Complex Analysis and Operator Theory (2019) 13:3457–3478
https://doi.org/10.1007/s11785-019-00956-5

Complex Analysis and Operator Theory

The Polyanalytic Reproducing Kernels

Hicham Hachadi[1] · El Hassan Youssfi[2]

Received: 7 January 2019 / Accepted: 9 September 2019 / Published online: 16 September 2019

Abstract

Let ν be a rotation invariant Borel probability measure on the complex plane having moments of all orders. Given a positive integer q, it is proved that the space of ν-square integrable q-analytic functions is the closure of q-analytic polynomials, and in particular it is a Hilbert space. We establish a general formula for the corresponding polyanalytic reproducing kernel. New examples are given and all known examples, including those of the analytic case are covered. In particular, weighted Bergman and Fock type spaces of polyanalytic functions are introduced. Our results have a higher dimensional generalization for measure on $\mathbb{C}^p$ which are in rotation invariant with respect to each coordinate.

Keywords Reproducing kernel · Polyanalytic function · Bergman space · Fock space

Mathematics Subject Classification Primary 32A23 · 32A36 · 30A94

1 Introduction

In their recent work Haimi and Hedenmalm [16,17], established asymptotics for the Bergman–Fock type space of polyanalytic functions with respect to a given weight and mentioned that in general finding explicit formula for these kernels is difficult ([17], p. 4668). This problem was also addressed by Alpay ([3], p. 479). The main goal of this paper is to answer this question in the general context of rotation invariant

Communicated by Nikolai Vasilevski.

✉ El Hassan Youssfi
el-hassan.youssfi@univ-amu.fr

Hicham Hachadi
hachadi.hi82@gmail.com

1 Collège Idriss Benzekri, Mehdya, Kenitra, Marocco

2 I2M, U.M.R. C.N.R.S. 7373, CMI, Université d'Aix-Marseille, 39 Rue F-Joliot-Curie, 13453 Marseille Cedex 13, France

Borel probability measure on the complex plane having moments of all orders. More precisely, given a positive integer q, we shall establish the formula for the reproducing kernel for Hilbert spaces of square q-analytic functions with respect to a rotation invariant Borel probability measure. The reduction of our formula to the unit disc $\mathbb{D}$ gives an explicit formula for the weighted Bergman spaces of polyanalytic functions on $\mathbb{D}$, which in turn, reduces to the result of Koshelev [20] when the weight is trivial. We point out that the result of Koshelev is proved by a very specific method based on integration by parts which does not work for the weighted case. Other applications are given to provide new results on other Bargmann–Fock type spaces of polyanalytic functions on $\mathbb{C}$ and related projections.

We recall that a function $f(z)$ is called a polyanalytic function of order q (or just q-analytic) in the domain $\Omega \subseteq \mathbb{C}$ if in this domain it satisfies the generalized Cauchy–Riemann equation

$$\frac{\partial^q f}{\partial \bar{z}^q} = 0. \tag{1.1}$$

Polyanalytic functions inherit some of the properties of analytic functions and the simplest case is the so-called bianalytic functions. However, as in the theory of several complex variables, many of the properties break down once we leave the analytic setting. They are naturally related to polyharmonic functions see [7,18] and [25] for further results.

The properties of these functions have been studied by several authors see Balk and Zuev [9], Balk [8] and Dzhuraev [14] and the references therein. It is well known that any q-analytic function in the domain Ω can be uniquely expressed as

$$f(z) = \sum_{j=0}^{q-1} \bar{z}^j \phi_j(z). \tag{1.2}$$

where $\phi_j(z)$ are holomorphic in Ω. This representation was used to study the boundary behavior and integral representation of polyanalytic functions. Hilbert spaces of polyanalytic functions and related projections were considered for the case of the unit disc by Koshelev [20] and later by Vasin [33] and Ramazanov [23] and [24]. In the latter reference a representation of the space of polyanalytic functions as direct sum of orthogonal subspaces is given and applied to rational approximation. The case of the Bargmann–Fock and Bergman spaces of polyanalytic functions was studied by N. L. Vasilevski [29–31], Sánchez-Nungaray and Vasilevski [32], and later by Abreu [1,2] in connection with Gabor and time-frequency analysis. A deep study of the general case of weighted Bargmann–Fock space of polyanalytic functions was considered by A. Haimi and H. Hedenmalm [16,17], where they obtain the asymptotic expansion of the polyanalytic Bergman kernel as well as the asymptotic behavior of the generating kernel and the asymptotic in the bulk for the q-analytic Bergman spaces in the setting of the weights e^{-2mQ} (see [17]). Their approach relies on the study of polyanalytic Ginibre ensembles and appeals to the connection with random normal matrix theory and Landau levels.

Polyanalytic functions of several variables were considered by Avanissian and Traoré [4] and [5]. They are defined in an analogous way. Namely, a function $f(z)$

is called a polyanalytic function of order $q = (q_1, \dots, q_p) \in \mathbb{N}_0^p$ (or just q-analytic) in the domain $\Omega \subset \mathbb{C}^p$ if in this domain it satisfies the generalized Cauchy–Riemann equation

$$\frac{\partial^{q_1 + \dots q_p} f}{\partial \bar{z}_1^{q_1} \dots \partial \bar{z}_p^{q_p}} = 0. \tag{1.3}$$

These functions can be uniquely expressed as

$$f(z) = \sum_{j=(0,\dots,0)}^{(q_1-1,\dots,q_p-1)} \bar{z}^j \phi_j(z) \tag{1.4}$$

where $\phi_j(z)$ are holomorphic in Ω where for $j = (j_1, \dots, j_p), k = (k_1, \dots, k_p) \in \mathbb{N}_0^p$ and $z = (z_1, \dots z_p) \in \mathbb{C}^p$, the inequality $j \leq k$ means that $j_l \leq k_l$ for all $l = 1, \dots, p$ and $z^j := z_1^{j_1} \dots z_p^{j_p}$.

However, few results are available in this case.

2 Statements of the Main Results

In this section we will state the main results in the one dimensional case. The higher dimensional analogs will be stated at the end of the paper.

The setting is the following. We recall that a sequence $s = (s_d)$, $d \in \mathbb{N}_0$, is said to be a Stieltjes moment sequence if it has the form

$$s_d = \int_0^{+\infty} t^d d\mu(t),$$

where μ is a non-negative measure on $[0, +\infty[$, called a representing measure for s. These sequences have been characterized by Stieltjes [27,28] in terms of some positive definiteness conditions. We denote by $\mathcal{S}$ the set of such sequences and if $s \in \mathcal{S}$ we let $\mathcal{M}(s)$ the convex cone of the representing measures of s. It follows from the above integral representation that each $s \in \mathcal{S}$ is either non-vanishing; that is, $s_d > 0$ for all d, or else $s_d = \delta_0$ for all d. We denote by $\mathcal{S}_q^*$ the set of all non-vanishing elements of $\mathcal{S}$ having a representing measure μ with support containing at least q strictly positive elements. Fix an element $s = (s_d) \in \mathcal{S}_q^*$ and let $\mu \in \mathcal{M}(s)$. It is known [12] that the sequence $(s_d^{\frac{1}{2d}})$ converges to limit $R_s \in]0, +\infty]$, where R_s is the supremum over all $t > 0$ such that t is in the support of μ. We denote by $\mathbb{D}_s$ the disc in $\mathbb{C}$ centered at the origin with radius R_s and $\mathbb{D}_s = \mathbb{C}$ when $R_s = +\infty$.

For each pair of non-negative integers (d, n) such that $n \leq q - 1$, let $\mathcal{P}_n(\mu)$ be the subspace of the Hilbert $L^2(x^d d\mu(x))$ consisting of all polynomials with degree at most n furnished with the real inner product

$$\langle f, g \rangle := \int_0^{+\infty} f(x) g(x) x^d d\mu(x), \quad f, g \in \mathcal{P}_n(\mu)$$

and denote by $Q_{d,n} : (0,+\infty)\times(0,+\infty) \to \mathbb{C}$ the corresponding reproducing kernel.

