Complex Analysis and Operator Theory (2021) 15:55
https://doi.org/10.1007/s11785-021-01097-4

Complex Analysis and Operator Theory

Arithmetic and Analysis of the Series $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}$

Ahmed Sebbar[1,2] · **Roger Gay**[2]

Received: 22 July 2020 / Accepted: 4 March 2021 / Published online: 8 April 2021

Abstract

In this paper we connect a celebrated theorem of Nyman and Beurling on the equivalence between the Riemann hypothesis and the density of some functional space in $L^2(0, 1)$ to a trigonometric series considered first by Hardy and Littlewood (see (3.4)). We highlight some of its curious analytical and arithmetical properties.

Keywords Hardy–Littlewood function · Franel integral · Beurling's theorem · Arithmetic functions

Mathematics Subject Classification 11M32 · 11M38 · 11K70 · 11K65

1 Introduction

The main purpose of this work is to bring to light a new relationship between two facets of Riemann's zeta function: On the one hand a functional analysis approach to the Riemann hypothesis due to Nymann and Beurling, and on the other hand a trigonometric series first studied by Hardy and Littlewood [16], and then followed by Flett [15], Segal [25] and Delange [13]. The trigonometric series in question is

To the memory of our friend Carlos Berenstein

Communicated by Irene Sabadini.

This article is part of the topical collection "In memory of Carlos A. Berenstein (1944–2019)" edited by Irene Sabadini and Daniele Struppa.

✉ Ahmed Sebbar
sebbar@chapman.edu; ahmed.sebbar@math.u-bordeaux.fr

Roger Gay
roger.gay@math.cnrs.fr

1 Chapman University, One University Drive, Orange, CA 92866, USA

2 IMB, UMR 5251, Université de Bordeaux, 33405 Talence, France

$$\mathfrak{f}(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}. \tag{1.1}$$

It differs from the finite sum $\sum_{n \le x} \frac{1}{n} \sin \frac{x}{n}$, as x tends to ∞, by

$$\sum_{n>1} \frac{1}{n} \sin \frac{x}{n} = O\left(\sum_{n>x} \frac{x}{n^2}\right) = O(1).$$

Hardy and Littlewood proved [16] that, as x tends to ∞,

$$\mathfrak{f}(x) = O\left((\log x)^{\frac{3}{4}} (\log\log x)^{\frac{3}{4}+\epsilon}\right)$$

and that

$$\mathfrak{f}(x) = \Omega\left((\log\log x)^{\frac{3}{4}}\right)$$

from the fact that for $x \ge 5$, the number of $n \le x$ whose prime divisors are equivalent to 1 modulo 4 is $C\frac{x}{(\log x)^{\frac{1}{2}}}$, where C is a constant. Delange [13] showed that $\mathfrak{f}(x)$ is not bounded on the real line only from the following result on the reciprocals of primes in arithmetic progressions

$$\sum_{\substack{p \text{ prime}, \\ p \equiv 1 \pmod 4}} \frac{1}{p} = \infty$$

and obtained the Ω-result of Hardy and Littlewood just because

$$\sum_{\substack{p \text{ prime} \le x, \\ p \equiv 1 \pmod 4}} \frac{1}{p} = \frac{1}{2} \log\log x + c + o(1).$$

This trigonometric series, despite its simplicity, has many similarities with the Riemann zeta function [15] and deep relation to the divisor functions through the sawtooth function

$$\{t\} = -\frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\sin 2m\pi t}{m} = \begin{cases} t - \lfloor t \rfloor - \frac{1}{2} & \text{if} \quad t \ne \lfloor t \rfloor \\ \quad 0 & \text{if} \quad t = \lfloor t \rfloor. \end{cases} \tag{1.2}$$

For $s \in \mathbb{C}$ we define

$$\sigma^s(n) = \sum_{d|n} d^s, \quad \sigma_s(n) = \sum_{d|n} d^{-s}$$

so that $n^s \sigma_s(n) = \sigma^s(n)$. For example if we define

$$S_1(x) = \sum_{n \le x} \sigma_1(n), \quad S^1(x) = \sum_{n \le x} \sigma^1(n)$$

and

$$\rho(x) = \sum_{n \le x} \frac{1}{n} \left\{ \frac{x}{n} \right\} = \sum_{n \le x} \frac{1}{n} \left(\frac{x}{n} - \left\lfloor \frac{x}{n} \right\rfloor - \frac{1}{2} \right)$$

then the divisors and the fractional parts functions are related by

$$\begin{aligned} S_1(x) &= \sum_{n \le x} \frac{1}{n} \left\lfloor \frac{x}{n} \right\rfloor = x \sum_{n \le x} \frac{1}{n^2} - \rho(x) \\ &= \frac{\pi^2}{2} x - \frac{1}{2} \log x - \rho(x) + O(1). \end{aligned}$$

Similarly [32] (p.70):

$$S^1(x) = \frac{\pi^2}{12} x^2 - x\rho(x) + O(x).$$

We will see ((3.5) with $f(2\pi x) = \sin x$) an integral representation of the partial sums of $\mathfrak{f}(x)$, using the sawtooth function.

2 Nyman–Beurling Criterion for the Riemann Hypothesis

2.1 Nyman–Beurling Theorem

For $x > 0$, let $\rho(x)$ be the fractional part of x so that $\rho(x) = x - \lfloor x \rfloor$. To each $0 < \theta \le 1$ we associate the function $\rho_\theta(x) = \rho(\frac{\theta}{x})$. Then $0 \le \rho_\theta(x) \le 1$ and $\rho_\theta(x) = \frac{\theta}{x}$ if $\theta < x$. We introduce, as in [4–7,14,22,29] and the more recent book [23]

$$\mathcal{M} = \left\{ f,\, f(x) = \sum_{n=1}^{N} a_n \rho(\frac{\theta_n}{x}),\ a_n \in \mathbb{R}, \theta_n \in (0,1],\ \sum_{n=1}^{N} a_n \theta_n = 0,\ N \ge 1 \right\}.$$

Each function in $\mathcal{M}$ has at most a countable set of points of discontinuity, and is identically zero for $x > 0$.

Theorem 2.1 (Nyman–Beurling) *Let $1 < p \le \infty$. The subspace $\mathcal{M}$ is dense in the Banach space $L^p(0,1)$ if and only if the Riemann zeta function $\zeta(s)$ has no zero in the right half plane* $\mathrm{Re}s > \frac{1}{p}$.

The fundamental relations in the proof of this theorem are

$$\int_0^1 \rho(\frac{\theta}{x}) x^{s-1}\, dx = -\frac{\theta}{1-s} - \theta^s \frac{\zeta(s)}{s}, \quad \mathrm{Re}s > 1, \tag{2.1}$$

which is just a variant of the classical representation

$$\zeta(s) = \frac{s}{s-1} - s\int_0^\infty \frac{u - \lfloor u \rfloor}{(u+1)^{s+1}}\, du. \tag{2.2}$$

It follows from (2.1) that for $f(x) \in \mathcal{M}$

$$\int_0^1 f(x)x^{s-1}\, dx = -\frac{\zeta(s)}{s}\sum_{k=1}^{N} a_k \theta_k^s.$$

The study of the function $\mathfrak{f}(x)$ is intimately linked to that of following function

$$\{t\} = \begin{cases} t - \lfloor t \rfloor - \frac{1}{2} & \text{if} \quad t \neq \lfloor t \rfloor \\ 0 & \text{if} \quad t = \lfloor t \rfloor. \end{cases}$$

We have the formal Fourier series expansion [11,12]

$$\sum_{n=1}^{\infty} \frac{a_n}{n}\{n\theta\} = -\frac{1}{\pi}\sum_{n=1}^{\infty} \frac{A_n}{n}\{\sin 2\pi n\theta\} \tag{2.3}$$

where

$$A_n = \sum_{d|n} a_d.$$

Davenport considered the cases of

$$a_n = \mu(n);\;\; a_n = \lambda(n);\;\; a_n = \Lambda(n);\;\; a_{n^2} = \mu(n), a_n = 0,\; n \neq m^2.$$

These arithmetical functions have their usual number-theoretic meanings. For example if $\omega(n)$ is the number of distinct prime factors of n or, in other terms, $\omega(n) = \sum_{b|n} 1$ and $\omega(1) = 0$, then the Möbius function $\mu(n)$ is defined by

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ is divisible by a perfect square} > 1 \\ (-1)^{\omega(n)} & \text{otherwise} \end{cases}$$

and the Von Mangoldt function $\Lambda(n)$ is defined by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^\alpha \text{ for a prime } p \text{ and some } \alpha \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

In the case of the Möbius function $a_n = \mu(n)$, Davenport uses Vinogradov's method, a refinement of Weyl's method on estimating trigonometric sums, to prove that for any fixed h

$$\sum_{n \leq y} \mu(n) e^{2i\pi nx} = O\left(y(\log y)^{-h}\right) \tag{2.4}$$

uniformly in $x \in \mathbb{R}/\mathbb{Z}$. The implied constants are not effective. There have been several results justifying (2.3) for other particular sequences (a_n). The most general problem is considered in [17].

