



Geometric Formalities Along the Chern-Ricci Flow

Daniele Angella¹ · Tommaso Sferruzza¹

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Abstract

In this paper, we study how the notions of geometric formality according to Kotschick and other geometric formalities adapted to the Hermitian setting evolve under the action of the Chern-Ricci flow on class VII surfaces, including Hopf and Inoue surfaces, and on Kodaira surfaces.

Keywords Formality · Kotschick geometric formality · Hermitian geometric formalities · Bott–Chern cohomology · Chern-Ricci flow

Mathematics Subject Classification 53C55 · 53C44 · 32J15 · 57T10

1 Introduction

This note is related to the problem of understanding the algebraic structure of cohomologies of complex manifolds.

On a differentiable manifold X , the differential complex of forms has a structure of algebra. The wedge product induces an algebra structure in de Rham cohomology, but *not in a uniform way*. We mean that, in general, it is not possible to choose

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✉ Daniele Angella
daniele.angella@gmail.com; daniele.angella@unifi.it
Tommaso Sferruzza
tommaso.sferruzza@stud.unifi.it

¹ Dipartimento di Matematica e Informatica “Ulisse Dini”, Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy

a system of representatives having itself a structure of algebra. This is evident if we want to choose harmonic representatives with respect to a Riemannian metric on a compact differentiable manifold: indeed, in the words of Sullivan, there is an “incompatibility of wedge products and harmonicity of forms” [55, page 326]. In general, on a compact differentiable manifold X , the choice of representatives for the de Rham cohomology yields just a structure of A_∞ -algebra in the sense of Stasheff [51], by the Homotopy Transfer Principle by Kadeishvili [28], see e.g. [40,63]. We refer to [15,39] for understanding the relationship between the higher multiplications and the Massey products. When such an A_∞ -algebra is actually an algebra, we say that X is *formal* in the sense of Sullivan [55]. In this case, the differential graded algebra of forms and the dga of de Rham cohomology share the same minimal model, which contains information on the rational homotopy: therefore the rational homotopy type of X “can be computed formally from” the cohomology ring $H_{dR}^\bullet(X; \mathbb{R})$; see [55] for details. As a special case, when we can fix a Riemannian metric g such that the *harmonic* representatives with respect to g have a structure of algebra, say that X is *geometrically formal* in the sense of Kotschick [31].

On a complex manifold X , the double complex of forms $(\wedge^{\bullet,\bullet} X, \partial, \bar{\partial})$ has a structure of bi-differential bi-graded algebra. Then it is possible to investigate analogous notions for the Dolbeault cohomology. Neisendorfer and Taylor introduced the notion of (*strictly*) *Dolbeault formality* and “complex homotopy groups” in [42]; for Hermitian manifolds, the notion of (*strictly*) *geometric-Dolbeault formality* has been investigated by Tomassini and Torelli in [59].

Besides Dolbeault cohomology, Bott–Chern $H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial \bar{\partial}}$ [13] and Aeppli $H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}}$ [1] cohomologies provide further cohomological invariants on complex manifolds. They are directly related by morphisms to both the Dolbeault and the de Rham cohomology, and allow to numerically characterize the strong Hodge decomposition given by the $\partial\bar{\partial}$ -Lemma [6,7]. The notion of *geometrically-Bott–Chern formality* for Hermitian manifolds is introduced and studied in [8]. Here, *Bott–Chern-harmonic* forms are in the kernel of the fourth-order Bott–Chern Laplacian introduced in [30,50]. On the other side, it is not clear how to study a notion of “Bott–Chern and Aeppli formality”. However, the Bott–Chern cohomology has a structure of algebra, while the Aeppli cohomology is just a $H_{BC}(X)$ -module. We refer to the discussion in [3,6].

In this note, we focus on geometric formalities of complex manifolds and its dependence on the Hermitian metric. In [56,59], the authors study the behaviour of Dolbeault formality, respectively geometric-Bott–Chern formality, under small deformations of the complex structure. Here, we keep the complex structure fixed, and we study geometric formalities with respect to Hermitian metrics evolving along a geometric flow. More precisely, we consider the *Chern-Ricci flow* [22,60] that evolves Hermitian structures $\omega(t)$ by the Chern-Ricci form,

$$\frac{\partial}{\partial t} \omega = -\text{Ric}^{Ch}(\omega),$$

and we study the possible algebra structure on the space of (de Rham, Dolbeault, Bott–Chern, Aeppli) harmonic forms with respect to $\omega(t)$ varying t .

We study in details geometric formality according to Kotschick for a whole class of surfaces evolving by the Chern-Ricci flow, *i.e.* compact complex non-Kähler surfaces with Kodaira dimension $\text{Kod}(X) = -\infty$ and first Betti number $b_1(X) = 1$, known as *class VII* of the Enriques-Kodaira classification. In particular, we first rule out class VII surfaces with second Betti number $b_2 > 0$ by applying arguments as in [31]. Then, we exploit the structure of quotients of Lie groups with invariant complex and Hermitian structure on the only class VII surfaces with $b_2 = 0$, that is Hopf and Inoue surfaces see [11,29,37,57], in order to reduce the description of harmonic forms and the equation of the Chern-Ricci flow of such surfaces at the level of invariant forms and thus make explicit computations. We obtain the following.

Theorem 4.1. *On class VII surfaces, the Chern-Ricci flow preserves the notion of geometric formality according to Kotschick starting at initial invariant metrics.*

We also study the evolution of geometric formality according to Kotschick on other compact complex non-Kähler surfaces that are diffeomorphic to solvmanifolds, *e.g.* Kodaira surfaces. Since any complex structures on such surfaces is left-invariant, see [24, Theorem 1], we focus on invariant forms also in this case.

Proposition 4.2 *On Kodaira surfaces, geometric formality according to Kotschick is preserved by the Chern-Ricci flow starting at initial invariant metrics.*

We note that, also in this case, it is possible to rule out primary Kodaira surfaces by the obstructions in [31] or [23], and therefore we focus on secondary Kodaira surfaces with initial invariant metrics.

Regarding Dolbeault and Bott–Chern geometric formalities evolving by the Chern-Ricci flow, by applying the analogous procedure on Hopf, Inoue, and Kodaira surfaces, we have reached results as follows. We also checked how the algebraic structures of Aeppli cohomology and its harmonic representatives are modified along the Chern-Ricci flow.

Proposition 5.1 *On Hopf, Inoue, and Kodaira surfaces, the notions of Dolbeault geometric formality and Bott–Chern geometric formality are all preserved by the Chern-Ricci flow starting at initial invariant metrics. Moreover the properties of Aeppli-harmonic forms having a structure of algebra or a structure of H_{BC} -module are all preserved by the Chern-Ricci flow starting at initial invariant metrics.*

Throughout this note, we give a complete description of harmonic forms on such compact complex surfaces depending on the invariant Hermitian metrics. We made computations with the aid of SageMath [49].

We ask whether for Dolbeault and Bott–Chern geometric formalities there exist obstructions (such as the ones found in [31]) which would help complete the picture for geometric formalities for class VII surfaces. We also ask whether the behaviour we observed is more general or there exist counterexamples.

2 Preliminaries on Hermitian Manifolds

2.1 Cohomologies of Compact Complex Manifolds

For a complex compact manifold X , one can define its real and complex de Rham cohomology groups by

$$H_{dR}^k(X; \mathbb{R}) := \frac{\ker(d: \wedge^k(X) \rightarrow \wedge^{k+1}(X))}{\text{im}(d: \wedge^{k-1}(X) \rightarrow \wedge^k(X))}$$

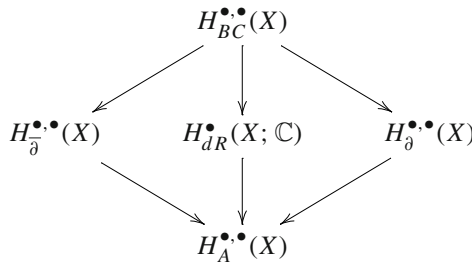
$$H_{dR}^k(X; \mathbb{C}) := \frac{\ker(d: \wedge_{\mathbb{C}}^k(X) \rightarrow \wedge_{\mathbb{C}}^{k+1}(X))}{\text{im}(d: \wedge_{\mathbb{C}}^{k-1}(X) \rightarrow \wedge_{\mathbb{C}}^k(X))}$$

which correspond, by the universal coefficients theorem.

In the case of non-Kähler complex compact manifolds, further invariants regarding the complex structure are given by other cohomologies, such as Dolbeault, Bott–Chern and Aeppli cohomologies, defined respectively by

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) := \frac{\ker \bar{\partial}}{\text{im} \bar{\partial}} \quad H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im}(\partial \bar{\partial})} \quad H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\text{im} \partial + \text{im} \bar{\partial}}.$$

In general, the homomorphisms induced by the identity between those cohomologies are not necessarily injective nor surjective. However, for compact manifolds satisfying the $\partial \bar{\partial}$ -lemma, such as compact Kähler manifolds, these natural maps that are induced by the identity are all isomorphisms, see [19]:



where $H_{\partial}^{\bullet,\bullet}(X)$ is the conjugate of the Dolbeault cohomology.

Once fixed a Hermitian metric g on X , the Hodge- $*$ -operator with respect to such metric induces isomorphisms, see e.g. [50, §2.b, §2.c]:

$$H_{dR}^k(X) \xrightarrow{\sim} H_{dR}^{2n-k}(X), \quad H_{\bar{\partial}}^{p,q}(X) \xrightarrow{\sim} H_{\partial}^{n-q,n-p}(X),$$

$$H_{BC}^{p,q}(X) \xrightarrow{\sim} H_A^{n-q,n-p}(X), \tag{2.1}$$

where $\dim_{\mathbb{R}} X = 2n$, $1 \leq k \leq 2n$, and $1 \leq p, q \leq n$.

2.2 Hodge Theory

For a compact Hermitian manifold (X^n, g) , we denote by $\mathcal{H}_{\square}^k(X, g)$, with $k \in \{1, \dots, 2n\}$, $\square \in \{dR, \bar{\partial}, BC, A\}$ the set of harmonic k -forms with respect to the regular Laplacian, Dolbeault Laplacian, Bott–Chern Laplacian, Aeppli Laplacian, namely the following self-adjoint, elliptic operators, see [50, §2.b, §2.c]:

$$\begin{aligned} \Delta_{dR} &= dd^* + d^*d, & \Delta_{\bar{\partial}} &= \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial} \\ \Delta_{BC} &= \partial\bar{\partial}\bar{\partial}^*\partial^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \bar{\partial}^*\partial\partial^*\bar{\partial} + \partial^*\bar{\partial}\bar{\partial}^*\partial + \bar{\partial}^*\bar{\partial} + \partial^*\partial \\ \Delta_A &= \partial\partial^* + \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\partial^*\partial\bar{\partial} + \partial\bar{\partial}\bar{\partial}^*\partial^* + \partial\bar{\partial}^*\partial\partial^* + \bar{\partial}\bar{\partial}^*\partial\bar{\partial}^*. \end{aligned}$$

Clearly, those operators depend on the fixed metric g . For $\square \in \{\bar{\partial}, BC, A\}$, we can define also $\mathcal{H}_{\square}^{p,q}(X, g)$, *i.e.* the set of harmonic (p, q) -forms with respect to Dolbeault, Bott–Chern and Aeppli Laplacians.