Consider the following function

$$F_{q,s}(\lambda, x, y) := \sum_{d=0}^{+\infty} \lambda^d Q_{d,q-1}(x, y) + \sum_{d=1}^{q-1} \bar{\lambda}^d Q_{d,q-1-d}(x, y). \tag{2.1}$$

where λ is a complex number and $(x, y) \in [0,+\infty[\times[0,+\infty[$. Our first result is the following:

Theorem A *For all fixed non-negative real numbers x and y, the series $\lambda \mapsto F_{q,s}(\lambda, x, y)$ converges uniformly on compact subsets of the disc centered at* 0 *with radius R_s^2.*

Next, let $\mu \in \mathcal{M}(s)$ and ν denote the image measure on $\mathbb{C}$ of $\mu \otimes \sigma$ under the map $(t, \xi) \mapsto \sqrt{t}\xi$ from $[0, +\infty[\times\mathbb{T}$ onto $\mathbb{C}$, where σ is the rotation invariant probability measure on the unit circle $\mathbb{T}$ in $\mathbb{C}$. Then ν is rotation invariant. Conversely, it is known [10] that any rotation invariant Borel probability ν on $\mathbb{C}$ is of this form. Since μ is supported in the interval $[0, +R_s]$, it follows that the support of ν is contained in closure $\overline{\mathbb{D}}_s$ of the open disc $\mathbb{D}_s$.

We consider the Hilbert space $L^2(\nu)$ of square integrable complex-valued functions in $\overline{\mathbb{D}}_s$ with respect to the measure ν. We denote by $\mathcal{A}^2_{\nu,q}$ the space of those q-analytic functions on $\mathbb{D}_s$ which are square integrable with respect to ν. The natural inner product inherited from that of $L^2(\nu)$ turns $\mathcal{A}^2_{\nu,q}$ into a pre-Hilbert space. We are now prepared to state our second main result.

Theorem B *The space $\mathcal{A}^2_{\nu,q}$ is a Hilbert space which coincides with the closure of the q-analytic polynomials in $L^2(\nu)$. Moreover, for each compact set $K \subset \mathbb{D}_s$ we have that*

$$\sup_{z\in K} |f(z)| \le C \parallel f \parallel_{L^2(\nu)}$$

for all q-analytic polynomials $f \in L^2(\nu)$, where

$$C = C(K) := \sup_{z\in K} \sqrt{F_{q,s}(|z|^2, |z|^2, |z|^2)}.$$

Furthermore, the reproducing kernel of $\mathcal{A}^2_{\nu,q}$ is given by

$$K_{\nu,q}(z, w) = F_{q,s}(z\bar{w}, |z|^2, |w|^2), \quad z, w \in \mathbb{D}_s.$$

Remark C When the measure μ has a finite support and the number of points on the support of μ is q, then Theorem B gives a Cauchy type formula for polyanalytic functions. Such an example can be obtained using the Kroutchouk measure and related orthogonal polynomials.

A first application of our results provides the weighted polyanalytic Bergman kernel of the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. More precisely, for $\alpha > -1$, we consider the space $\mathcal{A}^2_{\alpha,q}$ of all square integrable of q-analytic functions with respect to the measure $d\nu_\alpha(z) := (1-|z|^2)^\alpha \frac{dA(z)}{\pi}$, where $dA(z)$ is the Lebesgue measure on $\mathbb{D}$. We will prove the following

Theorem D *The space $\mathcal{A}^2_{\alpha,q}$ is a Hilbert space which coincides with the closure of the q-analytic polynomials in $L^2(\nu_\alpha)$ and its reproducing kernel is given by*

$$K_{\alpha,q}(z,w) = q\frac{(1-\bar{z}w)^{q-1}}{(1-z\bar{w})^{\alpha+q+1}} \sum_{j=0}^{q-1}(-1)^j \binom{q-1}{j}\binom{\alpha+q+j}{\alpha+q-1}\frac{|z-w|^{2j}}{|1-z\bar{w}|^{2j}}.$$

for all $z, w \in \mathbb{D}$.

We point out that when $\alpha = 0$, this result was established by Koshelev [20] by a different method limited to the case $\alpha = 0$, but does not work for $\alpha \neq 0$.

A second application of our results provides the weighted polyanalytic Bergman kernel for the weighted Fock space. Namely, let $\alpha > 0$, and denote by $\mathcal{F}_{\alpha,q}(\mathbb{C})$ the space of all square integrable of q-analytic functions with respect to the measure $d\nu_\alpha(z) := |z|^{2\alpha}e^{-|z|^2}\frac{dA(z)}{\pi}$, where $dA(z)$ is the Lebesgue measure on $\mathbb{C}$. We will establish the following

Theorem E *The space $\mathcal{F}_{\alpha,q}(\mathbb{C})$ is a Hilbert space which coincides with the closure of the q-analytic polynomials in $L^2(\nu_\alpha)$ and its reproducing kernel is given by*

$$\begin{aligned} K_{\alpha,q}(z,w) &= \sum_{k=0}^{q-1} E_{\alpha,k}(z\bar{w}) L^{\alpha}_{q-1-k}\left(|z-w|^2\right) \\ &+ \sum_{r=1}^{q-1}\sum_{d=0}^{r-1}\left(\frac{1}{d!\Gamma(r+\alpha+1)} - \frac{1}{r!\Gamma(d+\alpha+1)}\right)(\bar{z}w)^r(z\bar{w})^d L^{d+\alpha+r+1}_{q-1-r}\left(|z|^2+|w|^2\right) \end{aligned}$$

for all $z, w \in \mathbb{C}$, where L^{α}_{q-1} is the classical weighted Laguerre polynomial of degree $q-1$ and weight α and $E_{\alpha,k}$ is the generalized Mittag-Leffler's function defined by

$$E_{\alpha,k}(z) := \frac{e^z}{k!}\frac{d^k}{dz^k}\left(z^k e^{-z}E_\alpha(z)\right) \tag{2.2}$$

where E_α is the Mittag-Leffler's function defined by

$$E_\alpha(z) = \sum_{d=0}^{+\infty}\frac{z^d}{\Gamma(d+\alpha+1)}. \tag{2.3}$$

We point out that when $\alpha = 0$, then $E_{\alpha,k}(z) = e^z$ and hence

$$K_{0,q}(z,w) = e^{z\bar{w}}L^1_{q-1}\left(|z-w|^2\right).$$

This result was established by Haimi and Hedenmalm [16] using different methods which do not go through for $\alpha \neq 0$. This particular case was also treated in a very recently work by Maximenko and Tellería-Romero [22].

3 Preliminary Results

We collect a few preliminary results from [13] or [19]. Let $s = (s_n)$ be a Stieltjes moment sequence and $\mu \in \mathcal{S}$ be a representing measure of s. We assume in this section that the support of μ has $N(\mu) \geq q$ elements. For each non-negative integers $n \leq N(\mu) - 1$ set

$$D_{\mu,n} := \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ s_n & s_{n+1} & \cdots & s_{2n} \end{vmatrix}$$

and for $x \in \mathbb{C}$, let

$$D_{\mu,n}(x) := \begin{vmatrix} s_0 & s_1 & \dots & s_n \\ s_1 & s_2 & \dots & s_{n+1} \\ \vdots & \vdots & \dots & \vdots \\ s_{n-1} & s_n & \cdots & s_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix}$$

It is well-known that the sequence $(P_{\mu,n})_{n=0}^{N(\mu)-1}$ of orthogonal polynomials with respect to the measure $d\mu(x)$ is given by

$$P_{\mu,n}(x) = \frac{D_{\mu,n}(x)}{\sqrt{D_{\mu,n-1} D_{\mu,n}}}. \tag{3.1}$$

so that the reproducing kernel $Q_{\mu,n}$ is given by

$$Q_{\mu,n}(x, y) = \sum_{j=0}^{n} \frac{D_{\mu,j}(x) D_{\mu,j}(y)}{D_{\mu,j-1} D_{\mu,j}}. \tag{3.2}$$

