \\ &= \frac{n\left[1 + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right] P(\bar{z})}{\bar{z}P'(\bar{z})} - 1. \end{aligned}$$

Therefore
$$\left| \frac{n\left[1 + \frac{|a_n| - |a_0|}{|a_n| + |a_0|}\right] P(\bar{z})}{\bar{z}P'(\bar{z})} - 1 \right| = \left|\frac{R(z)}{S(z)}\right| \text{ on } |z| = 1. \tag{3.1}$$

Since $P(z)$ has all its zeros in $|z| < 1$, from Lemma 2.2, we have on $|z| = 1$,

$$Re\left(\frac{zP'(z)}{P(z)}\right) > \frac{n}{2} + \frac{|a_n| - |a_0|}{2(|a_n| + |a_0|)} = \frac{n}{2}\left[1 + \frac{|a_n| - |a_0|}{n(|a_n| + |a_0|)}\right].$$

Equivalently

$$Re\left(\frac{zP'(z)}{n\left[1+\frac{|a_n|-|a_0|}{n(|a_n|+|a_0|)}\right]P(z)}\right) > \frac{1}{2} \text{ on } |z|=1. \tag{3.2}$$

It is again a simple exercise to verify that if $Re(z) > \frac{1}{2}$, then $\left|\frac{1}{z}-1\right| < 1$. Now using this property in (3.2) we obtain

$$\left|\frac{n\left[1+\frac{|a_n|-|a_0|}{n(|a_n|+|a_0|)}\right]P(z)}{zP'(z)}-1\right| < 1 \text{ on } |z|=1.$$

Replacing z by $\overline{z}$ in the last inequality, we get on $|z|=1$,

$$\left|\frac{n\left[1+\frac{|a_n|-|a_0|}{n(|a_n|+|a_0|)}\right]P(\overline{z})}{\overline{z}P'(\overline{z})}-1\right| < 1. \tag{3.3}$$

The equations (3.1) and (3.3) together give

$$\left|\frac{R(z)}{S(z)}\right| < 1 \text{ on } |z|=1,$$

which completes the proof. □

Proof of Theorem 1.3 We have

$$|T(z)| < |S(z)| \text{ in } |z| \leq 1. \tag{3.4}$$

So, $S(z) \neq 0$ in $|z| \leq 1$. But then $z^{n-1}S\left(\frac{1}{z}\right)$ has all its zeros in $|z| < 1$. Since for every complex number z on $|z| = 1$, the complex number $\frac{1}{z}$ is also on $|z| = 1$, it follows from (3.4) that

$$\left|T\left(\frac{1}{z}\right)\right| < \left|S\left(\frac{1}{z}\right)\right|,$$

holds for all z on $|z| = 1$, which further implies

$$\left|z^{n-1} T\left(\frac{1}{z}\right)\right| < \left|z^n S\left(\frac{1}{z}\right)\right| \tag{3.5}$$

on $|z| = 1$.

Since the polynomial $z^n S\left(\frac{1}{z}\right)$ has all its zeros in $|z| < 1$, from (3.5) and using Rouché's Theorem it follows that the polynomial $z^{n-1} T\left(\frac{1}{z}\right) + z^n S\left(\frac{1}{z}\right)$ has all its zeros in $|z| < 1$. A simple calculation shows that

$$\begin{aligned} z^{n-1} T\left(\frac{1}{z}\right) + z^n S\left(\frac{1}{z}\right) &= nP(z) - zP'(z) + zP'(z) \\ &= nP(z). \end{aligned}$$

Therefore $P(z)$ has all its zeros in $|z| < 1$, and thus the proof is complete. □

Proof of Theorem 1.5 Since $P(z) \neq 0$ in $|z| < 1$, it follows from Lemma 2.3 that $cP(z) + (\gamma - 1)z \sum_{k=1}^{n} \alpha_k P_k(z) \neq 0$ in $|z| < 1$, for any real number $c \geq 1$ and any complex number γ with $|\gamma| \leq 1$. But then $\gamma z \sum_{k=1}^{n} \alpha_k P_k(z) \neq z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)$ for $|\gamma| \leq 1$ and $|z| < 1$. For any fixed z we can appropriately choose the argument of γ to get $|\gamma z| |\sum_{k=1}^{n} \alpha_k P_k(z)| \neq |z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)|$. Thus we have

$$|\gamma z| \left|\sum_{k=1}^{n} \alpha_k P_k(z)\right| < \left|z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)\right| \tag{3.6}$$

for $|z| < 1$ and $|\gamma| \leq 1$. Otherwise, if $|\gamma z| |\sum_{k=1}^{n} \alpha_k P_k(z)| \geq |z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)|$ for $|z| < 1$ and $|\gamma| \leq 1$, then small values of γ would contradict our claim. Hence (3.6) holds and taking $|\gamma| = 1$ and $|z| \to 1$ in (3.6) we get

$$\left|\sum_{k=1}^{n} \alpha_k P_k(z)\right| \leq \left|z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)\right| \tag{3.7}$$

for $|z| = 1$.

Firstly, let us prove that (3.7) is true for $|z| \leq 1$, whenever $P(z)$ has no zeros in $|z| \leq 1$. From (3.6), we have $z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z) \neq 0$ in $|z| < 1$, whenever $P(z)$ has no zeros in $|z| < 1$. Suppose $P(z) \neq 0$ on $|z| = 1$ also. Then $z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z) \neq 0$ on $|z| = 1$, because if z_0 were a zero of $z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)$ on $|z| = 1$ then by (3.7), we would have $\sum_{k=1}^{n} \alpha_k P_k(z_0) = 0$, in which case $P(z_0) = 0$, a contradiction to our hypothesis. Therefore $z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z) \neq 0$ in $|z| \leq 1$, whenever $P(z) \neq 0$ in $|z| \leq 1$. Now since

$$g(z) = \frac{\sum_{k=1}^{n} \alpha_k P_k(z)}{z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)}$$

is analytic in $|z| < 1$ and continuous on $|z| = 1$, by Maximum Modulus Principle, (3.7) holds for all z such that $|z| \leq 1$, whenever $P(z)$ has no zeros in $|z| \leq 1$. The continuity of the terms on both sides of (3.7) with respect to the polynomial $P(z)$ ensures that (3.7) continues to hold for all z such that $|z| \leq 1$, even if we restrict $P(z)$ to have no zeros in the open unit disc and allow $P(z)$ to have zeros on $|z| = 1$. This is because if this were not true for any possible zero w of $z\sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)$ on $|z| = 1$ becoming a pole of $g(z)$, then (3.7) would be violated for the points on the unit circle that are $\epsilon-$ close to w.

Therefore (3.7) holds for all z in the disc $|z| \leq 1$, if $P(z)$ has no zeros in $|z| < 1$ which clearly implies

$$\left|\alpha \sum_{k=1}^{n} \alpha_k P_k(z)\right| < \left|z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)\right| \tag{3.8}$$

for any α with $|\alpha| < 1$ and $|z| < 1$. Therefore we have

$$\alpha \sum_{k=1}^{n} \alpha_k P_k(z) \neq z \sum_{k=1}^{n} \alpha_k P_k(z) - cP(z)$$

whenever $|\alpha| < 1$ and $|z| < 1$, and hence the proof is complete. □

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