Both the de Rham $(\bigwedge_{\mathbb{C}}^{\bullet}(X), d)$ and Dolbeault complexes $(\bigwedge^{\bullet,\bullet}(X), \partial, \bar{\partial})$ of a compact Hermitian manifold (X, g) , via harmonic forms, admit decompositions that yield the following isomorphisms of vector spaces between harmonic forms and cohomology classes:

$$\begin{aligned} \mathcal{H}_{dR}^k(X, g) &\xrightarrow{\sim} H_{dR}^k(X), & \mathcal{H}_{\bar{\partial}}^{p,q}(X, g) &\xrightarrow{\sim} H_{\bar{\partial}}^{p,q}(X), \\ \mathcal{H}_{BC}^{p,q}(X, g) &\xrightarrow{\sim} H_{BC}^{p,q}(X), & \mathcal{H}_A^{p,q}(X, g) &\xrightarrow{\sim} H_A^{p,q}(X). \end{aligned} \tag{2.2}$$

We note that isomorphisms as in (2.1) still hold at the level of harmonic forms.

2.3 Geometric Formalities

In general, whereas de Rham, Dolbeault and Bott–Chern cohomologies have a structure of algebra induced by the cup product of cohomology classes, the mentioned isomorphisms (2.2) do not preserve the structure of algebra, expected by the wedge product. For Aeppli cohomology, especially, we do not have a structure of algebra *a priori*, but a structure of $H_{BC}(X)$ -module; the isomorphism in (2.2) could not preserve this structure. This remark makes non-trivial the following definitions.

Definition 2.1 Let X be a compact complex manifold, g a fixed Hermitian metric on X . The metric g is said to be

- (1) *geometrically formal according to Kotschick* if $\mathcal{H}_{dR}^{\bullet}(X, g)$ has a structure of algebra induced by the wedge product;
- (2) *Dolbeault-geometrically formal* if $\mathcal{H}_{\bar{\partial}}^{\bullet,\bullet}(X, g)$ has a structure of algebra induced by the wedge product;
- (3) *Bott–Chern-geometrically formal* if $\mathcal{H}_{BC}^{\bullet,\bullet}(X, g)$ has a structure of algebra induced by the wedge product.

2.4 Chern-Ricci Flow

The Chern-Ricci flow (introduced in [22] and studied in [60]) is a parabolic geometric flow that preserves the Hermitian condition of the initial given metric. The equations that describe such flow on a Hermitian manifold (X^n, J, g_0) are

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}^{Ch}(\omega(t)), \quad \omega(0) = \omega_0,$$

where $\omega_0, \omega(t)$ are the fundamental forms associated, respectively, to the Hermitian initial metric g_0 and the evolution metric $g(t)$ by the usual relation $\omega(\cdot, \cdot) = g(J(\cdot), \cdot)$. For an arbitrary real $(1, 1)$ -form ω , $\text{Ric}^{Ch}(\omega)$ is the Chern-Ricci form of ω . The first Chern-Ricci curvature Ric^{Ch} is defined starting from ∇^{Ch} , the Chern connection on (X, J, g) , i.e. the unique connection ∇ on the holomorphic tangent bundle $T^{1,0}X$ such that ∇ is compatible with both g and J and $\nabla^{0,1} = \bar{\partial}$. In a holomorphic chart, the curvature tensor R^{Ch} of such connection has components $R_{i\bar{j}k\bar{l}}^{Ch}$, for $i, j, k, l \in \{1, \dots, n\}$. The Chern-Ricci tensor is obtained by contracting the last two indices via the metric

$$\text{Ricci}_{i\bar{j}}^{Ch} := g^{k\bar{l}} R_{i\bar{j}k\bar{l}}^{Ch},$$

where $(g^{k\bar{l}})$ is the inverse of the matrix $(g_{k\bar{l}})$ representing in local coordinates the metric g . The Chern-Ricci form is defined by

$$\text{Ric}^{Ch} := \text{Ricci}^{Ch}(J(\cdot), \cdot).$$

Such form has important properties, among which a very simple form in local coordinates:

$$\text{Ric}^{Ch}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det(g),$$

from which we can deduce that $\text{Ric}^{Ch}(\omega)$ is a ∂ -, $\bar{\partial}$ -closed form, hence it defines a cohomology class in $H_{BC}^{1,1}(X)$. Such class is a holomorphic invariant, denoted by $c_1^{BC}(X)$, which plays a fundamental role in the classification of complex manifolds.

3 Preliminaries on Compact Complex Surfaces and Quotients of Lie Groups

In this section, we analyze in details complex structures, cohomologies, and Chern-Ricci flow on non-Kähler compact complex surfaces that can be described as quotients of Lie groups G endowed with invariant complex structure [24], namely Hopf, Inoue, and Kodaira surfaces. Here, *invariant* tensors are meant to be defined on the covering Lie group and invariant by the action of the Lie group on itself by left-translations, that is, they can be considered as linear tensors over the associated Lie algebra \mathfrak{g} .

3.1 Complex Structure

More precisely, we describe the complex structure J by a coframe of invariant $(1, 0)$ -forms $\{\varphi^1, \varphi^2\}$ and their conjugates, and by their structure equations

$$d\varphi^I = -c^I_{HK} \varphi^H \wedge \varphi^K,$$

equivalently, by the dual frame $\{\varphi_1, \varphi_2\}$ of $(1, 0)$ -vector fields and their conjugates, with structure equations $[\varphi_H, \varphi_K] = c^I_{HK} \varphi_I$. (Capital letters here vary in the ordered set $(1, 2, \bar{1}, \bar{2})$ and refer to the corresponding component. The Einstein summation is assumed, for increasing indices in case of forms. Hereafter, we shorten e.g. $\varphi^{2\bar{1}} := \varphi^2 \wedge \bar{\varphi}^1$.)

3.2 Hermitian Structure

The arbitrary invariant Hermitian metric $g := \omega(\cdot, J(\cdot))$ has associated $(1, 1)$ -form

$$2\omega = \sqrt{-1} \sum_{I, J=1}^2 g_{I\bar{J}} \varphi^I \wedge \bar{\varphi}^{\bar{J}} = \sqrt{-1} r^2 \varphi^{1\bar{1}} + \sqrt{-1} s^2 \varphi^{2\bar{2}} + u \varphi^{1\bar{2}} - \bar{u} \varphi^{2\bar{1}} \tag{3.1}$$

where the coefficients satisfy

$$r^2 > 0, \quad s^2 > 0, \quad r^2 s^2 > |u|^2.$$

That is to say, the Hermitian matrix

$$(g_{KL})_{K,L} = \frac{1}{2} \cdot \begin{pmatrix} r^2 & -\sqrt{-1}u \\ \sqrt{-1}\bar{u} & s^2 \end{pmatrix} \in \text{GL}(\mathfrak{g})$$

is positive-definite. Its inverse is

$$(g^{KL})_{K,L} := (g_{KL})_{K,L}^{-1} = \frac{2}{r^2 s^2 - |u|^2} \cdot \begin{pmatrix} s^2 & \sqrt{-1}u \\ -\sqrt{-1}\bar{u} & r^2 \end{pmatrix}.$$

The Christoffel symbols of the Chern connection can be computed as follows, see e.g. [45]:

$$(\Gamma^{Ch})^K_{IH} = \frac{1}{2} c^K_{IH} - \frac{1}{2} g^{KA} g_{BI} c^B_{HA} - \frac{1}{2} g^{KA} g_{BH} c^B_{IA} + \frac{1}{2} g^{KL} C_{IHL},$$

where

$$C_{IHL} = d\omega(J\varphi_I, \varphi_H, \varphi_L).$$

We can then express the $(4, 0)$ -Riemannian curvature of the Chern connection as

$$(R^{Ch})_{IHKL} = g_{AL}(\Gamma^{Ch})_{HK}^B(\Gamma^{Ch})_{IB}^A - g_{AL}(\Gamma^{Ch})_{IK}^B(\Gamma^{Ch})_{HB}^A - g_{AL}(\Gamma^{Ch})_{IH}^B(\Gamma^{Ch})_{BK}^A,$$

and the Chern-Ricci tensor as

$$(\text{Ricci}^{Ch})_{IH} = g^{KL}(R^{Ch})_{IHKL}.$$

Then the Chern-Ricci form is

$$\text{Ric}^{Ch} = \text{Ricci}^{Ch}(J(\cdot), \cdot) \in c_1^{BC}(X) \in H_{BC}^{1,1}(X; \mathbb{R}).$$

Finally, we collect here some explicit description of the Hodge-star-operator on forms for the arbitrary Hermitian metric associated to the form (3.1), in order to describe harmonicity, see also [48, Lemma 2]. It is straightforward to check that:

$$\begin{aligned} *_g \varphi^1 &= \frac{\sqrt{-1}}{2} \bar{u} \varphi^{12\bar{1}} + \frac{1}{2} s^2 \varphi^{12\bar{2}}, \\ *_g \varphi^2 &= -\frac{1}{2} r^2 \varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2} u \varphi^{12\bar{2}}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} *_g \bar{\varphi}^1 &= -\frac{\sqrt{-1}}{2} u \varphi^{1\bar{1}\bar{2}} + \frac{1}{2} s^2 \varphi^{2\bar{1}\bar{2}}, \\ *_g \bar{\varphi}^2 &= -\frac{1}{2} r^2 \varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2} \bar{u} \varphi^{2\bar{1}\bar{2}}, \end{aligned} \tag{3.3}$$

$$*_g \varphi^{12} = \varphi^{12}, \quad *_g \varphi^{\bar{1}\bar{2}} = \varphi^{\bar{1}\bar{2}} \tag{3.4}$$

$$V *_g \varphi^{1\bar{1}} = |u|^2 \varphi^{1\bar{1}} - \sqrt{-1} u s^2 \varphi^{1\bar{2}} + \sqrt{-1} \bar{u} s^2 \varphi^{2\bar{1}} + s^4 \varphi^{2\bar{2}}, \tag{3.5}$$

$$V *_g \varphi^{1\bar{2}} = -\sqrt{-1} \bar{u} r^2 \varphi^{1\bar{1}} - r^2 s^2 \varphi^{1\bar{2}} + \bar{u}^2 \varphi^{2\bar{1}} - \sqrt{-1} \bar{u} s^2 \varphi^{2\bar{2}}$$

$$V *_g \varphi^{2\bar{1}} = \sqrt{-1} u r^2 \varphi^{1\bar{1}} + u^2 \varphi^{1\bar{2}} - r^2 s^2 \varphi^{2\bar{1}} + \sqrt{-1} u s^2 \varphi^{2\bar{2}}$$

$$V *_g \varphi^{2\bar{2}} = r^4 \varphi^{1\bar{1}} - \sqrt{-1} u r^2 \varphi^{1\bar{2}} + \sqrt{-1} \bar{u} r^2 \varphi^{2\bar{1}} + |u|^2 \varphi^{2\bar{2}},$$

$$V *_g \varphi^{12\bar{1}} = -2\sqrt{-1} u \varphi^1 + 2s^2 \varphi^2, \quad V *_g \varphi^{12\bar{2}} = -2r^2 \varphi^1 - 2\sqrt{-1} \bar{u} \varphi^2,$$

$$V *_g \varphi^{1\bar{1}\bar{2}} = 2\sqrt{-1} \bar{u} \bar{\varphi}^1 + 2s^2 \bar{\varphi}^2, \quad V *_g \varphi^{2\bar{1}\bar{2}} = -2r^2 \bar{\varphi}^1 + 2\sqrt{-1} u \bar{\varphi}^2; \tag{3.6}$$

where we set $V = g^{1\bar{1}} g^{2\bar{2}} - g^{1\bar{2}} g^{2\bar{1}} = r^2 s^2 - |u|^2$.