We recall the following classical theorem of Heine a proof of which can be found in [13]

Lemma 3.1 *The determinants $D_{\mu,n}$ and $D_{\mu,n}(x)$ have the integral representations*

$$D_{\mu,n} = \frac{1}{(n+1)!} \int_{[0,+\infty[^{n+1}} \prod_{1 \leq j < k \leq n+1} (x_j - x_k)^2 d\mu(x_1) \dots d\mu(x_{n+1}) \tag{3.3}$$

$$D_{\mu,n}(x)=\frac{1}{n!}\int_{[0,+\infty[^n}\prod_{i=1}^{n}(x-x_i)\prod_{1\leq j<k\leq n}(x_j-x_k)^2d\mu(x_1)\dots d\mu(x_n). \tag{3.4}$$

In what follows we shall fix the measure μ, and for each $d\in\mathbb{N}_0$, we consider determinants and orthogonal polynomials with respect to the measure $x^d d\mu(x)$. Then we simply set

$$D_{x^d\mu(x),n}:=D_{d,n}\ \text{ and }\ D_{x^d\mu(x),n}(x):=D_{d,n}(x). \tag{3.5}$$

The sequence of orthogonal polynomials with respect to the measure $x^d d\mu(x)$ will be then denoted by $(P_{d,m})_{m=0}^{N(\mu)-1}$ and it is given by

$$P_{d,n}(x)=\frac{D_{d,n}(x)}{\sqrt{D_{d,n-1}D_{d,n}}}, \tag{3.6}$$

so that the corresponding reproducing kernel $Q_{d,n}$ is given by

$$Q_{d,n}(x,y)=\sum_{j=0}^{n}\frac{D_{d,j}(x)D_{d,j}(y)}{D_{d,j-1}D_{d,j}}. \tag{3.7}$$

Lemma 3.2 *Suppose that the support of μ is unbounded. Then for any positive integer n and $x\in[0,+\infty[$, there exist $t_x>0$ and a constant $C_x>0$ such that*

$$|P_{d,n}(x)|\mu\left([t_x,+\infty[\right)\leq\tfrac{C_x}{t^{d/2}} \tag{3.8}$$

for all $t\geq t_x$.

Proof In view of Lemma 3.1 by Cauchy–Schwarz inequality we see that

$$\left|D_{d,n}(x)\right|^2\leq D_{d,n-1}\int_{[0,+\infty[^n}\prod_{i=1}^{n}(x-x_i)^2\prod_{1\leq j<k\leq n}(x_j-x_k)^2(x_1\dots x_n)^d d\mu(x_1)\dots d\mu(x_n).$$

Since the degree n of the polynomial $D_{d,n}(x)$ is positive, there is $t_x>0$ such that

$$|D_{d,n}(x)|\leq|D_{d,n}(t)|,\quad \text{for all } t\geq t_x$$

and thus by Lemma 3.1 we get

$$\begin{aligned}\frac{\left|D_{d,n}(x)\right|^2}{D_{d,n-1}}\int_{t_x}^{+\infty}x_{n+1}^d d\mu(x_{n+1}) &\leq \int_{t_x}^{+\infty}\left|D_{d,n}(x_{n+1})\right|^2x_{n+1}^d d\mu(x_{n+1})\\ &\leq (n+1)!D_{d,n}\end{aligned}$$

Taking $C=\sqrt{(n+1)!}$ and using (3.6) completes the proof. □

4 Orthogonal Polynomials with Respect to Rotation Invariant Measures

Throughout this section, fix an element $s = (s_d) \in \mathcal{S}_q^*$ and let $\mu \in \mathcal{M}(s)$. For each pair of non-negative integers (d, n), with d arbitrary and $n \leq q-1$, let $(P_{d,k}), k \in \{0, \dots, n\}$ be a sequence of orthonormal polynomials of Hilbert space $\mathcal{P}_n(x^d \mu)$ equipped with the $L^2(x^d d\mu(x))$ inner product. For all integers $m, n \in \mathbb{N}_0$, set

$$m \wedge n := \min(m, n). \tag{4.1}$$

and for all $z = r\xi \in \mathbb{C}, r \geq 0, |\xi| = 1,$

$$H_{m,n}(z) := r^{|m-n|} \xi^m \bar{\xi}^n P_{|m-n|, m\wedge n}(r^2). \tag{4.2}$$

Lemma 4.1 *The family $(H_{m,n})$ forms an orthogonal system in $L^2(\nu)$.*

Proof Let $(m, n), (m', n') \in \mathbb{N}_0^2$. We first observe that for $m + n' = m' + n$, we have

$$\int_0^{+\infty} H_{m,n}(r^{1/2}) \overline{H_{m',n'}(r^{1/2})} d\mu(r) = \int_0^{+\infty} r^{|m-n|} P_{|m-n|, m\wedge n}(r) P_{|m-n|, m'\wedge n'}(r) d\mu(r)$$
$$= \delta_{m\wedge n, m'\wedge n'}.$$

By the change of variables formula, we see that

$$\begin{aligned} \int_{\mathbb{D}_s} H_{m,n}(z) \overline{H_{m',n'}(z)} d\nu(z) &= \int_0^{+\infty} \int_{\mathbb{T}} H_{m,n}(r^{1/2}\xi) \overline{H_{m',n'}(r^{1/2}\xi)} d\sigma(\xi) d\mu(r) \\ &= \int_0^{+\infty} H_{m,n}(r^{1/2}) \overline{H_{m',n'}(r^{1/2})} d\mu(r) \int_{\mathbb{T}} \xi^{m+n'} \overline{\xi^{m'+n}} d\sigma(\xi) \\ &= \delta_{m+n', m'+n} \delta_{m\wedge n, m'\wedge n'} \\ &= \delta_{(m,n),(m',n')}. \end{aligned}$$

This completes the proof. □

Lemma 4.2 *Let n and d be positive integers such that $n \leq q-1$. Consider a polynomial f in n-variables such that $f(x_1, \dots, x_n) > 0$ for all pairwise distinct elements $x_1, \dots, x_n$ of $[0, +\infty[$, and set*

$$\gamma_{d,n}(f) := \int_0^{+\infty} \cdots \int_0^{+\infty} f(x_1, \dots, x_n)(x_1 \dots x_n)^d d\mu(x_1) \dots d\mu(x_n). \tag{4.3}$$

Then

$$\lim_{d\to+\infty} \frac{\gamma_{d+1,n}(f)}{\gamma_{d,n}(f)} = \left(\lim_{d\to+\infty} \frac{s_{d+1}}{s_d} \right)^n \tag{4.4}$$

Proof Let η be the image of the measure on $[0, +\infty[$ of $f(x_1, \ldots, x_n)d\mu(x_1)\ldots d\mu(x_n)$ under the map $(x_1, \ldots, x_n) \mapsto x_1 \ldots x_n$. Then

$$\gamma_{d,n}(f) := \int_0^{+\infty} x^d d\eta(x) \tag{4.5}$$

so that by [12] we see that

$$\lim_{d\to+\infty} \frac{\gamma_{d+1,n}(f)}{\gamma_{d,n}(f)} = R, \tag{4.6}$$

where R is the supremum over all $t > 0$ such that t is in the support of η. Moreover, it can be easily checked that if $t > 0$, then t is in the support of η if and only if $t^{\frac{1}{n}}$ is in the support of μ. This completes the proof. □

Now we can prove Theorem A.

