Complex Analysis and Operator Theory (2020) 14:65
https://doi.org/10.1007/s11785-020-01023-0

Complex Analysis and Operator Theory

On the Inequalities Concerning Polynomials

Prasanna Kumar[1]

Received: 22 February 2020 / Accepted: 12 August 2020 / Published online: 19 August 2020

Abstract
If $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a complex polynomial of degree n having all its zeros in $|z| \le K$, $K \ge 1$ then Aziz (Proc Am Math Soc 89:259–266, 1983) proved that

$$\max_{|z|=1} |P'(z)| \ge \frac{2}{1+K^n} \sum_{j=1}^{n} \frac{K}{K+|z_j|} \max_{|z|=1} |P(z)|. \tag{0.1}$$

In this paper we sharpen the inequality (0.1) and further extend the obtained result to the polar derivative of a polynomial. As a consequence we also derive two results on the generalization of Erdös–Lax type inequality for the class of polynomials having no zeros in the disc $|z| < K$, $K \le 1$.

Keywords Inequalities · Polynomials · Zeros

Mathematics Subject Classification 30A10

1 Introduction and Statement of Results

If $P(z)$ is a polynomial of degree n then from a well-known inequality due to Bernstein [3], we have

$$\max_{|z|=1} |P'(z)| \le n \max_{|z|=1} |P(z)|. \tag{1.1}$$

Communicated by Dan Volok.

✉ Prasanna Kumar
prasannak@goa.bits-pilani.ac.in

1 Department of Mathematics, Birla Institute of Technology and Science Pilani K K Birla Goa Campus, Sancoale, Goa 403726, India

The inequality (1.1) is sharp and equality holds, if $P(z)$ has all its zeros at the origin. If $P(z)$ is a polynomial of degree n having no zeros in $|z| < 1$, then Erdös conjectured and later Lax [12] proved that

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

The inequality (1.2) is best possible and equality holds for $P(z) = a + bz^n$, where $|a| = |b|$. If $P(z)$ is a polynomial of degree n having all its zeros in $|z| \leq 1$, then Turán [17] proved that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.3}$$

Aziz [1] considered the modulus of each zero of the underlying polynomial in the bound and generalized the inequality (1.3) to the class of polynomials having all their zeros in a closed disc of finite radius greater than or equal to unit length by proving that, if $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ is a complex polynomial of degree n with $|z_j| \leq K$, $K \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{2}{1 + K^n} \sum_{j=1}^{n} \frac{K}{K + |z_j|} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Recently Govil and Kumar [8] generalized the inequality (1.3) to the class of polynomials having all their zeros in the disc $|z| \leq K$, $K \geq 1$, by including the information from leading and constant coefficients of the polynomial, but it did not capture the modulus of each individual zero, which plays a crucial role in sharpening the bound. We prove a generalization of (1.3) to the class of polynomials having all their zeros in the disc $|z| \leq K$, $K \geq 1$, by obtaining the bound which involves the modulus of each zero of the underlying polynomial, and at the same time our result sharpens (1.4) and also several of the earlier results considerably.

Theorem 1.1 *If* $P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n = a_n \prod_{j=1}^{n}(z - z_j)$ *is a polynomial of degree* n *which has all its zeros in the disk* $|z| \leq K$, $K \geq 1$, *then*

$$\max_{|z|=1} |P'(z)| \geq \left(\frac{2}{1+K^n} + \frac{(|a_n|K^n - |a_0|)(K-1)}{(1+K^n)(|a_n|K^n + K|a_0|)} \right) \sum_{j=1}^{n} \frac{K}{K+|z_j|} \max_{|z|=1} |P(z)|. \tag{1.5}$$

The result is best possible and equality in (1.5) holds for the polynomial $P(z) = z^n + K^n$.

It is clearly seen that Theorem 1.1 includes Aziz's inequality (1.4), and hence Theorem 1.1 also includes the Turán's inequality (1.3) as the case $K = 1$.

We have not seen any generalization of (1.2) to the class of polynomials having no zeros in $|z| < K$, $K \leq 1$ except for few special cases. Using the above Theorem

1.1, we can establish a result that deals with a class of polynomials having no zeros in $|z| < K, K \leq 1$ satisfying the property that the modulus of the derivative of the polynomial and the modulus of the derivative of the conjugate reciprocal of the polynomial attain maximum on the unit circle at a same point. The obtained result is stated below.