It should be noted that the Davenport or Hardy Littlewood estimates admit a common analysis. For the convenience of the reader we gather together a few classical results on exponential sums. Let $\mathbf{I}$ be an interval of length at most $N \geq 1$ and let $f : \mathbf{I} \to \mathbb{R}$ be a smooth function satisfying the estimates $x \in \mathbf{I}$, $2 \leq N \ll T$, $j \geq 1$

$$|f^{(j)}(x)| = \exp\left(O(j^2)\right) \frac{T}{N^j}$$

then with $f(x) = e^x$

(1) Van der Corput estimate: For any natural number $k \geq 2$, we have

$$\frac{1}{N} \sum_{n \in \mathbf{I}} e(f(n)) = O\left(\frac{T}{N^k}^{\frac{1}{2^k - 2}} \log^{\frac{1}{2}}(2 + T)\right) \tag{2.5}$$

(2) Vinogradov estimate: For some absolute constant $c > 0$.

$$\frac{1}{N} \sum_{n \in \mathbf{I}} e(f(n)) \ll N^{-\frac{c}{k^2}}. \tag{2.6}$$

2.2 The Functions $\lfloor x \rfloor$, $\rho(x)$ and $\{x\}$

The Hardy–Littlewood–Flett function $\mathfrak{f}(x)$ is related, in many ways, to the three functions $\lfloor x \rfloor$, $\rho(x)$ and $\{x\}$. The floor function $\lfloor x \rfloor$ is related to the divisor function $d(n) = 1 \star 1(n)$, the multiplicative square convolution product of the constant function 1, through the Dirichlet hyperbola method. More generally if g, h are two multiplicative functions and $f = g \star h$. The Dirichlet hyperbola method is just the evaluation of a sum in two different ways:

$$\sum_{n \leq x} f(n) = \sum_{n \leq x} \sum_{ab=n} g(a)h(b) = \sum_{a \leq \sqrt{x}} \sum_{b \leq \frac{x}{a}} g(a)h(b) + \sum_{b \leq \sqrt{x}} \sum_{a \leq \frac{x}{a}} g(a)h(b)$$
$$- \sum_{a \leq \sqrt{x}} \sum_{b \leq \sqrt{x}} g(a)h(b).$$

If $g = h$, then

$$\sum_{n\leq x} f(n) = 2\sum_{a\leq\sqrt{x}}\sum_{b\leq\frac{x}{a}} g(a)h(b) - \left(\sum_{a\leq\sqrt{x}} g(a)\right)^2.$$

As an application we have the estimate [28] (p. 262) for the divisor function $d = 1\star 1$:

$$d(x) = x\log x + (2\gamma - 1)x + O(x^{\frac{1}{2}}).$$

The importance of the functions $\{x\}$ and $\rho(x)$ lies in the integral representations of the Riemann zeta-function:

$$\zeta(s) = -s\int_0^\infty \frac{\{x\} - \frac{1}{2}}{x^{s+1}}\,dx = -s\int_0^\infty \frac{\rho(x)}{x^{s+1}}\,dx$$

valid for $-1 < \mathrm{Re}s < 0$. Making the change of variable $x = \dfrac{1}{u}$ and applying Mellin inversion formula gives

$$\rho(\frac{1}{u}) = -\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)}{s} u^{-s}\,ds.$$

For later use, we give some details on the case considered by Davenport in (2.3). From (2.4) we obtain for $-1 < c < 0$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\rho(nx) = -\frac{1}{2i\pi}\int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)}{s\zeta(1-s)} x^s\,ds.$$

By the functional equation of the Riemann ζ-function and the functional equation of the Γ-function we obtain for $0 < a < 1$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\rho(nx) = -\frac{1}{2i\pi^2}\int_{a-i\infty}^{a+i\infty} \Gamma(s)\sin\left(\frac{1}{2}\pi s\right)(2\pi x)^{-s}\,ds = -\frac{1}{\pi}\sin(2\pi x).$$

Using the classical equivalent formulation of the Prime Number Theorem that $\sum_{n=1}^{\infty} \dfrac{\mu(n)}{n} = 0$ we obtain Davenport's relation

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n}\{nx\} = -\frac{1}{\pi}\sin(2\pi x) \tag{2.7}$$

where the convergence is uniform by Davenport estimate (2.4). We will need two important properties of the function $\{x\}$:

Kubert identity:

$$\sum_{l \bmod m} \left\{ x + \frac{l}{m} \right\} = \{mx\} \tag{2.8}$$

Franel formula:

$$\int_0^1 \{ax\} \{bx\}\, dx = \frac{\mathrm{lcm}(a,b)}{12ab}. \tag{2.9}$$

Kubert identity and Franel's formula are interesting features shared by many functions. Let $B_r(x)$ be the Bernoulli polynomial defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{r=0}^{\infty} B_r(x) t^r, \quad |t| < 2\pi,$$

so that

$$B_1(x) = x - \frac{1}{2}, \quad 2!B_2(x) = x^2 - x + \frac{1}{6}, \cdots$$

If $r \geq 2$ is even then for $0 \leq x \leq 1$

$$(2\pi)^r B_r(x) = (-1)^{\frac{1}{2}-1} \sum_{l=1}^{\infty} \frac{2\cos(2l\pi x)}{l^r}$$

with absolute convergence of the series. The Hurwitz zeta function $\zeta(s,x)$ is defined for $\mathrm{Re}s > 1$ by

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^s}.$$

Then $B_r(x)$ and $\zeta(s,x)$ both satisfy the functional equation [21]

$$f(x) + f(x + \frac{1}{k}) + \cdots + f(x + \frac{k-1}{k}) = f^{(k)} f(kx),$$

where $f^{(k)} = k^{1-n}$ if $f(x) = B_n(x)$ and $f^{(k)} = k^s$ if $f(x) = \zeta(s,x)$. Furthermore, if a, b denote arbitrary positive integers and $(a,b) = \gcd(a,b)$, $[a,b] = \mathrm{lcm}(a,b)$ the greatest common divisor and least common multiple respectively of a and b, then [21]:

$$\int_0^1 B_r(ax) B_r(bx)\, dx = (-1)^{r-1} \frac{B_{2r}}{(2r)!} \left(\frac{(a,b)}{[a,b]} \right)^r$$

and for $\mathrm{Re}s > \frac{1}{2}$

$$\int_0^1 \zeta(1-s, ax) \zeta(1-s, bx)\, dx = \frac{2\Gamma^2(s)\zeta(2s)}{(2\pi)^{2s}} \left(\frac{(a,b)}{[a,b]} \right)^s.$$

Similarly to (2.2) we have

$$\zeta(s,w)=\frac{1}{(s-1)w^{s-1}}+\frac{1}{w^s}-s\int_0^\infty \frac{u-\lfloor u\rfloor}{(u+w)^{s+1}}\,du,$$

and the function $\zeta(s,w)-\dfrac{1}{(s-1)w^{s-1}}$ is analytic in $\{\mathrm{Re}s>0\}$. In the next section we use two summation formulas.

If F is an antiderivative of f, then, formally [1]

$$\int_0^1 \rho(\frac{\theta}{t})f(t)dt=\theta\int_0^1 \frac{f(t)}{t}\,dt-\sum_{n=1}^{\infty} n\left(F(\frac{\theta}{n})-F(\frac{\theta}{n+1})\right) \tag{2.10}$$

and if μ is the Möbius function and if $0<\theta,\ x\leq 1$, we have, pointwise [2]

$$\sum_{n=1}^{\infty}\mu(n)\left\{\rho\left(\frac{\theta}{nx}\right)-\frac{1}{n}\rho\left(\frac{\theta}{x}\right)\right\}=-\chi_{]0,\theta]}(x).$$

3 From Beurling's Theorem to Hardy–Littlewood–Flett Function $\mathfrak{f}(x)$

3.1 The Emergence of Franel Integral Type

To show that the constant function $1\in\overline{\mathcal{M}}$ one has, as in [2], to minimize the norms in $L^2([0,\ 1])$

$$\|1+\sum_{j=1}^{N}a_j\rho\left(\frac{\alpha_j}{\cdot}\right)\| \tag{3.1}$$

which brings back to the evaluation of integrals of Franel type, computed in [2]:

$$J(\beta)=\int_0^1 \rho\left(\frac{1}{x}\right)\rho\left(\frac{\beta}{x}\right)\,dx,\quad \beta\in[0,1].$$

To show that the function $\sin x\in\overline{\mathcal{M}}$ one has, this time, to minimize the norms

$$\|\sin x+\sum_{j=1}^{N}a_j\rho\left(\frac{\alpha_j}{\cdot}\right)\| \tag{3.2}$$

Using (2.7) the minimization problem reduces to evaluation of the scalar products in $L^2\,(0,\ 1)$ giving the Fourier sine series of the function $\{\frac{\theta}{x}\}$, that is

$$a_n=(\{\frac{\theta}{x}\}\big|\sqrt{2}\sin(n\pi x))=\sqrt{2}\int_0^1\{\frac{\theta}{x}\}\sin(n\pi x)dx=-\pi\sqrt{2}\sum_{j\geq 1}\frac{\mu_j}{j}\int_0^1\{\frac{\theta}{x}\}\{\frac{jnx}{2}\}dx$$

and then to the evaluation of $\int_0^1 \{\frac{a}{x}\}\{bx\}dx$, another kind of integrals of Franel type. We compute these integrals in the case $a = m,\ b = n$, m and n being integers.