3.3 Cohomologies

Consider the inclusion of invariant forms into the double complex of forms, $\iota: (\wedge^{\bullet, \bullet} \mathfrak{g}^\vee, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet, \bullet} X, \partial, \bar{\partial})$. By choosing an invariant Hermitian metric, (the easier finite-dimensional version of) elliptic Hodge theory also applies at the level of invariant forms; in particular, any cohomology class of invariant forms admits a

unique invariant harmonic representative. It follows that the above inclusion induces injective maps in de Rham ι_{dR} , Dolbeault $\iota_{\bar{\partial}}$, Bott–Chern ι_{BC} , Aeppli ι_A cohomology, see [18, Lemma 9]. We claim that they are in fact isomorphisms.

The de Rham cohomology of Hopf, Inoue, Kodaira surfaces is well known, and it happens that the above maps ι_{dR} are actually isomorphisms, that is, any de Rham class admits an invariant representative. In fact, the Hopf surface is diffeomorphic to the product $\mathbb{S}^1 \times \mathbb{S}^3$ of two compact Lie groups, so one can use the Künneth formula and e.g. [20, Theorem 1.28]; the primary Kodaira surface is a nilmanifold, so one can use the Nomizu theorem [43]; the secondary Kodaira surfaces are quotients of primary Kodaira surfaces by finite groups; the Inoue surface of type S^\pm is a completely-solvable solvmanifold, so one can use the Hattori theorem [25]; and the de Rham cohomology of the Inoue surface of type S_M can be computed by exploiting their number-theoretic construction as [44] does in a more general setting.

As for the Dolbeault cohomology, for compact complex surfaces, we know that the Frölicher spectral sequence degenerates at the first page, see e.g. [9], that is, $\dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}) = \sum_{p+q=k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X)$ for any k . By explicitly computing the Dolbeault cohomology of invariant forms [4], one then notice that the above maps $\iota_{\bar{\partial}}$ are actually isomorphisms.

Finally, Bott–Chern cohomology of compact complex surfaces is well-understood since [58]. By [5, Theorem 1.3, Proposition 2.2], (that fits in the general theory later developed by [52],) also ι_{BC} are isomorphisms. Explicit computations can be found in [4]. Finally, ι_A are isomorphisms thanks to the Schweitzer duality between Bott–Chern and Aeppli cohomologies, where one can use the Hodge-star-operator with respect to an invariant Hermitian metric.

Finally, by uniqueness of the harmonic representative in a cohomology class, we also deduce that harmonic representatives with respect to invariant metrics are invariant.

3.4 Chern-Ricci Flow

Recall that the Chern-Ricci form represents the first Bott–Chern class $c_1^{BC}(X) \in H_{BC}^{1,1}(X)$. Since a class in $H_{BC}^{1,1}(X)$ contains only one invariant representative, the Chern-Ricci form $\text{Ric}^{Ch}(\omega)$ does not depend on the invariant Hermitian metric ω . In particular, the Chern-Ricci flow starting at ω_0 reduces to

$$\frac{\partial}{\partial t} \omega(t) = -\text{Ric}^{Ch}(\omega_0), \quad \omega(0) = \omega_0. \tag{CRF}$$

We notice that the solution of the Chern-Ricci flow starting at an invariant metric remains invariant for any existence time. Indeed, by short existence and uniqueness assured by parabolicity, the symmetry group is preserved along the flow (and possibly increases in the limit, see e.g. [35]).

Denote by ρ_r, ρ_s, ρ_u the coefficients of the Chern-Ricci form,

$$2\text{Ric}^{Ch} = \sqrt{-1}\rho_r\varphi^{1\bar{1}} + \sqrt{-1}\rho_s\varphi^{2\bar{2}} + \rho_u\varphi^{1\bar{2}} - \bar{\rho}_u\varphi^{2\bar{1}},$$

and let the initial metric ω_0 be of the form

$$2\omega_0 = \sqrt{-1}r_0^2\varphi^{1\bar{1}} + \sqrt{-1}s_0^2\varphi^{2\bar{2}} + u_0\varphi^{1\bar{2}} - \bar{u}_0\varphi^{2\bar{1}}, \tag{3.7}$$

where $r_0, s_0 \in \mathbb{R} \setminus \{0\}$ and $u_0 \in \mathbb{C}$ such that $r_0^2s_0^2 - |u_0|^2 > 0$. The solution $\omega(t)$ of the Chern-Ricci flow starting at ω_0 is then

$$2\omega(t) = \sqrt{-1}(r_0^2 - t\rho_r)\varphi^{1\bar{1}} + \sqrt{-1}(s_0^2 - t\rho_s)\varphi^{2\bar{2}} + (u_0 - t\rho_u)\varphi^{1\bar{2}} - (\bar{u}_0 - t\bar{\rho}_u)\varphi^{2\bar{1}},$$

defined for times t such that $r(t)^2 = r_0^2 - t\rho_r > 0$, $s(t) = s_0^2 - t\rho_s > 0$, $u(t) = u_0 - t\rho_u \in \mathbb{C}$ such that $r(t)^2s(t)^2 - |u(t)|^2 > 0$.

4 Geometric Formality According to Kotshick

In this section we state the main theorem of this note, regarding class VII surfaces of the Enriques-Kodaira classification of compact complex surfaces.

Theorem 4.1 *On class VII surfaces of the Enriques-Kodaira classification, geometric formality according to Kotshick is preserved by the Chern-Ricci flow starting at initial invariant Hermitian metrics.*

Proof Let X be a class VII surface, that is, $\text{Kod}(X) = -\infty$ and $b_1(X) = 1$. By [31, Theorem 6], for a compact oriented Kotschick-geometrically formal 4-manifold X , the first Betti number satisfies $b_1(X) \in \{0, 1, 2, 4\}$. Since all class VII surfaces are non-Kähler, they must have odd first Betti number by [14,33], that is, $b_1(X) = 1$. By [31, Theorem 9], the Euler characteristic of such manifolds vanishes, implying that $b_2(X) = 0$. Since the characterization result by [11,29,37,57], class VII surfaces with $b_2(X) = 0$ are necessarily Hopf or Inoue surfaces, then we see that the only Kotschick-geometrically formal class VII surfaces can be Hopf and Inoue surfaces. Therefore, Chern-Ricci flow starting at any metric cannot produce geometrically formal metrics on class VII surfaces other than Hopf and Inoue surfaces: we will then check the statement for those surfaces.

4.1 Case 1: Hopf Surfaces

Hopf surfaces X are compact complex surfaces in class VII defined as a quotient of $\mathbb{C}^2 \setminus \{0\}$ by a free action of a discrete group generated by a holomorphic contraction $\gamma(z, w) = (\alpha z + \lambda w^p, \beta w)$ where $\alpha, \beta, \lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ are such that $0 < |\alpha| \leq |\beta| < 1$ and $(\alpha - \beta^p)\lambda = 0$, see [34], [53, page 820], see [62, Remark 1].

The diffeomorphism type is $\mathbb{S}^1 \times \text{SU}(2)$, and the complex structure is a special case of the Calabi-Eckmann complex structure on product of spheres [16]. See also [47, Theorem 4.1]. In terms of a coframe (φ^1, φ^2) of $(1, 0)$ -forms, they are described as

$$d\varphi^1 = \sqrt{-1}\varphi^1 \wedge \varphi^2 + \sqrt{-1}\varphi^1 \wedge \bar{\varphi}^2, \quad d\varphi^2 = -\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1.$$

The de Rham cohomology of Hopf surfaces is

$$H_{dR}^\bullet(X; \mathbb{C}) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

Harmonic representatives for cohomologies We look at how harmonic representatives of de Rham cohomology change with respect to the invariant Hermitian metric, and in particular whether their product is still harmonic.

We notice that, varying the invariant Hermitian metric, the harmonic representatives are

$$\begin{aligned} H_{dR}^\bullet(X; \mathbb{R}) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C} \\ &\left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}} + \frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle. \end{aligned}$$

Indeed, it suffices to check that the harmonic representative of the class $[\varphi^2 - \bar{\varphi}^2]$ does not depend on the invariant metric. This is because harmonic representatives are invariant, and the class $[\varphi^2 - \bar{\varphi}^2] = \{\varphi^2 - \bar{\varphi}^2 + dc : c \in \mathbb{R}\}$ contains only one invariant representative, which is then harmonic with respect to any metric. Then we compute the harmonic representative of the dual class in $H_{dR}^3(X; \mathbb{R})$ by applying the Hodge-star-operator to $\varphi^2 - \bar{\varphi}^2$ with respect to the arbitrary Hermitian metric. In any case, the product of an invariant 1-form and an invariant 3-form is either zero or a scalar multiple of the volume form. It follows that *any invariant metric on the Hopf surface is geometrically formal in the sense of Kotschick*.

Chern-Ricci flow Clearly, on the Hopf surface with invariant Hermitian metrics, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow. Nonetheless, for completeness and for later use, we compute the Chern-Ricci form and the Chern-Ricci flow on X .