Theorem 1.2 *Let $P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n = a_n \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in $|z| < K, K \leq 1$, and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)|$$

$$\leq \left[n - \left(\frac{2K^n}{1+K^n} + \frac{K^n(|a_0|-|a_n|K^n)(1-K)}{(1+K^n)(|a_0|K+|a_n|K^n)} \right) \sum_{j=1}^{n} \frac{|z_j|}{|z_j| + K} \right] \max_{|z|=1} |P(z)|. \tag{1.6}$$

The result is best possible and equality in (1.6) holds for the polynomial $P(z) = z^n + K^n$.

The above Theorem 1.2 gives the following immediate corollary, which was independently proved by Govil [5], and thus one can observe considerable improvement of the result of Govil [5] in Theorem 1.2.

Corollary 1.3 *If $P(z)$ is a polynomial of degree n having no zeros in $|z| < K, K \leq 1$, and $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{1 + K^n} \max_{|z|=1} |P(z)|, \tag{1.7}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

If $P(z)$ is a polynomial of degree n then the *polar derivative of $P(z)$ with respect to a complex number α* is defined as

$$D_\alpha\{P(z)\} = nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha\{P(z)\}$ is a polynomial of degree atmost $n-1$, and it is a 'generalization' of the ordinary derivative in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha\{P(z)\}}{\alpha} = P'(z)$$

uniformly with respect to z for $|z| \leq R, \ R > 0$. For more information on polar derivatives of polynomials one can refer monographs by Rahman and Schmeisser [16] or Milovanović et al. [13] and also a recently published book chapter authored by Govil and Kumar [6].

Bernstein-type inequalities on complex polynomials have been extended extensively from 'ordinary derivative' to 'polar derivative' of complex polynomials. For the latest publications in this direction one can refer some of the papers of this author [7,10,11]. In this context it is quite natural to seek an extension of Theorem 1.1 involving ordinary derivative of a restricted polynomial to the one in more generalized form involving polar derivative of a polynomial with the same restrictions which is stated below.

Theorem 1.4 *Let $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n = a_n \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having all its zeros in $|z| \leq K$, $K \geq 1$. Then for any complex number α with $|\alpha| \geq K$,*

$$\max_{|z|=1} |D_\alpha\{P(z)\}| \geq \left(2\frac{(|\alpha| - K)}{1 + K^n} + (|\alpha| - K)\frac{(|a_n|K^n - |a_0|)(K - 1)}{(1 + K^n)(|a_n|K^n + K|a_0|)}\right) \sum_{j=1}^{n} \frac{K}{K + |z_j|} \max_{|z|=1} |P(z)|. \tag{1.8}$$

Remark 1.5 If we divide (1.8) by $|\alpha|$ and take $|\alpha| \to \infty$, we get the inequality (1.5), and thus Theorem 1.4 contains Theorem 1.1.

In the same way let us extend Theorem 1.2 also to the polar derivative of a polynomial as follows.

Theorem 1.6 *Let $P(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n = a_n \prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in $|z| < K$, $K \leq 1$, and $Q(z) = z^n \overline{P(1/\overline{z})}$. If $|P'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z| = 1$, then for any complex number α with $|\alpha| \geq 1$,*

$$\max_{|z|=1} |D_\alpha\{P(z)\}| \leq \left[n|\alpha| - (|\alpha| - 1)\left(\frac{2K^n}{1 + K^n} + \frac{K^n(|a_0| - |a_n|K^n)(1 - K)}{(1 + K^n)(|a_0|K + |a_n|K^n)}\right) \sum_{j=1}^{n} \frac{|z_j|}{|z_j| + K} \right] \max_{|z|=1} |P(z)|. \tag{1.9}$$

Remark 1.7 If we divide (1.9) by $|\alpha|$ and take $|\alpha| \to \infty$, we get (1.6), and thus Theorem 1.6 includes Theorem 1.2 as a special case.

2 Lemmas

Our first Lemma is the generalization of well-known Schwarz Lemma and due to Osserman [14].

Lemma 2.1 *Let $f(z)$ be analytic in $|z| < 1$ such that $|f(z)| < 1$ for $|z| < 1$ and $f(0) = 0$. Then*

$$|f(z)| \leq |z|\frac{|z| + |f'(0)|}{1 + |f'(0)||z|}$$

for $|z| < 1$.

The next lemma is proved by Aziz and Mohammad [2].

