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A Note on Koliha-Drazin Invertible Operators and a-Browder's Theorem

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Abstract

Let T be a bounded linear operator on a Banach space X. We prove certain inclusions and equalities between different parts of the spectrum of T and then apply them to study Koliha-Drazin invertible operators and operators satisfying a-Browder's theorem.

Keywords Koliha-Drazin invertible operator · A-Browder's theorem · Isolated point · Interior point · Decomposition

Mathematics Subject Classification 47A53 · 47A10

1 Introduction and preliminaries

Throughout, X is an infinite dimensional complex Banach space, T is a bounded linear operator acting on X, and $L(X)$ is the algebra of all bounded linear operators defined on X. A subspace M of X is said to be T-invariant if $T(M) \subset M$. We define $T_M : M \to M$ as $T_M x = Tx$, $x \in M$. Clearly, T_M is linear and bounded. If M and N are two closed T-invariant subspaces of X such that $X = M \oplus N$, we say that T is completely reduced by the pair (M, N) and it is denoted by $(M, N) \in Red(T)$. In this case we write $T = T_M \oplus T_N$ and we say that T is the direct sum of T_M and T_N.

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In this note we establish some relationships between different parts of the spectrum (Theorem 2.3) and apply that result to Koliha-Drazin invertible operators and a-Browder's theorem. Drazin and Koliha-Drazin invertible operators have been widely studied [5,6,14,20]. For instance, $T \in L(X)$ is Koliha-Drazin invertible if and only if T admits the following decomposition: $T = T_1 \oplus T_2$, T_1 is invertible and T_2 is quasinilpotent [14, Theorem 7.1]. In Sect. 3 we give new characterizations that involve some decomposition properties of T as well (Theorem 3.1). In particular, we prove that $T \in L(X)$ is Koliha-Drazin invertible if and only if 0 is not an interior point of its descent spectrum and $T = T_1 \oplus T_2$ where T_1 is upper semi-Weyl and T_2 is quasinilpotent. Moreover, a-Browder's theorem is also studied. We extend [2, Theorem 2.3] in a sense that we show that the assertion (iv) of [2, Theorem 2.3] can be reversed if the spectrum and the approximate point spectrum of T coincide (Theorem 4.1). We also prove that the converse of the assertion (iii) of [2, Theorem 2.3] holds if the spectrum of T is equal to the surjective spectrum of T (Theorem 4.5). In addition, we give several examples that serve to illustrate our results.

In what follows we will recall some necessary facts and give the auxiliary results. Let $N(T)$ and $R(T)$ be the null space and range of T, respectively. Denote by $\alpha(T)$ and $\beta(T)$, the dimension of $N(T)$ and the codimension of $R(T)$, respectively.

Definition 1.1 Let $T \in L(X)$. Then:

(i) T is Kato if $R(T)$ is closed and $N(T) \subset R(T^n)$ for all $n \in \mathbb{N}_0$;
(ii) T is of Kato type if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is nilpotent;
(iii) T admits a generalized Kato decomposition (GKD for short) if there exists a pair $(M, N) \in Red(T)$ such that T_M is Kato and T_N is quasinilpotent.

If T is Kato, then T^n is Kato for all $n \in \mathbb{N}$ [17, Theorem 12.7], and hence $R(T^n)$ is closed. We define $k(T)$, the lower bound of T, to be $k(T) = \inf\{\|Tx\| : x \in X \text{ with } \|x\| = 1\}$. We say that T is bounded below if $k(T) > 0$. It may be shown that $T \in L(X)$ is bounded below if and only if T is injective and $R(T)$ is closed. The approximate point spectrum of $T \in L(X)$, denoted by $\sigma_{ap}(T)$, is the set of all $\lambda \in \mathbb{C}$ such that the operator $T - \lambda I$ is not bounded below. The surjective spectrum $\sigma_{su}(T)$ is defined as the set of all $\lambda \in \mathbb{C}$ such that $R(T - \lambda I) \neq X$. It is well known that $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ are compact subsets of $\mathbb{C}$ that contain the boundary of $\sigma(T)$. Evidently, bounded below operators and surjective operators are examples of Kato operators.

We say that $T \in L(X)$ is semi-Fredholm if $R(T)$ is closed and either $\alpha(T) < \infty$ or $\beta(T) < \infty$. In such a case we define the index of T as $\text{ind}(T) = \alpha(T) - \beta(T)$. It is well known that every semi-Fredholm operator is of Kato type with N finite dimensional [17, Theorem 16.21]. An operator $T \in L(X)$ is said to be upper semi-Fredholm (resp. lower semi-Fredholm, Fredholm) if $R(T)$ is closed and $\alpha(T) < \infty$ (resp. $\beta(T) < \infty$, $\alpha(T) < \infty$ and $\beta(T) < \infty$).