Proof of Theorem A We only need to prove that for all non-negative real numbers x and y, the series

$$S_{s,q}(\lambda) = \sum_{n=1}^{q-1} \sum_{m=0}^{+\infty} \lambda^m P_{m,n}(x)\overline{P_{m,n}(y)}, \tag{4.7}$$

converges uniformly on compact sets of $\mathbb{D}_s$. We shall distinguish two cases. First, assume that the support of μ is bounded; that is R_s is finite. In view (3.6), the latter series can be written in the form

$$S_{s,q}(\lambda) := \sum_{n=1}^{q-1} \sum_{m=0}^{+\infty} \lambda^m \frac{D_{m,n}(x)D_{m,n}(y)}{D_{m,n-1}D_{m,n}}.$$

Using the integral expressions (3.3) and (3.4) with respect to the measure $r^m d\mu(r)$ instead of $d\mu(r)$, we see that $D_{m,n}(x)$ is a finite sum of terms of the form $x^j \gamma_{m,n}(f)$ where $j \in \mathbb{N}_0$ and f is a function of the form

$$f(x) = x_1^{k_1} \cdots x_n^{k_n} \prod_{1\le j<k\le n} (x_j - x_k)^2. \tag{4.8}$$

The same holds for $D_{m,n}(y)$ with y instead of x. Finally, we observe that

$$D_{m,n} = \gamma_{m,n}(g), \quad \text{and} \quad D_{m,n-1} = \gamma_{m,n-1}(g)$$

where

$$g(x) := \prod_{1\le j<k\le n} (x_j - x_k)^2. \tag{4.9}$$

Therefore the series $S_{s,q}$ is a linear combination of series of the form

$$S_{s,q,j,l}(\lambda) := \sum_{n=1}^{q-1}\sum_{m=0}^{+\infty} \lambda^m x^j y^l \frac{\gamma_{m,n}(f)\gamma_{m,n}(h)}{\gamma_{m,n}(g)\gamma_{m,n-1}(g)}$$

where f and h are of the form (4.8) and g is given by (4.9). Appealing to Lemma 4.2 and using D'Alembert's rule yields that the series $S_{s,q,j,l}(\lambda)$ converges as long as $|\lambda| < R_s^2$. From this it is also clear that the series converges uniformly on compact sets of $\mathbb{D}_s$.

Next, suppose that $R_s = +\infty$. Let x, y be arbitrary non-negative real numbers. Then by Lemma 3.2, there is $t_{x,y}$ such that

$$\left|P_{m,n}(x)P_{m,n}(y)\right| \le \frac{(n+1)!}{t^m},$$

for all $t \ge t_{x,y}$. This proves that the series (4.7) convergence absolutely. This completes the proof. □

Next, we denote by $\mathcal{A}^2(s)$ the subspace of $L^2(\nu)$ consisting of all functions of the form

$$f(z) = \sum_{n=0}^{q-1}\sum_{m=0}^{+\infty} a_{m,n} H_{m,n}(z)$$

on $\mathbb{D}_s$ that satisfy

$$\sum_{n=1}^{q-1}\sum_{m=0}^{+\infty} \left|a_{m,n}\right|^2 < +\infty.$$

We equip the space $\mathcal{A}^2(s)$ with the natural inner product

$$\langle f, g\rangle_s := \sum_{n=0}^{q-1}\sum_{m=0}^{+\infty} a_{m,n}\overline{b_{m,n}}, \tag{4.10}$$

for all members $f(z) = \sum_{n=0}^{q-1}\sum_{m=0}^{+\infty} a_{m,n}H_{m,n}(z)$ and $g(z) = \sum_{n=0}^{q-1}\sum_{m=0}^{+\infty} b_{m,n}$ $H_{m,n}(z)$ of $\mathcal{A}^2(s)$. It is standard that this is a Hilbert space which contains all q-analytic polynomials, which is contained in $L^2(\nu)$ and its inner product coincides with the scalar product inherited from the scalar product of $L^2(\nu)$. Indeed, we have

Theorem 4.3 *The space $\mathcal{A}^2(s)$ consists of q-analytic functions and its reproducing kernel $K_{s,q}$ is given by*

$$K_{s,q}(z, w) = F_{s,q}(z\bar{w}, |z|^2, |w|^2),\ z, w \in \mathbb{D}_s,$$

where $F_{s,q}$ is the function defined by (2.1).

Proof By virtue of Theorem A, the series

$$K_{s,q}(z,w)=\sum_{n=0}^{q-1}\sum_{m=0}^{+\infty}H_{m,n}(z)\overline{H_{m,n}(w)}. \tag{4.11}$$

converges uniformly for $z\bar{w}$ lying in a compact subset of $\mathbb{D}_s$. Since the system $(H_{m,n}), m,n\in\mathbb{N}_0, n\le q-1$ forms an orthonormal basis of $\mathcal{A}^2(s)$, a little computing shows that

$$\begin{aligned}K_{s,q}(z,w)&=\sum_{n=0}^{q-1}\sum_{m=0}^{+\infty}H_{m,n}(z)\overline{H_{m,n}(w)}\\&=\sum_{n=0}^{q-1}\sum_{m=0}^{+\infty}(z\bar{w})^m P_{m,n}(|z|^2)P_{m,n}(|w|^2)\\&\quad+\sum_{n=1}^{q-1}\sum_{m=0}^{n-1}(\bar{z}w)^m P_{n-m,m}(|z|^2)P_{n-m,m}(|w|^2).\end{aligned}$$

When $q=1$, we are in the analytic case. Since $P_{m,0}$ is constant, the latter sum gives

$$K_{s,1}(z,w)=\sum_{m=0}^{+\infty}(z\bar{w})^m P_{m,0}P_{m,0}.$$

However, , when $q\ge 2$, we have

$$\begin{aligned}K_{s,q}(z,w)&=\sum_{m=0}^{+\infty}(z\bar{w})^m\sum_{n=0}^{q-1}P_{m,n}(|z|^2)P_{m,n}(|w|^2)\\&\quad+\sum_{n=1}^{q-1}\sum_{m=0}^{n-1}(\bar{z}w)^{n-m}P_{n-m,m}(|z|^2)P_{n-m,m}(|w|^2)\\&=F_{s,q}(z\bar{w},|z|^2,|w|^2).\end{aligned}$$

Now each element f of $\mathcal{A}^2(s)$ admits a unique representation

$$f(z)=\sum_{n=0}^{q-1}\sum_{m=0}^{+\infty}a_{m,n}H_{m,n}(w).$$

By Cauchy–Schwarz inequality, it follows that this series converges uniformly on compact sets of $\mathbb{D}_s$ and hence it defines a q-analytic function. Moreover, it can be easily checked that

$$f(z)=\langle f,K_{s,q}(\cdot,z)\rangle.$$

for all $z \in \mathbb{D}_s$. This completes the proof. □

Now we are ready to prove Theorem B.

Proof of Theorem B It suffices to show that each q-analytic function which belongs to $L^2(\nu)$ is an element of the space $\mathcal{A}^2(s)$. Now let f be a q-analytic function which belongs to $L^2(\nu)$. By (1.4) we know that f has a unique representation of the form

$$f(z) = \sum_{n=0}^{q-1} \overline{z}^n f_n(z), \ z \in \mathbb{D}_s,$$

where the functions f_n are analytic on $\mathbb{D}_s$. Therefore, f can be written in the form

$$f(z) = \sum_{n=0}^{q-1} \sum_{m=0}^{+\infty} \overline{z}^n f_m(z), \ z \in \mathbb{D}_s,$$

where f_m are analytic polynomials and the series converges uniformly on compact sets of $\mathbb{D}_s$. In view of Theorem 4.3, we see that f admits a unique representation of the form

$$f(z) = \sum_{n=0}^{q-1} \sum_{m=0}^{+\infty} c_{m,n} H_{m,n}, \ z \in \mathbb{D}_s,$$

where $c_{m,n}$ are complex coefficients and the series converges uniformly on compact sets of $\mathbb{D}_s$. Since f is in $L^2(\nu)$ it follows that

$$\sum_{n=0}^{q-1} \sum_{m=0}^{+\infty} |c_{m,n}|^2 < +\infty,$$

showing that $f \in \mathcal{A}^2(s)$.