Lemma 2.2 *If* $P(z)$ *is a polynomial of degree n then for any* $R \geq 1$ *and* $0 \leq \theta \leq 2\pi$,

$$|P(Re^{i\theta})| + |Q(Re^{i\theta})| \leq (1 + R^n) \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 2.3 *If* $P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$ *is a polynomial of degree* $n \geq 1$ *having no zeros in* $|z| < 1$, *then for any* $R \geq 1$

$$\max_{|z|=R} |P(z)| \leq \left(\frac{(1 + R^n)(|a_0| + R|a_n|)}{(1 + R)(|a_0| + |a_n|)}\right) \max_{|z|=1} |P(z)|. \tag{2.1}$$

Proof This result is proved in a paper due to Dubinin [4, Corollary 1]. But for the sake of completeness we present the proof. Note that since $P(z)$ has no zeros in the disk $|z| < 1$, its conjugate reciprocal polynomial $Q(z) := z^n \overline{P(1/\overline{z})}$ has all its zeros in $|z| \leq 1$. Then $\frac{zQ(z)}{P(z)}$ satisfies the hypotheses of Lemma 2.1, and hence we have for $|z| < 1$,

$$|zQ(z)| \leq |z|\frac{|z||a_0| + |a_n|}{|a_0| + |a_n||z|}|P(z)|,$$

which is nothing but

$$|Q(z)| \leq \frac{|z||a_0| + |a_n|}{|a_0| + |a_n||z|}|P(z)|. \tag{2.2}$$

Replacing z by $1/z$ in the above inequality (2.2) we have for $|z| > 1$,

$$|P(z)| \leq \frac{|a_0| + |z||a_n|}{|a_0||z| + |a_n|}|Q(z)|. \tag{2.3}$$

Note that the inequality (2.3) is true for all z on $|z| = 1$ also, and therefore for any $R \geq 1$, and $0 \leq \theta \leq 2\pi$,

$$|P(Re^{i\theta})| \leq \frac{|a_0| + R|a_n|}{|a_0|R + |a_n|}|Q(Re^{i\theta})|. \tag{2.4}$$

Using (2.4) in Lemma 2.2, we obtain the required inequality (2.1), and hence the proof of Lemma 2.3 is complete. □

Lemma 2.4 *If $P(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq K$, $K \geq 1$, then*

$$\max_{|z|=K} |P(z)| \geq \left(\frac{2K^n}{1+K^n} + \frac{K^n(|a_n|K^n - |a_0|)(K-1)}{(1+K^n)(|a_n|K^n + K|a_0|)} \right) \max_{|z|=1} |P(z)|. \tag{2.5}$$

Proof Since $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, the polynomial $G(z) = P(Kz)$ has all its zeros in the unit disc $|z| \leq 1$. Let $H(z) = z^n G(1/z)$. Then $H(z)$ is a polynomial of degree atmost n having no zeros in $|z| < 1$. Therefore using Lemma 2.3 we have for $K \geq 1$,

$$\max_{|z|=K} |H(z)| \leq \frac{(1+K^n)(|a_n|K^n + K|a_0|)}{(1+K)(|a_n|K^n + |a_0|)} \max_{|z|=1} |H(z)|.$$

Since $|H(z)| = |G(z)|$ on $|z| = 1$,

$$\max_{|z|=1} |G(z)| \geq \frac{(1+K)(|a_n|K^n + |a_0|)}{(1+K^n)(|a_n|K^n + K|a_0|)} \max_{|z|=K} |H(z)|. \tag{2.6}$$

But $H(z) = z^n G(1/z) = z^n P(K/z)$ and so,

$$\max_{|z|=K} |H(z)| = K^n \max_{|z|=1} |P(z)|. \tag{2.7}$$

Using (2.7) in (2.6) we get

$$\max_{|z|=1} |G(z)| \geq K^n \frac{(1+K)(|a_n|K^n + |a_0|)}{(1+K^n)(|a_n|K^n + K|a_0|)} \max_{|z|=1} |P(z)|. \tag{2.8}$$

Using the facts

$$\max_{|z|=1} |G(z)| = \max_{|z|=K} |P(z)|$$

and

$$K^n \frac{(1+K)(|a_n|K^n + |a_0|)}{(1+K^n)(|a_n|K^n + K|a_0|)} = \frac{2K^n}{1+K^n} + \frac{K^n(|a_n|K^n - |a_0|)(K-1)}{(1+K^n)(|a_n|K^n + K|a_0|)},$$

in the inequality (2.8) we get the desired inequality (2.5). □

Lemma 2.4 sharpens the following result which was independently proved first by Aziz [1], and recently by Govil and Kumar [8] with a shorter and direct proof.