Other important classes of operators in Fredholm theory are the classes of upper and lower semi-Weyl operators. These classes are defined as follows: $T \in L(X)$ is said to be upper semi-Weyl (resp. lower semi-Weyl) if it is upper semi-Fredholm and $\text{ind}(T) \leq 0$ (resp. lower semi-Fredholm and $\text{ind}(T) \geq 0$). The upper semi-Weyl spectrum is the set

$$\sigma_{uw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Weyl}\},$$

while the lower semi-Weyl spectrum is defined by

$$\sigma_{lw}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Weyl}\}.$$

Recall that the ascent of an operator $T \in L(X)$ is defined as the smallest nonnegative integer $p = p(T)$ such that $N(T^p) = N(T^{p+1})$. If such an integer does not exist, then $p(T) = \infty$. Similarly, the descent of T is defined as the smallest nonnegative integer $q = q(T)$ such that $R(T^q) = R(T^{q+1})$, and if such an integer does not exist, we put $q(T) = \infty$. The descent spectrum is the set

$$\sigma_{desc}(T) = \{\lambda \in \mathbb{C} : q(T - \lambda I) = \infty\}.$$

The set $\sigma_{desc}(T)$ is compact but possibly empty (see Corollary 1.3 and Theorem 1.5 in [8]). For example, if T is the zero operator on X, then $\sigma_{desc}(T) = \emptyset$.

An operator $T \in L(X)$ is upper (lower) semi-Browder if T is upper (lower) semi-Fredholm and $p(T) < \infty$ ($q(T) < \infty$), while an operator $T \in L(X)$ is Browder if it is Fredholm of finite ascent and descent. We consider the following spectra generated by these classes:

$$\sigma_{ub}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not upper semi-Browder}\},$$
$$\sigma_{lb}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not lower semi-Browder}\},$$
$$\sigma_{b}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder}\}.$$

The sets $\sigma_{ub}(T)$, $\sigma_{lb}(T)$ and $\sigma_b(T)$ are the upper-Browder spectrum, the lower-Browder spectrum and the Browder spectrum of T, respectively.

Drazin invertibility was initially introduced in rings [10]. In the case of Banach spaces, an operator $T \in L(X)$ is Drazin invertible if and only if $p(T) = q(T) < \infty$ [13, Theorem 4], and this is exactly when $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is nilpotent [17, Theorem 22.10]. The concept of Drazin invertibility for bounded operators may be extended as follows.

Definition 1.2 $T \in L(X)$ is said to be left Drazin invertible if $p = p(T) < \infty$ and $R(T^{p+1})$ is closed.

The Drazin spectrum and the left Drazin spectrum are defined respectively by

$$\sigma_{D}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\},$$
$$\sigma_{LD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not left Drazin invertible}\}.$$

In particular, we note that the left Drazin spectrum is a closed subset of the complex plane [16]. J. Koliha also extended the concept of Drazin invertibility [14]. An operator $T \in L(X)$ is generalized Drazin invertible (or Koliha-Drazin invertible) if and only if zero is not an accumulation point of its spectrum, and this is equivalent to saying that $T = T_1 \oplus T_2$, where T_1 is invertible and T_2 is quasinilpotent [14, Theorems 4.2 and 7.1].

From now on, if $F \subset \mathbb{C}$ then the set of isolated points of F, the set of accumulation points of F, the boundary of F and the interior of F will be denoted by iso F, acc F, ∂F and int F, respectively.

The quasinilpotent part of an operator $T \in L(X)$ is defined by

$$H_0(T) = \left\{ x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0 \right\}.$$

Clearly, $H_0(T)$ is a subspace of X not necessarily closed. An operator $T \in L(X)$ has the single-valued extension property at $\lambda_0 \in \mathbb{C}$, SVEP at λ_0, if for every open disc D_{λ_0} centered at λ_0 the only analytic function $f : D_{\lambda_0} \to X$ which satisfies

$$(T - \lambda I) f(\lambda) = 0 \quad \text{for all} \quad \lambda \in D_{\lambda_0},$$

is the function $f \equiv 0$. The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$. The following implication follows immediately from the definition of the localized SVEP and from the identity theorem for analytical functions:

$$\lambda_0 \notin \text{int}\, \sigma_{ap}(T) \Longrightarrow T \text{ has the SVEP at } \lambda_0. \tag{1}$$

Consequently,

$$\lambda_0 \notin \text{int}\, \sigma_{su}(T) \Longrightarrow T' \text{ has the SVEP at } \lambda_0, \tag{2}$$

since $\sigma_{ap}(T') = \sigma_{su}(T)$, where T' is the adjoint operator of T. In particular, from (1) and (2) we easily obtain that

$$\lambda_0 \notin \text{acc}\, \sigma(T) \Longrightarrow \text{ both } T \text{ and } T' \text{ has the SVEP at } \lambda_0.$$

Under the assumption that T admits a GKD, we have the following result (see [11, Theorem 3.5] and [3, Theorem 2.6]).