3.2 The Second Kind of Franel Type Integrals $I_{n,m} = \int_0^1 \{nx\}\{\frac{m}{x}\}dx,\ \ n, m \in \mathbb{N}^*$

The values of the integrals $I_{n,m}$ are given by the following

Theorem 3.1 *For positive integers m, n, the modified Franel integrals are given by*

$$I_{n,m} = \frac{n}{m} + m\log m + m(n-1)\log(mn) - m(\log((n-1)!))$$
$$-\frac{n(n-1)}{2} - \frac{nm^2}{2}(\zeta(2) - \sum_{1\le j\le m}(1-\frac{m}{j})) + \sum_{1\le k\le n-1, mn\ge jk}(1-\frac{mn}{jk}).$$

Let us first give few examples:

$$I_{(2,1)} = \frac{5}{2} - \log(2) - \zeta(2); \qquad I_{(3,1)} = \frac{25}{6} + \log(2) - 2\log(3) - \frac{3}{2}\zeta(2)$$
$$I_{(4,1)} = \frac{35}{6} - 5\log(2) + \log(3) - 2\zeta(2); \qquad I_{(5,1)} = \frac{35}{6} - 5\log(2) + \log(3) - 2\zeta(2)$$
$$I_{(1,2)} = \frac{7}{2} - 2\zeta(2); \qquad I_{(1,3)} = \frac{61}{8} - \frac{9}{2}\zeta(2)$$
$$I_{(1,4)} = \frac{5989}{288} - \frac{25}{2}\zeta(2); \qquad I_{(2,2)} = \frac{49}{6} - 2\log(2) - 4\zeta(2)$$
$$I_{(2,3)} = \frac{171}{10} - 3\log(2) - 9\zeta(2); \qquad I_{(2,4)} = \frac{18469}{630} - 4\log(2) - 16\zeta(2)$$
$$I_{(2,5)} = \frac{15059}{336} - 5\log(2) - 25\zeta(2); \qquad I_{(3,2)} = \frac{196}{15} + 2\log(2) - 4\log(3) - 6\zeta(2)$$

We observe that in all these examples the factor $\zeta(2) = \frac{\pi^2}{6}$ is present.

For the proof we consider the two functions defined on $]0, +\infty[$

$$f(x) = f_n(x) = x\chi_{[0,1]}(x)\{nx\}, \quad g(x) = \{x\}\chi_{[1,+\infty]}(x)$$

and their multiplicative convolution

$$(f\star g)(a) = \int_0^{+\infty} f(x)g(\frac{a}{x})\frac{dx}{x}, \quad (f\star g)(m) = I_{n,m}.$$

We split the computations in several steps. A natural method is to use first the Mellin transform with its property $\mathcal{M}(f\star g)(s) = \mathcal{M}(f)(s)\mathcal{M}(g)(s)$, followed by an inversion. The main idea is the decomposition formula (2.10), valid if $\int_0^1 \frac{|f(x)|}{x}dx$ is finite:

$$\int_0^1 \rho_\theta(x) f(x)dx = \sum_{n=1}^{\infty} \int_{\frac{\theta}{n+1}}^{\frac{\theta}{n}} (\frac{\theta}{x} - n) f(x)dx + \int_\theta^1 \frac{\theta}{x} f(x)dx,$$

or in an generalized function form,

$$\rho_\theta(x) = \sum_{n=1}^{\infty} \left(\frac{\theta}{x} - n\right) \chi_{[\frac{\theta}{n+1}, \frac{\theta}{n}]}(x) + \frac{\theta}{x} \chi_{[\theta,1]}$$

where χ_B denotes the characteristic function of the set B.

3.2.1 Computations of Different Integrals

(1) Computation of $F(s) = M(f)(s)$ For $\sigma = \mathrm{Re}\, s > -2$ we have

$$\begin{aligned}
F(s) &= \int_0^1 \{nx\} x^s dx \\
&= \sum_{0 \le k \le n-1} \int_{\frac{k}{n}}^{\frac{k+1}{n}} (nx - k) x^s dx \\
&= n \int_0^1 x^{s+1} dx - \sum_{1 \le k \le n-1} k \int_{\frac{k}{n}}^{\frac{k+1}{n}} x^s dx \\
&= \frac{n}{s+2} - \frac{1}{(s+1)n^{s+1}} \sum_{1 \le k \le n-1} k((k+1)^{s+1} - k^{s+1}) \\
&= \frac{n}{s+2} - \frac{1}{(s+1)n^{s+1}} \{n^{s+2} - (1 + 2^{s+1} + 3^{s+1} + \cdots n^{s+1})\}
\end{aligned}$$

(2) Computation of $G(s) = M(g)(s)$ For $-2 < \sigma = \mathrm{Re} s < -1$ we have

$$\begin{aligned}
G(s) &= \int_1^{+\infty} \{x\}^{s-1} dx \\
&= \sum_{k \ge 1} \int_k^{k+1} (x - k) x^{s-1} dx \\
&= \int_1^{+\infty} x^s dx - \sum_{k \ge 1} k \int_k^{k+1} x^{s-1} dx \\
&= \int_1^{+\infty} x^s dx - \frac{1}{s} \sum_{k \ge 1} k((k+1)^s - k^s) \\
&= \frac{1}{s+1} - \frac{\zeta(-s)}{s} \quad \sigma < -1
\end{aligned}$$

Hence for $-2 < c < -1$ we can write

$$I_{n,m} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{1}{s+1} - \frac{\zeta(-s)}{s} \right) \left(\frac{n}{s+2} - \frac{1}{(s+1)n^{s+1}} (n^{s+1} -(1 + 2^{s+1} + 3^{s+1} + \cdots (n-1)^{s+1}) \right) \frac{ds}{m^s}.$$

and, by changing s to $-s$, we get for $1 < c < 2$

$$I_{n,m} = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \left(\frac{1}{1-s} + \frac{\zeta(s)}{s} \right) \left(\frac{n}{2-s} - \frac{1}{(1-s)n^{1-s}} (n^{1-s} -(1 + 2^{1-s} + 3^{1-s} + \cdots (n-1)^{1-s}) \right) \frac{ds}{m^{-s}}.$$

By expanding we find:

$$\begin{aligned} I_{n,m} &= \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{n.m^s}{(1-s)(2-s)} ds \\ &- \frac{1}{2i\pi} \int_{c-i\infty}^{c+\infty} \frac{m^s}{(1-s)^2 n^{1-s}} (n^{1-s} - (1 + 2^{1-s} + 3^{1-s} + \cdots (n-1)^{1-s})) ds \\ &+ \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{\zeta(s)}{s} (\frac{n}{2-s} - \frac{1}{(1-s)n^{1-s}} (n^{1-s} - (1 + 2^{1-s} + \cdots \\ &+ (n-1)^{1-s})) m^s ds \end{aligned}$$

We treat the last integral by expanding the ζ function in Dirichlet series. We will treat each type of integrals appearing separately. Then we proceed to the necessary groupings in order to conclude.

In the following we write $\int_{(c)}$ instead of $\int_{c-i\infty}^{c+i\infty}$, with $1 < c < 2$.

(3) Computation of $\frac{n}{2i\pi} \int_{(c)} \frac{m^s}{(1-s)(2-s)} ds$. We set $f(x) = -\frac{1}{x}$ for $0 < x \le 1$ and $f(x) = -\frac{1}{x^2}$ for $x > 1$. Its Mellin transform is $\frac{1}{(1-s)(2-s)}$ for $1 < \sigma < 2$. We obtain $\frac{n}{m}$ for $m \ge 1$.

(4) Computation of $-\frac{1}{2i\pi} \int_{(c)} \frac{m^s}{(1-s)^2} ds$. We take $f(x) = \frac{\log x}{x}$ for $0 < x < 1$ and 0 for $x \ge 1$. Its Mellin transform is $-\frac{1}{(s-1)^2}$ for $\sigma > 1$. Here we obtain $m \log m$ for $m \ge 1$.

(5) Computation of $\frac{k}{n} \int_{(c)} \frac{(m\,n)^s ds}{(1-s)^2 k^s}$. As before we find $m \log(\frac{m\,n}{k})$ if $mn \ge k$ and 0 if $mn < k$.

(6) Computation of $\frac{n}{2i\pi} \int_{(c)} \frac{m^s ds}{s(2-s) j^s}$, $j \ge 1$. We take $f(x) = -\frac{1}{2}$ for $0 < x \le 1$ and $f(x) = -\frac{1}{2x^2}$ for $x > 1$. We get $-\frac{n}{2}$ if $j \le m$ and $-\frac{nm^2}{2j^2}$ if $j > m$.

(7) Computation of $-\dfrac{1}{2i\pi}\displaystyle\int_{(c)} \dfrac{m^s ds}{s(1-s)j^s}$, $j \geq 1$. Here we take $f(x) = 1 - \dfrac{1}{x}$ if $0 < x \leq 1$ and 0 for $x > 1$. We obtain $1 - \dfrac{m}{j}$ if $m \geq j$ and 0 otherwise.