We start by computing the Chern-Riemann curvature of an invariant Hermitian metric. We follow notation as in [45, Section 2] (see also [38, Section 6] for another argument). With respect to the frame $(\varphi_1, \varphi_2, \varphi_3, \bar{\varphi}_1, \bar{\varphi}_2, \bar{\varphi}_3)$ and to the dual coframe $(\varphi^1, \varphi^2, \varphi^3, \bar{\varphi}^1, \bar{\varphi}^2, \bar{\varphi}^3)$, we set the structure constants

$$[\varphi_I, \varphi_H] =: c_{IH}^K \varphi_K.$$

Here, capital letters vary in $\{1, 2, 3, \bar{1}, \bar{2}, \bar{3}\}$, and the Einstein summation is assumed. In our case, the non-trivial structure constants are

$$\begin{aligned} c_{12}^1 &= -\sqrt{-1}, & c_{11}^2 &= \sqrt{-1}, & c_{11}^{\bar{2}} &= \sqrt{-1}, & c_{12}^{\bar{1}} &= -\sqrt{-1}, \\ c_{21}^1 &= \sqrt{-1}, & c_{21}^{\bar{1}} &= -\sqrt{-1}, & c_{11}^2 &= -\sqrt{-1}, & c_{11}^{\bar{2}} &= -\sqrt{-1}, \\ c_{12}^{\bar{1}} &= \sqrt{-1}, & c_{12}^{\bar{2}} &= \sqrt{-1}, & c_{21}^1 &= \sqrt{-1}, & c_{21}^{\bar{1}} &= -\sqrt{-1}. \end{aligned}$$

Recall that the Christoffel symbols of the Levi-Civita connections (with respect to the above non-commutative frame) can be computed as

$$\begin{aligned} (\Gamma^{LC})_{IH}^K &= \frac{1}{2} g^{KL} (g([\varphi_I, \varphi_H], \varphi_L) - g([\varphi_H, \varphi_L], \varphi_I) - g([\varphi_I, \varphi_L], \varphi_H)) \\ &= \frac{1}{2} c_{IH}^K - \frac{1}{2} g^{KA} g_{BI} c_{HA}^B - \frac{1}{2} g^{KA} g_{BH} c_{IA}^B. \end{aligned}$$

Set $V = r^2 s^2 - |u|^2$ for simplicity. In our case, up to conjugation, the non-trivial ones are

$$\begin{aligned} (\Gamma^{LC})_{11}^1 &= -s^2 u V^{-1}, & (\Gamma^{LC})_{11}^2 &= -\sqrt{-1} u^2 V^{-1}, \\ (\Gamma^{LC})_{12}^1 &= \frac{1}{2} (-\sqrt{-1} s^4 + \sqrt{-1} |u|^2) V^{-1}, & (\Gamma^{LC})_{12}^2 &= -\frac{1}{2} (r^2 - s^2) u V^{-1}, \\ (\Gamma^{LC})_{1\bar{1}}^1 &= \frac{1}{2} s^2 \bar{u} V^{-1}, & (\Gamma^{LC})_{1\bar{1}}^2 &= \frac{1}{2} \sqrt{-1} r^2 s^2 V^{-1}, \\ (\Gamma^{LC})_{1\bar{1}}^{\bar{1}} &= \frac{1}{2} s^2 u V^{-1}, & (\Gamma^{LC})_{1\bar{1}}^{\bar{2}} &= \frac{1}{2} (\sqrt{-1} r^2 s^2 - 2\sqrt{-1} |u|^2) V^{-1}, \\ (\Gamma^{LC})_{12}^{\bar{1}} &= -\frac{1}{2} \sqrt{-1} s^4 V^{-1}, & (\Gamma^{LC})_{12}^{\bar{2}} &= \frac{1}{2} s^2 u V^{-1}, \\ (\Gamma^{LC})_{1\bar{2}}^{\bar{1}} &= \frac{1}{2} \sqrt{-1} u^2 V^{-1}, & (\Gamma^{LC})_{1\bar{2}}^{\bar{2}} &= \frac{1}{2} r^2 u V^{-1}, \\ (\Gamma^{LC})_{21}^1 &= \frac{1}{2} (2\sqrt{-1} r^2 s^2 - \sqrt{-1} s^4 - \sqrt{-1} |u|^2) V^{-1}, & (\Gamma^{LC})_{21}^2 &= -\frac{1}{2} (r^2 - s^2) u V^{-1}, \\ (\Gamma^{LC})_{22}^1 &= -s^2 \bar{u} V^{-1}, & (\Gamma^{LC})_{22}^2 &= -\sqrt{-1} |u|^2 V^{-1}, \\ (\Gamma^{LC})_{2\bar{1}}^1 &= -\frac{1}{2} \sqrt{-1} \bar{u}^2 V^{-1}, & (\Gamma^{LC})_{2\bar{1}}^2 &= \frac{1}{2} r^2 \bar{u} V^{-1}, \\ (\Gamma^{LC})_{2\bar{1}}^{\bar{1}} &= \frac{1}{2} (-2 * \sqrt{-1} r^2 s^2 + \sqrt{-1} s^4 + 2\sqrt{-1} |u|^2) V^{-1}, & (\Gamma^{LC})_{2\bar{1}}^{\bar{2}} &= \frac{1}{2} s^2 \bar{u} V^{-1}, \\ (\Gamma^{LC})_{2\bar{2}}^1 &= -\frac{1}{2} s^2 \bar{u} V^{-1}, & (\Gamma^{LC})_{2\bar{2}}^2 &= -\frac{1}{2} \sqrt{-1} |u|^2 V^{-1}, \\ (\Gamma^{LC})_{2\bar{2}}^{\bar{1}} &= -\frac{1}{2} s^2 u V^{-1}, & (\Gamma^{LC})_{2\bar{2}}^{\bar{2}} &= \frac{1}{2} \sqrt{-1} |u|^2 V^{-1}. \end{aligned}$$

We can now compute the Christoffel symbols $(\Gamma^{Ch})_{IH}^K$ of the Chern connection by the formula [45, Equation (7)]:

$$(\Gamma^{\varepsilon, \rho})_{IH}^K = (\Gamma^{LC})_{IH}^K + \varepsilon g^{KL} T_{IHL} + \rho g^{KL} C_{IHL},$$

by setting $(\varepsilon, \rho) = (0, \frac{1}{2})$, where

$$T_{IHL} = -d\omega(J\varphi_I, J\varphi_H, J\varphi_L), \quad C_{IHL} = d\omega(J\varphi_I, \varphi_H, \varphi_L).$$

We get

$$\begin{aligned} (\Gamma^{Ch})_{21}^2 &= -r^2uV^{-1}, & (\Gamma^{Ch})_{21}^1 &= \sqrt{-1}r^2s^2V^{-1}, \\ (\Gamma^{Ch})_{1\bar{1}}^2 &= \sqrt{-1}, & (\Gamma^{Ch})_{2\bar{1}}^1 &= -\sqrt{-1}, \\ (\Gamma^{Ch})_{12}^1 &= -\sqrt{-1}s^4V^{-1}, & (\Gamma^{Ch})_{12}^2 &= s^2uV^{-1}, \end{aligned}$$

the others being equal to the corresponding Levi-Civita symbols or deduced by conjugation. We can compute the (4, 0)-Riemannian curvature of $\nabla^{\varepsilon,\rho}$ as

$$\begin{aligned} (R^{\varepsilon,\rho})_{IHKL} &= g_{AL}(\Gamma^{\varepsilon,\rho})_{HK}^B(\Gamma^{\varepsilon,\rho})_{IB}^A - g_{AL}(\Gamma^{\varepsilon,\rho})_{IK}^B(\Gamma^{\varepsilon,\rho})_{HB}^A \\ &\quad - g_{AL}c_{IH}^B(\Gamma^{\varepsilon,\rho})_{BK}^A. \end{aligned}$$

By using the symmetries for the Chern curvature $(R^{Ch})_{IHKL} = -(R^{Ch})_{HIKL} = -(R^{Ch})_{IHKL}$ and the conjugation, we get that the only non-zero components are

$$\begin{aligned} (R^{Ch})_{1\bar{1}\bar{1}\bar{1}} &= \frac{1}{2}(2r^4s^2 - r^2s^4 - 2(r^2 - s^2)|u|^2)V^{-1}, \\ (R^{Ch})_{1\bar{1}\bar{1}\bar{2}} &= \frac{1}{2}(\sqrt{-1}|u|^2u + (-\sqrt{-1}r^2s^2 - \sqrt{-1}s^4)u)V^{-1}, \\ (R^{Ch})_{1\bar{1}\bar{2}\bar{1}} &= \frac{1}{2}(-\sqrt{-1}|u|^2\bar{u} - (-\sqrt{-1}r^2s^2 - \sqrt{-1}s^4)\bar{u})V^{-1}, \\ (R^{Ch})_{1\bar{1}\bar{2}\bar{2}} &= \frac{1}{2}s^6V^{-1}, \\ (R^{Ch})_{1\bar{2}\bar{1}\bar{1}} &= \frac{1}{2}(-\sqrt{-1}r^2s^2u + 2\sqrt{-1}|u|^2u)V^{-1}, \\ (R^{Ch})_{1\bar{2}\bar{1}\bar{2}} &= \frac{1}{2}s^2u^2V^{-1}, \\ (R^{Ch})_{1\bar{1}\bar{2}\bar{1}} &= -\frac{1}{2}s^2|u|^2V^{-1}, \\ (R^{Ch})_{1\bar{2}\bar{2}\bar{2}} &= \frac{1}{2}\sqrt{-1}s^4uV^{-1}, \\ (R^{Ch})_{2\bar{1}\bar{1}\bar{1}} &= \frac{1}{2}(\sqrt{-1}r^2s^2\bar{u} - 2\sqrt{-1}|u|^2\bar{u})V^{-1}, \\ (R^{Ch})_{2\bar{1}\bar{1}\bar{2}} &= -\frac{1}{2}s^2|u|^2V^{-1}, \\ (R^{Ch})_{2\bar{1}\bar{2}\bar{1}} &= \frac{1}{2}s^2\bar{u}^2V^{-1}, \\ (R^{Ch})_{2\bar{1}\bar{2}\bar{2}} &= -\frac{1}{2}\sqrt{-1}s^4\bar{u}V^{-1}, \\ (R^{Ch})_{2\bar{2}\bar{1}\bar{1}} &= \frac{1}{2}r^2|u|^2V^{-1}, \\ (R^{Ch})_{2\bar{2}\bar{1}\bar{2}} &= -\frac{1}{2}\sqrt{-1}r^2s^2uV^{-1}, \end{aligned}$$

$$\begin{aligned} (R^{Ch})_{2\bar{2}2\bar{1}} &= \frac{1}{2}\sqrt{-1}r^2s^2\bar{u}V^{-1}, \\ (R^{Ch})_{2\bar{2}2\bar{2}} &= \frac{1}{2}s^2|u|^2V^{-1}. \end{aligned}$$

Finally, we can compute the (first) Chern-Ricci curvature by tracing on the third and fourth indices:

$$(\text{Ric}^{Ch})_{IH} = g^{KL}(\text{R}^{Ch})_{IHKL};$$

then the Chern-Ricci form can be defined as

$$\text{Ric}^{Ch} = (\text{Ric}^{Ch})_{ih}\sqrt{-1}dz^i \wedge d\bar{z}^h.$$

In our case, the only non-trivial coefficients are

$$(\text{Ric}^{Ch})_{1\bar{1}} = 2$$

and the corresponding $(\text{Ric}^{Ch})_{\bar{1}1} = -(\text{Ric}^{Ch})_{1\bar{1}}$. Therefore the Chern-Ricci form of any invariant Hermitian metric is

$$\text{Ric}^{Ch}(\omega) = 2\sqrt{-1}\varphi^1 \wedge \bar{\varphi}^1.$$

Therefore the solution of the Chern-Ricci flow starting at ω_0 of the form (3.7) is

$$2\omega(t) = \sqrt{-1}(r_0^2 - t)\varphi^{1\bar{1}} + \sqrt{-1}s_0^2\varphi^{2\bar{2}} + u_0\varphi^{1\bar{2}} - \bar{u}_0\varphi^{2\bar{1}}, \tag{4.1}$$

defined as long as $t < \frac{r_0^2 s_0^2 - |u_0|^2}{s_0^2} < r_0^2$.