Lemma 1.3 *Suppose that $T \in L(X)$ admits a GKD. Then the following properties are equivalent:*

(i) *$H_0(T)$ is closed;*
(ii) *T has the* SVEP *at* 0*;*
(iii) 0 *is not an interior point of $\sigma_{ap}(T)$.*

For more comprehensive study of the SVEP we refer the reader to [1] and [15].

2 Different Parts of the Spectrum

In this section we are concerned with the relationships between different parts of the spectrum. We start with the following well known lemma.

Lemma 2.1 *Let $F \subset \mathbb{C}$ be closed. Then $\lambda \in \text{iso}\, F$ if and only if $\lambda \in \text{iso}\, \partial F$.*

Using Lemma 2.1 we prove the next simple lemma.

Lemma 2.2 *Let E and F be compact sets of the complex plane such that $\partial F \subset E \subset F$. If $\lambda_0 \in \partial F$, then the following statements are equivalent:*

(i) $\lambda_0 \in \text{iso}\, E$;
(ii) $\lambda_0 \in \text{iso}\, F$.

Proof If $\lambda_0 \in \text{iso}\, F$, then $\lambda_0 \in \partial F \subset E$. It follows that $\lambda_0 \in \text{iso}\, E$ since $E \subset F$. To prove the opposite implication suppose that $\lambda_0 \in \text{iso}\, E \cap \partial F$. Evidently, $\lambda_0 \in \text{iso}\, \partial F$. Now, we obtain $\lambda_0 \in \text{iso}\, F$ by Lemma 2.1. □

We are now ready to prove Theorem 2.3 which is very important for what will follow.

Theorem 2.3 *Let $T \in L(X)$. Then*

(i) $\text{iso}\, \sigma_{ap}(T) \subset \text{iso}\, \sigma(T) \cup \text{int}\, \sigma_{desc}(T)$;
(ii) $\text{iso}\, \sigma_{su}(T) \subset \text{iso}\, \sigma(T) \cup \text{int}\, \sigma_{LD}(T)$;
(iii) $\text{iso}\, \sigma_{uw}(T) \subset \text{iso}\, \sigma_b(T) \cup \text{int}\, \sigma_{desc}(T)$;
(iv) $\text{iso}\, \sigma_{lw}(T) \subset \text{iso}\, \sigma_b(T) \cup \text{int}\, \sigma_{LD}(T)$;
(v) $\sigma(T) = \sigma_{ap}(T) \cup \text{int}\, \sigma_{desc}(T)$;
(vi) $\sigma(T) = \sigma_{su}(T) \cup \text{int}\, \sigma_{LD}(T)$;
(vii) $\sigma_b(T) = \sigma_{uw}(T) \cup \text{int}\, \sigma_{desc}(T)$;
(viii) $\sigma_b(T) = \sigma_{lw}(T) \cup \text{int}\, \sigma_{LD}(T)$.

Proof (i). Let $\lambda_0 \in \text{iso}\, \sigma_{ap}(T) \setminus \text{int}\, \sigma_{desc}(T)$. Then there exists a sequence (λ_n) converging to λ_0 such that $T - \lambda_n I$ is bounded below and $q(T - \lambda_n I)$ is finite for all $n \in \mathbb{N}$. From [1, Theorem 3.4, part (ii)] it follows that $T - \lambda_n I$ is invertible for every $n \in \mathbb{N}$ and that λ_0 is a boundary point of $\sigma(T)$. Now, applying Lemma 2.2 with $F = \sigma(T)$ and $E = \sigma_{ap}(T)$ gives $\lambda_0 \in \text{iso}\, \sigma(T)$.

(ii). Use $\partial\sigma(T) \subset \sigma_{su}(T) \subset \sigma(T)$ and [1, Theorem 3.4, part (i)], and proceed analogously to the proof of (i).

(iii). Let $\lambda_0 \in \text{iso}\, \sigma_{uw}(T) \setminus \text{int}\, \sigma_{desc}(T)$. Then there exists a sequence (λ_n), $\lambda_n \to \lambda_0$ as $n \to \infty$, such that $q(T - \lambda_n I)$ is finite and $T - \lambda_n I$ is upper semi-Weyl for all $n \in \mathbb{N}$. Consequently, $\alpha(T - \lambda_n I) \leq \beta(T - \lambda_n I)$, $n \in \mathbb{N}$. Further, we have $\beta(T - \lambda_n I) \leq \alpha(T - \lambda_n I)$ by [1, Theorem 3.4, part (ii)], so $\alpha(T - \lambda_n I) = \beta(T - \lambda_n I) < \infty$. We conclude from [1, Theorem 3.4, part (iv)] that $p(T - \lambda_n I)$ is finite, hence that $T - \lambda_n I$ is Browder, and finally that $\lambda_0 \in \partial\sigma_b(T)$. Since $\partial\sigma_b(T) \subset \sigma_{uw}(T) \subset \sigma_b(T)$ [19, Corollary 2.5], $\lambda_0 \in \text{iso}\, \sigma_b(T)$ by Lemma 2.2.