Corollary 2.5 *If $P(z)$ is a polynomial of degree n which has all its zeros in the disk $|z| \leq K$, $K \geq 1$, then*

$$\max_{|z|=K} |P(z)| \geq \frac{2K^n}{1+K^n} \max_{|z|=1} |P(z)|. \tag{2.9}$$

One can observe that, for the polynomials satisfying the hypothesis of Lemma 2.4 and having some zeros within the circle $|z| = K$, inequality (2.5) shows considerable amount of improvement over the inequality (2.9).

Lemma 2.6 *If $P(z)$ is a polynomial of degree n then on $|z| = 1$,*

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1} |P(z)|,$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$.

Lemma 2.6 is due to Govil and Rahman [9].

3 Proofs of Theorems

Proof of Theorem 1.1. Since $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, the polynomial $G(z) = P(Kz) = a_n K^n \prod_{j=1}^{n}(z - \frac{z_j}{K})$ has all its zeros in the closed unit disc $|z| \leq 1$. Since for all z on $|z| = 1$ for which $G(z) \neq 0$,

$$\frac{zG'(z)}{G(z)} = \sum_{j=1}^{n} \frac{z}{z - (z_j/K)},$$

we have

$$Re\left(\frac{zG'(z)}{G(z)}\right) \geq \sum_{j=1}^{n} \frac{1}{1 + |z_j/K|} = \sum_{j=1}^{n} \frac{K}{K + |z_j|}.$$

But then

$$\left|\frac{zG'(z)}{G(z)}\right| \geq \sum_{j=1}^{n} \frac{K}{K + |z_j|},$$

for all z on $|z| = 1$ for which $G(z) \neq 0$. Therefore

$$\max_{|z|=1} |G'(z)| \geq \sum_{j=1}^{n} \frac{K}{K + |z_j|} \max_{|z|=1} |G(z)|, \tag{3.1}$$

or equivalently

$$K \max_{|z|=1} |P'(Kz)| \geq \sum_{j=1}^{n} \frac{K}{K + |z_j|} \max_{|z|=1} |P(Kz)|. \tag{3.2}$$

Using the fact $K^{n-1} \max_{|z|=1} |P'(z)| \geq |P'(Kz)|$ (see [15, problem 269, p. 137] and Lemma 2.4 in (3.2), we get

$$K^n \max_{|z|=1}|P'(z)| \geq \sum_{j=1}^{n} \frac{K}{K+|z_j|}\left(\frac{2K^n}{1+K^n}+\frac{K^n(|a_n|K^n-|a_0|)(K-1)}{(1+K^n)(|a_n|K^n+K|a_0|)}\right)\max_{|z|=1}|P(z)|,$$

and thereby the proof is complete.

Proof of Theorem 1.2 Since $P(z)$ has no zeros in $|z| < K$, $K \leq 1$, $Q(z) = z^n\overline{P(1/\overline{z})}$ has all its zeros in $|z| \leq \frac{1}{K}$, $\frac{1}{K} \geq 1$. Hence by Theorem 1.1 and using the fact that $\max_{|z|=1}|Q(z)| = \max_{|z|=1}|P(z)|$, we have

$$\max_{|z|=1}|Q'(z)| \geq$$

$$\left(\frac{2}{1+1/K^n}+\frac{1}{(1+(1/K^n))}\frac{(|a_0|/K^n)-|a_n|)(1/K-1)}{(|a_0|/K^n)+(|a_n|/K)}\right) \times \sum_{j=1}^{n}\frac{|z_j|}{|z_j|+K}\max_{|z|=1}|P(z)|. \tag{3.3}$$

From Lemma 2.6 we have on $|z| = 1$,

$$|P'(z)| + |Q'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{3.4}$$

Since $|P'(z)|$ and $|Q'(z)|$ attain the maximum at the same point, we have

$$\max_{|z|=1}\{|P'(z)| + |Q'(z)|\} = \max_{|z|=1}|P'(z)| + \max_{|z|=1}|Q'(z)|. \tag{3.5}$$