(8) Computation of $\dfrac{k}{n}\dfrac{1}{2i\pi}\displaystyle\int_{c)} \dfrac{(nm)^s}{s(1-s)(jk)^s}$. Here we obtain $1 - \dfrac{nm}{jk}$ if $mn \geq jk$ and 0 otherwise.

By putting together these partial results we end the proof of Theorem (3.1).

3.3 Second Approach $\left\{\frac{\theta}{x}\right\}$

The most interesting approach for the evaluation of the integral $\displaystyle\int_0^1 \left\{\frac{\theta}{t}\right\} \sin(n\pi t)dt$ is to use (2.10):

$$\int_0^1 \left\{\frac{\theta}{t}\right\} \sin(n\pi t)dt = \int_0^\theta \left\{\frac{\theta}{t}\right\} \sin(n\pi t)dt + \theta \int_\theta^1 \frac{1}{t} \sin(n\pi t)dt.$$

Moreover

$$\begin{aligned}
\int_0^\theta \left\{\frac{\theta}{t}\right\} \sin(n\pi t)dt &= \sum_{p\geq 1} \int_{\frac{\theta}{p+1}}^{\frac{\theta}{p}} \sin(n\pi t)(\frac{\theta}{t} - p)dt \\
&= \theta \sum_{p\geq 1} \int_{\frac{\theta}{p+1}}^{\frac{\theta}{p}} \frac{\sin(n\pi t)}{t} dt - \sum_{p\geq 1} p \int_{\frac{\theta}{p+1}}^{\frac{\theta}{p}} \sin(n\pi t)dt \\
&= \theta \int_0^\theta \frac{\sin(n\pi t)}{t} dt + \frac{1}{n\pi} \sum_{p\geq 1} p\Big(\cos\frac{n\pi\theta}{p} - \cos\frac{n\pi\theta}{p+1}\Big)
\end{aligned}$$

Hence the n-th Fourier coefficient

$$a_n = (\{\frac{\theta}{x}\}\big|\sqrt{2}\sin(n\pi x)) = \sqrt{2}\int_0^1 \{\frac{\theta}{x}\} \sin(n\pi x)dx$$

is also

$$a_n = \sqrt{2}\left(\theta \int_0^1 \frac{\sin(n\pi t)}{t} dt + \frac{1}{n\pi} \sum_{p\geq 1} p\Big(\cos\frac{n\pi\theta}{p} - \cos\frac{n\pi\theta}{p+1}\Big)\right).$$

Seeking for the coefficient corresponding to $f(x) = \sum_{1\leq\nu\leq N} c_\nu \{\frac{\theta_\nu}{x}\}$ the first integral does not matter since $\sum_{1\leq\nu\leq N} c_\nu\theta_\nu = 0$. It remains to compute

$$A = \sum_{p\geq 1} p\Big(\cos\frac{n\pi\theta}{p} - \cos\frac{n\pi\theta}{p+1}\Big).$$

A depends on n and θ. We first consider the finite sum

$$A_N = \sum_{1 \le p \le N} p\Big(\cos \frac{n\pi\theta}{p} - \cos \frac{n\pi\theta}{p+1}\Big)$$

and set $x = n\pi\theta$. We have by a partial summation

$$\begin{aligned} A_N &= \cos \frac{x}{1} + \cdots + \cos \frac{x}{N} - N \cos \frac{x}{N+1} \\ &= (\cos \frac{x}{1} - 1) + \cdots + (\cos \frac{x}{N} - 1) + N(1 - \cos \frac{x}{N+1}) \\ &= -2 \sum_{k=1}^{N} \sin^2 \frac{x}{k} + 2N \sin^2 \frac{x}{N+1}. \end{aligned}$$

Hence

$$\lim_{N \to +\infty} A_N = -2 \sum_{k=1}^{\infty} \sin^2 \frac{x}{k}. \tag{3.3}$$

We thus obtain one of our main results: the n-th Fourier coefficient a_n of the fundamental function $\{\frac{\theta}{\bullet}\}$ is related to the value at n of the antiderivative of the function $\mathfrak{f}(x)$ given in (1.1)

$$a_n = \sqrt{2} \left(\theta \int_0^1 \frac{\sin(n\pi t)}{t} dt - \frac{1}{n\pi} \sum_{k=1}^{\infty} \sin^2 \frac{n\pi\theta}{k} \right), \tag{3.4}$$

bearing in mind that the derivative of $\sum_{k=1}^{\infty} \sin^2 \frac{u}{k}$ is $\mathfrak{f}(2u)$.

To give some useful integral representations we adapt an interesting method, due to Delange [13], and use a result of Saffari and Vaughan [24]. First we introduce for $0 < \alpha \le 1$

$$c_\alpha(u) = \begin{cases} 1 \text{ if } & u - \lfloor u \rfloor = \rho(u) < \alpha \\ 0 & \text{otherwise} \end{cases}$$

Furthermore for $x > 0,\ y > 1$ let

$$\vartheta_{x,y}(u) = \frac{1}{\log y} \sum_{n \le y} \frac{1}{n} c_\alpha(\frac{x}{n}).$$

According to [24] we have

Lemma 3.1 *We have the estimate*

$$\vartheta_{x,y}(u) = u + O\left((\log x)^{\frac{2}{3}} (\log y)^{-1} \right),$$

the O is uniform in u.

If f is continuously differentiable function on [0, 1]

$$f(2\pi\frac{x}{n}) = -2\pi\int_{\{\frac{x}{n}\}}^{1} f'(2\pi u)\,du = -2\pi\int_0^1 c_u(\frac{x}{n})f'(2\pi u)\,du.$$

Hence

$$\sum_{n\le x}\frac{1}{n}f(2\pi\frac{x}{n}) = -2\pi(\log x)\int_0^1 \vartheta_{x,x}(u)f'(2\pi u), \tag{3.5}$$

since

$$\int_0^1 \vartheta_{x,x}(u)f'(2\pi u)\,du = \int_0^1 f'(2\pi u)\,du + \int_0^1 \left(\vartheta_{x,x}(u) - u\right) f'(2\pi u)\,du$$
$$= \int_0^1 \left(\vartheta_{x,x}(u) - u\right) f'(2\pi u)\,du.$$

From the Lemma (3.1) we get, since f' is bounded on (0, 1)

$$\sum_{n\le x}\frac{1}{n}f(2\pi\frac{x}{n}) = O\,(\log x)^{\frac{2}{3}}\,.$$

A natural example is to consider a Dirichlet character modulo N, χ. In this case

$$\sum_{n\le x}\frac{\chi(n)}{n}\sin(2\pi\frac{x}{n}) = O\,(\log x)^{\frac{2}{3}}\,.$$

We shall not try to give sufficient conditions to justify the process here. The main interest of the remark is that it suggests a method of dealing with various other sums than $\mathfrak{f}(x)$.

4 Almost Periodicity

The goal of this section is to show, by elementary methods, that the Hardy–Littelwood–Flett function $\mathfrak{f}(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{x}{n}$ is not bounded on the real line. First we recall two fundamental results on Bohr-almost periodic functions [8] (p.39, 44, and 58).

Theorem 4.1 (The Mean value theorem) *For every almost periodic function $f(x)$, there exists a mean value*

$$\lim_{T\to+\infty}\frac{1}{T}\int_0^T f(x)dx = M\{f(x)\}$$

and

$$\lim_{T\to+\infty} \frac{1}{T} \int_a^{a+T} f(x)dx = M\{f(x)\}$$

uniformly with respect to a. In particular if f is an odd almost periodic function, then its mean $M\{f(x)\}$ is zero.

Theorem 4.2 (The antiderivative theorem) *The integral $F(x) = \int_0^x f(t)dt$ of an almost-periodic function $f(x)$ is almost-periodic if and only if it is bounded.*

Let

$$\mathfrak{F}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \cos \frac{x}{n}.$$

The series defining $\mathfrak{F}(x)$ is uniformly convergent on the real line. The partial sums

$$\mathfrak{F}_n(x) = \sum_{p=1}^{n} \frac{1}{p^2} \cos \frac{x}{p}$$

are almost periodic [8] (Corollary, p.38), and then $\mathfrak{F}(x)$ is also almost periodic [8] (Theorem IV, p.38). It is interesting to note that $\mathfrak{F}_n$ is periodic of period $p_n = \operatorname{lcm}(1, 2, \cdots, n) = e^{\psi(n)}$, with $\psi(x)$ is the Chebyshev function, given by $\psi(x) = \sum_{p\le x} \Lambda(p)$, where $\Lambda(n)$ is the Mangoldt function.

The prime number theorem asserts that $p_n = e^{n(1+o(1))}$ as $n \to \infty$ [26] (p.261). Actually $p_n \le 3^n$.