4.2 Case 2: Inoue Surfaces

Inoue–Bombieri surfaces [12,27] X are compact complex surfaces in class VII with second Betti number equal to zero and with no holomorphic curves [10,11,36,37,57]. Their universal cover is $\mathbb{C} \times \mathbb{H}$, where \mathbb{H} denotes the upper half-plane. They are divided into three families, S_M , $S_{N,p,q,r;t}^+$, and $S_{N,p,q,r}^-$, depending on parameters.

4.3 Case 2.1: Inoue–Bombieri Surface of Type S_M

We focus now on the case S_M : it has a structure of fibre bundle over \mathbb{S}^1 , where the fibre is a 3-dimensional torus.

Inoue–Bombieri surfaces of type S_M admit a description as quotients of solvable Lie groups with invariant complex structure [24], that we now describe. We can fix a coframe (φ^1, φ^2) of $(1, 0)$ -forms with structure equations

$$\begin{aligned}
 d\varphi^1 &= \frac{\alpha - \sqrt{-1}\beta}{2\sqrt{-1}}\varphi^1 \wedge \varphi^2 - \frac{\alpha - \sqrt{-1}\beta}{2\sqrt{-1}}\varphi^1 \wedge \bar{\varphi}^2, \\
 d\varphi^2 &= -\sqrt{-1}\alpha\varphi^2 \wedge \bar{\varphi}^2,
 \end{aligned}$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$. The de Rham cohomology of X can be explicitly described [58], see [4, Theorem 4.1]:

$$H_{dR}^\bullet(X; \mathbb{R}) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}2} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

Harmonic representatives for cohomologies We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (3.1):

$$\begin{aligned}
 H_{dR}^\bullet(X; \mathbb{R}) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C} \\
 &\left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}} + \frac{1}{2}r^2\varphi^{1\bar{1}2} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle \quad (4.2)
 \end{aligned}$$

We conclude that: *any invariant Hermitian metric on an Inoue surface of type S_M is geometrically formal in the sense of Kotschick.*

Chern-Ricci flow The Chern-Ricci form of any invariant Hermitian metric is

$$2\text{Ric}^{Ch}(\omega) = -\sqrt{-1}\alpha^2\varphi^2 \wedge \bar{\varphi}^2,$$

whence the solution of the Chern-Ricci flow (CRF) is given by

$$2\omega(t) = \sqrt{-1}r_0^2\varphi^{1\bar{1}} + \sqrt{-1}\left(s_0^2 + \alpha^2t\right)\varphi^{2\bar{2}} + u_0\varphi^{1\bar{2}} - \bar{u}_0\varphi^{2\bar{1}}, \quad (4.3)$$

defined for any non-negative time $t \geq 0$.

Clearly, on an Inoue surface of type S_M with invariant Hermitian metrics, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow.

4.4 Case 2.2: Inoue Surfaces of Class S^\pm

In this subsection, we focus on the case of Inoue surfaces of type S^\pm . Inoue–Bombieri surfaces of type S^- have an unramified double cover of type S^+ : we can then restrict to Inoue–Bombieri surfaces of type S^+ , which have a structure of fibre bundle over \mathbb{S}^1 , where the fibre is a compact quotient of the 3-dimensional Heisenberg group.

Also Inoue–Bombieri surfaces of type S^+ admit a description as quotients of solvable Lie groups [24], that we now describe. We can fix a coframe (φ^1, φ^2) of

(1, 0)-forms with structure equations

$$\begin{aligned}
 d\varphi^1 &= \frac{1}{2\sqrt{-1}}\varphi^1 \wedge \varphi^2 + \frac{1}{2\sqrt{-1}}\varphi^2 \wedge \bar{\varphi}^1 + \frac{q\sqrt{-1}}{2}\varphi^2 \wedge \bar{\varphi}^2, \\
 d\varphi^2 &= \frac{1}{2\sqrt{-1}}\varphi^2 \wedge \bar{\varphi}^2,
 \end{aligned}$$

where $q \in \mathbb{R}$. The de Rham cohomology of X can be explicitly described [58], see [4, Theorem 4.1]:

$$H_{dR}^\bullet(X; \mathbb{C}) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

Harmonic representatives for cohomologies The situation is exactly as in (4.2). We conclude that: *any invariant Hermitian metric on an Inoue surface of type S^\pm is geometrically formal in the sense of Kotschick.*

Chern-Ricci flow The Chern-Ricci form of any invariant Hermitian metric is

$$2\text{Ric}^{Ch}(\omega) = -\sqrt{-1}\varphi^{2\bar{2}},$$

whence the solution of the Chern-Ricci flow (CRF) is given by

$$2\omega(t) = \sqrt{-1}r_0^2\varphi^{1\bar{1}} + \sqrt{-1}(s_0^2 + t)\varphi^{2\bar{2}} + u_0\varphi^{1\bar{2}} - \bar{u}_0\varphi^{2\bar{1}}, \tag{4.4}$$

defined for any non-negative time $t \geq 0$.

Clearly, on an Inoue surface of type S^\pm with invariant Hermitian metrics, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow. □

We also analyze in details primary and secondary Kodaira surfaces resulting in the following proposition, for which we give explicit computations.

Proposition 4.2 *On any Kodaira surface, the properties of geometric formality in the sense of Kotschick is preserved along the Chern-Ricci flow starting at initial invariant Hermitian metrics.*

Proof We will look at each case separately.

4.5 Case 1: Primary Kodaira Surface

Kodaira surfaces X are compact complex surfaces of Kodaira dimension $\text{Kod}(X) = 0$ and first Betti number $b_1(X) = 3$. Primary Kodaira surfaces have trivial canonical bundle.

We note that, by [31, Theorem 6], primary Kodaira surfaces are never Kotschick-geometrically formal, not even with regards to non-invariant metrics, by having $b_1 = 3$: hence *Chern-Ricci flow preserves geometric formality according to Kotschick*. The same conclusion follows by [23, Theorem 1] stating that non-tori nilmanifolds are never formal, therefore never geometrically formal in the sense of Kotschick. Nevertheless we give explicit computations for this fact.

We recall the description of primary Kodaira surfaces as quotients of solvable Lie groups [24]. There exists a coframe (φ^1, φ^2) of $(1, 0)$ -forms with structure equations

$$d\varphi^1 = 0, \quad d\varphi^2 = \frac{\sqrt{-1}}{2}\varphi^1 \wedge \bar{\varphi}^1.$$

The de Rham cohomology of X can be explicitly described:

$$\begin{aligned} H_{dR}^\bullet(X; \mathbb{R}) = & \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^{\bar{1}}, \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{\bar{1}\bar{2}} \rangle \\ & \oplus \mathbb{C}\langle \varphi^{12\bar{2}}, \varphi^{2\bar{1}\bar{2}}, \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (3.1):

$$\begin{aligned} H_{dR}^\bullet(X; \mathbb{R}) = & \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^{\bar{1}}, \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C} \\ & \left\langle \varphi^{12}, \varphi^{1\bar{2}} + \frac{\sqrt{-1}\bar{u}}{s^2}\varphi^{1\bar{1}}, \varphi^{2\bar{1}} - \frac{\sqrt{-1}u}{s^2}\varphi^{1\bar{1}}, \varphi^{\bar{1}\bar{2}} \right\rangle \\ & \oplus \mathbb{C} \left\langle \frac{1}{2}s^2\varphi^{12\bar{2}} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{12\bar{1}}, \frac{1}{2}s^2\varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}u\varphi^{1\bar{1}\bar{2}}, \right. \\ & \left. -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}} + \frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \end{aligned}$$

We explicitly notice that, *on primary Kodaira surfaces, an invariant Hermitian metric is never geometrically formal in the sense of Kotschick*: indeed, $\varphi^1 \wedge \varphi^{\bar{1}\bar{2}}$ is never harmonic.

As for Chern-Ricci flow, the primary Kodaira surface has trivial canonical bundle, therefore $\text{Ric}^{Ch}(\omega) = 0$. Then, clearly, the Chern-Ricci flow does not evolve invariant Hermitian metrics.

4.6 Case 2: Secondary Kodaira Surface

Secondary Kodaira surfaces X are quotients of primary Kodaira surfaces by finite groups; they have torsion non-trivial canonical bundle.

We recall the description of secondary Kodaira surfaces as quotients of solvable Lie groups [24]. There exists a coframe (φ^1, φ^2) of $(1, 0)$ -forms with structure equations

$$d\varphi^1 = -\frac{1}{2}\varphi^1 \wedge \varphi^2 + \frac{1}{2}\varphi^1 \wedge \bar{\varphi}^2, \quad d\varphi^2 = \frac{\sqrt{-1}}{2}\varphi^1 \wedge \bar{\varphi}^1.$$

The cohomologies of X can be explicitly described:

$$H_{dR}^\bullet(X; \mathbb{R}) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}2} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle,$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

As for the harmonic representatives for de Rham cohomology, the situation is very similar to the Inoue case. We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (3.1):

$$H_{dR}^\bullet(X; \mathbb{R}) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 - \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}} + \frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle.$$

We conclude that *any invariant Hermitian metric on an secondary Kodaira is geometrically formal in the sense of Kotschick.*

As for the Chern-Ricci flow, the secondary Kodaira surface has torsion canonical bundle, therefore $\text{Ric}^{Ch}(\omega) = 0$. Therefore, the Chern-Ricci flow does not evolve invariant Hermitian metrics. □

5 Dolbeault and Bott–Chern Geometric Formalities

As for Dolbeault or Bott–Chern geometric formality, the situation is clear for Hopf, Inoue and Kodaira surfaces, as we now describe. We also make computations regarding Aepli cohomology and harmonic representatives with respect to the Aepli Laplacian.