Therefore from (3.3), (3.4) and (3.5) we have,

$$\max_{|z|=1}|P'(z)| + \left(\frac{2K^n}{1+K^n}+\frac{K^n(|a_0|-|a_n|K^n)(1-K)}{(1+K^n)(|a_0|K+|a_n|K^n)}\right)\sum_{j=1}^{n}\frac{|z_j|}{|z_j|+K}\max_{|z|=1}|P(z)| \leq n\max_{|z|=1}|P(z)|. \tag{3.6}$$

Simple rearrangements in (3.6) yields the required inequality. □

Proof of Theorem 1.4 Since $P(z)$ has all its zeros in $|z| \leq K$, $K \geq 1$, all the zeros of $G(z) = P(Kz)$ lie in $|z| \leq 1$. Now therefore for $|\alpha|/K \geq 1$, it is a straight forward exercise to obtain

$$\max_{|z|=1}|D_{\alpha/K}G(z)| \geq \frac{(|\alpha|-K)}{K}\max_{|z|=1}|G'(z)|,$$

which is nothing but

$$\max_{|z|=1}|nP(Kz) + (\alpha/K - z)KP'(Kz)| \geq \frac{(|\alpha|-K)}{K}\max_{|z|=1}|G'(z)|.$$

Using the definition of polar derivative and the inequality (3.1) we have

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \sum_{j=1}^{n} \frac{K}{K + |z_j|} \max_{|z|=1} |G(z)|,$$

which is equivalent to

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \sum_{j=1}^{n} \frac{K}{K + |z_j|} \max_{|z|=K} |P(z)|.$$

Now applying Lemma 2.4 in the right hand side of the above inequality, we get

$$\max_{|z|=K} |D_\alpha P(z)| \geq \frac{(|\alpha| - K)}{K} \sum_{j=1}^{n} \frac{K}{K + |z_j|}$$

$$\left(\frac{2K^n}{1 + K^n} + \frac{K^n(|a_n|K^n - |a_0|)(K - 1)}{(1 + K^n)(|a_n|K^n + K|a_0|)} \right) \max_{|z|=1} |P(z)|.$$

From the fact that $\max_{|z|=K} |D_\alpha P(z)| \leq K^{n-1} \max_{|z|=1} |D_\alpha P(z)|$, we have

$$K^{n-1} \max_{|z|=1} |D_\alpha P(z)|$$

$$\geq \frac{(|\alpha| - K)}{K} \sum_{j=1}^{n} \frac{K}{K + |z_j|} \left(\frac{2K^n}{1 + K^n} + \frac{K^n(|a_n|K^n - |a_0|)(K - 1)}{(1 + K^n)(|a_n|K^n + K|a_0|)} \right) \max_{|z|=1} |P(z)|. \tag{3.7}$$

By a simplification of terms in the above inequality (3.7), we get the desired inequality, and hence the proof is complete. □

Proof of Theorem 1.6 Note that for any complex number α with $|\alpha| \geq 1$, we have on $|z| = 1$

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &= |nP(z) - zP'(z) + \alpha P'(z)| \\ &\leq |nP(z) - zP'(z)| + |\alpha||P'(z)| \\ &= |Q'(z)| + |\alpha||P'(z)| \\ &\leq n \max_{|z|=1} |P(z)| - |P'(z)| + |\alpha||P'(z)|, \text{ by Lemma 2.6} \\ &= n \max_{|z|=1} |P(z)| + (|\alpha| - 1)|P'(z)|. \end{aligned}$$

Therefore using Theorem 1.2 we have

$$\max_{|z|=1}|D_\alpha P(z)| \le n \max_{|z|=1}|P(z)|$$
$$+(|\alpha|-1)\left[n-\left(\frac{2K^n}{1+K^n}+\frac{K^n(|a_0|-|a_n|K^n)(1-K)}{(1+K^n)(|a_0|K+|a_n|K^n)}\right)\sum_{j=1}^{n}\frac{|z_j|}{|z_j|+K}\right]\max_{|z|=1}|P(z)|.$$

With a minor simplification in the right hand side of the above inequality we get the required inequality and thus the proof is complete. □

Remark 3.1 If $P(z)$ is a polynomial of degree n with all coefficients non-negative or all coefficients non-positive, then $|P'(z)|$ and $|Q'(z)|$ attain the maximum at $z = 1$ and therefore Theorems 1.2 and 1.6 hold true for such class of polynomials having no zeros in $|z| \le K,\ K \ge 1$. Let us state this spacial case as a corollary.

