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On the Operator Hermite–Hadamard Inequality

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Abstract
The main target of this paper is to discuss operator Hermite–Hadamard inequality for convex functions, without appealing to operator convexity. Several forms of this inequality will be presented and some applications including norm and mean inequalities will be shown too.

Keywords Hermite–Hadamard inequality · Mond–Pečarić method · Self adjoint operator · Convex function

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1 Introduction and Preliminaries

Let $\mathcal{B}(\mathcal{H})$ be the C^*–algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. As usual, we reserve m, M for scalars and $\mathbf{1}_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. A self adjoint operator A is said to be positive (written $A \geq 0$) if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$,

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while it is said to be strictly positive (written $A > 0$) if A is positive and invertible. If A and B are self adjoint, we write $B \geq A$ in case $B - A \geq 0$.

The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$–isomorphism between the C^*–algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self adjoint operator A and the C^*–algebra generated by A and the identity operator $\mathbf{1}_{\mathcal{H}}$. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$. This is called the functional calculus for the operator A.

A real valued continuous function f defined on the interval J is said to be operator convex if $f((1-v)A + vB) \leq (1-v)f(A) + vf(B)$ for every $0 < v < 1$ and for every pair of bounded self adjoint operators A and B whose spectra are both in J. One of the most important examples is the power function $t \mapsto t^p$ for $1 \leq p \leq 2$.

The Hermite–Hadamard inequality, named after Charles Hermite and Jacques Hadamard, states that if a function $f : J \to \mathbb{R}$ is convex, then the following chain of inequalities hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2}, \quad (a, b \in J,\ a < b). \tag{1.1}$$

Since (see, e.g. [4] [Lemma 2.1])

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \int_0^1 f((1-t)a + tb)\,dt = \int_0^1 f((1-t)b + ta)\,dt,$$

we can rewrite (1.1) in the following form

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f((1-t)a + tb)\,dt \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

The Hermite–Hadamard inequality plays an essential role in research on inequalities and has quite a sizeable technical literature; as one can see in [1,2,5,8–11].

Obtaining operator inequalities corresponding to certain scalar inequalities have been an active research area in operator theory. Dragomir [3] gave an operator version of Hermite–Hadamard inequality and proved that

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)\,dt \leq \frac{f(A)+f(B)}{2}, \tag{1.3}$$

whenever $f : J \to \mathbb{R}$ is an operator convex and A, B are two self adjoint operators with spectra in J.

We emphasize here that the assumption *operator convexity* is essential to obtain (1.3). For example, if

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f(t) = t^3,$$

then simple computations show that

$$f\left(\frac{A+B}{2}\right)=\begin{pmatrix}17/4 & 7/4\\ 7/4 & 3/4\end{pmatrix},\quad \frac{f(A)+f(B)}{2}=\begin{pmatrix}7 & 4\\ 4 & 5/2\end{pmatrix}$$

and

$$\int_0^1 f((1-t)A+tB)\,dt=\begin{pmatrix}31/6 & 5/2\\ 5/2 & 4/3\end{pmatrix}.$$

It is easily seen that

$$f\left(\frac{A+B}{2}\right)\nleq\int_0^1 f((1-t)A+tB)\,dt\nleq\frac{f(A)+f(B)}{2}.$$

So, even though $f(t)=t^3$ is convex (not operator convex), (1.3) does not hold; showing that operator convexity cannot be dropped.

It is then natural to ask about which conditions one should have so that the inequalities in (1.3) are valid for any convex function.

In [7], it is shown that convex functions satisfy (1.3) if some empty intersection conditions are imposed on the spectra of A, B. We also refer the reader to [12]. In this article, we present several forms of (1.3) using the Mond–Pečarić method for convex functions. For example, we show that for appropriate constants α, β,

$$\int_0^1 f((1-t)A+tB)\,dt\le\beta\mathbf{1}_{\mathcal{H}}+\alpha\left(\frac{g(A)+g(B)}{2}\right),\tag{1.4}$$

when $m\mathbf{1}_{\mathcal{H}}\le A, B\le M\mathbf{1}_{\mathcal{H}}$ and f, g are certain functions. Then several converses and variants of (1.4) are presented. See Theorem 2.1 and the results that follow for the details.

In the end, we present other forms using properties of inner product; without appealing to the Mond–Pečarić method. Our results generalize some known inequalities presented in [3,9].