(iv). Using [19, Corollary 2.5] and a standard duality argument we obtain $\partial\sigma_b(T) \subset \sigma_{lw}(T) \subset \sigma_b(T)$. Now, the result follows by the same method as in (iii).

(v). Since $\sigma_{ap}(T) \cup \text{int}\, \sigma_{desc}(T) \subset \sigma(T)$ is clear, it suffices to show the opposite inclusion. Let $\lambda_0 \in \sigma(T)$ and $\lambda_0 \notin \sigma_{ap}(T) \cup \text{int}\, \sigma_{desc}(T)$. As in the proof of (i), we conclude $\lambda_0 \in \partial\sigma(T)$, and hence $\lambda_0 \in \sigma_{ap}(T)$, which is a contradiction.

(vii). The inclusion $\sigma_{uw}(T) \cup \operatorname{int} \sigma_{desc}(T) \subset \sigma_b(T)$ is clear. Let $\lambda_0 \in \sigma_b(T)$ and $\lambda_0 \notin \sigma_{uw}(T) \cup \operatorname{int} \sigma_{desc}(T)$. Then analysis similar to that in the proof of (iii) shows that $\lambda_0 \in \partial\sigma_b(T)$. This is impossible since $\partial\sigma_b(T) \subset \sigma_{uw}(T)$.

(vi) and (viii). Apply an argument similar to the one used in the proof of statements (v) and (vii), respectively.

□

3 Koliha-Drazin Invertible Operators

We apply Theorem 2.3 to obtain new characterizations of Koliha-Drazin invertible operators.

Theorem 3.1 *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is Koliha-Drazin invertible;*
(ii) *$0 \notin \operatorname{int} \sigma_{desc}(T)$ and there exists $(M, N) \in Red(T)$ such that T_M is upper semi-Weyl and T_N is quasinilpotent;*
(iii) *$0 \notin \operatorname{int} \sigma_{LD}(T)$ and there exists $(M, N) \in Red(T)$ such that T_M is lower semi-Weyl and T_N is quasinilpotent.*

Proof (i) $\Longrightarrow$ (ii). There exists a pair $(M, N) \in Red(T)$ such that T_M is invertible and T_N is quasinilpotent. Clearly, T_M is upper semi-Weyl and $0 \notin \operatorname{acc} \sigma(T)$. Since $\operatorname{int} \sigma_{desc}(T) \subset \operatorname{int} \sigma(T) \subset \operatorname{acc} \sigma(T)$ then $0 \notin \operatorname{int} \sigma_{desc}(T)$.
(ii) $\Longrightarrow$ (i). According to [9, Theorem 3.4], T admits a GKD and $0 \notin \operatorname{acc} \sigma_{uw}(T)$. We will prove that $0 \notin \operatorname{acc} \sigma_b(T)$. If $0 \in \operatorname{iso} \sigma_{uw}(T)$, then Theorem 2.3, part (iii), implies $0 \in \operatorname{iso} \sigma_b(T)$. If $0 \notin \sigma_{uw}(T)$, then $0 \notin \sigma_b(T)$ by Theorem 2.3, part (vii). Applying [9, Theorem 3.9] we obtain the desired conclusion.
(i) $\Longleftrightarrow$ (iii). Apply an argument similar to the one used in the proof of the equivalence (i) $\Longleftrightarrow$ (ii), and use parts (iv) and (viii) of Theorem 2.3. □

Recently, P. Aiena and S. Triolo proved that if $H_0(T)$ is closed then $0 \in \operatorname{iso} \sigma_{su}(T)$ if and only if $0 \in \operatorname{iso} \sigma(T)$ [4, Theorem 2.4]. In what follows we consider the case where the surjective spectrum is replaced by the lower Weyl spectrum of T.

Corollary 3.2 *Let $T \in L(X)$ and $0 \in \sigma_{lw}(T)$. Suppose that T admits a GKD and that $H_0(T)$ is closed. Then:*

$$0 \in \operatorname{iso} \sigma_{lw}(T) \Longleftrightarrow 0 \in \operatorname{iso} \sigma(T).$$

Proof If $0 \in \operatorname{iso} \sigma(T)$ it follows immediately that $0 \notin \operatorname{acc} \sigma_{lw}(T)$. Since $0 \in \sigma_{lw}(T)$, $0 \in \operatorname{iso} \sigma_{lw}(T)$.

Suppose that $0 \in \operatorname{iso} \sigma_{lw}(T)$. From [9, Theorem 3.4] we see that T can be represented as $T = T_M \oplus T_N$ with T_M lower semi-Weyl and T_N quasinilpotent. On the other hand, $0 \notin \operatorname{int} \sigma_{ap}(T)$ by Lemma 1.3. Obviously, $0 \notin \operatorname{int} \sigma_{LD}(T)$, and from Theorem 3.1 it follows that T is Koliha-Drazin invertible, and hence $0 \in \operatorname{iso} \sigma(T)$.