Corollary 3.2 *Let $P(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1} + a_nz^n = a_n\prod_{j=1}^{n}(z - z_j)$ be a polynomial of degree n having no zeros in $|z| < K$, $K \le 1$ with all the coefficients either non-positive or non-negative, and α be any complex number with $|\alpha| \ge 1$. Then*

$$\max_{|z|=1}|P'(z)|$$
$$\le\left[n-\left(\frac{2K^n}{1+K^n}+\frac{K^n(|a_0|-|a_n|K^n)(1-K)}{(1+K^n)(|a_0|K+|a_n|K^n)}\right)\sum_{j=1}^{n}\frac{|z_j|}{|z_j|+K}\right]\max_{|z|=1}|P(z)|.$$

and

$$\max_{|z|=1}|D_\alpha\{P(z)\}|$$
$$\le\left[n|\alpha|-(|\alpha|-1)\left(\frac{2K^n}{1+K^n}+\frac{K^n(|a_0|-|a_n|K^n)(1-K)}{(1+K^n)(|a_0|K+|a_n|K^n)}\right)\sum_{j=1}^{n}\frac{|z_j|}{|z_j|+K}\right]$$
$$\max_{|z|=1}|P(z)|.$$

We believe that Theorems 1.2 and 1.6 are possibly the best available partial answers so far in an attempt towards the problem of deriving the Erdös–Lax inequality for the class of polynomials having no zeros in the disc $|z| < K,\ K \le 1$. The general problem of this kind without any additional hypothesis is still open.

Acknowledgements The author is indebted to the anonymous referee and the editor for their valuable suggestions. The work in the paper was supported by research grants from the National Board for Higher Mathematics India, and Council for Scientific and Industrial Research India.

References

1. Aziz, A.: Inequalities for the derivative of a polynomial. Proc. Am. Math. Soc. **89**, 259–266 (1983)
2. Aziz, A., Mohammad, Q.G.: Growth of polynomials with zeros outside a circle. Proc. Am. Math. Soc. **81**, 549–553 (1981)

3. Bernstein, S.N.: Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle. Gauthier-Villars, Paris (1926)
4. Dubinin, V.N.: Applications of Schwarz lemma to inequalities for entire functions with constraints on zeros. J. Math. Sci. **143**, 3069–3076 (2007)
5. Govil, N.K.: On a theorem of Bernstein. Proc. Natl. Acad. Soc. India **50**, 50–52 (1980)
6. Govil, N.K., Kumar, P.: On Bernstein-type inequalities for the polar derivative of a polynomial. Prog. Approx. Theory Appl. Complex Anal. Springer Optim. Appl. **117**, 41–74 (2017)
7. Govil, N.K., Kumar, P.: On L^p inequalities involving polar derivative of a polynomial. Acta Math. Hungar. **152**(1), 130–139 (2017)
8. Govil, N.K., Kumar, P.: On sharpening of an inequality of Turán. Appl. Anal. Discrete Math. **13**(3), 711–720 (2019)
9. Govil, N.K., Rahman, Q.I.: Functions of exponential type not vanishing in a half-plane and related polynomials. Trans. Am. Math. Soc. **137**, 501–517 (1969)
10. Kumar, P.: Some integral inequalities for the polar derivative of polynomials. Publ. Inst. Math. **106**(120), 85–94 (2019)
11. Kumar, P.: On Zygmund-type inequalities involving polar derivative of a lacunary-type polynomial. Bull. Math. Soc. Sci. Math. Roumanie **61**(110), 155–164 (2019)
12. Lax, P.D.: Proof of a conjecture due to Erdös on the derivative of a polynomial. Bull. Am. Math. Soc. **50**, 509–513 (1944)
13. Milovanović, G.V., Mitrinović, D.S., Rassias, ThM: Topics in Polynomials: Extremal Properties, Inequalities, Zero. World scientific Publishing Co., Singapore (1994)
14. Osserman, R.: A sharp Schwarz inequality on the boundary. Proc. Am. Math. Soc. **128**, 3513–3517 (2000)
15. Pólya, G., Szegö, G.: Aufgaben und Lehrsatze aus der Analysis. Springer, Berlin **1925**(81), 549–553 (1981)
16. Rahman, Q.I., Schmeisser, G.: Analytic Theory of Polynomials. Oxford University Press, New York (2002)
17. Turán, P.: Über die Ableitung von Polynomen. Compositio Mathematica **7**, 89–95 (1939)