Finally, the inequality in Theorem B follows by Cauchy–Schwarz inequality. The remaining equality in the theorem is straightforward. The proof is now complete. □

5 The Polyanalytic Bergman Space on the Unit Disc

In this section we apply our approach to different classes of orthogonal polynomials to give natural examples of Hilbert spaces of polyanalytic functions.

We start with the weighted polyanalytic Bergman space on $\mathbb{D}$. Consider the weighted Lebesgue measure on $\mathbb{D}$ given by

$$dA_\alpha(z) := \left(1 - |z|^2\right)^\alpha \frac{dA(z)}{\pi}, \ \ \alpha > -1,$$

where $dA(z)$ is the Lebesgue measure on $\mathbb{D}$. We denote by $\mathcal{A}_q^\alpha(\mathbb{D})$, the weighted q-polyanalytic Bergman space on $\mathbb{D}$ where $q \in \mathbb{N}_0$ and $\alpha > -1$. This is the space of all q-polyanalytic functions f on $\mathbb{D}$ which are square integrable with respect to $dA_\alpha(z)$.

It can be easily checked that the measure ν is the image measure in $\mathbb{D}$ of $\mu \otimes \sigma$ under the map $(t, \xi) \mapsto \sqrt{t}\xi$ from $[0, 1[\times\mathbb{T}$ onto $\mathbb{D}$ where μ is the measure $[0, 1[$ given by

$$d\mu(t) := (1-t)^\alpha \, dt.$$

The corresponding moment sequence is

$$s_d = \int_0^1 t^d \, (1-t)^\alpha \, dt = \frac{\Gamma\,(d+1)\,\Gamma\,(\alpha+1)}{\Gamma\,(d+\alpha+2)}. \tag{5.1}$$

Lemma 5.1 *Suppose that φ is an automorphism of the unit disc. The Bergman kernel $K_{q,\alpha}$ of $\mathcal{A}_q^\alpha(\mathbb{D})$ follows the transformation rule*

$$K_{q,\alpha}(z, \xi) = \frac{\left(\varphi'(z)\overline{\varphi'(\xi)}\right)^{(\alpha+q+1)/2}}{\left(\overline{\varphi'(z)}\varphi'(\xi)\right)^{(q-1)/2}} K_{q,\alpha}(\varphi(z), \varphi(\xi)) \tag{5.2}$$

for all $z, \xi \in \mathbb{D}$.

Proof It is sufficient to assume that $\varphi \circ \varphi(z) = z$, for all $z \in \mathbb{D}$. We recall that the measure $\frac{dA(z)}{(1-|z|^2)^2}$ is invariant under the action of the automorphism group of the unit disc. We also observe that for any fixed $\xi \in \mathbb{D}$, the function $z \mapsto \frac{(\varphi')(z))^{(\alpha+q+1)/2}}{(\overline{\varphi'(z)})^{(q-1)/2}} K_{q,\alpha}(\varphi(z), \xi)$ is an element of $\mathcal{A}_q^\alpha(\mathbb{D})$. By the reproducing property and change of variables formula we see that

$$\begin{aligned}
\frac{\left(\varphi'(z)\right)^{(\alpha+q+1)/2}}{\left(\overline{\varphi'(z)}\right)^{(q-1)/2}} K_{q,\alpha}(\varphi(z), \xi) &= \int_{\mathbb{D}} \frac{\left(\varphi'(w)\right)^{(\alpha+q+1)/2}}{\left(\overline{\varphi'(w)}\right)^{(q-1)/2}} K_{q,\alpha}(\varphi(w), \xi) K_{q,\alpha}(z, w) dA_\alpha(w) \\
&= \int_{\mathbb{D}} \frac{\left(\overline{\varphi'(w)}\right)^{(\alpha+q+1)/2}}{(\varphi'(w))^{(q-1)/2}} K_{q,\alpha}(w, \xi) K_{q,\alpha}(z, \varphi(w)) dA_\alpha(w) \\
&= \overline{\int_{\mathbb{D}} \frac{(\varphi'(w))^{(\alpha+q+1)/2}}{\left(\overline{\varphi'(w)}\right)^{(q-1)/2}} K_{q,\alpha}(\xi, w) K_{q,\alpha}(\varphi(w), z) dA_\alpha(w)} \\
&= \frac{\left(\overline{\varphi'(\xi)}\right)^{(\alpha+q+1)/2}}{(\varphi'(\xi))^{(q-1)/2}} K_{q,\alpha}(z, \varphi(\xi)).
\end{aligned}$$

Replacing ξ by $\varphi(\xi)$ the latter equalities yield

$$\frac{\left(\varphi'(z)\overline{\varphi'(\xi)}\right)^{(\alpha+q+1)/2}}{\left(\overline{\varphi'(z)}\varphi'(\xi)\right)^{(q-1)/2}} K_{q,\alpha}(\varphi(z),\varphi(\xi)) = K_{q,\alpha}(z,\xi).$$

This completes the proof. □

We shall make use of the classical Jacobi polynomials $P_n^{(\alpha,d)}$ with parameters (α, d) and degree n. An explicit formula for these polynomials is given by

$$P_n^{(\alpha,d)}(x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{\alpha+n}{k}\binom{d+n}{n-k}(x-1)^{n-k}(x+1)^k. \tag{5.3}$$

It is well-known by formula (3.96) in ([26], p. 71) that these polynomials verify the equality

$$P_n^{(\alpha,d)}(1-2x) = \frac{\Gamma(n+\alpha+1)}{n!\Gamma(n+\alpha+d+1)}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\frac{\Gamma(n+j+\alpha+d+1)}{\Gamma(j+\alpha+1)}x^j. \tag{5.4}$$

The Jacobi polynomials satisfy the orthogonality condition

$$\int_0^1 P_n^{(\alpha,d)}(2x-1)P_{n'}^{(\alpha,d)}(2x-1)x^d(1-x)^\alpha dx = \delta_{n,n'}h_n^{\alpha,d} \tag{5.5}$$

where

$$h_n^{\alpha,d} := \frac{\Gamma\left(\alpha+n+1\right))\,\Gamma\left(d+n+1\right)}{n!\Gamma\left(\alpha+d+n+1\right)\left(\alpha+d+2n+1\right)}. \tag{5.6}$$

and hence for each non-negative integer d, the reproducing kernel of the space of polynomials of degree at most $q-1$ with respect to the L^2-inner product associated to the measure $t^d d\mu(t)$ is then

$$\begin{aligned} Q_{d,q-1}(x,y) &= \sum_{n=0}^{q-1}\frac{P_n^{\alpha,d}(2x-1)P_n^{\alpha,d}(2y-1)}{h_n^{\alpha,d}} \\ &= \sum_{n=0}^{q-1}\frac{P_n^{d,\alpha}(1-2x)P_n^{d,\alpha}(1-2y)}{h_n^{\alpha,d}}. \end{aligned}$$

By the identity (3.114) in ([26], p. 75) we see that

$$Q_{0,q-1}(x,0) = (q+\alpha)P_{q-1}^{1,\alpha}(1-2x) \tag{5.7}$$

so that by (5.4) we obtain

$$\begin{aligned} F_{q,s}(0,x,0) &= Q_{0,q-1}(x,0) \\ &= q\sum_{j=0}^{q-1}(-1)^j\binom{q-1}{j}\binom{\alpha+q+j}{\alpha+q-1}x^j \end{aligned}$$

We observe that if $z\in\mathbb{D}$, then $K_{q,\alpha}(z,0)=F_{q,s}(0,|z|^2,0)$. For $z,w\in\mathbb{D}$ let

$$\varphi_w(z):=\frac{z-w}{1-z\bar{w}}.$$

By Lemma 5.1, we have

$$K_{q,\alpha}(z,w)=\frac{\left(\varphi_w'(z)\overline{\varphi_w'(w)}\right)^{(\alpha+q+1)/2}}{\left(\overline{\varphi_w'(z)}\varphi_w'(w)\right)^{(q-1)/2}}K_{q,\alpha}(\varphi_w(z),0).$$