Lemma 4.1 *We have*

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{n=1}^{\infty} \sin^2 \frac{x}{n} = \frac{\pi}{2}.$$

Let $x > 0$ and $n_x = \left\lfloor \frac{2x}{\pi} \right\rfloor$. The function $h : x \to \sin^2 \frac{1}{x}$, being bounded on $[0, \frac{\pi}{2}]$ and continuous on each $[\alpha, \frac{\pi}{2}]$, is Riemann-integrable on $[0, \frac{\pi}{2}]$, so by considering Riemann sums

$$\lim_{x\to+\infty} \frac{1}{x} \sum_{n=1}^{n_x} \sin^2 \frac{x}{n} = \int_0^{\frac{2}{\pi}} \sin^2 \frac{1}{t}\, dt = \int_{\frac{\pi}{2}}^{\infty} \frac{\sin^2 u}{u^2}\, du. \tag{4.1}$$

For $x > 0$ the function of $g(t) = \sin^2 \frac{x}{t}$ is decreasing on $(\dfrac{2x}{\pi},\ +\infty)$ and thus

$$\left| \sum_{n=n_x+1}^{\infty} \sin^2 \frac{x}{n} - \int_{\frac{2x}{\pi}}^{\infty} \sin^2 \frac{x}{t}\, dt \right| \leq 1. \tag{4.2}$$

Since $\displaystyle\int_{\frac{2x}{\pi}}^{\infty} \sin^2 \frac{x}{t}\, dt = x \int_0^{\frac{\pi}{2}} \frac{\sin^2 u}{u^2}\, du$ we deduce the lemma from (4.8) (4.9) and the relations

$$\int_0^{\infty} \frac{\sin^2 u}{u^2}\, du = \int_0^{\infty} \frac{\sin u}{u}\, du = \frac{\pi}{2}.$$

Corollary 4.1 *The function* $\mathfrak{f}(x) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}$ *is not bounded on the real line.*

Proof Assume that $\mathfrak{f}(x)$ is bounded on $\mathbb{R}$ then it would be almost periodic by the antiderivative theorem (4.2) and the remark that $\mathfrak{f}'(x) = \mathfrak{F}(x)$. Since $\mathfrak{f}(x)$ is odd its mean is zero. This is in contradiction with the limit $\dfrac{\pi}{2}$ given by the Lemma (4.1). □

Remark 4.1 The same analysis applies to the series $\sum_{n=1}^{\infty} \dfrac{\chi(n)}{n} \sin(\dfrac{x}{n})$, χ being a Dirichlet character modulo N.

We will need to consider some Bessel functions. We recall that for $\mathrm{Re} s > 0$ the Γ-function is

$$\Gamma(s) = \int_0^{\infty} u^{s-1} e^{-u}\, du.$$

By Fubini's theorem

$$\Gamma^2(s) = \int_0^{\infty} \int_0^{\infty} (uv)^{s-1} e^{-(u+v)}\, du dv = \int_0^{\infty} u^{s-1} \xi_0(u)\, du,$$

where

$$\xi_0(u) = \int_1^{\infty} \frac{2e^{2t\sqrt{u}}}{\sqrt{t^2-1}}\, dt.$$

More generally the iterated integrals [30,31]

$$\xi_1(x) = \int_x^{\infty} \xi_0(t)\, dt, \ldots, \xi_k(x) = \int_x^{\infty} \xi_{k-1}(t)\, dt.$$

satisfy the differential equation of Bessel type:

$$x \frac{d^2 \xi_k(x)}{dx^2} + (1-k) \frac{d\xi_k(x)}{dx} - \xi_k(x) = 0.$$

The ordinary Bessel function of order ν is

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \quad I_\nu(z) = i^{-\nu} J_\nu(iz), \quad |z| < \infty.$$

The K-Bessel function of order ν, for ν not an integer, is

$$K_\nu(z) = \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \pi \nu}.$$

When ν is an integer we take the limiting value. It could be also defined by

$$K_\nu(z) = \frac{1}{2} \int_0^\infty t^{\nu-1} e^{-z/2(t+1/t)} dt, \quad \mathrm{Re}\nu \geq 0. \tag{4.3}$$

The Mellin transform of the J_0-Bessel function is:

$$\int_0^\infty J_0(\sqrt{x}) x^{s-1} dx = 4^s \frac{\Gamma(s)}{\Gamma(1-s)}.$$

We will need two Mellin transforms, due essentially to Voronoi

$$\int_0^\infty x^{s-1} K_0(4\pi\sqrt{x})\, dx = \frac{1}{2} (2\pi)^{-2s} \Gamma^2(s),$$
$$\int_0^\infty x^{s-1} Y_0(4\pi\sqrt{x})\, dx = -\frac{1}{\pi} (2\pi)^{-2s} \cos(\pi s) \Gamma^2(s).$$

4.1 Summations Formulas and Beyond

Various classical summation formulas, as Poisson summation formula, Voronoi summation formula or Hardy–Ramanujan summation formula can all be given a unified formulation. The following Generalized Poisson summation formula is proved in [9]

Theorem 4.3 *Let $a = a(n)$ be an arithmetic function with moderate growth. We define the Dirichlet series*

$$L(a,s) = \sum_{n=1}^{\infty} a(n) n^{-s}, \quad \mathrm{Re} s > 1$$

and we suppose that $L(a,s)$ has an analytic continuation to $\mathbb{C}$ with only a possible pole at $s = 1$. We suppose also that there are positive constants $A, a_1, \ldots, a_g$ such *that with the* Γ*-factors*

$$\gamma(s) = A^s \prod_{j=1}^{g} \Gamma(a_j s)$$

$L(a, s)$ *satisfies the functional equation*

$$\gamma(s)L(a, s) = \gamma(1-s)L(a, 1-s).$$

Furthermore for $f \in \mathcal{S}(\mathbb{R})$*, the Schwartz space, we define a very special Hankel's transform:*

$$g(x) = \int_0^\infty f(y)K(xy)dy, \quad \text{with} \quad K(x) = \int_{\mathrm{Re}s=\frac{3}{2}} \frac{\gamma(s)}{\gamma(1-s)} x^{-s} ds.$$

Then,

$$\sum_{n=1}^\infty a(n)f(n) = f(0)L(a, 0) + \mathrm{Res}_{s=1}\mathcal{M}(f)(s)L(a, s) + \sum_{n=1}^\infty a(n)g(n)$$

where $\mathrm{Res}_{s=1}$ *is the evaluation of the residue at* $s = 1$.

4.2 2 Classical Choices

(1) For $a(n) = 1$ we have $L(a, s) = \zeta(s)$ and

$$\gamma(s) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}), \quad K(x) = 2\cos(2\pi x).$$

We recover Poisson summation formula for even functions in $f(x) \in \mathcal{S}(\mathbb{R})$:

$$\sum_{n=1}^\infty f(n) = -\frac{1}{2}f(0) + \int_0^\infty f(x)dx + 2\sum_{n=1}^\infty \int_0^\infty f(y)\cos(2\pi ny)dy. \quad (4.4)$$

(2) If $a(n) = d(n)$ we have $L(d, s) = \zeta^2(s)$ and

$$\gamma(s) = \pi^{-s}\Gamma(\frac{s}{2})^2, \quad \frac{\gamma(s)}{\gamma(1-s)} = (2\pi)^{-2s}(2 + 2\cos\pi s)\Gamma(s)^2$$

and

$$K(x) = 4K_0(4\pi\sqrt{x}) - 4Y_0(4\pi\sqrt{x}).$$

We recover Voronoi summation formula

$$\sum_{n=1}^{\infty} f(n)d(n) = \frac{1}{4} f(0) + \int_0^{\infty} f(x)\,(2\gamma + \log x)\,dx + \\ \sum_{n=1}^{\infty} d(n) \int_0^{\infty} f(y) \left(4K_0\left(4\pi (ny)^{\frac{1}{2}}\right) - 2\pi Y_0\left(4\pi (ny)^{\frac{1}{2}}\right)\right) dy.$$

As a consequence we have Koshliakov's formula valid for $a > 0$:

$$\sqrt{a}\left(\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n) K_0(2\pi a n)\right) \\ = \frac{1}{\sqrt{a}}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n) K_0\left(\frac{2\pi n}{a}\right)\right).$$

This formula was proved by Ramanujan about ten years earlier (He did not appeal to Voronoi's formula) and by many authors later.

4.3 Another Function of Hardy and Littlewood

Hardy and Littlewood gave in [16] (p.269) the following relation

$$\mathfrak{F}(z) = \sum_{n=1}^{\infty} \frac{1}{n}\left(1 - e^{-z/n}\right) = 2\log z + 2\gamma \\ - 2\sum_{n=1}^{\infty}\left\{K_0\left(\sqrt{2n\pi i z}\right) + K_0\left(\sqrt{-2n\pi i z}\right)\right\} \quad (4.5)$$

where $\mathrm{Re} z > 0$, γ is Euler's constant.