Proposition 5.1 *On Hopf, Inoue, and Kodaira surfaces, the property of Dolbeault geometric formality and of Bott–Chern geometric formality is preserved along the Chern-Ricci flow starting at invariant metrics. In the same situation, the properties of having a structure of algebra or a structure of H_{BC} -module for harmonic-Aepli forms are all preserved by the Chern Ricci flow.*

Proof We refer to the complex structures used in Theorem 4.1 and Proposition 4.2, for the computation of Dolbeault, Bott–Chern, and Aepli cohomologies.

5.1 Hopf Surfaces

The Dolbeault cohomology of Hopf surfaces is explicitly described in [26, Appendix II, Theorem 9.5], and see [4, Section 3.1] for the Bott–Chern cohomology:

$$\begin{aligned} H_{\bar{\partial}}^{\bullet, \bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{2}} \rangle, \\ H_{BC}^{\bullet, \bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{2}\bar{2}} \rangle, \\ H_A^{\bullet, \bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle, \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We look at how the harmonic representatives of such cohomologies change with respect to the invariant Hermitian metric, and in particular when their product is still harmonic.

We summarize them as follows:

$$\begin{aligned} H_{\bar{\partial}}^{\bullet, \bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{2}\bar{2}} \rangle, \\ H_{BC}^{\bullet, \bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \\ &\quad \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{2}\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{1\bar{2}\bar{2}} \right\rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{2}\bar{2}} \rangle, \\ H_A^{\bullet, \bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \\ &\quad \oplus \mathbb{C}\langle s^4\varphi^{2\bar{2}} + |u|^2\varphi^{1\bar{1}} - \sqrt{-1}s^2u\varphi^{1\bar{2}} + \sqrt{-1}s^2\bar{u}\varphi^{2\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle, \end{aligned}$$

Let us focus first on the Dolbeault cohomology. Here, the only Dolbeault-harmonic representative that changes is for the generator in $H_{\bar{\partial}}^{1,2}(X)$. We conclude that *any invariant Hermitian metric on the Hopf surface is geometrically-Dolbeault formal*.

As regards the Bott–Chern cohomology, to our aim, that is, studying harmonicity of products of Bott–Chern-harmonic forms, the only case of interest is the product $[\varphi^{1\bar{1}}] \smile [\varphi^{1\bar{1}}]$, the products with the class [1] being trivial and the other products being zero because of degree reasons. Since the harmonic representatives with respect to invariant metrics are invariant, the Bott–Chern class $[\varphi^{1\bar{1}}] = \{\varphi^{1\bar{1}} + \partial\bar{\partial}c : c \in \mathbb{R}\}$ contains only one invariant representative, that is also harmonic with respect to any invariant Hermitian metric. Again, we have that *any invariant Hermitian metric on the Hopf surface is geometrically-Bott–Chern formal*.

We consider the Aeppli cohomology. On the one side, we can consider the products between Aeppli-harmonic forms: the only possibly non-trivial products concern the classes $[\varphi^2]$ and $[\bar{\varphi}^2]$, $[\varphi^2]$ and $[\varphi^2 \wedge \bar{\varphi}^2]$, $[\bar{\varphi}^2]$ and $[\varphi^2 \wedge \bar{\varphi}^2]$. Since the classes $[\varphi^2]$ and $[\bar{\varphi}^2]$ contain only one invariant representative, we are reduced to study how the harmonic representative of the Aeppli cohomology class $[\varphi^2 \wedge \bar{\varphi}^2]$ depends on the invariant Hermitian metric. The arbitrary representative in the Aeppli cohomology

class $[\varphi^{2\bar{2}}]$ is

$$\begin{aligned} h &:= \varphi^2 \wedge \bar{\varphi}^2 + \partial \left(\lambda_1 \bar{\varphi}^1 + \lambda_2 \bar{\varphi}^2 \right) + \bar{\partial} \left(\lambda_3 \varphi^1 + \lambda_4 \varphi^2 \right) \\ &= \varphi^{2\bar{2}} - \sqrt{-1} (\lambda_2 + \lambda_4) \varphi^{1\bar{1}} + \sqrt{-1} \lambda_1 \varphi^{2\bar{1}} + \sqrt{-1} \lambda_3 \varphi^{1\bar{2}}, \end{aligned}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{C}$. By (3.5), we compute

$$\begin{aligned} V \cdot *_g h &= V \cdot \left(*_g \varphi^{2\bar{2}} + \sqrt{-1} \lambda_1 *_g \varphi^{2\bar{1}} - \sqrt{-1} (\lambda_2 + \lambda_4) *_g \varphi^{1\bar{1}} + \sqrt{-1} \lambda_3 *_g \varphi^{1\bar{2}} \right) \\ &= \left(r^4 - \lambda_1 u r^2 - \sqrt{-1} (\lambda_2 + \lambda_4) |u|^2 + \lambda_3 \bar{u} r^2 \right) \varphi^{1\bar{1}} \\ &\quad + \left(-\sqrt{-1} u r^2 + \sqrt{-1} \lambda_1 u^2 - (\lambda_2 + \lambda_4) u s^2 - \sqrt{-1} \lambda_3 r^2 s^2 \right) \varphi^{1\bar{2}} \\ &\quad + \left(\sqrt{-1} \bar{u} r^2 - \sqrt{-1} \lambda_1 r^2 s^2 + (\lambda_2 + \lambda_4) \bar{u} s^2 + \sqrt{-1} \lambda_3 \bar{u}^2 \right) \varphi^{2\bar{1}} \\ &\quad + \left(|u|^2 - \lambda_1 u s^2 - \sqrt{-1} (\lambda_2 + \lambda_4) s^4 + \lambda_3 \bar{u} s^2 \right) \varphi^{2\bar{2}}. \end{aligned}$$

By using the structure equations, we now compute

$$\begin{aligned} \partial(*_g h) &= \sqrt{-1} \frac{-\sqrt{-1} u r^2 + \sqrt{-1} \lambda_1 u^2 - (\lambda_2 + \lambda_4) u s^2 - \sqrt{-1} \lambda_3 r^2 s^2}{r^2 s^2 - |u|^2} \varphi^{12\bar{2}} \\ &\quad - \sqrt{-1} \frac{|u|^2 - \lambda_1 u s^2 - \sqrt{-1} (\lambda_2 + \lambda_4) s^4 + \lambda_3 \bar{u} s^2}{r^2 s^2 - |u|^2} \varphi^{12\bar{1}}, \\ \bar{\partial}(*_g h) &= \sqrt{-1} \frac{\sqrt{-1} \bar{u} r^2 - \sqrt{-1} \lambda_1 r^2 s^2 + (\lambda_2 + \lambda_4) \bar{u} s^2 + \sqrt{-1} \lambda_3 \bar{u}^2}{r^2 s^2 - |u|^2} \varphi^{21\bar{2}} \\ &\quad - \sqrt{-1} \frac{|u|^2 - \lambda_1 u s^2 - \sqrt{-1} (\lambda_2 + \lambda_4) s^4 + \lambda_3 \bar{u} s^2}{r^2 s^2 - |u|^2} \varphi^{11\bar{2}}. \end{aligned}$$

Therefore the Aeppli-harmonic conditions $\partial \bar{\partial} h = \partial *_g h = \bar{\partial} *_g h = 0$ yield

$$\begin{pmatrix} \sqrt{-1} u^2 & -u s^2 & -\sqrt{-1} r^2 s^2 & -u s^2 \\ -u s^2 & -\sqrt{-1} s^4 & \bar{u} s^2 & -\sqrt{-1} s^4 \\ -\sqrt{-1} r^2 s^2 & \bar{u} s^2 & \sqrt{-1} \bar{u}^2 & \bar{u} s^2 \\ -u s^2 & -\sqrt{-1} s^4 & \bar{u} s^2 & -\sqrt{-1} s^4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} \sqrt{-1} u r^2 \\ -|u|^2 \\ -\sqrt{-1} \bar{u} r^2 \\ -|u|^2 \end{pmatrix},$$

where the rank of the first matrix is 3 thanks to the condition $r^2 s^2 - |u|^2 > 0$. By solving the system, we get

$$\begin{aligned} \lambda_1 &= \frac{\bar{u}}{s^2}, \\ \lambda_2 &= \frac{\sqrt{-1} |u|^2}{s^4} - \lambda, \\ \lambda_3 &= -\frac{u}{s^2}, \end{aligned}$$

$$\lambda_4 = \lambda,$$

varying $\lambda \in \mathbb{C}$. We finally get that the harmonic representative of $[\varphi^{2\bar{2}}]$ with respect to g is

$$h = \frac{|u|^2}{s^4} \varphi^{1\bar{1}} - \frac{\sqrt{-1}u}{s^2} \varphi^{1\bar{2}} + \frac{\sqrt{-1}\bar{u}}{s^2} \varphi^{2\bar{1}} + \varphi^{2\bar{2}}.$$

At the end of the day, we get that: *Aeppli-hamornic forms have a structure of algebra if and only if the metric (3.1) is diagonal, namely, $u = 0$.*

Finally, we consider the Aeppli cohomology as a Bott–Chern-cohomology-module. By the Stokes theorem, there is no invariant exact 4-form other than the zero form; in particular, any invariant (2, 2)-form is harmonic with respect to any Hermitian invariant metric. This reduces to consider only the products $[\varphi^{1\bar{1}}]_{BC} \smile [\varphi^2]_A$ and $[\varphi^{1\bar{1}}]_{BC} \smile [\bar{\varphi}^2]_A$. By the argument above, both $[\varphi^{1\bar{1}}]_{BC}$ and $[\varphi^2]_A$, respectively $[\bar{\varphi}^2]_A$, contain only one invariant representative that is harmonic with respect to any Hermitian metric. Therefore: *for any invariant Hermitian metric on the Hopf surface, Aeppli-harmonic forms have a structure of module over Bott–Chern-harmonic forms.*

As regards the Chern-Ricci flow, we already have an expression for it computed in (4.1). Clearly, then, on the Hopf surface with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott–Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-hamornic forms having a structure of module over Bott–Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

5.2 Inoue–Bombieri Surfaces of Type S_M

The cohomologies of Inoue–Bombieri surfaces of type S_M can be explicitly described [58], see [4, Theorem 4.1]:

$$\begin{aligned} H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\ H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\ H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $2\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (3.1):

$$\begin{aligned} H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\ H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \end{aligned}$$

$$\begin{aligned} & \oplus \mathbb{C} \left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}} \right\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}\bar{2}}\rangle, \\ H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle\varphi^2\rangle \oplus \mathbb{C}\langle\varphi^{\bar{2}}\rangle \\ & \oplus \mathbb{C} \left\langle \varphi^{1\bar{1}} - \frac{\sqrt{-1}u}{r^2}\varphi^{1\bar{2}} + \frac{\sqrt{-1}\bar{u}}{r^2}\varphi^{2\bar{1}} + \frac{|u|^2}{r^4}\varphi^{2\bar{2}} \right\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}\bar{2}}\rangle, \end{aligned} \quad (5.1)$$

We conclude that: *any invariant Hermitian metric on an Inoue surface of type S_M is geometrically-Dolbeault formal, is geometrically-Bott–Chern formal, and the Aeppli-harmonic forms have a structure of module over Bott–Chern-harmonic forms. On the other hand, Aeppli-harmonic forms have a structure of algebra if and only if the metric is diagonal.*

The Chern-Ricci flow has expression as in (4.3). Clearly, we can state that on an Inoue surface of type S_M with invariant Hermitian metrics, *the properties of Dolbeault-geometric formality, of Bott–Chern-geometric formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-harmonic forms having a structure of module over Bott–Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

5.3 Inoue Surfaces of Type S^\pm

The cohomologies of Inoue surfaces of type S^\pm can be explicitly described [58], see [4, Theorem 4.1]:

$$\begin{aligned} H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle\varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}}\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}\bar{2}}\rangle, \\ H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle\varphi^{2\bar{2}}\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}}\rangle \oplus \mathbb{C}\langle\varphi^{1\bar{1}\bar{2}}\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}\bar{2}}\rangle, \\ H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle\varphi^2\rangle \oplus \mathbb{C}\langle\varphi^{\bar{2}}\rangle \oplus \mathbb{C}\langle\varphi^{1\bar{1}}\rangle \oplus \mathbb{C}\langle\varphi^{12\bar{1}\bar{2}}\rangle, \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

As for the harmonic representatives of Dolbeault, Bott–Chern and Aeppli cohomologies, the situation is exactly as (5.1).