□

The following examples demonstrate that Corollary 3.2 can not be proved under the weaker hypothesis. We recall that it is said that $T \in L(H)$ is normal if $TT^\star = T^\star T$, where H is a complex Hilbert space and $T^\star$ is the Hilbert-adjoint of T. We also note that $T \in L(X)$ is Riesz if $T - \lambda I$ is Fredholm for $\lambda \in \mathbb{C} \setminus \{0\}$.

Example 3.3 Let $T \in L(H)$ be a compact normal operator such that $\sigma(T)$ is an infinite subset of $\mathbb{C}$. Since every compact operator is Riesz, T does not admit a GKD (see Remark 5.3 in [9]). Moreover, $H_0(T) = N(T)$ [1, Example 3.9], so $H_0(T)$ is closed. Further, $0 \in \operatorname{iso} \sigma_{lw}(T)$, but $0 \in \operatorname{acc} \sigma(T)$.

A standard way to construct such an operator is as follows. Let H be a separable Hilbert space with scalar product $(\,,\,)$ and let $\{e_n : n \in \mathbb{N}\}$ be a total orthonormal system in H. If (λ_n) is a sequence of complex numbers such that $(|\lambda_n|)$ is a decresing sequence converging to zero, then we define

$$Tx = \sum_{n=1}^{\infty} \lambda_n (x, e_n) e_n \quad \text{for all} \quad x \in H.$$

It is not too hard to check that T is a compact normal operator and that $\sigma(T) = \{\lambda_n : n \in \mathbb{N}\} \cup \{0\}$.

Example 3.4 Define $T = L \oplus O$, where L and O are the unilateral left shift and the zero operator on the Hilbert space $\ell_2(\mathbb{N})$, respectively. The spectrum and the surjective spectrum of L are given by $\sigma(L) = \mathbb{D}$ and $\sigma_{su}(L) = \partial\mathbb{D}$, where $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Since L is surjective and O is quasinilpotent, then T admits a GKD. In addition, $H_0(T)$ is not closed (see Remark 2.6 in [4]). We have

$$\sigma_{lw}(T) \subset \sigma_{su}(T) = \sigma_{su}(L) \cup \sigma_{su}(O) = \partial\mathbb{D} \cup \{0\}. \tag{3}$$

Further, $\beta(T) = \beta(L) + \beta(O) = 0 + \infty = \infty$, and hence $0 \in \sigma_{lw}(T)$. From this and from (3) we deduce that $0 \in \operatorname{iso} \sigma_{lw}(T)$. On the other hand, $\sigma(T) = \sigma(L) \cup \sigma(O) = \mathbb{D}$, and so $0 \in \operatorname{acc} \sigma(T)$.

At the end of this section we characterize Drazin invertible operators.

Corollary 3.5 *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T is Drazin invertible;*
(ii) *$q(T) < \infty$ and there exists $(M, N) \in Red(T)$ such that T_M is upper semi-Weyl and T_N is quasinilpotent.*

Proof The implication (i) $\Longrightarrow$ (ii) is clear.
(ii) $\Longrightarrow$ (i). Since $0 \notin \sigma_{desc}(T)$, T is Koliha-Drazin invertible by Theorem 3.1. We have that $T = T_M \oplus T_N$ where T_M is invertible and T_N is quasinilpotent with finite descent. It follows that T_N is nilpotent and thus T is Drazin invertible. $\square$

4 a-Browder's Theorem

It is said that $T \in L(X)$ satisfies a-Browder's theorem if $\sigma_{uw}(T) = \sigma_{ub}(T)$, or equivalently, if $\operatorname{acc}\sigma_{ap}(T) \subset \sigma_{uw}(T)$ [2]. P. Aiena et al. proved the following implication [2, Theorem 2.3]:

$$T' \text{ has SVEP at every } \lambda \notin \sigma_{uw}(T) \Rightarrow \text{a-Browder's theorem holds for } T. \tag{4}$$

Moreover, it was also observed in [2] that the implication (4) can not be reversed in general. Further, it is known that $\sigma(T) = \sigma_{ap}(T)$ if T' has SVEP [15, Proposition 1.3.2]. M. Berkani et al. proved a more general implication [7, Lemma 2.1]:

$$T' \text{ has SVEP at every } \lambda \notin \sigma_{uw}(T) \Rightarrow \sigma(T) = \sigma_{ap}(T). \tag{5}$$

Our main contribution here is the implication (ii) $\Rightarrow$ (i) of Theorem 4.1, i.e. we prove that the implication (4) can be reversed if $\sigma(T) = \sigma_{ap}(T)$. For the sake of compliteness, we also include the proofs of (4) and (5).