Since

$$\frac{\left(\varphi_w'(z)\overline{\varphi_w'(w)}\right)^{(\alpha+q+1)/2}}{\left(\overline{\varphi_w'(z)}\varphi_w'(w)\right)^{(q-1)/2}}=\frac{(1-\bar{z}w)^{q-1}}{(1-z\bar{w})^{\alpha+q+1}}$$

and

$$|\varphi_w(z)|^{2j}=\frac{[z-w|^{2j}}{|1-z\bar{w}|^{2j}}$$

it follows that

$$K_{q,\alpha}(z,w)=q\frac{(1-\bar{z}w)^{q-1}}{(1-z\bar{w})^{\alpha+q+1}}\sum_{j=0}^{q-1}(-1)^j\binom{q-1}{j}\binom{\alpha+q+j}{\alpha+q-1}\frac{|z-w|^{2j}}{|1-z\bar{w}|^{2j}}.$$

6 Weighted Polyanalytic Fock Spaces

The second example is the weighted measure defined on $\mathbb{C}$ by

$$d\nu(z):=|z|^{2\alpha}e^{-|z|^2}dA(z),\quad \alpha>-1,$$

where $dA(z)$ is the normalized Lebesgue measure on $\mathbb{C}$. We denote by $\mathcal{A}_q^\alpha(\mathbb{C})$ the weighted q-polyanalytic Fock space on $\mathbb{C}$ where q is a positive integer. This is the space of all q-analytic functions f on $\mathbb{C}$ which are square integrable with respect to

$d\nu(z)$. The measure ν is the image measure in $\mathbb{C}$ of $\mu\otimes\sigma$ under the map $(t,\xi)\mapsto t^{1/2}\xi$ from $[0,+\infty[\times\mathbb{T}$ onto $\mathbb{C}$ where μ is the measure on $[0,+\infty[$ given by

$$d\mu(t):=\frac{1}{\Gamma(\alpha+1)}t^{\alpha}e^{-t}dt.$$

The corresponding moment sequence is

$$s_d=\int_0^{+\infty}t^d d\mu(t)=\frac{\Gamma(\alpha+d+1)}{\Gamma(\alpha+1)}. \tag{6.1}$$

We will use the classical weighted Laguerre polynomials L_n^{α} of degree n and weight α. These polynomials satisfy,

$$\int_0^{+\infty}L_n^{d+\alpha}(x)\,L_{n'}^{d+\alpha}(x)\,x^{d+\alpha}e^{-x}dx=\frac{\Gamma(d+\alpha+n+1)}{n!}\delta_{n,n'}. \tag{6.2}$$

They have the following explicit representation

$$L_n^{d+\alpha}(x)=\sum_{l=0}^{n}\frac{(n+d+\alpha)!}{r!\Gamma(n+d+\alpha+1-r)}\frac{(-x)^{n-r}}{(n-r)!}. \tag{6.3}$$

Laguerre polynomials enjoy the following product formula due to Bailey [6]

$$L_n^{d+\alpha}(x)\,L_n^{d+\alpha}(y)=\frac{\Gamma(d+\alpha+n+1)}{n!}\sum_{l=0}^{n}\frac{(xy)^{n-l}\,L_l^{d+\alpha+2n-2l}(x+y)}{(n-l)!\Gamma(d+\alpha+n+1-l)}. \tag{6.4}$$

and the following transfer equalities due to [15]

$$L_n^{\beta}(x-y)=\sum_{r=0}^{n}\frac{y^r}{r!}L_{n-r}^{\beta+r}(x)=e^{-y}\sum_{r=0}^{+\infty}\frac{y^r}{r!}L_n^{\beta+r}(x). \tag{6.5}$$

We recall some other useful formulas

$$L_n^{\alpha+\beta+1}(x+y)=\sum_{k=0}^{n}L_{n-k}^{\alpha}(x)L_k^{\beta}(y). \tag{6.6}$$

In particular, if $y=\beta=0$, then

$$L_n^{\alpha+1}=\sum_{j=0}^{n}L_j^{\alpha}. \tag{6.7}$$

which can be found in ([19], p. 104).

Finally, a little computing shows that the generalized -Mittag-Leffler's function $E_{\alpha,k}$ defined by (2.2) has the explicit form

$$E_{\alpha,k}(z) = \sum_{d=0}^{+\infty} \frac{z^d L_k^d(z)}{\Gamma(d+\alpha+1)} \tag{6.8}$$

for all $z \in \mathbb{C}$.

Next, we prove Theorem E. For $z, w \in \mathbb{C}$, we shall set $\lambda := z\bar{w}, x =: |z|^2$ and $y := |w|^2$ so that

$$\frac{xy}{\lambda} := \bar{\lambda} \quad \text{and} \quad \frac{xy}{\bar{\lambda}} = \lambda. \tag{6.9}$$

To compute the series $F_{q,s}(\lambda, x, y)$ in this case, it is sufficient to calculate the following expressions

$$S_{\alpha,q}(\lambda) = \sum_{n=0}^{q-1} \sum_{d=0}^{+\infty} \frac{n!\lambda^d}{\Gamma(n+d+\alpha+1)} L_n^{d+\alpha}(x) L_n^{d+\alpha}(y).$$

$$S'_{\alpha,q}(\lambda) = \sum_{d=1}^{q-1} \sum_{n=0}^{q-1-d} \frac{n!\lambda^d}{\Gamma(n+d+\alpha+1)} L_n^{d+\alpha}(x) L_n^{d+\alpha}(y), \quad \text{when } q \geq 2.$$

Using first (6.4) and then (6.5) and (6.6) we have

$$\begin{aligned}
S_{\alpha,q}(\lambda) &= \sum_{n=0}^{q-1} n! \sum_{d=0}^{+\infty} \frac{\lambda^d L_n^{d+\alpha}(x) L_n^{d+\alpha}(y)}{\Gamma(d+\alpha+n+1)} \\
&= \sum_{n=0}^{q-1} \sum_{d=0}^{+\infty} \sum_{r=0}^{n} \frac{\lambda^d (xy)^r}{r!\Gamma(d+\alpha+r+1)} L_{n-r}^{d+\alpha+2r}(x+y) \\
&= \sum_{r=0}^{q-1} \sum_{d=0}^{+\infty} \frac{\lambda^d (xy)^r}{r!\Gamma(d+\alpha+r+1)} \sum_{n=r}^{q-1} L_{n-r}^{d+\alpha+2r}(x+y) \\
&= \sum_{r=0}^{q-1} \sum_{d=0}^{+\infty} \frac{\lambda^d (xy)^r}{r!\Gamma(d+\alpha+r+1)} \frac{d^r}{dx^r} \left(\sum_{n=0}^{q-1} L_n^{d+\alpha+r}(x+y) \right) \\
&= \sum_{r=0}^{q-1} \sum_{d=0}^{+\infty} \frac{\lambda^d (xy)^r}{r!\Gamma(d+\alpha+r+1)} L_{q-1-r}^{d+\alpha+1+2r}(x+y) \\
&= \sum_{r=0}^{q-1} \sum_{d=r}^{+\infty} \frac{\lambda^d \bar{\lambda}^r}{r!\Gamma(d+\alpha+1)} L_{q-1-r}^{d+\alpha+1+r}(x+y)
\end{aligned}$$

$$= \sum_{r=0}^{q-1} \sum_{d=0}^{+\infty} \frac{\lambda^d \bar{\lambda}^r}{r! \Gamma(d+\alpha+1)} L_{q-1-r}^{d+\alpha+1+r}(x+y)$$
$$- \sum_{r=1}^{q-1} \sum_{d=0}^{r-1} \frac{\lambda^d \bar{\lambda}^r}{r! \Gamma(d+\alpha+1)} L_{q-1-r}^{d+\alpha+1+r}(x+y).$$