We conclude that: *any invariant Hermitian metric on an Inoue surface of type S^\pm is geometrically-Dolbeault formal, is geometrically-Bott–Chern formal, and the Aeppli-harmonic forms have a structure of module over Bott–Chern-harmonic forms. On the other hand, Aeppli-harmonic forms have a structure of algebra if and only if the metric is diagonal.*

The Chern-Ricci flow has expression as in (4.4). Hence, we have that on an Inoue surface of type S^\pm with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott–Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-harmonic forms having a structure of module over Bott–Chern-harmonic forms, are all preserved along the Chern-Ricci flow.*

5.4 Primary Kodaira Surfaces

The cohomologies of primary Kodaira surfaces can be explicitly described [58], see [4, Theorem 4.1]:

$$\begin{aligned}
 H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}}, \varphi^{2\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
 &\quad \oplus \mathbb{C}\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
 H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
 &\quad \oplus \mathbb{C}\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
 H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
 &\quad \oplus \mathbb{C}\langle \varphi^{12\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle,
 \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (3.1):

$$\begin{aligned}
 H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \\
 &\quad \oplus \mathbb{C}\langle \varphi^{1\bar{2}} - \sqrt{-1}s\varphi^{1\bar{1}}, \varphi^{2\bar{1}} + \sqrt{-1}s\varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
 &\quad \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}}, \frac{1}{2}s^2\varphi^{12\bar{2}} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{12\bar{1}} \right\rangle \\
 &\quad \oplus \mathbb{C}\left\langle \frac{1}{2}s^2\varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}u\varphi^{1\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
 H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
 &\quad \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{12\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{12\bar{2}}, \frac{\sqrt{-1}}{2}\bar{u}\varphi^{12\bar{1}} + \frac{1}{2}s^2\varphi^{12\bar{2}} \right\rangle \\
 &\quad \oplus \mathbb{C}\left\langle \frac{1}{2}s^2\varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}u\varphi^{1\bar{1}\bar{2}}, -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle, \\
 H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^1, \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{12} \rangle \oplus \mathbb{C}\langle \varphi^{\bar{1}\bar{2}} \rangle \\
 &\quad \oplus \mathbb{C}\langle s^2\varphi^{1\bar{2}} + \sqrt{-1}\bar{u}\varphi^{1\bar{1}}, s^2\varphi^{2\bar{1}} - \sqrt{-1}u\varphi^{1\bar{1}}, s^4\varphi^{2\bar{2}} - |u|^2\varphi^{1\bar{1}} \rangle \\
 &\quad \oplus \mathbb{C}\left\langle \frac{1}{2}s^2\varphi^{12\bar{2}} + \frac{\sqrt{-1}}{2}\bar{u}\varphi^{12\bar{1}} \right\rangle \oplus \mathbb{C}\left\langle \frac{1}{2}s^2\varphi^{2\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}u\varphi^{1\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{12\bar{1}\bar{2}} \rangle,
 \end{aligned}$$

We notice that for primary Kodaira surfaces *an invariant Hermitian metric is never geometrically-Dolbeault formal*, e.g. $\varphi^1 \wedge \bar{\varphi}^1$ is never Dolbeault-harmonic. In fact, Cattaneo and Tomassini noticed in [17, Example 4.3] that primary Kodaira surfaces have a non-vanishing Dolbeault-Massey triple product, whence they are not Dolbeault formal in the sense of [42]. Also, it is *never geometrically-Bott–Chern formal*, e.g.

$\varphi^1 \wedge \varphi^{\bar{1}\bar{2}}$ is never Bott–Chern-harmonic. *The space of Aeppli-harmonic forms is never an algebra, e.g. $\varphi^1 \wedge \bar{\varphi}^1$ is never Aeppli-harmonic, neither a module over the space of Bott–Chern harmonic forms, e.g. $\varphi^1 \wedge \bar{\varphi}^1$ is never Aeppli-harmonic.*

The primary Kodaira surface has trivial canonical bundle, therefore $\text{Ric}^{Ch}(\omega) = 0$. Then the Chern–Ricci flow does not evolve invariant Hermitian metrics.

Then clearly on a primary Kodaira surface with invariant Hermitian metrics, *the properties of geometric–Dolbeault formality, of geometric–Bott–Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-harmonic forms having a structure of module over Bott–Chern-harmonic forms, are all preserved along the Chern–Ricci flow.*

5.5 Secondary Kodaira Surfaces

The cohomologies of secondary Kodaira surfaces can be explicitly described [58], see [4, Theorem 4.1]:

$$\begin{aligned} H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle, \\ H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle, \\ H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{2\bar{2}} \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle, \end{aligned}$$

where we have listed the harmonic representatives with respect to the Hermitian metric with fundamental form $\omega = \sqrt{-1}\varphi^{1\bar{1}} + \sqrt{-1}\varphi^{2\bar{2}}$ instead of their classes.

As for the harmonic representatives for Dolbeault, Bott–Chern and Aeppli cohomologies, the situation is very similar to the Inoue case, only the computations for the class $[\varphi^{2\bar{2}}] \in H_A^{1,1}(X)$ being slightly different.

We list the harmonic representatives with respect to the arbitrary Hermitian metric as in (3.1):

$$\begin{aligned} H_{\bar{\partial}}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle \\ H_{BC}^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^{1\bar{1}} \rangle \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{1}\bar{2}} - \frac{\sqrt{-1}}{2}\bar{u}\varphi^{2\bar{1}\bar{2}} \right\rangle \\ &\quad \oplus \mathbb{C}\left\langle -\frac{1}{2}r^2\varphi^{1\bar{2}\bar{1}} + \frac{\sqrt{-1}}{2}u\varphi^{1\bar{2}\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle, \\ H_A^{\bullet,\bullet}(X) &= \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle \varphi^2 \rangle \oplus \mathbb{C}\langle \varphi^{\bar{2}} \rangle \\ &\quad \oplus \mathbb{C}\left\langle |u|^2\varphi^{1\bar{1}} - \sqrt{-1}s^2u\varphi^{1\bar{2}} + \sqrt{-1}s^2\bar{u}\varphi^{2\bar{1}} + s^4\varphi^{2\bar{2}} \right\rangle \oplus \mathbb{C}\langle \varphi^{1\bar{2}\bar{1}\bar{2}} \rangle. \end{aligned}$$

We conclude that: *any invariant Hermitian metric on a secondary Kodaira surface is geometrically–Dolbeault formal, is geometrically–Bott–Chern formal, and the Aeppli-harmonic forms have a structure of module over Bott–Chern-harmonic forms. On the*

Table 1 Summary of Theorem 4.1 and Propositions 4.2 and 5.1 concerning geometric formalities (for Kotschick, Dolbeault, Bott–Chern) and the structure of Aepli-harmonic forms with respect to Hermitian metrics, respectively invariant Hermitian metrics on Hopf, Inoue, Kodaira surfaces

Surface	Kotschick geom. form.	Dolbeault geom. form.	Bott–Chern geom. form.	Aepli harm. f. as algebra	Aepli harm. f. as BC-module
Class VII ^{b₂ > 0}	Never	?	?	?	?
Hopf (invariant metrics)	Always	Always	Always	Diagonal	Always
Inoue–Bombieri S_M (invariant metrics)	Always	Always	Always	Diagonal	Always
Inoue S^\pm (invariant metrics)	Always	Always	Always	Diagonal	Always
Primary Kodaira	Never	Never	Never	Never	Never
Secondary Kodaira (invariant metrics)	Always	Always	Always	Diagonal	Always

other hand, Aeppli-harmonic forms have a structure of algebra if and only if the metric is diagonal.

The secondary Kodaira surface has torsion canonical bundle, therefore $\text{Ric}^{Ch}(\omega) = 0$. Then the Chern-Ricci flow does not evolve invariant Hermitian metrics.

Then clearly on a secondary Kodaira surface with invariant Hermitian metrics, *the properties of geometric-Dolbeault formality, of geometric-Bott-Chern formality, of the Aeppli-harmonic forms having a structure of algebra, of the Aeppli-harmonic forms having a structure of module over Bott-Chern-harmonic forms, are all preserved along the Chern-Ricci flow.* \square

We summarize the results in the last two sections in Table 1.

In view of further study, we notice that:

- in any mentioned cases, the Chern-Ricci flow starting at an invariant metric clearly preserves each one of the above properties, since it preserves diagonal metrics. (Compare also [32, Proposition 3], showing that, for certain G -homogeneous spaces, every G -invariant metrics is geometrically formal.) We ask whether this behaviour is more general, or whether there exists a counterexample for which the Chern-Ricci flow does *not* preserve the geometric formality in the sense of Kotschick, or any other of the geometric Hermitian formalities discussed above. We notice that the above invariant metrics are Gauduchon, that is pluriclosed (also known as SKT) being defined on four-dimensional manifolds. Therefore, as the Referee kindly suggested to us, it may be interesting to further investigate the 6-dimensional nilmanifolds admitting invariant SKT metrics as classified in [21].
- Clearly, holomorphically-parallelizable manifolds do not provide such counterexamples when restricting to invariant metrics, since they have holomorphically-trivial canonical bundle, whence invariant Hermitian metrics are Chern-Ricci-flat. Our attempts on four-dimensional Lie groups (possibly not admitting compact quotients), as in [46] and references therein, or small deformations of the Iwasawa manifold [2,41] still have not provided further examples.
- The same question may be addressed for other geometric flows other than the Chern-Ricci flow, for example the Hermitian curvature flows in [54] or in particular the one studied in [61].
- It could be interesting to further investigate Massey triple products and Dolbeault Massey products, see [17,59], or other Massey products, in particular on class VII surfaces with $b_2 > 0$ and on primary Kodaira surfaces.