In our proofs, we will frequently use the basic inequality [6, Theorem 1.2]

$$f(\langle Ax, x\rangle)\le\langle f(A)x, x\rangle\tag{1.5}$$

valid for the convex function $f: J\to\mathbb{R}$, the self adjoint operator A with spectrum in J and the unit vector $x\in\mathcal{H}$.

2 Main Results

We present our main results in this section; where the Mond–Pečarić method is discussed first. Throughout this section, we use the following two standard notations for the function $f:[m, M]\to\mathbb{R}$;

$$a_f = \frac{f(M) - f(m)}{M - m} \;\&\; b_f = \frac{Mf(m) - mf(M)}{M - m}.$$

2.1 Hermite–Hadamard Inequalities Using the Mond–Pečarić Method

Our first convex (not operator convex) version of (1.3) reads as follows.

Theorem 2.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two self adjoint operators satisfying $m\mathbf{1}_{\mathcal{H}} \leq A, B \leq M\mathbf{1}_{\mathcal{H}}$ and let $f, g : [m, M] \to \mathbb{R}$ be two continuous functions. If f and g are both convex functions, then for a given $\alpha \geq 0$,*

$$\int_0^1 f((1-t)A + tB)\,dt \leq \beta\mathbf{1}_{\mathcal{H}} + \alpha\left(\frac{g(A) + g(B)}{2}\right), \tag{2.1}$$

where $\beta = \max_{m \leq x \leq M}\left\{a_f x + b_f - \alpha g(x)\right\}.$

Proof It follows from the convexity of $f : [m, M] \to \mathbb{R}$ that

$$f(x) \leq a_f x + b_f \tag{2.2}$$

for any $m \leq x \leq M$. Since $m\mathbf{1}_{\mathcal{H}} \leq A, B \leq M\mathbf{1}_{\mathcal{H}}$, then $m\mathbf{1}_{\mathcal{H}} \leq (1-t)A + tB \leq M\mathbf{1}_{\mathcal{H}}$. Applying functional calculus for the operator $T = (1-t)A + tB$ in (2.2) implies

$$f((1-t)A + tB) \leq a_f((1-t)A + tB) + b_f\mathbf{1}_{\mathcal{H}}.$$

Integrating the inequality over $t \in [0, 1]$, we get

$$\int_0^1 f((1-t)A + tB)\,dt \leq a_f\left(\frac{A+B}{2}\right) + b_f\mathbf{1}_{\mathcal{H}}.$$

Now, let $x \in \mathcal{H}$ be a unit vector. One can write

$$\begin{aligned}
&\left\langle\left(\int_0^1 f((1-t)A + tB)\,dt\right)x, x\right\rangle - \alpha\left\langle\left(\frac{g(A) + g(B)}{2}\right)x, x\right\rangle \\
&\leq a_f\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle + b_f - \alpha\left\langle\left(\frac{g(A) + g(B)}{2}\right)x, x\right\rangle \\
&= a_f\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle + b_f - \alpha\left(\frac{\langle g(A)x, x\rangle + \langle g(B)x, x\rangle}{2}\right) \\
&\leq a_f\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle + b_f - \alpha\left(\frac{g(\langle Ax, x\rangle) + g(\langle Bx, x\rangle)}{2}\right) \qquad (2.3) \\
&\leq a_f\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle + b_f - \alpha g\left(\frac{\langle Ax, x\rangle + \langle Bx, x\rangle}{2}\right) \qquad (2.4) \\
&= a_f\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle + b_f - \alpha g\left(\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle\right) \\
&\leq \max_{m \leq x \leq M}\left\{a_f x + b_f - \alpha g(x)\right\}
\end{aligned}$$

where in (2.3) we used (1.5), and (2.4) follows directly from convexity of g. Consequently,

$$\left\langle \left(\int_0^1 f\left((1-t)A + tB\right) dt \right) x, x \right\rangle \leq \beta + \alpha \left\langle \left(\frac{g(A) + g(B)}{2} \right) x, x \right\rangle$$

for any unit vector $x \in \mathcal{H}$. This completes the proof of inequality (2.1). □

Now we present some applications of Theorem 2.1.