Applying (6.5), (6.6) and (6.8) yields

$$\sum_{r=0}^{q-1} \sum_{d=0}^{+\infty} \frac{\lambda^d \bar{\lambda}^r}{r! \Gamma(d+\alpha+1)} L_{q-1-r}^{d+\alpha+1+r}(x+y) = \sum_{d=0}^{+\infty} \frac{\lambda^d}{\Gamma(d+\alpha+1)} L_{q-1}^{d+\alpha+1}\left(x+y-\bar{\lambda}\right)$$
$$= \sum_{k=0}^{q-1} \sum_{d=0}^{+\infty} \frac{\lambda^d L_k^d(\lambda)}{\Gamma(d+\alpha+1)} L_{q-1-k}^{\alpha}\left(x+y-\bar{\lambda}-\lambda\right)$$
$$= \sum_{k=0}^{q-1} E_{\alpha,k}(\lambda) L_{q-1-k}^{\alpha}\left(x+y-\bar{\lambda}-\lambda\right).$$

On the other hand, when $q \geq 2$, then using (6.4), (6.5), (6.6) and (6.9), we see that

$$S'_{\alpha,q}(\lambda) = \sum_{d=1}^{q-1} \sum_{n=0}^{q-1-d} \frac{n! \bar{\lambda}^d}{\Gamma(n+d+\alpha+1)} L_n^{d+\alpha}(x) L_n^{d+\alpha}(y)$$
$$= \sum_{d=1}^{q-1} \sum_{n=0}^{q-1-d} \sum_{r=0}^{n} \frac{\bar{\lambda}^{d+r} \lambda^r}{r! \Gamma(d+\alpha+r+1)} L_{n-r}^{d+\alpha+2r}(x+y)$$
$$= \sum_{r=0}^{q-2} \sum_{n=r}^{q-1} \sum_{d=1}^{q-1-n} \frac{\bar{\lambda}^{d+r} \lambda^r}{r! \Gamma(d+\alpha+r+1)} L_{n-r}^{d+\alpha+2r}(x+y)$$
$$= \sum_{r=0}^{q-2} \sum_{d=r+1}^{q-1} \sum_{n=r}^{q-1+r-d} \frac{\bar{\lambda}^{d} \lambda^r}{r! \Gamma(d+\alpha+1)} L_{n-r}^{d+\alpha+r}(x+y)$$
$$= \sum_{r=0}^{q-2} \sum_{d=r+1}^{q-1} \sum_{n=0}^{q-1-d} \frac{\bar{\lambda}^{d} \lambda^r}{r! \Gamma(d+\alpha+1)} L_{n}^{d+\alpha+r}(x+y)$$
$$= \sum_{r=0}^{q-2} \sum_{d=r+1}^{q-1} \frac{\bar{\lambda}^{d} \lambda^r}{r! \Gamma(d+\alpha+1)} L_{q-1-d}^{d+\alpha+r+1}(x+y)$$
$$= \sum_{d=1}^{q-1} \sum_{r=0}^{d-1} \frac{\bar{\lambda}^{d} \lambda^r}{r! \Gamma(d+\alpha+1)} L_{q-1-d}^{d+\alpha+r+1}(x+y)$$
$$= \sum_{r=1}^{q-1} \sum_{d=0}^{r-1} \frac{\bar{\lambda}^{r} \lambda^d}{d! \Gamma(r+\alpha+1)} L_{q-1-r}^{d+\alpha+r+1}(x+y).$$

Hence

$$\begin{aligned} F_{q,s}(\lambda, x, y) &= S_{\alpha,q}(\lambda) + S'_{\alpha,q}(\lambda) \\ &= \sum_{k=0}^{q-1} E_{\alpha}^{(k)}(\lambda) L_{q-1-k}^{\alpha}\left(x + y - \bar{\lambda} - \lambda\right) \\ &\quad + \sum_{r=1}^{q-1}\sum_{d=0}^{r-1}\left(\frac{\bar{\lambda}^r\lambda^d}{d!\Gamma(r+\alpha+1)} - \frac{\bar{\lambda}^r\lambda^d}{r!\Gamma(d+\alpha+1)}\right) L_{q-1-r}^{d+\alpha+r+1}(x+y). \end{aligned}$$

In particular, when $\alpha = 0$, we have

$$F_{q,s}(\lambda, x, y) = e^{\lambda} L_{q-1}^{1}\left(x + y - \lambda - \frac{xy}{\lambda}\right). \tag{6.10}$$

7 The Higher Dimensional Case

Consider n Stieltjes moment sequences $s(1) \in \mathcal{S}_{q_1}^*, \ldots, s(n) \in \mathcal{S}_{q_n}^*$, where $q_1, \ldots, q_n$ are positive integers and for each j let $\mu_j \in \mathcal{M}(s(j))$ and denote by ν_j denote the image measure on $\mathbb{C}$ of $\mu_j \otimes \sigma$ under the map $(t, \xi) \mapsto \sqrt{t}\xi$ from $[0, +\infty[\times\mathbb{T}$ onto $\mathbb{C}$, where σ is the rotation invariant probability measure on the unit circle $\mathbb{T}$ in $\mathbb{C}$. Then the support of each ν_j is contained in the closure of the disc $\mathbb{D}_j$ centered at 0 with radius $R_{s(j)}$. Then we set $\nu := \nu_1 \otimes \cdots \otimes \nu_n$ and consider the Hilbert space $L^2(\nu)$ of square integrable complex-valued functions in $\overline{\mathbb{D}}_1 \times \cdots \times \overline{\mathbb{D}}_n$ with respect to the measure ν. Let $q = (q_1, \ldots, q_n)$ and denote by $\mathcal{A}_{\nu,q}^2$ the space of those q-analytic functions on $\mathbb{D}_1 \times \cdots \times \mathbb{D}_n$ which are square integrable with respect to ν. The natural inner product inherited from that of $L^2(\nu)$ turns $\mathcal{A}_{\nu,q}^2$ into a pre-Hilbert space. We are now prepared to state the higher dimensional analog of Theorem B.

Theorem B′ *The space $\mathcal{A}_{\nu,q}^2$ is a Hilbert space which coincides with the closure of the q-analytic polynomials in $L^2(\nu)$. Moreover, for each set compact $K \subset \mathbb{D}_s$ we have that*

$$\sup_{z\in K}|f(z)| \le C \parallel f \parallel_{L^2(\nu)}$$

for all q-analytic polynomials $f \in L^2(\nu)$, where

$$C = C(K) := \sup_{z\in K}\prod_{j=1}^{n}\sqrt{F_{q_j,s(j)}(|z_j|^2, |z_j|^2, |z_j|^2)}.$$

Furthermore, the reproducing kernel of $\mathcal{A}_{\nu}^2$ is given by

$$K_{\nu,q}(z, w) = \prod_{j=1}^{n} F_{q_j,s(j)}(z_j\bar{w}_j, |z_j|^2, |w_j|^2), \quad z, w \in \mathbb{D}_1 \times \cdots \times \mathbb{D}_n.$$

Remark C′ As in Remark C, when each of the measures μ_j has a finite support with exacly q_j elements, Theorem B' provides the polyanalytic Cauchy type kernel of the unit polydisc

$$\mathbb{D}^n := \{z = (z_1, \ldots, z_n) \in \mathbb{C} : \max_{j=1,\ldots,p} |z_j| < 1\}.$$

In a similar manner, from Theorem B' we obtain the weighted polyanalytic Bergman kernel of the unit polydisc $\mathbb{D}^n$. More precisely, for $\alpha > -1$, we consider the space $\mathcal{A}^2_{\alpha,n,q}$ of all square integrable q-analytic functions with respect to the measure $d\nu_{\alpha,n}(z) := \frac{1}{\pi^n} \prod_{j=1}^n (1-|z_j|^2)^\alpha dV(z)$ on $\mathbb{D}^n$, where $dV(z)$ is the Lebesgue measure on $\mathbb{C}^n$. We obtain the following

Theorem D′ *The space $\mathcal{A}^2_{\alpha,n,q}$ is a Hilbert space which coincides with the closure of the q-analytic polynomials in $L^2(\nu_\alpha)$ and its reproducing kernel is given by*

$$K_{\alpha,n,q}(z, w) = \prod_{j=1}^n K_{\alpha,q_j}(z_j, w_j)$$

for all $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{D}^n$, where K_{α,q_j} is the weighted polyanalytic Bergman kernel of the unit disc.