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References

1. Aeppli, A.: On the cohomology structure of Stein manifolds. In: Proceedings of Conference Complex Analysis (Minneapolis, Minn., 1964), pp. 58–70. Springer, Berlin (1965)
2. Angella, D.: The cohomologies of the Iwasawa manifold and of its small deformations. *J. Geom. Anal.* **23**(3), 1355–1378 (2013)
3. Angella, D.: On the Bott–Chern and Aeppli cohomology, [arXiv:1507.07112](https://arxiv.org/abs/1507.07112). In: Bielefeld Geometry & Topology Days, <https://www.math.uni-bielefeld.de/sfb701/suppls/ssfb15001.pdf>, 2015
4. Angella, D., Dloussky, G., Tomassini, A.: On Bott–Chern cohomology of compact complex surfaces. *Ann. Mat. Pura Appl.* **195**(1), 199–217 (2016)
5. Angella, D., Kasuya, H.: Bott–Chern cohomology of solvmanifolds. *Ann. Global Anal. Geom.* **52**(4), 363–411 (2017)
6. Angella, D., Tardini, N.: Quantitative and qualitative cohomological properties for non-Kähler manifolds. *Proc. Amer. Math. Soc.* **145**(1), 273–285 (2017)
7. Angella, D., Tomassini, A.: On the $\partial\bar{\partial}$ -Lemma and Bott–Chern cohomology. *Invent. Math.* **192**(1), 71–81 (2013)
8. Angella, D., Tomassini, A.: On Bott–Chern cohomology and formality. *J. Geom. Phys.* **93**, 52–61 (2015)
9. Barth, W.P., Hulek, K., Peters, C.A.M., Van de Ven, A.: *Compact Complex Surfaces*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, 3, 2nd edn. Springer, Berlin (2004)
10. Bogomolov, F.A.: Classification of surfaces of class VII_0 with $b_2 = 0$. *Izv. Akad. Nauk SSSR Ser. Mat.* **40**(2), 273–288, 469 (1976)
11. Bogomolov, F.A.: Surfaces of class VII_0 and affine geometry. *Izv. Akad. Nauk SSSR Ser. Mat.* **46**(4), 710–761, 896 (1982)
12. Bombieri, E.: Letter to Kodaira (1973)
13. Bott, R., Chern, S.S.: Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. *Acta Math.* **114**(1), 71–112 (1965)
14. Buchdahl, N.: On compact Kähler surfaces. *Ann. Inst. Fourier (Grenoble)* **49**(1), vii, xi, 287–302 (1999)
15. Buijs, U., Moreno-Fernández, J.M., Murillo, A.: A_∞ structures and Massey products. *Mediterr. J. Math.* **17**(1), Paper No. 31 (2020)
16. Calabi, E., Eckmann, B.: A class of compact, complex manifolds which are not algebraic. *Ann. Math.* **58**(3), 494–500 (1953)
17. Cattaneo, A., Tomassini, A.: Dolbeault–Massey triple products of low degree. *J. Geom. Phys.* **98**, 300–311 (2015)
18. Console, S., Fino, A.: Dolbeault cohomology of compact nilmanifolds. *Transform. Groups* **6**(2), 111–124 (2001)
19. Deligne, P., Griffiths, PhA, Morgan, J., Sullivan, D.P.: Real homotopy theory of Kähler manifolds. *Invent. Math.* **29**(3), 245–274 (1975)
20. Félix, Y., Oprea, J., Tanré, D.: *Algebraic Models in Geometry*, *Oxford Graduate Texts in Mathematics*, 17. Oxford University Press, Oxford (2008)
21. Fino, A., Parton, M., Salamon, S.: Families of strong KT structures in six dimensions. *Comment. Math. Helv.* **79**(2), 317–340 (2004)
22. Gill, M.: Convergence of the parabolic complex Monge–Ampère equation on compact Hermitian manifolds. *Commun. Anal. Geom.* **19**(2), 277–304 (2011)
23. Hasegawa, K.: Minimal models of nilmanifolds. *Proc. Amer. Math. Soc.* **106**(1), 65–71 (1989)
24. Hasegawa, K.: Complex and Kähler structures on compact solvmanifolds, Conference on Symplectic Topology. *J. Symplectic Geom.* **3**(4), 749–767 (2005)
25. Hattori, A.: Spectral sequence in the de Rham cohomology of fibre bundles. *J. Fac. Sci. Univ. Tokyo Sect. I* **8**(1960), 289–331 (1960)
26. Hirzebruch, F.: *Topological methods in algebraic geometry*, Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition, *Classics in Mathematics*, Springer, Berlin, (1995)
27. Inoue, M.: On surfaces of Class VII_0 . *Invent. Math.* **24**, 269–310 (1974)
28. Kadeishvili, T.V.: On the homology theory of fibre spaces. *Russian Math. Surv.* **35**(3), 231–238 (1980)
29. Kodaira, K.: On the structure of compact complex analytic surfaces. I. *Amer. J. Math.* **86**, 751–798 (1964)

30. Kodaira, K., Spencer, D.C.: On deformations of complex analytic structures. III. Stability theorems for complex structures. *Ann. Math.* **71**(1), 43–76 (1960)
31. Kotschick, D.: On products of harmonic forms. *Duke Math. J.* **107**(3), 521–531 (2001)
32. Kotschick, D., Terzić, S.: Geometric formality of homogeneous spaces and of biquotients. *Pac. J. Math.* **249**(1), 157–176 (2011)
33. Lamari, A.: Courants kählériens et surfaces compactes. *Ann. Inst. Fourier (Grenoble)* **49**(1), vii, x, 263–285 (1999)
34. Lattés, S.: Sur les formes réduites des transformations ponctuelles à deux variables. Application à une classe remarquable de *Taylor*. *C. Rend.* **152**, 1566–1569 (1911)
35. Lauret, J.: Geometric flows and their solitons on homogeneous spaces. *Rend. Semin. Mat. Univ. Politec. Torino* **74**(1), 55–93 (2016)
36. Li, J., Yau, S.-T., Zheng, F.: A simple proof of Bogomolov’s theorem on class VII_0 surfaces with $b_2 = 0$. *Illinois J. Math.* **34**(2), 217–220 (1990)
37. Li, J., Yau, S.-T., Zheng, F.: On projectively flat Hermitian manifolds. *Commun. Anal. Geom.* **2**(1), 103–109 (1994)
38. Liu, K.-F., Yang, X.-K.: Geometry of Hermitian manifolds. *Int. J. Math.* **23**(6), 1250055-40 (2012)
39. Lu, D.-M., Palmieri, J.H., Wu, Q.-S., Zhang, J.J.: A -infinity structure on Ext-algebras. *J. Pure Appl. Algebra* **213**(11), 2017–2037 (2009)
40. Merkulov, S.A.: Strong homotopy algebras of a Kähler manifold. *Int. Math. Res. Notices* **1999**(3), 153–164 (1999)
41. Nakamura, I.: Complex parallelisable manifolds and their small deformations. *J. Differ. Geom.* **10**(1), 85–112 (1975)
42. Neisendorfer, J., Taylor, L.: Dolbeault homotopy theory. *Trans. Amer. Math. Soc.* **245**, 183–210 (1978)
43. Nomizu, K.: On the cohomology of compact homogeneous spaces of nilpotent Lie groups. *Ann. Math.* **59**(3), 531–538 (1954)
44. Oeljeklaus, K., Toma, M.: Non-Kähler compact complex manifolds associated to number fields. *Ann. Inst. Fourier (Grenoble)* **55**(1), 161–171 (2005)
45. Otal, A., Ugarte, L., Villacampa, R.: Invariant solutions to the Strominger system and the heterotic equations of motion. *Nuclear Phys. B* **920**, 442–474 (2017)
46. Ovando, G.: Complex, symplectic and Kähler structures on four dimensional Lie groups. *Rev. Un. Mat. Argentina* **45**(2), 55–67 (2004)
47. Parton, M.: Explicit parallelizations on products of spheres and Calabi-Eckmann structures. *Rend. Istit. Mat. Univ. Trieste* **35**(1–2), 61–67 (2003)
48. Picard, S.: Calabi-Yau Manifolds with Torsion and Geometric Flows, to appear in *Complex non-Kähler Geometry*, Lecture Notes in Mathematics, CIME Subseries, Springer (2019)
49. *SageMath, the Sage Mathematics Software System (Version 7.3)*, The Sage Developers, 2016, <http://www.sagemath.org>
50. Schweitzer, M.: Autour de la cohomologie de Bott-Chern, Prépublication de l’Institut Fourier 703, [arXiv:0709.3528](https://arxiv.org/abs/0709.3528)
51. Stasheff, J.D.: Homotopy associativity of H -spaces. I, II, *Trans. Amer. Math. Soc.* **108**(2), 275–292, 293–312 (1963)
52. Stelzig, J.: On the Structure of Double Complexes, [arXiv:1812.00865](https://arxiv.org/abs/1812.00865)
53. Sternberg, S.: Local contractions and a theorem of Poincaré. *Amer. J. Math.* **79**, 809–824 (1957)
54. Streets, J., Tian, G.: Hermitian curvature flow. *J. Eur. Math. Soc. (JEMS)* **13**(3), 601–634 (2011)
55. Sullivan, D.: Infinitesimal computations in topology. *Inst. Hautes Études Sci. Publ. Math.* **47**(1977), 269–331 (1978)
56. Tardini, N., Tomassini, A.: On geometric Bott-Chern formality and deformations. *Ann. Mat. Pura Appl.* (4) **196**(1), 349–362 (2017)
57. Teleman, A.: Projectively flat surfaces and Bogomolov’s theorem on class VII_0 -surfaces. *Int. J. Math.* **5**, 253–264 (1994)
58. Teleman, A.: The pseudo-effective cone of a non-Kählerian surface and applications. *Math. Ann.* **335**(4), 965–989 (2006)
59. Tomassini, A., Torelli, S.: On Dolbeault formality and small deformations. *Int. J. Math.* **25**(11), 9 (2014). Article ID 1450111
60. Tosatti, V., Weinkove, B.: On the evolution of a Hermitian metric by its Chern-Ricci form. *J. Differ. Geom.* **99**(1), 125–163 (2015)
61. Ustinovskiy, Y.: Hermitian curvature flow on complex homogeneous manifolds, [arXiv:1706.07023](https://arxiv.org/abs/1706.07023)

62. Wehler, J.: Versal deformation of Hopf surfaces. *J. Reine Angew. Math.* **1981**(328), 22–32 (1981)
63. Zhou, J.: Hodge theory and A_∞ -structures on cohomology. *Int. Math. Res. Notices* **2000**(2), 71–78 (2000)

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