Corollary 2.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two self adjoint operators satisfying $m\mathbf{1}_{\mathcal{H}} \leq A, B \leq M\mathbf{1}_{\mathcal{H}}$ and let $f, g : [m, M] \to \mathbb{R}$ be two continuous functions. If f and $g > 0$ are convex, then*

$$\int_0^1 f\left((1-t)A + tB\right) dt \leq \alpha \left(\frac{g(A) + g(B)}{2} \right), \tag{2.5}$$

where $\alpha = \max\limits_{m \leq x \leq M} \left\{ \frac{a_f x + b_f}{g(x)} \right\}$.

Further,

$$\int_0^1 f\left((1-t)A + tB\right) dt \leq \beta \mathbf{1}_{\mathcal{H}} + \frac{g(A) + g(B)}{2},$$

where $\beta = \max\limits_{m \leq x \leq M} \left\{ a_f x + b_f - g(x) \right\}$

Proof Notice that when $\alpha = \max\limits_{m \leq x \leq M} \left\{ \frac{a_f x + b_f}{g(x)} \right\}$, then $a_f x + b_f - \alpha g(x) \leq 0$. Therefore, from Theorem 2.1, $\beta \leq 0$ and (2.1) implies (2.5). The other inequality follows similarly from Theorem 2.1. □

Remark 2.1 Setting $f = g > 0$ the inequality (2.5) implies

$$\int_0^1 f\left((1-t)A + tB\right) dt \leq \alpha \left(\frac{f(A) + f(B)}{2} \right) \tag{2.6}$$

where $\alpha = \max\limits_{m \leq x \leq M} \left\{ \frac{a_f x + b_f}{f(x)} \right\}$. We remark that a similar result as in (2.6) was shown in [9, Theorem 3.9]. Therefore, Theorem 2.1 can be considered as an extension of [9, Theorem 3.9].

Notice that Theorem 2.1 and its consequences above present operator order inequalities. In the next result, we obtain operator norm inequalities. Here, $|A| = (A^* A)^{1/2}$, where A^* is the adjoint operator of A.

Proposition 2.1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two self adjoint operators satisfying $m\mathbf{1}_{\mathcal{H}} \leq |A|, |B| \leq M\mathbf{1}_{\mathcal{H}}$ and let $f : [m, M] \to \mathbb{R}$ be a nonnegative continuous increasing convex function. Then for a given $\alpha \geq 0$,*

$$f\left(\left\| \frac{A + B}{2} \right\| \right) \leq \left\| \int_0^1 f\left((1-t)|A| + t|B|\right) dt \right\| \leq \beta + \alpha \left\| \frac{f(|A|) + f(|B|)}{2} \right\|$$

where $\beta = \max\limits_{m \le x \le M} \left\{ a_f x + b_f - \alpha f(x) \right\}$.

Proof Recall that if $T \in \mathcal{B}(\mathcal{H})$ is a self adjoint operator, then $\|T\| = \sup\limits_{\|x\|=1} |\langle Tx, x\rangle|$. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned} f\left(\left|\left\langle\left(\frac{A+B}{2}\right)x, x\right\rangle\right|\right) &= f\left(\left|\frac{\langle Ax, x\rangle + \langle Bx, x\rangle}{2}\right|\right) \\ &\le f\left(\frac{|\langle Ax, x\rangle| + |\langle Bx, x\rangle|}{2}\right) \quad \text{(by the triangle inequality)} \\ &\le f\left(\frac{\langle |A| x, x\rangle + \langle |B| x, x\rangle}{2}\right) \quad \text{(by (1.5))} \\ &\le \int_0^1 f\left((1-t)\langle |A| x, x\rangle + t\langle |B| x, x\rangle\right) dt \quad \text{(by (1.2))} \\ &= \int_0^1 f\left(\langle ((1-t)|A| + t|B|)x, x\rangle\right) dt \\ &\le \int_0^1 \langle f((1-t)|A| + t|B|)x, x\rangle \, dt \quad \text{(by (1.5))} \\ &= \left\langle \left(\int_0^1 f((1-t)|A| + t|B|)\, dt\right) x, x\right\rangle \\ &\le \left\| \int_0^1 f((1-t)|A| + t|B|)\, dt \right\|. \end{aligned} \tag{2.7}$$

Now, by taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$ in (2.7) and noting that f is increasing,

$$\begin{aligned} f\left(\left\|\frac{A+B}{2}\right\|\right) &\le \left\| \int_0^1 f((1-t)|A| + t|B|)\, dt \right\| \\ &\le \left\| \beta 1_H + \alpha\left(\frac{f(|A|) + f(|B|)}{2}\right)\right\| \\ &\le \beta + \alpha \left\| \frac{f(|A|) + f(|B|)}{2} \right\| \end{aligned}$$

thanks to (2.1). This completes the proof. □

We end this section by giving the weighted generalization of operator Hermite–Hadamard inequality. For convenience, we use $A\nabla_\lambda B$ to denote $(1-\lambda)A + \lambda B$. We then show that Theorem 2.2 is a generalization of (1.3).