We also obtain by similar arguments the weighted polyanalytic Bergman kernel for the weighted Fock space in $\mathbb{C}^n$. Namely, let $\alpha > 0$, and denote by $\mathcal{F}_{\alpha,q}(\mathbb{C}^n)$ the space of all square integrable q-analytic functions with respect to the measure

$$d\nu_\alpha(z) := |z|^{2\alpha} e^{-|z|^2} \frac{dV(z)}{\pi^n}, \alpha > -1$$

where $dV(z)$ is the Lebesgue measure on $\mathbb{C}^n$. We will establish the following

Theorem E′ *The space $\mathcal{F}_{\alpha,q}(\mathbb{C}^n)$ is a Hilbert space which coincides with the closure of the q-analytic polynomials in $L^2(\nu_\alpha)$ and its reproducing kernel is given by*

$$K_{\alpha,n,q}(z, w) = \prod_{j=1}^n K_{\alpha,q_j}(z_j, w_j)$$

for all $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, where $K_{\alpha,q_j}(z_j, w_j)$ is the one reproducing kernel of the one dimensional space $\mathcal{F}_{\alpha,q_j}(\mathbb{C})$. In particular, when $\alpha = 0$, the expression of reproducing kernel $K_{0,n,q}(z, w)$ reduces to

$$K_{0,n,q}(z, w) = e^{\langle z,w\rangle} \prod_{j=1}^n L^1_{q_j-1}(|z_j - w_j|^2),$$

where $\langle z, w\rangle := \sum_{j=1}^n z_j \bar{w}_j$.

Acknowledgements The authors would like to thank the anonymous referees for valuable comments and pointing out a computational error in the proof of Theorem E.

References

1. Abreu, L.D.: Sampling and interpolation in Bargmann–Fock spaces of polyanalytic functions. Appl. Comput. Harmon. Anal. **29**, 287–302 (2010)
2. Abreu, L.D.: On the structure of Gabor and super Gabor spaces. Monatsh Math. **161**, 237–253 (2010)
3. Alpay, D.: An Advanced Complex Analysis Problem Book. Topological Vector Spaces, Functional Analysis, and Hilbert Spaces of Analytic Functions. Birkhauser, Basel (2015)
4. Avanissian, V., Traoré, A.: Sur les fonctions polyanalytiques de plusieurs variables. C.R. Acad. Sci. Paris Sér. **A–B 286**(17), A743–A746 (1978)
5. Avanissian, V., Traoré, A.: Extension des théorèmes de Hartogs et de Lindelöf aux fonctions polyanalytiques de plusieurs variables. C.R. Acad. Sci. Paris Sér. **A–B 291**(4), A263–A265 (1980)
6. Bailey, W.N.: On the product of two Legendre polynomials with different arguments. Proc. Lond. Math. Soc. **s2–41**(1), 215–220 (1936)
7. Balk, M.B.: Polyanalytic functions and their generalizations. Complex analysis I. Encycl. Math. Sci, **85**, 195–253.
8. Balk, M.B., Mazalov, M.Y.: On the Hayman uniqueness problem for polyharmonic functions. In: Clifford Algebras and Their Application in Mathematical Physics, Aachen, Germany, 1996. In: Fundam. Theor. Phys., vol. **94**, Kluwer, Dordrecht, Germany, pp. 11–16 (1998)
9. Balk, M.B., Zuev, M.F.: On polyanalytic functions. Russ. Math. Surv. **25**(5), 201–223 (1970)
10. Berg, C., Thill, M.: Rotation invariant moment problems. Acta Math. **167**, 207–227 (1991)
11. Bommier-Hato, H., Youssfi, E.H.: Hankel operators on weighted Fock spaces. Integr. Equ. Oper. Theory **59**, 1–17 (2007)
12. Bommier-Hato, H., Youssfi, E.H.: Hankel operators and the Stieltjes moment problem. J. Funct. Anal. **258**, 978–998 (2010)
13. Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables. Cambridge Univ. Press, Cambridge (2001)
14. Dzhuraev, A.: Multikernel functions of a domain, kernel operators, singular integral operators. Sov. Math. Dokl. **32**(1), 251–253 (1985)
15. Erdelyi, : Transformation of a certain series of products of confluent hypergeometric functions with applications to Laguerre and Charlier polynomials. Compos. Math. **7**, 340–352 (1939)
16. Haimi, A., Hedenmalm, Haakan: The polyanaytic Ginibre ensembles. J. Stat. Phys. **153**(1), 10–47 (2013)
17. Haimi, A., Hedenmalm, Haakan: Asymptotic expansion of polyanalytic Bergman kernels. J. Func. Anal. **267**(12), 4667–4731 (2014)
18. Hayman, W.K., Korenblum, B.: Representation and uniqueness theorems for polyharmonic functions. J. Anal. Math. **60**, 113–133 (1993)
19. Ismail, M.E.H.: Classical and Quantum Orthogonal Polynomials in One Variable. Cambridge University Press, Cambridge (2005)
20. Koshelev, A.D.: The kernel function of a Hilbert space of functions that are polyanalytic in the disc. Dokl. Akad. Nauk SSSR **232**(2), 277–279 (1977) (English translation: Soviet Math. Dokl. **18** (1977), no. 1, 59–62)
21. Lovera, S., Youssfi, E.H.: Spectral properties of the $\overline{\partial}$-canonical solution operator. J. Funct. Anal. **208**, 360–376 (2004)
22. Maximenko, E.A., Tellería-Romero, A.M.: Radial operators on polyanalytic Bargmann–Segal–Fock spaces. arXiv:1905.00978
23. Ramazanov, A.K.: Representation of the Space of polyanalytic functions as a direct sum of orthogonal subspaces. Application to rational approximations. Math. Not. **66**(5), (1999)
24. Ramazanov, A.K.: On the structure of spaces of polyanalytic functions. Mat. Zametki [Math. Not.] **72**(5), 750–764 (2002)
25. Render, H.: Real Bargmann spaces, Fischer decompositions, and sets of uniqueness for polyharmonic functions. Duke Math. J. **142**(2), 313–352 (2008)
26. Shen, J., Tang, T., Wang, L.: Spectral Methods: Algorithms, Analysis and Applications. Springer, New York (2011)
27. Stieltjes, T.: Recherches sur les fractions continues. Ann.Fac. Sci. Toulouse **8**, 1–122 (1894)
28. Stieltjes, T.: Recherches sur les fractions continues. Ann. Fac. Sci. Toulouse **9**, 5–47 (1895)
29. Vasilevski, N.L.: On the structure of Bergman and poly-Bergman spaces. Integr. Equ. Oper. Theory **33**, 471–488 (1999)

30. Vasilevski, N.L.: Poly-Fock spaces. Oper. Theory Adv. Appl. **117**, 371–386 (2000)
31. Vasilevski, N.: Poly-Bergman spaces and two-dimensional singular integral operators. Oper. Theory Adv. Appl. **171**, 349–359 (2007)
32. Sánchez-Nungaray, A., Vasilevski, N.: Toeplitz operators on the Bergman spaces with pseudodifferential defining symbols. Oper. Adv. Appl. **228**, 355–374 (2013)
33. Vasin, A.V.: Projections onto L^p-spaces of polyanalytic functions. J. Math. Sci. **71**(1), 2180–2191 (1991)
34. Zhu, K.: Operator Theory in Function Spaces. Marcel Dekker, New York (1990)