For $|z| < 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - e^{-\frac{z}{n}}\right) = -\sum_{n=1}^{\infty} \zeta(n+1)\frac{(-z)^n}{n!}. \quad (4.6)$$

An immediate consequence of this expansion is obtained by taking real and imaginary parts with $z = ix$, $x \in \mathbb{R}$, $|x| < 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\frac{x}{n}\right) = 2\sum_{n=1}^{\infty} \frac{1}{n}\sin^2\frac{x}{2n} = -\sum_{k\geq 0} \zeta(4k+1)\frac{x^{4k}}{(4k)!} + \sum_{k\geq 0} \zeta(4k+3)\frac{x^{4k+2}}{(4k+2)!} \\ = \sum_{k\geq 0} (-1)^{k-1}\zeta(2k+1)\frac{x^{2k}}{(2k)!}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{x}{n} = \sum_{k\geq 0} \zeta(4k+2)\frac{x^{4k+1}}{(4k+1)!} - \sum_{k\geq 0} \zeta(4k+4)\frac{x^{4k+3}}{(4k+3)!}.$$

More generally we define the series

$$G_\nu(z) = \sum_{n > \mathrm{Re}\nu+1} \zeta(n-\nu)\frac{(-z)^n}{n!}$$

which has a Mellin-Barnes type integral representation when $x > 0$, c is fixed with $\mathrm{Re}\nu + 1 < c < \mathrm{Re}\nu + 2$:

$$G_\nu(z) = \frac{1}{2i\pi}\int_{(c)} \Gamma(-s)\zeta(s-\nu)x^s ds.$$

The proof of the main equality results from the deformation of the path of integration and the fact that the pair

$$x^\nu K_\nu(x), \quad 2^{s+\nu-2}\Gamma(s/2)\Gamma(s/2+\nu), \quad \mathrm{Re}s > \max(0, -2\mathrm{Re}\nu)$$

is a pair of Mellin transforms [18,19].
The series (4.6) has many remarkable properties. It may be differentiated term by term to get $\mathfrak{G}(-x)$ where $\mathfrak{G}(x)$ is the function defined in [26] (p.243):

$$\mathfrak{G}(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} e^{\frac{x}{n}}. \tag{4.7}$$

The following formula is mentioned in [27]

$$\lim_{K\to\infty} \frac{1}{K}\sum_{k=1}^{K} \mathfrak{G}(2i\pi nk) = \sum_{d|n} \frac{1}{d^2}.$$

4.4 Laplace Transform of $a\sqrt{t}J_1(a\sqrt{t})$, $a > 0$, $t > 0$, Another Approach to Segal's Formula

In [25] (formula (12)) Segal proves the following result

Theorem 4.4 *If* $g(z) := \sum_{k\geq 1}(1-\cos\frac{z}{k})$ *then*

$$g(z) = \frac{\pi z}{2} - \frac{1}{2} + \frac{1}{4}\sum_{k\geq 1} \frac{2\sqrt{2k\pi}\sqrt{z}}{k} J_1(2\sqrt{2k\pi}\sqrt{z}).$$

This formula is interesting compared to (4.5), as we have for real z, $g(z) = \mathrm{Re}\mathfrak{F}(iz)$. The proof given in [25] uses a rather elaborated tools such the three Bessel functions J_1, J_2, J_3, the functional equation of the Riemann ζ-function etc. We give here a proof which we think is simpler.

Let

$$g_1(z) = \frac{\pi z}{2} - \frac{1}{2} + \frac{1}{4} \sum_{k \geq 1} \frac{2\sqrt{2k\pi}\sqrt{z}}{k} J_1(2\sqrt{2k\pi}\sqrt{z}).$$

In the Laplace transform

$$\mathcal{L}(a\sqrt{t}J_1(a\sqrt{t}))(p) = a \int_0^{+\infty} \sqrt{t}J_1(a\sqrt{t})e^{-tp}dt, \quad \text{Re } p > 0$$

we set $u^2 = t$ and obtain

$$\mathcal{L}(a\sqrt{t}J_1(a\sqrt{t}))(p) = 2a \int_0^{+\infty} J_1(au)e^{-pu^2}u^2du, \quad \text{Re } p > 0.$$

According to [33] (page 394, formula(4)) we have for with $|\text{Arg } p| < \frac{\pi}{4}$

$$\int_0^{+\infty} J_\nu(au)e^{-p^2u^2}u^{\nu+1}du = \frac{a^\nu}{(2p^2)^{\nu+1}}e^{-\frac{a^2}{4p^2}}.$$

Replacing p by $\sqrt{p}$ with $|\text{Arg } p| < \frac{\pi}{2}$ and taking $\nu = 1$ we obtain

$$\int_0^{+\infty} J_1(au)e^{-pu^2}u^2du = \frac{a}{4p^2}e^{-\frac{a^2}{4p}}.$$

Hence

$$\mathcal{L}(a\sqrt{t}J_1(a\sqrt{t}))(p) = \frac{a^2}{2p^2}e^{\frac{-a^2}{4p}} \quad \text{Re} p > 0.$$

Note that $a\sqrt{t}J_1(a\sqrt{t})$ is not in $L^2([0, +\infty[)$ since its Laplace transform is not bounded in the L^2-norm on the lines $\text{Re} p = c$. With $a = 2\sqrt{2k\pi}$ we get

$$\mathcal{L}(2\sqrt{2k\pi t}J_1(2\sqrt{2k\pi t}))(p) = \frac{4k\pi}{p^2}e^{-\frac{2k\pi}{p}}.$$

As we have

$$\mathcal{L}\left(\frac{\pi t - 1}{2}\right)(p) = \frac{\pi}{2p^2} - \frac{1}{2p}$$

and, by continuity, the Laplace transform of the sum in

$$g_1(t) = \frac{\pi t - 1}{2} + \frac{1}{4} \sum_{k \geq 1} \frac{2\sqrt{2k\pi t}}{k} J_1(2\sqrt{2k\pi t})$$

is

$$\frac{\pi}{p^2}\sum_{k\geq 1}(e^{-\frac{2\pi}{p}})^k$$

which converges in $\mathrm{Re}\, p > 0$, the Laplace transform of $g_1(t)$ is

$$\frac{\pi}{2p^2} - \frac{1}{p} + \frac{\pi}{p^2}\frac{e^{-\frac{2\pi}{p}}}{1-e^{-\frac{2\pi}{p}}} = \frac{\pi}{2p^2} - \frac{1}{p} + \frac{\pi}{p^2}\frac{1}{e^{\frac{2\pi}{p}}-1}.$$

On the other hand

$$\mathcal{L}(1-\cos\frac{t}{k})(p) = \frac{1}{p} - \frac{p}{p^2+k^{-2}} = \frac{1}{p(p^2k^2+1)}$$

and

$$\mathcal{L}(g)(p) = \sum_{k=1}^{\infty}\frac{1}{p(p^2k^2+1)}.$$

The equality $\mathcal{L}(g)(p) = \mathcal{L}(g_1)(p)$ is obtained by using the well known partial fractions decomposition

$$\frac{z}{e^z-1} = 1 - \frac{z}{2} + \sum_{k\geq 1}\frac{2z^2}{z^2+4k^2\pi^2}, \quad z \in \mathbb{C}\backslash 2i\pi\mathbb{Z},$$

where we have to set $z = \dfrac{2\pi}{p}$. Hence $g = g_1$ by injectivity of Laplace Transform.

4.5 Some Mellin Transforms and the Cube of Theta Functions

It has been noticed in [15] (p.14) that the function

$$R(t) = \sum_{n\leq t}\frac{1}{n}e^{\frac{it}{n}}$$

is very similar to $\zeta(1+it)$ in its asymptotic behaviour as $t \to +\infty$. This could suggest a link between this function and the theta series $\vartheta_3(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$. In this section, following a suggestion of Crandall [10], we would like to briefly show by considering Mellin transforms an unexpected link to the third power of the (fourth) Jacobi theta function $\vartheta_4(q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2},\ |q| < 1$. We define

$$\tilde{\chi}(s,t) = \sum_{n=1}^{\infty}\frac{e^{-\frac{t}{n}}}{n^s}, \tag{4.8}$$

$$\chi(s,t) = \sum_{n=1}^{\infty} (-1)^n \frac{e^{-\frac{t}{n}}}{n^s}. \tag{4.9}$$

These two functions are defined for $s \in \mathbb{C}$ and $\mathrm{Re}s > 1$ for $\tilde{\chi}(s,t)$, $\mathrm{Re}s > 0$ for $\chi(s,t)$. They are related by

$$\chi(s,t) = \frac{1}{2^{s-1}} \tilde{\chi}(\frac{s}{2}, t) - \tilde{\chi}(s,t).$$

We have

$$\int_0^{\infty} t^{s-1} \tilde{\chi}^2(s,t)\, dt = \Gamma(s) \sum_{n,m=1}^{\infty} \frac{1}{(n+m)^s} = \Gamma(s) \left(\zeta(s-1) - \zeta(s)\right)$$
$$\int_0^{\infty} t^{s-1} \chi^2(s,t)\, dt = \Gamma(s) \sum_{n,m=1}^{\infty} \frac{(-1)^{n+m}}{(n+m)^s} = \Gamma(s) \left(\eta(s-1) - \eta(s)\right),$$

where

$$\eta(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

is the Dirichlet η-function. Furthermore

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \tilde{\chi}^3(s,t)\, dt = \sum_{p,q,r=1}^{\infty} \frac{1}{(pq+qr+rp)^s}$$

and for $\chi(s,t)$ we have a more interesting result

$$\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \chi^3(s,t)\, dt = \sum_{p,q,r=1}^{\infty} \frac{(-1)^{p+q+r}}{(pq+qr+rp)^s}. \tag{4.10}$$

We have the following lemma due to Andrews [3] (p.124)