Theorem 2.2 *Let* $A, B \in \mathcal{B}(\mathcal{H})$ *be two self adjoint operators satisfying* $m\mathbf{1}_{\mathcal{H}} \le A, B \le M\mathbf{1}_{\mathcal{H}}$ *and let* $f : [m, M] \to \mathbb{R}$ *be an operator convex function. Then for any* $0 \le \lambda \le 1$,

$$\begin{aligned} f(A\nabla_\lambda B) &\le \int_0^1 f((A\nabla_\lambda B)\nabla_v A)\nabla_\lambda f((A\nabla_\lambda B)\nabla_v B)\, dv \\ &\le f(A)\nabla_\lambda f(B). \end{aligned}$$

Proof Since for $0 \leq \lambda, v \leq 1$,

$$A \nabla_{\lambda} B = \left(\left(A \nabla_{\lambda} B \right) \nabla_{v} A \right) \nabla_{\lambda} \left(\left(A \nabla_{\lambda} B \right) \nabla_{v} B \right)$$

holds, we infer from the operator convexity of f that

$$\begin{aligned} f\left(A \nabla_{\lambda} B\right) &= f\left(\left(\left(A \nabla_{\lambda} B\right) \nabla_{v} A\right) \nabla_{\lambda}\left(\left(A \nabla_{\lambda} B\right) \nabla_{v} B\right)\right) \\ &\leq f\left(\left(A \nabla_{\lambda} B\right) \nabla_{v} A\right) \nabla_{\lambda} f\left(\left(A \nabla_{\lambda} B\right) \nabla_{v} B\right) \\ &\leq \{f(A \nabla_{\lambda} B) \nabla_{v} f(A)\} \nabla_{\lambda} \{f(A \nabla_{\lambda} B) \nabla_{v} f(B)\} \\ &\leq \{(f(A) \nabla_{\lambda} f(B)) \nabla_{v} f(A)\} \nabla_{\lambda} \{(f(A) \nabla_{\lambda} f(B)) \nabla_{v} f(B)\} \\ &\leq f(A) \nabla_{\lambda} f(B). \end{aligned}$$

Integrating the inequality over $v \in [0, 1]$, we get

$$\begin{aligned} f\left(A \nabla_{\lambda} B\right) &\leq \int_{0}^{1} f\left(\left(A \nabla_{\lambda} B\right) \nabla_{v} A\right) \nabla_{\lambda} f\left(\left(A \nabla_{\lambda} B\right) \nabla_{v} B\right) dv \\ &\leq f(A) \nabla_{\lambda} f(B) \end{aligned}$$

which is the statement of the theorem. □

Remark 2.2 To show that Theorem 2.2 is a generalization of (1.3), put $\lambda = 1/2$. Thus

$$\begin{aligned} & f\left(\frac{A+B}{2}\right) \\ & \leq \frac{1}{2}\left[\int_{0}^{1} f\left((1-v)\left(\frac{A+B}{2}\right)+vA\right) dv+\int_{0}^{1} f\left((1-v)\left(\frac{A+B}{2}\right)+vB\right) dv\right] \\ & \leq \frac{f(A)+f(B)}{2}. \end{aligned} \tag{2.8}$$

On making use of the change of variable $v = 1 - 2t$ we have

$$\frac{1}{2} \int_{0}^{1} f\left((1-v)\left(\frac{A+B}{2}\right)+vA\right) dv=\int_{0}^{\frac{1}{2}} f((1-t) A+tB) dt. \tag{2.9}$$

and by the change of variable $v = 2t - 1$,

$$\frac{1}{2} \int_{0}^{1} f\left((1-v)\left(\frac{A+B}{2}\right)+vB\right) dv=\int_{\frac{1}{2}}^{1} f((1-t) A+tB) dt. \tag{2.10}$$

Relations (2.9) and (2.10), gives

$$\begin{aligned} & \frac{1}{2}\left[\int_{0}^{1} f\left((1-v)\left(\frac{A+B}{2}\right)+vA\right) dv+\int_{0}^{1} f\left((1-v)\left(\frac{A+B}{2}\right)+vB\right) dv\right] \\ & =\int_{0}^{1} f((1-t) A+tB) dt \end{aligned} \tag{2.11}$$

and the assertion follows by combining (2.8) and (2.11).