Theorem 4.1 *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T' has* SVEP *at every $\lambda \notin \sigma_{uw}(T)$;*
(ii) *a-Browder's theorem holds for T and $\sigma(T) = \sigma_{ap}(T)$.*

Proof (i) $\Longrightarrow$ (ii). Let $\lambda \notin \sigma_{uw}(T)$. Then $T - \lambda I$ is upper semi-Fredholm and [1, Theorem 3.17] ensures that $q(T - \lambda I) < \infty$. It follows that $\lambda \notin \sigma_{desc}(T)$ and hence $\operatorname{int}\sigma_{desc}(T) \subset \sigma_{desc}(T) \subset \sigma_{uw}(T) \subset \sigma_{ap}(T)$. From part (v) of Theorem 2.3 we obtain $\sigma(T) = \sigma_{ap}(T)$. Further, $\sigma_b(T) = \sigma_{uw}(T)$ by Theorem 2.3, part (vii). From this and from $\sigma_{uw}(T) \subset \sigma_{ub}(T) \subset \sigma_b(T)$ we deduce $\sigma_{uw}(T) = \sigma_{ub}(T)$, so a-Browder's theorem holds for T.

(ii) $\Longrightarrow$ (i). Since $\operatorname{acc}\sigma_{ap}(T) \subset \sigma_{uw}(T)$ and $\sigma(T) = \sigma_{ap}(T)$, $\operatorname{int}\sigma_{desc}(T) \subset \operatorname{acc}\sigma(T) = \operatorname{acc}\sigma_{ap}(T) \subset \sigma_{uw}(T)$. Let $\lambda \notin \sigma_{uw}(T)$. From Theorem 2.3, part (vii), we have that $T - \lambda I$ is Browder and thus $\lambda \notin \operatorname{acc}\sigma(T)$ by [17, Corollary 20.20]. Consequently, T' has SVEP at λ. □

As a consequence, if $\sigma_{ap}(T)$ is a proper subset of $\sigma(T)$, then the converse of the assertion (4) does not hold. We now apply this fact to weighted right shifts, isometries and *Cesáro operator*. For relevant background material concerning weighted right shifts, see [15, pp. 85-90].

Example 4.2 A weighted right shift T on $\ell_2(\mathbb{N})$ is defined by

$$T(x_1, x_2, \cdots) = (0, w_1x_1, w_2x_2, \cdots) \text{ for all } (x_1, x_2, \cdots) \in \ell_2(\mathbb{N}),$$

where (w_n) is a bounded sequence of strictly positive numbers ($0 < w_n \leq 1$ for all $n \in \mathbb{N}$). It is easily seen that T is a bounded linear operator on $\ell_2(\mathbb{N})$ and that T has no eigenvalues. It follows that T has SVEP, so every weighted right shift satisfies a-Browder's theorem by [2, Theorem 2.3 (i)].

The approximate point spectrum and the spectrum of T are described as follows:

$$\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : i(T) \leq |\lambda| \leq r(T)\},\ \sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| \leq r(T)\},$$

where $r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\} = \lim\limits_{n\to\infty} \|T^n\|^{1/n}$ is the spectral radius of T and $i(T) = \lim\limits_{n\to\infty} k(T^n)^{1/n}$.

It is possible to find a weight sequence (w_n) for which the corresponding weighted right shift T satisfies $0 < i(T) < r(T)$. For such an operator T P. Aiena et al. proved [2, Example 2.5] that $0 \notin \sigma_{uw}(T)$ and that T' does not have SVEP at 0, so the converse of (4) does not hold for T. On the other hand, using Theorem 4.1 we obtain the same conclusion in a rather direct way if we observe that $\sigma_{ap}(T) \subsetneq \sigma(T)$.

Example 4.3 Let $T \in L(X)$ be a non-invertible isometry. It is known that $\sigma(T) = \mathbb{D}$ and $\sigma_{ap}(T) = \partial\mathbb{D}$ [15, p. 80], where $\mathbb{D}$ is as in Example 3.4. Consequently, int $\sigma_{ap}(T)$ is empty set, so T has SVEP, and from [2, Theorem 2.3 (i)] we infer that a-Browder's theorem holds for T. Since $\sigma_{ap}(T)$ is a proper subset of $\sigma(T)$, the reverse of the assertion (4) does not hold for T.

Example 4.4 For the *Cesáro operator* C_p defined on the classical Hardy space $H_p(\mathbb{D})$, $\mathbb{D}$ is the open unit disc centered at 0, $1 < p < \infty$, by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^{\lambda} \frac{f(\mu)}{1-\mu}\, d\mu,$$

for all $f \in H_p(\mathbb{D})$ and $\lambda \in \mathbb{D}$, we have $\sigma(C_p) = \mathbb{D}_p$ and $\sigma_{ap}(C_p) = \partial\mathbb{D}_p$, where $\mathbb{D}_p$ is the closed disc centered at $p/2$ with radius $p/2$ [15, Example 3.7.9]. Clearly, $\sigma_{su}(C_p) = \mathbb{D}_p$. Let $\lambda \in \mathbb{C}$ be such that $|\lambda - p/2| < p/2$. We have

$$\text{ind}(C_p - \lambda I) = \alpha(C_p - \lambda I) - \beta(C_p - \lambda I) = -\beta(C_p - \lambda I) < 0.$$