Lemma 4.2 *For* $|q| < 1$

$$\vartheta_4^3(q) = \sum_{n \in \mathbb{Z}} (-1)^n r_3(n) q^n = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^n}{1+q^n} - 2 \sum_{\substack{n=1 \\ |j| \le n}}^{\infty} \frac{q^{n^2-j^2}(1-q^n)(-1)^j}{1+q^n} \tag{4.11}$$

where $r_3(n)$ is the number of representations of n as sum of three squares. According to a result of Fermat an integer is a sum of three squares if and only if it is not of form

$4^n(8m+7)$. There are some gaps in the expansion in power series of the left hand side of (4.11). Similarly to (4.10) we have

$$\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\vartheta_4^3(q)-1\right)\,dt=\sum_{p,q,r\in\mathbb{Z}}^{\infty}{}' \frac{(-1)^{p+q+r}}{(p^2+q^2+r^2)^s}. \tag{4.12}$$

A link between (4.10) and (4.12) is given by Crandall [10] (p.372) as a relation between two Epstein zeta functions associated with the two not equivalent ternary forms

$$q_1(u,v,w)=u^2+v^2+w^2,\quad q_2(u,v,w)=uv+vw+wu$$

in the form

$$\sum_{p,q,r\in\mathbb{Z}}^{\infty}{}' \frac{(-1)^{p+q+r}}{(p^2+q^2+r^2)^s}=-6(1-2^{1-s})^2\zeta^2(s)-4\sum_{p,q,r=1}^{\infty}\frac{(-1)^{p+q+r}}{(pq+qr+rp)^s}. \tag{4.13}$$

Next we establish a functional equation

Theorem 4.5 *For $t>0$ the function $\chi(\frac{1}{2},t)$ satisfies the following functional equation*

$$\chi(\frac{1}{2},t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}e^{-t/n}=\sqrt{i}\sum_{\mathcal{O}}\frac{e^{-\gamma\sqrt{2\pi dt}}}{\sqrt{d}}, \tag{4.14}$$

with $\gamma=1-i$ and $\mathcal{O}$ is the set of odd integers.

We could also seek for a result similar to (4.14) for

$$\tilde{\chi}(s,t)=\sum_{n=1}^{\infty}\frac{1}{n^s}e^{-t/n},\quad \mathrm{Re}s>1. \tag{4.15}$$

The relevance of this function lies in its relation to a Hardy–Littlewood–Flett like function:

$$\mathrm{Im}\tilde{\chi}(1,-it)=\mathrm{Im}\sum_{n=1}^{\infty}\frac{1}{n}e^{it/n}=\sum_{n=1}^{\infty}\frac{1}{n}\sin(\frac{t}{n}).$$

In order to prove (4.14) we consider, for a fixed $t>0$, the function

$$f(x)=e^{i\pi x}\frac{e^{-t/x}}{\sqrt{x}},\ x>0$$

extended to the origin by $f(0) = 0$ and to $\mathbb{R}$ as an even function. The obtained function is $\mathcal{C}^\infty$ on the real line to which we apply the Poisson summation formula (4.4) to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} e^{-t/n} = \int_0^\infty e^{i\pi x} \frac{e^{-t/x}}{\sqrt{x}} dx + 2 \sum_{n=1}^{\infty} \int_0^\infty e^{i\pi x} \frac{e^{-t/x}}{\sqrt{x}} \cos(2\pi n x) dx. \quad (4.16)$$

Remark 4.2 The function $y \mapsto \dfrac{e^{\frac{-t}{y}}}{\sqrt{y}}$ is continuous on $[0, \infty[$ and decreases to 0 at infinity, hence the proper integrals $\displaystyle\int_0^\infty f(y) \cos(2\pi n y) dy$, $n \geq 0$ are convergent.

We compute $\mathcal{F}(f)(n)$ as follows

$$\begin{aligned}
\mathcal{F}(f)(n) &= \int_0^\infty e^{i\pi y} \frac{e^{\frac{-t}{y}}}{\sqrt{y}} \cos(2n\pi y)\, dy \\
&= \frac{1}{2} \left\{ \int_0^\infty e^{(i\pi + 2i\pi n)y - t/y} y^{-1/2}\, dy + \int_0^\infty e^{(i\pi - 2i\pi n)y - t/y} y^{-1/2}\, dy \right\}.
\end{aligned} \quad (4.17)$$

We recall the modified Bessel function (4.3), written in the form

$$\int_0^\infty w^{\nu-1} e^{-w-a/w}\, dw = 2 \left(\frac{1}{a}\right)^{\nu/2} K_\nu\left(2\sqrt{a}\right)$$

that we use in the form

$$\int_0^\infty w^{\nu-1} e^{-bw-a/w}\, dw = 2 \left(\frac{a}{b}\right)^{\nu/2} K_\nu\left(2\sqrt{ab}\right). \quad (4.18)$$

Actually for (4.17) we need only the simplest case of

$$K_{\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}. \quad (4.19)$$

The first integral in the left hand side of (4.17), with

$$a = t, \quad -b = i\pi + 2i\pi n = i\pi(2n + 1)$$

is equal to

$$2 \left(\frac{t}{-i\pi(2n+1)}\right)^{1/4} K_{1/2}\left(2\sqrt{t\,(-i\pi(2n+1))}\right). \quad (4.20)$$

The second integral with

$$a = t, \quad -b = i\pi - 2i\pi n = i\pi(-2n + 1)$$

is equal to

$$2\left(\frac{t}{-i\pi(-2n+1)}\right)^{1/4} K_{1/2}\left(2\sqrt{t\,(-i\pi(-2n+1))}\right). \tag{4.21}$$

By using (4.19) we see that (4.17) is the sum of

$$\left(\frac{t}{-i\pi(2n+1)}\right)^{1/4}\sqrt{\frac{\pi}{4\sqrt{t\,(-i\pi(2n+1))}}}e^{-2\sqrt{t(-i\pi(2n+1))}}$$

and

$$\left(\frac{t}{-i\pi(-2n+1)}\right)^{1/4}\sqrt{\frac{\pi}{4\sqrt{t\,(-i\pi(-2n+1))}}}e^{-2\sqrt{t(-i\pi(-2n+1))}}.$$

As in [10] we denote by $\gamma = 1-i$, $d = \pm 2n+1$, $n \in \mathbb{N}^\star$ with $\sqrt{-|d|} = i\sqrt{|d|}$. Then d describes $\mathcal{O}\backslash\{1\}$ and

$$\left(\frac{t}{-i\pi d}\right)^{1/4}\sqrt{\frac{\pi}{4\sqrt{t\,(-i\pi d)}}}e^{-2\sqrt{-it\pi d}} = \frac{1}{2}\sqrt{i}\,\frac{e^{-\gamma\sqrt{t\pi d}}}{\sqrt{d}}.$$

Hence

$$2\sum_{n=1}^{\infty}\int_0^{\infty} f(y)\cos(2\pi ny)dy = \sqrt{i}\sum_{d\in\mathcal{O},\, d\neq 1}\frac{e^{-\gamma\sqrt{t\pi d}}}{\sqrt{d}}. \tag{4.22}$$

For the remaining term in (4.16) we use (4.20), with $n=0$, to obtain

$$\int_0^{\infty} f(x)dx = \mathcal{F}(f)(0) = \sqrt{i}\,e^{-\gamma\sqrt{2\pi t}}$$

which together with (4.22) gives finally (4.14):

$$\chi(\frac{1}{2}, t) = \sqrt{i}\sum_{\mathcal{O}}\frac{e^{-\gamma\sqrt{2\pi dt}}}{\sqrt{d}}.$$

In close analogy to Jacobi's transformation of Theta functions (4.14) appears as a convergence acceleration of a slowly convergent series.

Incidentally $\chi(\frac{1}{2}; \frac{t^2}{4})$ is a Fourier transform of a function of the Schwartz class. Indeed let $g(x) = \frac{1}{1+e^{x^2}}$, the reciprocity formulas are

$$\hat{g}(t) = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{1}{1+e^{x^2}}e^{itx}\,dx, \quad g(x) = \int_{\mathbb{R}}\hat{g}(t)e^{-itx}\,dt$$

with

$$\hat{g}(t) = -\frac{1}{2\pi} \sum_{n>0} (-1)^n \int_{\mathbb{R}} e^{itx} e^{-nx^2} dx$$
$$= -\frac{1}{2\sqrt{\pi}} \sum_{n>0} \frac{(-1)^n}{\sqrt{n}} e^{-\frac{t^2}{4n}} = -\frac{1}{2\sqrt{\pi}} \chi(\frac{1}{2}; \frac{t^2}{4}).$$

Remark 4.3 The convolution of three functions $f, g, h \in \mathcal{S}(\mathbb{R})$ is, as well known,

$$(f\star(g\star h))(x) = \int_{\mathbb{R}} f(y)(g\star h)(x-y)dy$$
$$= \int_{\mathbb{R}} f(y) \left(\int_{\mathbb{R}} g(z)h(x-y-z) \right) dy$$
$$= \int\int_{\mathbb{R}\times\mathbb{R}} f(y)g(z)h(x-y-z)dzdy.$$