2.2 Reverse Hermite–Hadamard Inequalities Using the Mond–Pečarić Method

In the forthcoming theorem, we give additive, and multiplicative type reverses for the first and the second inequalities in (1.3).

Theorem 2.3 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be two self adjoint operators satisfying $m\mathbf{1}_{\mathcal{H}} \leq A, B \leq M\mathbf{1}_{\mathcal{H}}$ and let $f, g : [m, M] \to \mathbb{R}$ be two continuous functions. If f is a convex function, then for a given $\alpha \geq 0$*

$$\int_0^1 f\left((1-t)A + tB\right) dt \leq \beta \mathbf{1}_{\mathcal{H}} + \alpha g\left(\frac{A+B}{2}\right), \tag{2.12}$$

and

$$\frac{f(A) + f(B)}{2} \leq \beta \mathbf{1}_{\mathcal{H}} + \alpha \int_0^1 g\left((1-t)A + tB\right) dt, \tag{2.13}$$

where $\beta = \max\limits_{m \leq x \leq M} \left\{a_f x + b_f - \alpha g(x)\right\}$.

Proof From (2.2) and by applying functional calculus for the operator $T = (1-t)A + tB$, we have

$$f\left((1-t)A + tB\right) \leq a_f\left((1-t)A + tB\right) + b_f \mathbf{1}_{\mathcal{H}}.$$

Integrating both sides of the above inequality over $t \in [0, 1]$, we have

$$\int_0^1 f\left((1-t)A + tB\right) dt \leq a_f\left(\frac{A+B}{2}\right) + b_f \mathbf{1}_{\mathcal{H}}.$$

Therefore,

$$\begin{aligned} &\int_0^1 f\left((1-t)A + tB\right) dt - \alpha g\left(\frac{A+B}{2}\right) \\ &\leq a_f\left(\frac{A+B}{2}\right) + b_f \mathbf{1}_{\mathcal{H}} - \alpha g\left(\frac{A+B}{2}\right) \\ &\leq \max_{m \leq x \leq M} \left\{a_f x + b_f - \alpha g(x)\right\} \mathbf{1}_{\mathcal{H}}. \end{aligned}$$

Consequently,

$$\int_0^1 f\left((1-t)A + tB\right) dt \leq \beta \mathbf{1}_{\mathcal{H}} + \alpha g\left(\frac{A+B}{2}\right)$$

which proves (2.12). To prove (2.13), notice that (2.2) implies, for $0 \leq t \leq 1$,

$$(1-t)f(A) \leq a_f(1-t)A + b_f(1-t)\mathbf{1}_{\mathcal{H}}, \tag{2.14}$$

$$tf(B) \leq a_f tB + b_f t\mathbf{1}_{\mathcal{H}}. \tag{2.15}$$

From (2.14) and (2.15) we infer that

$$(1-t)f(A)+tf(B)\le a_f((1-t)A+tB)+b_f\mathbf{1}_{\mathcal{H}}.$$

Therefore

$$\begin{aligned}&(1-t)f(A)+tf(B)-\alpha g((1-t)A+tB)\\&\le a_f((1-t)A+tB)+b_f\mathbf{1}_{\mathcal{H}}-\alpha g((1-t)A+tB)\\&\le \max_{m\le x\le M}\left\{a_f x+b_f-\alpha g(x)\right\}\mathbf{1}_{\mathcal{H}}.\end{aligned}$$

Thus,

$$(1-t)f(A)+tf(B)\le \beta\mathbf{1}_{\mathcal{H}}+\alpha g((1-t)A+tB). \tag{2.16}$$

Integrating both sides of (2.16) over [0, 1] we get (2.13) and the proof is complete. □

2.3 Operator Hermite–Hadamard Inequality Using the Gradient Inequality

In this subsection, we present versions of the operator Hermite–Hadamard inequality using the gradient inequality

$$f'(s)(t-s)+f(s)\le f(t), \tag{2.17}$$

where $f:J\to\mathbb{R}$ is convex differentiable and $s,t\in J$.