Moreover, $C_p - \lambda I$ is Fredholm [15, Example 3.7.9] and since its index is negative, $C_p - \lambda I$ is upper semi-Weyl. Now, the following inclusions are immediate:

$$\{\lambda \in \mathbb{C} : |\lambda - p/2| < p/2\} \subset \sigma_{lw}(C_p) \subset \sigma_w(C_p) \subset \mathbb{D}_p \tag{6}$$

and

$$\sigma_{uw}(C_p) \subset \partial\mathbb{D}_p. \tag{7}$$

From (6) it follows that $\sigma_{lw}(C_p) = \sigma_w(C_p) = \mathbb{D}_p$.

V. Rakočević proved [18, Theorem 1] that $\partial\sigma_w(T) \subset \sigma_{uw}(T)$ for $T \in L(X)$. Therefore, $\partial\sigma_w(C_p) \subset \sigma_{uw}(C_p)$ which together with (7) gives $\sigma_{uw}(C_p) = \partial\mathbb{D}_p$.

Since acc $\sigma_{ap}(C_p) = \partial\mathbb{D}_p = \sigma_{uw}(C_p)$, C_p satisfies a-Browder's theorem. Because $\sigma(C_p) \neq \sigma_{ap}(C_p)$, Theorem 4.1 shows that there exists $\lambda \notin \sigma_{uw}(C_p)$ such that C_p' does not have SVEP at λ. To illustrate this, observe that $p/2 \notin \sigma_{uw}(C_p) = \partial\mathbb{D}_p$. Clearly, $C_p - (p/2)I$ is bounded below and $p/2 \in \text{acc}\,\sigma_{su}(C_p) = \mathbb{D}_p$. Since $C_p - (p/2)I$ is of Kato type, C_p' does not have SVEP at $p/2$ by [1, Theorem 3.27].

The following result can be also found in [2, Theorem 2.3]:

$$T \text{ has SVEP at every } \lambda \notin \sigma_{lw}(T) \Rightarrow \text{a-Browder's theorem holds for } T'. \tag{8}$$

What is more, the converse of the assertion (8) does not hold in general [2, Example 2.5]. Our main goal is to show that (8) can be reversed if $\sigma(T) = \sigma_{su}(T)$. We remark that $\sigma(T) = \sigma_{su}(T)$ if T has SVEP [15, Proposition 1.3.2]. In addition, the implication

$$T \text{ has SVEP at every } \lambda \notin \sigma_{lw}(T) \Rightarrow \sigma(T) = \sigma_{su}(T) \tag{9}$$

was proved in [7, Lemma 2.1] and it is an improvement of [15, Proposition 1.3.2]. For the convenience of the reader we also prove (8) and (9), thus making our exposition self-contained.

Theorem 4.5 *Let $T \in L(X)$. The following conditions are equivalent:*

(i) *T has* SVEP *at every $\lambda \notin \sigma_{lw}(T)$;*
(ii) *a-Browder's theorem holds for T' and $\sigma(T) = \sigma_{su}(T)$.*

Proof (i) $\Longrightarrow$ (ii). Let $\lambda \notin \sigma_{lw}(T)$. Then $T-\lambda I$ is lower semi-Fredholm, so $T-\lambda I$ is of Kato type. By [1, Theorem 3.16] the SVEP of T at λ implies that $p = p(T-\lambda I) < \infty$. In addition, $R((T-\lambda I)^{p+1})$ is closed, so $\lambda \notin \sigma_{LD}(T)$. Consequently, $\text{int}\,\sigma_{LD}(T) \subset \sigma_{lw}(T) \subset \sigma_{su}(T)$ and we conclude by Theorem 2.3, parts (vi) and (viii), that $\sigma(T) = \sigma_{su}(T)$ and $\sigma_b(T) = \sigma_{lw}(T)$. It follows that $\sigma_b(T') = \sigma_{uw}(T') \subset \sigma_{ub}(T') \subset \sigma_b(T')$, and hence $\sigma_{uw}(T') = \sigma_{ub}(T')$, so a-Browder's theorem holds for T'.
(ii) $\Longrightarrow$ (i). Since $\sigma_{uw}(T') = \sigma_{ub}(T')$, $\sigma_{lw}(T) = \sigma_{lb}(T)$ and therefore [1, Theorem 3.65, part (v)] ensures that $\text{acc}\,\sigma_{su}(T) \subset \sigma_{lw}(T)$. We deduce that $\text{int}\,\sigma_{LD}(T) \subset \text{acc}\,\sigma(T) = \text{acc}\,\sigma_{su}(T) \subset \sigma_{lw}(T)$ and hence $\sigma_b(T) = \sigma_{lw}(T)$ by Theorem 2.3, part (viii). If $\lambda \notin \sigma_{lw}(T)$, then $\lambda \notin \sigma_b(T)$, and hence $\lambda \notin \text{acc}\,\sigma(T)$ [17, Corollary 20.20]. Consequently, T has SVEP at λ. □

From Theorem 4.5 we may deduce that the implication (8) can not be reversed if $\sigma_{su}(T)$ is a proper subset of $\sigma(T)$.