With $f(x) = g(x) = h(x) = \dfrac{1}{1+e^{x^2}}$ we have

$$(f\star(g\star h))(x) = \int\int_{\mathbb{R}^2} \frac{dt\,du}{(1+e^{y^2})(1+e^{z^2})(1+e^{(x-y-z)^2})}$$

so that for the Fourier transform

$$\widehat{f\star g\star h}(t) = (2\pi)^2 \hat{g}^3(t) = (2\pi)^2 \frac{1}{8\pi\sqrt{\pi}} \chi^3(\frac{1}{2}; \frac{t^2}{4}) = \frac{\sqrt{\pi}}{2} \chi^3(\frac{1}{2}; \frac{t^2}{4})$$

or

$$f\star g\star h(x) = \frac{\sqrt{\pi}}{2} \int_{\mathbb{R}} \chi^3(\frac{1}{2}; \frac{t^2}{4}) e^{-itx}\, dt.$$

Evaluating at $x = 0$ we obtain

$$\int\int_{\mathbb{R}^2} \frac{dy\,dz}{(1+e^{y^2})(1+e^{z^2})(1+e^{(-y-z)^2})}$$
$$= \int\int_{\mathbb{R}^2} \frac{dy\,dz}{(1+e^{y^2})(1+e^{z^2})(1+e^{(y-z)^2})}$$
$$= \frac{\sqrt{\pi}}{2} \int_{\mathbb{R}} \chi^3(\frac{1}{2}; \frac{t^2}{4})\, dt = \sqrt{\pi} \int_0^{\infty} \frac{1}{\sqrt{u}} \chi^3(\frac{1}{2}; u)\, du.$$

From (4.10), with $s = \frac{1}{2}$, we have (Compare with [10])

$$\sum_{p,q,r=1}^{\infty} \frac{(-1)^{p+q+r}}{(pq+qr+rp)^{\frac{1}{2}}} = -\frac{1}{\pi} \int\int_{\mathbb{R}^2} \frac{dy\,dz}{(1+e^{y^2})(1+e^{z^2})(1+e^{(y-z)^2})}. \quad (4.23)$$

We end this study by using an interesting integral representation due to Mellin [20] (p. 22, 23):

$$\frac{\Gamma(s)}{(w_0+w_1+\cdots+w_q)^s}=\frac{1}{(2i\pi)^q}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty}\cdots\int_{\kappa_q-i\infty}^{\kappa_q+i\infty}\frac{\Gamma(s-z_1\cdots z_q)}{w_0^{s-z_1\cdots z_q}}\frac{\Gamma(z_1)\cdots\Gamma(z_q)}{w_1^{z_1}\cdots w_q^{z_q}}dz_1\cdots dz_q,\tag{4.24}$$

$$\kappa_\nu>0,\ \nu=1,\cdots,q;\quad \mathrm{Re}s>\kappa_1+\cdots+\kappa_q>0.$$

In the case of $q=2$ we obtain at once, as in (4.23)

$$\sum_{p,q,r\geq 1}\frac{(-1)^{p+q+r}}{(pq+qr+rp)^s}=-\frac{1}{4\pi^2\,\Gamma(s)}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty}\int_{\kappa_2-i\infty}^{\kappa_2+i\infty}\Gamma(s-u_1u_2)\Gamma(u_1)\Gamma(u_2)$$

$$\sum_{p\geq 1}\frac{(-1)^p}{p^{s+u_1-u_1u_2}}\sum_{q\geq 1}\frac{(-1)^q}{p^{s+u_2-u_1u_2}}\sum_{r\geq 1}\frac{(-1)^r}{r^{u_1+u_2}}du_1du_2$$

$$=-\frac{1}{4\pi^2}\int_{\kappa_1-i\infty}^{\kappa_1+i\infty}\int_{\kappa_2-i\infty}^{\kappa_2+i\infty}\Gamma(s-u_1u_2)\Gamma(u_1)\Gamma(u_2)K(s;u_1,u_2)\,du_1du_2,$$

where

$$K(s;u_1,u_2)=\eta(s+u_1-u_1u_2)\eta(s+u_2-u_1u_2)\eta(u_1+u_2).$$

This representation of the Epstein zeta function of the ternary form $q_2(u,v,w)=uv+vw+wu$ in terms of the Dirichlet η-function and similar other representations can shed some light on their analytic continuation.

References

1. Alcantara-Bode, J.: An integral equation formulation of the Riemann Hypothesis. Integr. Equ. Oper. Theory **17**(2), 151–168 (1993)
2. Alcantara-Bode, J.: Some properties of the Beurling function. Pro-Mathematica, XIV, No. **27–28**, 1–11 (2000)
3. Andrew, G.E.: The fifth and seventh order mock theta functions. Trans. Am. Math. Soc. **293**(1), 113–134 (1986)
4. Báez-Duarte, L.: A strengthening of the Nyman–Beurling criterion for the Riemann hypothesis. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **14**(1), 5–11 (2003)
5. Balazard, M., Saias, E.: Notes sur la fonction ζ de Riemann, 1. Adv. Math. **139**, 310–321 (1998)
6. Balazard, M., Saias, E.: Notes sur la fonction ζ de Riemann, 4. Adv. Math. **188**, 69–86 (2004)
7. Beurling, A.: A closure problem related to the Riemann zeta-function. Proc. Natl. Acad. Sci. **41**, 312–314 (1955)
8. Bohr, H.: Almost Periodic Functions. Julius Springer, Berlin (1933) (Chelsea Publishing Company, N.Y., 1947)
9. Cohen, H.: Number theory, volume II: Analytic and modern tools, Graduate Texts in Mathematics, vol. 240, Springer, (2007)
10. Crandall, R.E.: New representations for the Madelung constant. Exp. Math. **8**, 367–379 (1999)
11. Davenport, H.: On some infinite series involving arithmetical functions. Q. J. Math **2**(8), 8–13 (1937)

12. Davenport, H.: On some infinite series involving arithmetical functions (II). Q. J. Math.(2) **8**, 313–320 (1937)
13. Delange, H.: Sur la fonction $f(x) = \sum 1/n \sin(x/n)$. Théorie analytique et élémentaire des nombres, Caen, 29-30, Septembre 1980, Journées mathématiques SMF-CNRS, (1980)
14. Donoghue, W.F.: Distributions and Fourier Transforms. Academic, N.Y. (1969)
15. Flett, T.M.: On the function $\sum(1/n)\sin(t/n)$. J. Lond. Math. Soc. **25**, 5–9 (1950)
16. Hardy, G.H., Littlewood, J.E.: Notes on the theory of series (XX): on Lambert series. Proc. Lond. Math. Soc. **2**(41), 257–270 (1936)
17. Hartman, P.: On a class of arithmetical Fourier series. Am. J. Math. **60**, 66–74 (1938)
18. Katsurada, M.: On Mellin-Barnes Type of Integrals and Sums associated With the Riemann Zeta function. Publ. Inst. Math.(Beograd) (Nouvelle Ser.) **62** (76), 13-25 (1997)
19. Katsurada, M.: Power series with the Riemann zeta-function in the coefficients. Proc. Japan Acad. **72**, 61–63 (1996)
20. Mellin, H.: Eine Formel fur den Logarithmus transcendenter Funktionen von endlichen Geschlecht. Acta Sot. Sci. Fennicae **29**, 3–49 (1900)
21. Mordell, L.J.: Integral formulae of arithmetical character. J. Lond. Math. Soc. **33**, 371–375 (1958)
22. Nikolski, N.: Distance formulae and invariant subspaces, with an application to localization of zeros of the Riemann ζ-function. Ann. Inst. Fourier (Grenoble) **45**(1), 143–159 (1995)
23. Nikolski, N.: Hardy Spaces (Cambridge Studies in Advanced Mathematics). Cambridge University Press, Cambridge (2019)
24. Safari, B., Vaughan, R.C.: On the fractional parts of $\frac{x}{n}$ and related sequences II. Ann. Inst. Fourier (Grenoble) **27**(2), 1–30 (1977)
25. Segal, S.L.: On $f(x) = \sum 1/n \sin(x/n)$, J. London Math. Soc., (2)4, 385-393(1972)
26. Tenenbaum, G.: Introduction to analytic and probabilistic number theory, volume 163, Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, third edition, 2015. Translated from the 2008 French edition by Patrick D. F. Ion
27. Tennenbaum, J.: On the function $\sum_{n=0}^{\infty} \frac{\zeta(n+2)}{n!} x^n$. Math. Scand. **41**, 242–248 (1977)
28. Titchmarsh, E.C.: The Theory of the Riemann Zeta-Function. Clarendon Press, Oxford (1951)
29. Vasyunin, V.I.: On a biorthogonal system related with the Riemann hypothesis. St. Petersburg Math. J. **7**, 405–419 (1996)
30. Voronoi, G.: Sur une fonction transcendante et ses applications à la sommation de quelques séries. Annales Scientifiques de l'École Normale Supérieure, troisième série **21**, 207–267 (1904)
31. Voronoi, G.: Sur une fonction transcendante et ses applications à la sommation de quelques séries, seconde partie. Annales Scientifiques de l'École Normale Supérieure, troisième série **21**, 459–533 (1904)
32. Walfisz, A.: Teilerprobleme. Math. Z. **26**, 66–88 (1927)
33. Watson, G.N.: A Treatise on the Theory of Bessel Functions. Cambridge University Press, Cambridge (1922)