Theorem 2.4 *Let $A,B\in\mathcal{B}(\mathcal{H})$ be self adjoint operators with spectra in the interval J and let $f:J\to\mathbb{R}$ be a differentiable convex function. Then*

$$f\left(\frac{A+B}{2}\right)\le\int_0^1 f((1-v)A+vB)\,dv+\delta\mathbf{1}_{\mathcal{H}}, \tag{2.18}$$

where

$$\delta=\sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}}\left\{\left\langle f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)x,x\right\rangle-\left\langle f'\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle\right\}.$$

Proof Since f is convex differentiable, (2.17) applies. By applying functional calculus for the operator $s=\frac{A+B}{2}$ we get

$$tf'\left(\frac{A+B}{2}\right)-f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)+f\left(\frac{A+B}{2}\right)\le f(t)\mathbf{1}_{\mathcal{H}}.$$

So, for any unit vector $x\in\mathcal{H}$,

$$t\left\langle f'\left(\frac{A+B}{2}\right)x,x\right\rangle-\left\langle f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)x,x\right\rangle+\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\le f(t).$$

Again, by applying functional calculus for the operator $t = (1-v)A + vB$ we get

$$\left\langle f'\left(\frac{A+B}{2}\right)x,x\right\rangle((1-v)A+vB) - \left\langle f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)x,x\right\rangle\mathbf{1}_{\mathcal{H}} + \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\mathbf{1}_{\mathcal{H}}$$
$$\le f\left(((1-v)A+vB)\right).$$

Integrating both sides over $t \in [0,1]$ implies

$$\left\langle f'\left(\frac{A+B}{2}\right)x,x\right\rangle\left(\frac{A+B}{2}\right) - \left\langle f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)x,x\right\rangle\mathbf{1}_{\mathcal{H}} + \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle\mathbf{1}_{\mathcal{H}}$$
$$\le \int_0^1 f\left((1-v)A+vB\right)dv.$$

Whence, for any unit vector $x \in \mathcal{H}$,

$$\left\langle f'\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle - \left\langle f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)x,x\right\rangle + \left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle$$
$$\le \left\langle\left(\int_0^1 f\left((1-v)A+vB\right)dv\right)x,x\right\rangle.$$

Thus,

$$\left\langle f\left(\frac{A+B}{2}\right)x,x\right\rangle \le \left\langle\left(\int_0^1 f\left((1-v)A+vB\right)dv\right)x,x\right\rangle + \delta$$

where

$$\delta = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}}\left\{\left\langle f'\left(\frac{A+B}{2}\right)\left(\frac{A+B}{2}\right)x,x\right\rangle - \left\langle f'\left(\frac{A+B}{2}\right)x,x\right\rangle\left\langle\left(\frac{A+B}{2}\right)x,x\right\rangle\right\}.$$

Therefore,

$$f\left(\frac{A+B}{2}\right) \le \int_0^1 f\left((1-v)A+vB\right)dv + \delta\mathbf{1}_{\mathcal{H}},$$

which completes the proof. □

Our last result in this direction is as follows.

Theorem 2.5 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be self adjoint operators with spectra in the interval J and let $f : J \to \mathbb{R}$ be a differentiable convex function. Then*

$$\int_0^1 f\left((1-v)A+vB\right)dv \le \frac{f(A)+f(B)}{2} + \xi\mathbf{1}_{\mathcal{H}}, \tag{2.19}$$

where

$$\xi = \sup_{\substack{x\in\mathcal{H}\\ \|x\|=1}} \left\{ \int_0^1 \left\langle f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right)x,x\right\rangle dv - \int_0^1 \left\langle f'\left(\left(1-v\right)A+vB\right)x,x\right\rangle \left\langle \left(\left(1-v\right)A+vB\right)x,x\right\rangle dv \right\}.$$

Proof By applying functional calculus for the operator $T=(1-v)A+vB$ in (2.17), we have

$$tf'\left(\left(1-v\right)A+vB\right) - f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right) + f\left(\left(1-v\right)A+vB\right) \le f\left(t\right)\mathbf{1}_{\mathcal{H}}.$$