Example 4.6 Let L be the unilateral left shift on X, where $X \in \{c_0(\mathbb{N}), c(\mathbb{N}), \ell_\infty(\mathbb{N}), \ell_p(\mathbb{N})\}$, $p \geq 1$. Then $\sigma(L) = \sigma_{ap}(L) = \sigma_{uw}(L) = \sigma_w(L) = \mathbb{D}$ and $\sigma_{su}(L) = \partial\mathbb{D}$ [21, Example 4.10], where $\mathbb{D}$ is as in Example 3.4. We apply [18, Theorem 1] to conclude

$$\partial\mathbb{D} = \partial\sigma_w(L) = \partial\sigma_w(L') \subset \sigma_{uw}(L') = \sigma_{lw}(L) \subset \sigma_{su}(L) = \partial\mathbb{D},$$

and hence $\sigma_{lw}(L) = \partial\mathbb{D}$. Further, $\sigma_{ap}(L') = \sigma_{su}(L) = \partial\mathbb{D}$ implies that

$$\text{acc}\,\sigma_{ap}(L') = \partial\mathbb{D} = \sigma_{lw}(L) = \sigma_{uw}(L'),$$

so L' satisfies a-Browder's theorem. Since $\sigma_{su}(L) \neq \sigma(L)$, according to Theorem 4.5 it follows that L does not have SVEP at every $\lambda \notin \sigma_{lw}(L)$. Clearly, $0 \notin \sigma_{lw}(L) = \partial\mathbb{D}$. We will show that L does not have SVEP at 0. Observe that L is surjective and that

$0 \in \operatorname{acc} \sigma_{ap}(L) = \mathbb{D}$. Since L is of Kato type, L does not have SVEP at 0 by [1, Theorem 3.23]. On the other hand, $\operatorname{int} \sigma_{su}(L) = \emptyset$, so L' has SVEP by (2) and it follows that L satisfies conditions of Theorem 4.1.

Let $R \in L(X)$ be the unilateral right shift where X is as in the preceding paragraph. Since $\sigma_{ap}(R) = \partial\mathbb{D}$, $\operatorname{int} \sigma_{ap}(R) = \emptyset$ and hence R has SVEP. We obtain that R satisfies conditions of Theorem 4.5. Furthermore, $\sigma(R) = \mathbb{D} \neq \sigma_{ap}(R)$ and we conclude that R fails to satisfy conditions of Theorem 4.1.

There are classes of operators that satisfy conditions of Theorem 4.1 as well as of Theorem 4.5. Indeed, if $T \in L(X)$ is such that $\sigma(T)$ is contained in a line or $\sigma(T)$ is at most countable, then $\operatorname{int} \sigma(T)$ is empty and hence both T and T' have SVEP by (1) and (2). Consequently, T satisfies conditions of Theorems 4.1 and 4.5. For example, the spectrum of a bilateral shift T (forward or backward) on $c_0(\mathbb{Z})$ or $\ell_p(\mathbb{Z})$, $p \geq 1$, is the unit circle. If T is a selfadjoint or unitary operator, then in the former case its spectrum is real and in the second case its spectrum is contained in the unit circle.

An operator $T \in L(X)$ is meromorphic if its non-zero spectral points are poles of the resolvent of T. For $T \in L(X)$ is said to be a polynomially Riesz (polynomially meromorphic) operator if there exists a non-zero polynomial p such that $p(T)$ is Riesz (meromorphic). Every Riesz operator is meromorphic and every polynomially Riesz operator is a polynomially meromorphic operator. If T is a polynomially meromorphic operator, then its spectrum is at most countable [12].

5 Conclusions

We conclude this paper by reviewing our results. The first main result is Theorem 2.3 where numerous relationships between different parts of the spectrum are given and in forthcoming Sects. 3 and 4 we frequently use Theorem 2.3. In Sect. 3 we give new characterizations of Koliha-Drazin invertible operators. In addition, we show that if $T \in L(X)$ admits a GKD and if its quasinilpotent part is closed then 0 is isolated in the lower Weyl spectrum of T if and only if 0 is isolated in the spectrum of T. In Sect. 4 we prove the following implications:

a-Browder's theorem holds for T and $\sigma(T) = \sigma_{ap}(T) \Rightarrow T'$ has SVEP at every $\lambda \notin \sigma_{uw}(T)$;

a-Browder's theorem holds for T' and $\sigma(T) = \sigma_{su}(T) \Rightarrow T$ has SVEP at every $\lambda \notin \sigma_{lw}(T)$.

Many examples are given in order to provide a better presentation of our results.

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