Hence for any unit vector $x\in\mathcal{H}$,

$$t\left\langle f'\left(\left(1-v\right)A+vB\right)x,x\right\rangle - \left\langle f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right)x,x\right\rangle + \left\langle f\left(\left(1-v\right)A+vB\right)x,x\right\rangle \le f\left(t\right).$$

Again, it follows from the functional calculus for $t=A$ and $t=B$, respectively

$$\left(1-v\right)\left\langle f'\left(\left(1-v\right)A+vB\right)x,x\right\rangle A - \left(1-v\right)\left\langle f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right)x,x\right\rangle \mathbf{1}_{\mathcal{H}} + \left(1-v\right)\left\langle f\left(\left(1-v\right)A+vB\right)x,x\right\rangle \mathbf{1}_{\mathcal{H}} \le \left(1-v\right)f\left(A\right), \tag{2.20}$$

and

$$v\left\langle f'\left(\left(1-v\right)A+vB\right)x,x\right\rangle B - v\left\langle f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right)x,x\right\rangle \mathbf{1}_{\mathcal{H}} + v\left\langle f\left(\left(1-v\right)A+vB\right)x,x\right\rangle \mathbf{1}_{\mathcal{H}} \le vf\left(B\right). \tag{2.21}$$

By combining (2.20) and (2.21) we obtain

$$\left\langle f'\left(\left(1-v\right)A+vB\right)x,x\right\rangle \left(\left(1-v\right)A+vB\right) - \left\langle f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right)x,x\right\rangle \mathbf{1}_{\mathcal{H}} + \left\langle f\left(\left(1-v\right)A+vB\right)x,x\right\rangle \mathbf{1}_{\mathcal{H}} \le \left(1-v\right)f\left(A\right)+vf\left(B\right).$$

This implies

$$\left\langle f'\left(\left(1-v\right)A+vB\right)x,x\right\rangle \left\langle \left(\left(1-v\right)A+vB\right)x,x\right\rangle - \left\langle f'\left(\left(1-v\right)A+vB\right)\left(\left(1-v\right)A+vB\right)x,x\right\rangle + \left\langle f\left(\left(1-v\right)A+vB\right)x,x\right\rangle \le \left\langle \left(\left(1-v\right)f\left(A\right)+vf\left(B\right)\right)x,x\right\rangle$$

for any unit vector $x\in\mathcal{H}$. Integrating both sides over $v\in[0,1]$ we get

$$\left\langle \left(\int_0^1 f\left(\left(1-v\right)A+vB\right)dv\right)x,x\right\rangle \le \left\langle \left(\frac{f\left(A\right)+f\left(B\right)}{2}\right)x,x\right\rangle + \xi$$

where

$$\xi = \sup_{\substack{x \in \mathcal{H} \\ \|x\|=1}} \left\{ \int_0^1 \left\langle f'\left((1-v)A + vB\right)\left((1-v)A + vB\right)x, x \right\rangle dv \right.$$
$$\left. - \int_0^1 \left\langle f'\left((1-v)A + vB\right)x, x \right\rangle \left\langle \left((1-v)A + vB\right)x, x \right\rangle dv \right\}.$$

Consequently,

$$\int_0^1 f\left((1-v)A + vB\right) dv \le \frac{f(A) + f(B)}{2} + \xi \mathbf{1}_{\mathcal{H}},$$

as desired. □

Remark 2.3 Notice that in both Theorems 2.4 and 2.5, a quantity of the form

$$\sup_{\|x\|=1} \left\{ \langle Af'(A)x, x\rangle - \langle Ax, x\rangle \langle f'(A)x, x\rangle \right\}$$

has been found as a refining term, for some self adjoint operator A. We show here that this quantity is always non-negative, when f is such a convex function.
Applying functional calculus for $s = A$ in (2.17), we obtain

$$f(A) - f(t)\mathbf{1}_{\mathcal{H}} \le Af'(A) - tf'(A),$$

which implies

$$\langle f(A)x, x\rangle - f(t) \le \langle Af'(A)x, x\rangle - t\langle f'(A)x, x\rangle, x \in \mathcal{H}, \|x\| = 1.$$

Now replacing t by $\langle Ax, x\rangle$ and noting (1.5), we obtain

$$\langle Af'(A)x, x\rangle - \langle Ax, x\rangle\langle f'(A)x, x\rangle \ge \langle f(A)x, x\rangle - f\left(\langle Ax, x\rangle\right) \ge 0,$$

as desired.

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